Integrability of difference Calogero-Moser systems

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A general class of n-particle difference Calogero–Moser systems with elliptic potentials is introduced. Besides the step size and two periods, the Hamiltonian depends on nine coupling constants. We prove the quantum integrability of the model for $n = 2$ and present partial results for $n \geq 3$. In degenerate cases (rational, hyperbolic, or trigonometric limit), the integrability follows for arbitrary particle number from previous work connected with the multivariable q-polynomials of Koornwinder and Macdonald. Liouville integrability of the corresponding classical systems follows as a corollary. Limit transitions lead to various well-known models such as the nonrelativistic Calogero–Moser systems associated with classical root systems and the relativistic Calogero–Moser system.

I. INTRODUCTION

The well-known Calogero–Moser (CM) systems and their generalizations related to root systems form a rich and widely studied class of finite-dimensional models.[1,2] In these systems the particles are one-dimensional and interact by means of inverse square potentials or (doubly) periodic generalizations thereof. For the root system A_{n-1}, which corresponds to an n-particle system with pair potentials, the Hamiltonian with elliptic potential reads

$$H_{CM} = 1/2 \sum_{1 \leq j \leq n} \theta_j^2 + g^2 \sum_{1 \leq j < k \leq n} \wp(x_j - x_k), \tag{1.1}$$

where $\wp(\cdot)$ denotes the Weierstrass $\wp$-function.[3] It has been shown that the Hamiltonian (1.1) is Liouville integrable[1] and that integrability is preserved after quantization.[2]

A relativistic generalization of H_{CM} (1.1) has been introduced by Ruijsenaars e. a.[4,5] The Hamiltonian of the relativistic CM system (henceforth abbreviated RCM) reads

$$H_{RCM} = \sum_{1 \leq j \leq n} \prod_{k \neq j} f(x_j - x_k) \, \mathrm{ch} \, \beta\theta_j, \tag{1.2}$$

with $f(\cdot) = (1 + \beta^2 g^2 \wp(\cdot))^{1/2}$. It was shown that also the RCM system is integrable; in fact, in the relativistic case explicit combinatorial expressions were found for a complete set of integrals in involution. An appropriate quantization of the integrals gives rise to commuting difference operators. This type of operators is referred to in the literature as analytic difference operators (AΔOs) or finite-difference operators (also q-difference operators). The reason AΔOs emerge after quantization instead of PDOs [as for the quantization of H_{CM} (1.1)], is that substituting $\theta_j \mapsto -i\hbar\partial_{x_j}$ in H_{RCM} (1.2) leads to exponentials of the form $\exp(\pm i\beta\hbar\partial_{x_j})$. The parameter β, which determines the step size of the differences, is proportional to the inverse of the speed of light. For $\beta \to 0$, which corresponds to the nonrelativistic limit, $\beta^{-2}(H_{RCM} - n)$ converges to H_{CM}. For more information on RCM systems and interesting connections with soliton PDEs and integrable QFTs the reader is referred to the survey in Ref. 6.

The remaining classical root systems (i.e., B_n, C_n, D_n, and BC_n) also give rise to n-particle CM systems. The latter differ from the A_{n-1} model in the sense that translational invariance is traded for symmetry under the reflections $x_j \mapsto -x_j$, and by the presence of various types of external fields. The precise nature of the external field depends on the type of root system. A Hamiltonian that contains the latter models as special cases was studied by Inozemtsev e. a.:[7,8]

$$H = \frac{1}{2} \sum_{1 \leqslant j \leqslant n} \theta_j^2 + g^2 \sum_{1 \leqslant j < k \leqslant n} (\wp(x_j - x_k) + \wp(x_j + x_k))$$

$$+ \sum_{1 \leqslant j \leqslant n} (g_0^2 \, \wp(x_j) + g_1^2 \, \wp(\omega_1 + x_j) + g_2^2 \, \wp(\omega_2 + x_j) + g_3^2 \, \wp(\omega_1 + \omega_2 + x_j)), \qquad (1.3)$$

where $2\omega_1$ and $2\omega_2$ are the primitive periods of the $\wp$-function. A Lax pair for the flow generated by H (1.3) was presented in Ref. 8. It follows from the Lax pair that the system has n independent conserved quantities, but it has not been shown yet that these are in involution. Nevertheless, it is very plausible that this is indeed the case and, therefore, that the Hamiltonian (1.3) is integrable.

The main purpose of this paper is to present a difference CM Hamiltonian generalizing the quantum version of H (1.3), and study its integrability. The generalized Hamiltonian is given by an AΔO with meromorphic coefficients. These coefficients govern the interaction felt by the particles. The model depends on nine coupling constants. (Actually we shall impose one linear constraint reducing the number of independent coupling constants by one.) By sending the step size to zero one obtains a PDO that is the quantization of H (1.3). It will turn out that also the quantum RCM system can be seen as a special case of the difference CM model considered here.

Let us outline the plan of the paper and describe its results in more detail. We begin in Sec. II with a discussion of certain n-particle difference CM systems with rational, hyperbolic, or trigonometric coefficients (potentials). Just as for the relativistic system, the integrals are given by explicit combinatorial formulas. Sending the step size to zero leads to a complete set of integrals for the quantum CM system associated with the root system BC_n. The material in this section hinges on previous work[9] connected with recently discovered families of multivariable q-polynomials.[10,11]

Section III concentrates on the special case of only two particles. By generalizing the model of Sec. II for $n = 2$, we find a difference CM system with elliptic potentials. Integrability is proved and it is shown that by sending the step size to zero one obtains a quantum integrable system that has the two-particle specialization of H (1.3) as classical version.

In Sec. IV some partial results pertaining to the case of arbitrary particle number have been collected. The two commuting AΔOs with elliptic potentials introduced in Sec. III (viz. the two-particle Hamiltonian and its additional integral) are generalized to $n > 2$. A conjecture is formulated regarding the existence and structure of a complete set of integrals for the resulting n-particle difference CM system. The highest order terms of the conjectured integrals are related to the quantum integrals of the RCM system via a gauge transformation.

The last section of the paper, Sec. V, is devoted to the classical version of our difference CM systems. The quantum integrability of the models of Secs. II and III implies the Liouville integrability of the corresponding classical systems. Furthermore, for special values of the coupling constants the classical counterpart of the n-particle difference CM Hamiltonian introduced in Sec. IV can be seen as a reduction of the $2n(+1)$-particle version of H_{RCM} (1.2). In these special cases, reducing the integrals of the RCM model leads to a complete set of integrals for the classical system; the form of these classical integrals is in agreement with the structure of the quantum integrals conjectured in Sec. IV.

Some technicalities have been relegated to three short appendices.

II. DIFFERENCE CM SYSTEMS WITH RATIONAL, HYPERBOLIC, OR TRIGONOMETRIC POTENTIALS

The Hamiltonian of the system is given by the AΔO

$$\hat{\mathcal{H}} = \sum_{\substack{1 \le j \le n \\ \varepsilon = \pm 1}} w(\varepsilon x_j) \prod_{k \ne j} v(\varepsilon x_j + x_k)\, v(\varepsilon x_j - x_k)(e^{-\varepsilon \beta \hat{\theta}_j} - 1), \tag{2.1}$$

with

I. Rational case

$$v(z) = \frac{(\mu + z)}{z}, \quad w(z) = \frac{(\mu_0 + z)}{z} \frac{(\mu_0' + \gamma + z)}{(\gamma + z)}; \tag{2.2}$$

II. Hyperbolic case

$$v(z) = \frac{\operatorname{sh} \alpha(\mu + z)}{\operatorname{sh}(\alpha z)}, \tag{2.3}$$

$$w(z) = \frac{\operatorname{sh} \alpha(\mu_0 + z)}{\operatorname{sh}(\alpha z)} \frac{\operatorname{ch} \alpha(\mu_1 + z)}{\operatorname{ch}(\alpha z)} \frac{\operatorname{sh} \alpha(\mu_0' + \gamma + z)}{\operatorname{sh} \alpha(\gamma + z)} \frac{\operatorname{ch} \alpha(\mu_1' + \gamma + z)}{\operatorname{ch} \alpha(\gamma + z)}; \tag{2.4}$$

III. Trigonometric case

$$v(z) = \frac{\sin \alpha(\mu + z)}{\sin(\alpha z)}, \tag{2.5}$$

$$w(z) = \frac{\sin \alpha(\mu_0 + z)}{\sin(\alpha z)} \frac{\cos \alpha(\mu_1 + z)}{\cos(\alpha z)} \frac{\sin \alpha(\mu_0' + \gamma + z)}{\sin \alpha(\gamma + z)} \frac{\cos \alpha(\mu_1' + \gamma + z)}{\cos \alpha(\gamma + z)}; \tag{2.6}$$

where $\alpha > 0$ and

$$\hat{\theta}_j \equiv \frac{\hbar}{i} \frac{\partial}{\partial x_j}, \quad \gamma \equiv i\beta\hbar/2. \tag{2.7}$$

The exponentials $\exp(\pm \beta \hat{\theta}_j)$ act on analytic functions by a complex shift of the argument:

$$(e^{\pm \beta \hat{\theta}_j} f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{j-1}, x_j \mp i\beta\hbar, x_{j+1}, \ldots, x_n). \tag{2.8}$$

The functions v and w are the potentials of the model; v is responsible for the interaction between the particles and w models an external field. The parameters μ, μ_r, and μ_r' ($r = 0, 1$) are coupling constants; after setting them equal to zero the particles become free ($v, w = 1$). In a previous paper[9] we have introduced explicit expressions for the quantum integrals of the system in the case of trigonometric potentials:

$$\hat{\mathcal{H}}_l = \sum_{\substack{J \subset \{1, \ldots, n\}, |J| \le l \\ \varepsilon_j = \pm 1, j \in J}} U_{J^c, l - |J|}\, W_{\varepsilon J}\, V_{\varepsilon J;\, J^c}\, e^{-\beta \hat{\theta}_{\varepsilon J}}, \quad l = 1, \ldots, n, \tag{2.9}$$

with

$$W_{\varepsilon J} = \prod_{j \in J} w(\varepsilon_j x_j), \tag{2.10}$$

$$V_{\varepsilon J;\, K} = \prod_{\substack{j,j' \in J \\ j < j'}} v(\varepsilon_j x_j + \varepsilon_{j'} x_{j'}) v(\varepsilon_j x_j + \varepsilon_{j'} x_{j'} + 2\gamma) \prod_{\substack{j \in J \\ k \in K}} v(\varepsilon_j x_j + x_k) v(\varepsilon_j x_j - x_k), \tag{2.11}$$

$$\hat{\theta}_{\varepsilon J} = \sum_{j \in J} \varepsilon_j \hat{\theta}_j, \tag{2.12}$$

and

$$U_{I,p} = \sum_{\substack{1 \le q \le p \\ \varepsilon_i = \pm 1,\ i \in I}} (-1)^q \sum_{\substack{\varnothing \subsetneq I_1 \subsetneq \cdots \subsetneq I_q \subset I \\ |I_q| = p}} W_{\varepsilon I_q} \prod_{1 \le q' \le q} V_{\varepsilon(I_{q'} \setminus I_{q'-1});\, \wedge\, I_{q'}} \tag{2.13}$$

($U_{I,0} = 1$, $I_0 = \varnothing$). For $l = 1$ the operator $\hat{\mathscr{H}}_l$ (2.9) coincides with the Hamiltonian $\hat{\mathscr{H}}$ (2.1).

The difference operators $\hat{\mathscr{H}}_1, \ldots, \hat{\mathscr{H}}_n$ with type III potentials are simultaneously diagonalized by Koornwinder's multivariable generalization of the Askey–Wilson polynomials (see Remark *iii.* below), and therefore commute with each other.[9] Because hyperbolic potentials are obtained from trigonometric ones by analytic continuation: $\alpha \mapsto i\alpha$, and the rational case follows from the limit $\alpha \to 0$, the commutativity of $\hat{\mathscr{H}}_l$ (2.9)–(2.13) for the potentials of type I and II follows as well.

Theorem 2.1 (quantum integrability): *The operators $\hat{\mathscr{H}}_1, \ldots, \hat{\mathscr{H}}_n$ commute.*

The operators $\hat{\mathscr{H}}_l$, $l = 1, \ldots, n$, are not Hermitian (with respect to Lebesgue measure). However, after the reparametrization

$$\mu = i\beta g, \quad \mu_r = i\beta g_r, \quad \mu'_r = i\beta g'_r, \tag{2.14}$$

$$\beta > 0, \qquad g, g_r, g'_r \ge 0,$$

conjugation with an appropriate function transforms $\hat{\mathscr{H}}_l$ into a Hermitian operator:

$$\hat{H}_l \equiv \Delta^{1/2} \hat{\mathscr{H}}_l \Delta^{-1/2} = \sum_{\substack{J \subset \{1, \ldots, n\},\ |J| \le l \\ \varepsilon_j = \pm 1,\ j \in J}} U_{J^c,\, l - |J|}\, W_{\varepsilon J}^{1/2}\, V_{\varepsilon J;\, J^c}^{1/2}\, e^{-\beta \hat{\theta}_{\varepsilon J}}\, V_{-\varepsilon J;\, J^c}^{1/2} W_{-\varepsilon J}^{1/2}. \tag{2.15}$$

The transition $\hat{\mathscr{H}}_l \mapsto \hat{H}_l$, which can be seen as a gauge transformation, and the function Δ are detailed in Appendix A. Note how the transformation only affects the part of the coefficient that does not commute with the exponential $\exp(-\beta \hat{\theta}_j)$ and leaves the commuting part $U_{J^c,\, l-|J|}$ unchanged. To see that the formal adjoint $\hat{H}_l^*$ of the resulting operator indeed coincides with $\hat{H}_l$, notice that for $\mu, \mu_r, \mu'_r \in i\mathbb{R}$ one has $\overline{v(x_j)} = v(-x_j)$, $\overline{w(x_j)} = w(-x_j)$, and consequently

$$\bar{V}_{\varepsilon J; K} = V_{-\varepsilon J; K}, \quad \bar{V}_{\varepsilon J} = W_{-\varepsilon J}, \quad \bar{U}_{I,p} = U_{I,p}. \tag{2.16}$$

Let us now explain the relation between the above difference operators and the quantum CM systems associated with classical root systems. Consider first the case of trigonometric potentials. It follows from Ref. 9 that a formal expansion in β of $\hat{\mathscr{H}}_l$ (2.9) with parameters (2.14) is of the form

$$\hat{\mathscr{H}}_l(\beta) = \hat{\mathscr{H}}_{l,0} \beta^{2l} + o(\beta^{2l}) \tag{2.17}$$

with

$$\hat{\mathscr{H}}_{l,0} = \sum_{\substack{J \subset \{1,\dots,n\} \\ |J|=l}} \prod_{j \in J} \hat{\theta}_j^2 + l.o. \tag{2.18}$$

($l.o.$ stands for terms of lower order in the partials $\hat{\theta}_j$). Expansion formulas similar to (2.17), (2.18), now hold with $\hat{\mathscr{H}}_l$ and $\hat{\mathscr{H}}_{l,0}$ replaced by $\hat{H}_l$ and the corresponding leading part $\hat{H}_{l,0}$. Because of the analytic dependence on α, all these expansion formulas are also valid for the systems with hyperbolic and rational potentials.

Explicit computation of $\hat{H}_{1,0}$ entails (hyperbolic version)

$$\hat{H}_{1,0} = \lim_{\beta \to 0} \beta^{-2} \hat{H}_1(\beta)$$

$$= \sum_{1 \leq j \leq n} \hat{\theta}_j^2 + g(g-\hbar)\alpha^2 \sum_{1 \leq j \neq k \leq n} (\mathrm{sh}^{-2}\alpha(x_j+x_k) + \mathrm{sh}^{-2}\alpha(x_j-x_k))$$

$$+ \alpha^2 \sum_{1 \leq j \leq n} \left(\frac{\tilde{g}_0(\tilde{g}_0 - \hbar)}{\mathrm{sh}^2(\alpha x_j)} - \frac{\tilde{g}_1(\tilde{g}_1 - \hbar)}{\mathrm{ch}^2(\alpha x_j)} \right) + 4\alpha^2(\rho,\rho) \tag{2.19}$$

with $\tilde{g}_0 = g_0 + g_0'$, $\tilde{g}_1 = g_1 + g_1'$, and $\rho_j = (n-j)g + (\tilde{g}_0 + \tilde{g}_1)/2$. The operator (2.19) coincides with the BC_n-type Calogero–Moser Hamiltonian.[2] Hence, we see that the integrability of the BC_n Calogero–Moser system with potentials of type I–III is an immediate consequence of Theorem 2.1 and Expansion (2.17), (2.18).

Theorem 2.2 (transition to the BC_n-type CM system): *The limits*

$$\hat{H}_{l,0} = \lim_{\beta \to 0} \beta^{-2l} \hat{H}_l(\beta), \quad l = 1,\dots,n, \tag{2.20}$$

exist and the resulting PDOs $\hat{H}_{1,0},\dots,\hat{H}_{n,0}$ commute.

Remarks: i. The difference operators $\beta^{-2l}\hat{H}_l(\beta)$ form a deformation of the BC_n-type Calogero–Moser PDOs $\hat{H}_{l,0}$. The deformation parameter β determines the step size of the AΔOs via Eq. (2.8). One can look upon β as a parameter that governs the (imaginary) period of the Hamiltonians in the momentumlike variables $\hat{\theta}_j$. The transition $\beta \to 0$ then amounts to sending this period to infinity.

ii. Observe that the operator $\hat{\mathscr{H}}_l$ is homogeneous of degree l in the external field potential w. If, before taking α to zero, $\hat{\mathscr{H}}_l$ is divided by α^{2l} and μ_1, μ_1' are shifted over half a period (turning $\cos(\mathrm{h})\alpha(\mu_1^{(\prime)}+z)$ into $\sin(\mathrm{h})\alpha(\mu_1^{(\prime)}+z)$), then one ends up with a more general rational system with w (2.2) replaced by

$$w(z) = \frac{(\mu_0+z)(\mu_1+z)}{z} \frac{(\mu_0'+\gamma+z)(\mu_1'+\gamma+z)}{(\gamma+z)}. \tag{2.21}$$

The potential w (2.2) is recovered after multiplying $\hat{\mathscr{H}}_l$ by $(\mu_1\mu_1')^{-l}$ and sending both μ_1 and μ_1' to infinity.

iii. In the case of trigonometric potentials the system has polynomial eigenfunctions; one has[9] (taking for convenience $\alpha = 1/2$, $\hbar = 1$):

$$\hat{\mathscr{H}}_l \, p_\lambda = E_{l,n}(\lambda+\rho) \, p_\lambda, \tag{2.22}$$

where p_λ denotes the Koornwinder polynomial[10] associated with the dominant weight vector $\lambda \in \mathbb{Z}^n$ ($\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$), and the eigenvalues are given by

$$E_{l,n}(\theta) = \sum_{\substack{J \subset \{1,\dots,n\} \\ |J| \leq l}} (-1)^{l+|J|} \prod_{j \in J} \operatorname{ch} \beta\theta_j \sum_{1 \leq i_1 \leq \cdots \leq i_{l-|J|} \leq n} \operatorname{ch} \beta\rho_{i_1} \cdots \operatorname{ch} \beta\rho_{i_{l-|J|}} \qquad (2.23)$$

[with the same ρ as in (2.19)].

    iv. For special values of the coupling constants $\hat{H}_{1,0}$ (2.19) reduces to the CM Hamiltonian associated with B_n ($\tilde{g}_1 = 0$), C_n ($\tilde{g}_0 = \tilde{g}_1$), or D_n ($\tilde{g}_0, \tilde{g}_1 = 0$). For arbitrary root system the quantum integrability of the corresponding CM system with type I–III potentials already follows from a construction involving certain differential-reflection operators known as 'Dunkl operators'.[12,13]

III. GENERALIZING TO ELLIPTIC POTENTIALS FOR TWO PARTICLES

    In this section it is explained how for $n = 2$ the results of Sec. II can be generalized to the case where the potentials are elliptic.

A. Preliminaries

    In the elliptic case we will express our potentials in terms of the Weierstrass σ-function.[3] In order to keep the treatment self-contained we first recapitulate some elementary properties of sigma functions. For more details the reader is referred to, e.g., Whittaker and Watson.[3]

    The σ-function is an entire, odd, and quasiperiodic function with two primitive quasiperiods $2\omega_1$, $2\omega_2$. It is convenient to distinguish a third (dependent) quasiperiod $2\omega_3 = -2\omega_1 - 2\omega_2$. One has

$$\sigma(z + 2\omega_s) = -\sigma(z) \, e^{2\eta_s(z+\omega_s)}, \quad s = 1,2,3, \qquad (3.1)$$

with $\eta_s \equiv \zeta(\omega_s)$, where $\zeta(z) = \sigma'(z)/\sigma(z)$ denotes the Weierstrass ζ-function. By working out the the r.h.s. of the identity $\sigma(z) = \sigma(z + 2\omega_1 + 2\omega_2 + 2\omega_3)$ one sees that consistency implies that

$$\eta_1 + \eta_2 + \eta_3 = 0. \qquad (3.2)$$

Occasionally we will also use Legendre's formula, which relates η_s to the half-periods ω_s:

$$\eta_1 \omega_2 - \eta_2 \omega_1 = \pi i/2. \qquad (3.3)$$

    Let Γ denote the half-period lattice:

$$\Gamma = \omega_1 \mathbf{Z} + \omega_2 \mathbf{Z}. \qquad (3.4)$$

The zeros of $\sigma(z)$ are located at the points of the period lattice 2Γ. It follows from the expansion $\sigma(z) = z + O(z^3)$ and Eq. (3.1) that these zeros are simple. By introducing a shift of the argument over the half-periods one arrives at three associated sigma functions:

$$\sigma_r(z) \equiv e^{-\eta_r z} \, \sigma(\omega_r + z)/\sigma(\omega_r), \quad r = 1,2,3. \qquad (3.5)$$

For the associated sigma functions the quasiperiodicity relations read

$$\sigma_r(z + 2\omega_s) = (-1)^{\delta_{r,s}} \sigma_r(z) \, e^{2\eta_s(z+\omega_s)}, \quad r,s = 1,2,3 \qquad (3.6)$$

(where $\delta_{r,s}$ denotes the Kronecker delta).

    We close this subsection with a duplication-formula for sigma functions and some relations with the Weierstrass $\wp$-function:

$$\sigma(2z) = 2 \, \sigma(z) \, \sigma_1(z) \, \sigma_2(z) \, \sigma_3(z), \qquad (3.7)$$

$$\frac{\sigma(\mu+z)\sigma(\mu-z)}{\sigma^2(z)\sigma^2(\mu)}=\wp(z)-\wp(\mu),\tag{3.8}$$

$$\wp(z+\omega_r)=-\frac{d^2}{dz^2}\ln\sigma_r(z),\quad r=0,1,2,3,\tag{3.9}$$

where we have introduced the convention $\sigma_0(z)\equiv\sigma(z)$ and $\omega_0\equiv0$.

Remark: In order to have $\overline{\sigma_r(z)}=\sigma_r(\bar{z})$ (such that the functions are real for real values of the argument), it will be assumed most of the time that

$$\omega_1\in R^+,\quad \omega_2\in i\,R^+.\tag{3.10}$$

Although this assumption is important in matters concerning real-valuedness and Hermiticity of the Hamiltonians below, it is not essential for the integrability of our models.

B. A special case: $\mu=\omega_s$

First let us write out the relevant operators $\hat{\mathcal{H}}_l$ (2.9)–(2.13) for $n=2$:

$$\hat{\mathcal{H}}_1=\sum_{\varepsilon\in\{1,-1\}}w(\varepsilon x_1)\,v(\varepsilon x_1+x_2)\,v(\varepsilon x_1-x_2)\,e^{-\beta\varepsilon\hat{\theta}_1}$$

$$+\sum_{\varepsilon\in\{1,-1\}}w(\varepsilon x_2)\,v(\varepsilon x_2+x_1)\,v(\varepsilon x_2-x_1)\,e^{-\beta\varepsilon\hat{\theta}_2}+U_{\{1,2\},1},\tag{3.11}$$

$$\hat{\mathcal{H}}_2=\sum_{\varepsilon,\varepsilon'\in\{1,-1\}}w(\varepsilon x_1)\,w(\varepsilon'x_2)\,v(\varepsilon x_1+\varepsilon'x_2)\,v(\varepsilon x_1+\varepsilon'x_2+2\gamma)\,e^{-\beta(\varepsilon\hat{\theta}_1+\varepsilon'\hat{\theta}_2)}$$

$$+U_{\{2\},1}\sum_{\varepsilon\in\{1,-1\}}w(\varepsilon x_1)\,v(\varepsilon x_1+x_2)\,v(\varepsilon x_1-x_2)\,e^{-\beta\varepsilon\hat{\theta}_1}$$

$$+U_{\{1\},1}\sum_{\varepsilon\in\{1,-1\}}w(\varepsilon x_2)\,v(\varepsilon x_2+x_1)\,v(\varepsilon x_2-x_1)\,e^{-\beta\varepsilon\hat{\theta}_2}+U_{\{1,2\},2},\tag{3.12}$$

with

$$U_{\{j\},1}=-w(x_j)-w(-x_j)\quad(j=1,2),\tag{3.13}$$

$$U_{\{1,2\},1}=-\sum_{\varepsilon\in\{1,-1\}}(w(\varepsilon x_1)\,v(\varepsilon x_1+x_2)\,v(\varepsilon x_1-x_2)+w(\varepsilon x_2)\,v(\varepsilon x_2+x_1)\,v(\varepsilon x_2-x_1)),\tag{3.14}$$

$$U_{\{1,2\},2}=-\sum_{\varepsilon,\varepsilon'\in\{1,-1\}}w(\varepsilon x_1)\,w(\varepsilon'x_2)\,v(\varepsilon x_1+\varepsilon'x_2)\,v(\varepsilon x_1+\varepsilon'x_2+2\gamma)$$

$$+(w(x_2)+w(-x_2))\sum_{\varepsilon\in\{1,-1\}}w(\varepsilon x_1)\,v(\varepsilon x_1+x_2)\,v(\varepsilon x_1-x_2)$$

$$+(w(x_1)+w(-x_1))\sum_{\varepsilon\in\{1,-1\}}w(\varepsilon x_2)\,v(\varepsilon x_2+x_1)\,v(\varepsilon x_2-x_1).\tag{3.15}$$

Requiring that the commutator $[\hat{\mathcal{H}}_1, \hat{\mathcal{H}}_2]$ is zero leads to functional equations for the functions $v(z)$ and $w(z)$. We know already from Theorem 2.1 that these functional equations admit non-trivial solutions given by the potentials (2.2)–(2.6). By writing out the relevant equations one finds that there exist also elliptic solutions (observation due to S. N. M. Ruijsenaars):

$$v(z)=\frac{\sigma_s(z)}{\sigma(z)}=e^{-\eta_s z}\,\frac{\sigma(\omega_s+z)}{\sigma(z)\sigma(\omega_s)}\,, \tag{3.16}$$

$$w(z)=\left(\prod_{0\leqslant r\leqslant 3}\frac{\sigma_r(\mu_r+z)}{\sigma_r(z)}\right)\left(\frac{\sigma(\mu_0'+\gamma+z)}{\sigma(\gamma+z)}\,\frac{\sigma_s(\mu_s'+\gamma+z)}{\sigma_s(\gamma+z)}\right), \tag{3.17}$$

where the coupling constants of the external field potentials must satisfy the linear relation

$$(\mu_0'+\mu_s')+\sum_{0\leqslant r\leqslant 3}\mu_r=0, \tag{3.18}$$

and $s=1,2$, or 3. Condition (3.18) ensures that the potential $w(z)$ is doubly periodic in z [cf. Eqs. (3.1), (3.6)]. If this condition is not fulfilled, then the commutativity of $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_2$ breaks down.

Thus, we have the following proposition.

Proposition 3.1: Let $\hat{\mathcal{H}}_1$, $\hat{\mathcal{H}}_2$ be of the form (3.11)–(3.15) with the potentials v and w given by (3.16) and (3.17). Then the operators $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_2$ commute if the coupling constants of the external field satisfy Condition (3.18).

C. The general case

A serious drawback of the difference CM system in the previous subsection compared to the ones in Sec. II is the absence of an independent parameter in $v(z)$ (3.16) playing the rôle of coupling constant. Furthermore, the symmetry between the primed and the unprimed coupling constants in the external field potentials of type I–III is broken in $w(z)$ (3.17). In trying to cure these diseases we are led to a more general ansatz for the potentials $v(z)$ and $w(z)$:

IV. Elliptic case

$$v(z)=\frac{\sigma(\mu+z)}{\sigma(z)}\,, \tag{3.19}$$

$$w(z)=\prod_{0\leqslant r\leqslant 3}\frac{\sigma_r(\mu_r+z)\,\sigma_r(\mu_r'+\gamma+z)}{\sigma_r(z)\,\sigma_r(\gamma+z)}\,. \tag{3.20}$$

Apart from an exponential factor in $v(z)$ (3.16), the potentials (3.16), (3.17) correspond to setting $\mu=\omega_s$ and $\mu_r'=0$ if $r\neq 0,s$. In contrast with v (3.16) [which is doubly periodic with (nonprimitive) period lattice $4\,\Gamma$], the potential v (3.19) is no longer periodic in general. The external field potential w (3.20), however, is again doubly periodic provided the parameters satisfy the condition

$$\sum_{0\leqslant r\leqslant 3}(\mu_r+\mu_r')=0. \tag{3.21}$$

In fact, by using the defining equation for $\sigma_r(z)$ [Eq. (3.5)], and Eqs. (3.1), (3.2), one readily verifies that $w(z)$ is covariant with respect to a shift of the argument over a half-period provided (3.21) holds:

$$z\mapsto z+\omega_s\Leftrightarrow\mu_r\mapsto\mu_{\pi_s(r)},\quad\mu_r'\mapsto\mu_{\pi_{s(r)}}'\,, \tag{3.22}$$

where we have introduced the permutations $\pi_0 = id$, $\pi_1 = (01)(23)$, $\pi_2 = (02)(13)$, and $\pi_3 = (03)(12)$.

Unfortunately it turns out that the operators $\hat{\mathscr{H}}_1$, $\hat{\mathscr{H}}_2$ (3.11)–(3.15) with potentials given by (3.19) and (3.20) do not commute if $\mu \neq 0$. (For $\mu = 0$ commutativity is trivial because in that case the two particles become independent.) However, as will be demonstrated next, it is possible to replace $U_{\{1,2\},1}$ (3.14) and $U_{\{1,2\},2}$ (3.15) by functions such that the resulting operators $\hat{\mathscr{H}}_1$ (3.11) and $\hat{\mathscr{H}}_2$ (3.12) with potentials $v(z)$ (3.19) and $w(z)$ (3.20) do commute if (3.21) holds. Specifically, these functions read

$$U_{\{1,2\},1} = \sum_{0 \leqslant r \leqslant 3} c_r \prod_{j=1,2} \frac{\sigma_r(\mu - \gamma + x_j)}{\sigma_r(-\gamma + x_j)} \frac{\sigma_r(\mu - \gamma - x_j)}{\sigma_r(-\gamma - x_j)}, \qquad (3.23)$$

where

$$c_r = \frac{2}{\sigma(\mu)\sigma(\mu - 2\gamma)} \prod_{0 \leqslant s \leqslant 3} \sigma_s(\mu_{\pi_r(s)} - \gamma) \; \sigma_s(\mu'_{\pi_r(s)}) \qquad (3.24)$$

[with π_r as in Eq. (3.22)], and

$$U_{\{1,2\},2} = \sum_{\varepsilon,\varepsilon' \in \{1,-1\}} w(\varepsilon x_1) \; w(\varepsilon' x_2) \; v(\varepsilon x_1 + \varepsilon' x_2) \; v(-\varepsilon x_1 - \varepsilon' x_2 - 2\gamma). \qquad (3.25)$$

Our proof of commutativity exploits a small variation on a standard technique used to demonstrate functional identities between elliptic functions: we compute the commutator of $\hat{\mathscr{H}}_1$ and $\hat{\mathscr{H}}_2$ and show that the coefficients of the resulting AΔO are given by entire functions, i.e., poles introduced by the denominators of v (3.19) and w (3.20) cancel each other in the commutator. Using this property, and some additional information involving the quasiperiodicity of the coefficients, it follows (Appendix B) that these functions vanish identically.

First we need a lemma; it basically says that a one-particle version of (3.23) differs from (3.13) only by an (irrelevant) additive constant.

Lemma 3.2: If the coupling constants satisfy (3.21), then the expression

$$w(z) + w(-z) + \sum_{0 \leqslant r \leqslant 3} c_r \, v(-\gamma + \omega_r + z) \, v(-\gamma - \omega_r - z) \qquad (3.26)$$

is constant in z [with v(z), w(z) given by (3.19), (3.20), and c_r defined by (3.24)].

Proof: Condition (3.21) implies that Expression (3.26) is doubly periodic in z. Recall that the σ-function is entire and that its zeros are simple and located at the points of the period lattice $2 \Gamma (= 2\omega_1 \mathbf{Z} + 2\omega_2 \mathbf{Z})$. For γ generic the zeros of the sigma functions induce simple poles in the terms of (3.26) by means of the denominators of v and w. These poles are congruent to one of

$$z = \omega_r, \quad z = \pm \gamma + \omega_r, \quad 0 \leqslant r \leqslant 3 \qquad (3.27)$$

(with the convention $\omega_0 \equiv 0$). We want to show that the total residue of (3.26) at the above poles vanishes. Because (3.26) is covariant with respect to shifts $z \to z + \omega_s$, i.e., such shifts amount to a permutation of the coupling constants μ_r and μ'_r [cf. (3.22)], it suffices to verify the cases $z = 0$ and $z = \pm \gamma$. Since (3.26) is even in z the vanishing of the residue at $z = 0$ is immediate and either $z = -\gamma$ or $z = \gamma$ remains to be checked. Using $\sigma(z) = z + O(z^3)$ [and the duplication-formula for sigma functions (3.7)], one infers that the latter residue vanishes too.

The upshot is that (3.26) is entire and bounded (because doubly periodic) in z. By Liouville's theorem it must then be a function independent of z.

$\square$

We are now ready to prove that $[\hat{\mathscr{H}}_1, \hat{\mathscr{H}}_2] = 0$.

Theorem 3.3: *Let $\hat{\mathscr{H}}_1$, $\hat{\mathscr{H}}_2$ be of the form (3.11), (3.12) with the functions $U_{I,p}$ defined by (3.13) and (3.23)–(3.25), and the potentials v and w given by (3.19) and (3.20). Then the operators $\hat{\mathscr{H}}_1$ and $\hat{\mathscr{H}}_2$ commute if the coupling constants of the external field satisfy Condition (3.21).*

Proof: A straightforward calculation entails

$$[\hat{\mathscr{H}}_1, \hat{\mathscr{H}}_2] = \sum_{\varepsilon,\varepsilon' \in \{1,-1\}} K_1(\varepsilon x_1, \varepsilon' x_2, \gamma)\, w_{\varepsilon 1}\, w_{\varepsilon' 2}\, v_{\varepsilon 1 + \varepsilon' 2}\, v_{\varepsilon 1 + \varepsilon' 2 + 2\gamma}\, e^{-\beta(\varepsilon \hat{\theta}_1 + \varepsilon' \hat{\theta}_2)}$$

$$+ \sum_{\varepsilon \in \{1,-1\}} K_2(\varepsilon x_1, x_2, \gamma)\, w_{\varepsilon 1}\, v_{\varepsilon 1 + 2}\, v_{\varepsilon 1 - 2}\, e^{-\beta \varepsilon \hat{\theta}_1}$$

$$+ \sum_{\varepsilon \in \{1,-1\}} K_2(\varepsilon x_2, x_1, \gamma)\, w_{\varepsilon 2}\, v_{\varepsilon 2 + 1}\, v_{\varepsilon 2 - 1}\, e^{-\beta \varepsilon \hat{\theta}_2} \tag{3.28}$$

with

$$K_1(x_1, x_2, \gamma) = \sum_{0 \leqslant r \leqslant 3} c_r \left(\prod_{j=1,2} v_{-\gamma + \omega_r + j}\, v_{-\gamma - \omega_r - j} - \prod_{j=1,2} v_{\gamma + \omega_r + j}\, v_{-3\gamma - \omega_r - j} \right)$$

$$+ (w_1 + w_{-1})\, v_{1-2-2\gamma}\, v_{-1+2} - (w_{1+2\gamma} + w_{-1-2\gamma})\, v_{1-2}\, v_{-1+2-2\gamma}$$

$$+ (w_2 + w_{-2})\, v_{-1+2-2\gamma}\, v_{1-2} - (w_{2+2\gamma} + w_{-2-2\gamma})\, v_{-1+2}\, v_{1-2-2\gamma} \tag{3.29}$$

and

$$K_2(x_1, x_2, \gamma) = w_{1+2\gamma}\, w_2\, v_{1+2+2\gamma}\, v_{-1-2-4\gamma} - w_1\, w_2\, v_{1+2}\, v_{-1-2-2\gamma}$$

$$+ w_{-1-2\gamma}\, w_2\, v_{-1+2-2\gamma}\, v_{1-2} - w_{-1}\, w_2\, v_{-1+2}\, v_{1-2-2\gamma}$$

$$+ w_{1+2\gamma}\, w_{-2}\, v_{1-2+2\gamma}\, v_{-1+2-4\gamma} - w_1 w_{-2}\, v_{1-2}\, v_{-1+2-2\gamma}$$

$$+ w_{-1-2\gamma}\, w_{-2}\, v_{-1-2-2\gamma}\, v_{1+2} - w_{-1}\, w_{-2}\, v_{-1-2}\, v_{1+2-2\gamma}$$

$$+ (w_2 + w_{-2})$$

$$\times \sum_{0 \leqslant r \leqslant 3} c_r (v_{\gamma + \omega_r + 1}\, v_{-3\gamma - \omega_r - 1} - v_{-\gamma + \omega_r + 1}\, v_{-\gamma - \omega_r - 1}) v_{-\gamma + \omega_r + 2}\, v_{-\gamma - \omega_r - 2}$$

$$+ w_{2-2\gamma}\, w_{-2}\, (v_{1+2-2\gamma}\, v_{-1-2} - v_{-1+2-4\gamma}\, v_{1-2+2\gamma})$$

$$+ w_{-2-2\gamma}\, w_2\, (v_{1-2-2\gamma}\, v_{-1+2} - v_{-1-2-4\gamma}\, v_{1+2+2\gamma}), \tag{3.30}$$

where we have introduced the short-hand notation $w_{\varepsilon j} = w(\varepsilon x_j)$ and $v_{\varepsilon 1 + \varepsilon' 2 + 2\gamma + \omega_r} = v(\varepsilon x_1 + \varepsilon' x_2 + 2\gamma + \omega_r)$, etc.

By considering (3.29) and (3.30) as function of γ and μ we will see that K_1 and K_2 are identically zero.

It follows from (3.1), (3.6) and Condition (3.21) that K_1 and K_2 are quasiperiodic in γ

$$K_1(x_1, x_2, \gamma + 2\omega_s) = e^{-2\eta_s(2\mu + \Sigma_{0 \leqslant r \leqslant 3} \mu_r)} K_1(x_1, x_2, \gamma), \tag{3.31}$$

$$K_2(x_1,x_2,\gamma+2\omega_s)=e^{-4\eta_s(\mu+\Sigma 0\leq r\leq 3\mu_r)}K_2(x_1,x_2,\gamma). \tag{3.32}$$

The idea is now to show that K_1 and K_2 are entire in γ, μ, and then apply Lemma B.1 of Appendix B.

If x_1, x_2 and μ, μ_r, μ_r' are fixed in general position, then the zeros of the sigma functions induce simple poles in the terms of (3.29) and (3.30) via the denominators of v, w, and c_r. For $K_1(\gamma)$ these poles are located at

$$\gamma=\pm(x_1-x_2)/2 \mod \Gamma, \tag{3.33}$$

$$\gamma=\pm x_j \mod \Gamma, \quad \gamma=-x_j/2 \mod \Gamma/2, \quad \gamma=-x_j/3 \mod \Gamma/3, \tag{3.34}$$

$$\gamma=\mu/2 \mod \Gamma \tag{3.35}$$

[with the lattice Γ given by (3.4)]. Using the double periodicity of the function w (period lattice $2\,\Gamma$), one sees that the poles at (3.33) in (3.29) cancel manifestly. As regards the poles at (3.34), it suffices to consider only the cases that γ is actually equal to $\pm x_j$, $-x_j/2$, or $-x_j/3$, because K_1 is covariant with respect to simultaneous translation of the positions over the half-periods [cf. (3.22)]. At $\gamma=-x_j/2$ the residues in (3.29) cancel again manifestly. Using Lemma 3.2 one infers that the total residue at $\gamma=\pm x_j$ and at $\gamma=-x_j/3$ vanishes, too. Finally, the poles (3.35), which are caused by the denominator of c_r (3.24), are compensated by zeros in the part between brackets in the first line of (3.29) [the latter zeros stem from the identity $v(-\mu/2+z)\,v(-\mu/2-z)=1$].

Consequently, K_1 is an entire function of γ. A similar analysis of the residues reveals that also K_2 is entire in γ. Furthermore, both expressions are entire in μ too [poles caused by c_r (3.24) are again compensated by zeros in the parts between brackets].

It now follows from Lemma B.1 that the functions K_1 and K_2 are zero.

$\square$

D. Further properties of the system

In the elliptic case $\hat{\mathscr{H}}_1$ and $\hat{\mathscr{H}}_2$ are transformed into Hermitian operators $\hat{H}_1$ and $\hat{H}_2$ in the same way as the operators of Sec. II. Using parameters as in (2.14) and conjugating with the appropriate function from Appendix A entails

$$\hat{H}_1=\sum_{\varepsilon\in\{1,-1\}} w_{\varepsilon 1}^{1/2}\, v_{\varepsilon 1+2}^{1/2}\, v_{\varepsilon 1-2}^{1/2}\, e^{-\beta\varepsilon\hat{\theta}_1}\, v_{-\varepsilon 1+2}^{1/2}\, v_{-\varepsilon 1-2}^{1/2}\, w_{-\varepsilon 1}^{1/2}$$

$$+\sum_{\varepsilon\in\{1,-1\}} w_{\varepsilon 2}^{1/2}\, v_{\varepsilon 2+1}^{1/2}\, v_{\varepsilon 2-1}^{1/2}\, e^{-\beta\varepsilon\hat{\theta}_2}\, v_{-\varepsilon 2+1}^{1/2}\, v_{-\varepsilon 2-1}^{1/2}\, w_{-\varepsilon 2}^{1/2}+U_{\{1,2\},1}, \tag{3.36}$$

$$\hat{H}_2=\sum_{\varepsilon,\varepsilon'\in\{1,-1\}} (w_{\varepsilon 1}^{1/2}\, w_{\varepsilon'2}^{1/2}\, v_{\varepsilon 1+\varepsilon'2}^{1/2}\, v_{\varepsilon 1+\varepsilon'2+2\gamma}^{1/2}\, e^{-\beta(\varepsilon\hat{\theta}_1+\varepsilon'\hat{\theta}_2)}$$

$$\times v_{-\varepsilon 1-\varepsilon'2+2\gamma}^{1/2} v_{-\varepsilon 1-\varepsilon'2}^{1/2}\, w_{-\varepsilon 1}^{1/2}\, w_{-\varepsilon'2}^{1/2})$$

$$+U_{\{2\},1}\sum_{\varepsilon\in\{1,-1\}} w_{\varepsilon 1}^{1/2}\, v_{\varepsilon 1+2}^{1/2}\, v_{\varepsilon 1-2}^{1/2}\, e^{-\beta\varepsilon\hat{\theta}_1}\, v_{-\varepsilon 1+2}^{1/2}\, v_{-\varepsilon 1-2}^{1/2}\, w_{-\varepsilon 1}^{1/2}$$

$$+U_{\{1\},1}\sum_{\varepsilon\in\{1,-1\}} w_{\varepsilon 2}^{1/2}\, v_{\varepsilon 2+1}^{1/2}\, v_{\varepsilon 2-1}^{1/2}\, e^{-\beta\varepsilon\hat{\theta}_2}\, v_{-\varepsilon 2+1}^{1/2}\, v_{-\varepsilon 2-1}^{1/2}\, w_{-\varepsilon 2}^{1/2}+U_{\{1,2\},2}, \tag{3.37}$$

with v, w, and $U_{I,p}$ as in Sec. III B. [Eqs. (3.16), (3.17), and (3.13)–(3.15)] or as in Sec. III C. [Eqs. (3.19), (3.20), and (3.13), (3.23)–(3.25)]. The Hermiticity of $\hat{H}_1$ and $\hat{H}_2$ hinges on Assumption (3.10) and Condition (2.14) [verification as in Sec. II: $\overline{v(x_j)} = v(-x_j)$, $\overline{w(x_j)} = w(-x_j)$ and $\bar{U}_{I,p} = U_{I,p}$].

By comparing the Hermitian counterparts of the operators in Sec. III B and III C it follows that the former model is a special case of the latter. More precisely, setting $\mu = \omega_s$ and $\mu'_r = 0$, $r \neq 0, s$, in the operators $\hat{H}_1$ (3.36) and $\hat{H}_2$ (3.37) with potentials from Sec. III C leads, up to multiplicative and additive constants, to the corresponding operators with potentials from Sec. III B. To verify this, notice that exponential factors in $v^{1/2}_{\pm 1 \pm 2}$, if $v(z)$ is given by (3.16), give rise to multiplicative constants in $\hat{H}_1$ and $\hat{H}_2$ after commuting them through the difference operators $\exp(\pm \beta \hat{\theta}_j)$, $\exp \beta(\pm \hat{\theta}_1 \pm \hat{\theta}_2)$. Furthermore, the respective formulas for $U_{\{1,2\}, p}$ in Sec. III B and III C can be compared by an analysis of residues and by invoking the Liouville theorem similar to the proof of Lemma 3.2.

Let us now clarify the connection between our two-particle difference CM system and a quantization of the Inozemtsev Hamiltonian (1.3). By expanding in β [and using Eq. (3.9)] one infers that [cf. (2.17)–(2.19)]

$$\hat{H}_1(\beta) = \text{const} + \hat{H}_{1,0}\beta^2 + o(\beta^2), \tag{3.38}$$

$$\hat{H}_2(\beta) = \hat{H}_{2,0}\beta^4 + o(\beta^4), \tag{3.39}$$

with $\text{const} = 4 + \Sigma_r g'_r(2g_r - \hbar)/g(g - \hbar)$ and

$$\hat{H}_{1,0} = \hat{\theta}_1^2 + \hat{\theta}_2^2 + 2g(g - \hbar)(\wp(x_1 + x_2) + \wp(x_1 - x_2))$$

$$+ \sum_{0 \leq r \leq 3} \tilde{g}_r(\tilde{g}_r - \hbar)(\wp(\omega_r + x_1) + \wp(\omega_r + x_2)), \tag{3.40}$$

$$\hat{H}_{2,0} = \hat{\theta}_1^2 \, \hat{\theta}_2^2 + l.o. \tag{3.41}$$

Just as in Sec. II a confluence of the parameters g_r and g'_r into a single coupling constant $\tilde{g}_r = g_r + g'_r$ occurs for $\beta \to 0$. It follows from Theorem 3.3 and the above asymptotics that the PDOs $\hat{H}_{1,0}$ and $\hat{H}_{2,0}$ commute, which proves the (quantum) integrability of the Hamiltonian $\hat{H}_{1,0}$ (if $\Sigma_r \tilde{g}_r = 0$). For the corresponding classical two-particle Hamiltonian of the type (1.3) the Liouville integrability follows of course already from Ref. 8 (without any restrictions on $\tilde{g}_r$).

Another point of interest is the question whether Condition (3.21) is really essential for integrability. It is not difficult to deduce from the quasiperiodicity of the sigma functions that (3.21) guarantees that $U_{I,p}$ is doubly periodic in the positions. The part of the coefficient that does not commute with the translator is quasiperiodic but double periodicity is restored after gauging to Hermitian form (3.36), (3.37). It can be seen from the proofs that the double periodicity of the relevant functions $U_{I,p}$, and thus Condition (3.21), is essential for the commutativity of $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_2$. However, notice that $U_{\{1,2\},1}$ (3.23) remains doubly periodic even if (3.21) is not fulfilled, and that, because of Lemma 3.2, also $U_{\{j\},1}$ (3.13) can be rewritten in a form that is doubly periodic for arbitrary values of the parameters. Probably there also exists a doubly periodic version of $U_{\{1,2\},2}$ [which coincides with (3.25) if (3.21) holds] such that the operator $\hat{\mathcal{H}}_2$ (3.12) commutes with $\hat{\mathcal{H}}_1$ (3.11) without further restrictions on the coupling constants.

IV. PARTIAL RESULTS FOR ARBITRARY PARTICLE NUMBER

It seems plausible that one can generalize the difference CM systems of Sec. III to arbitrary particle number. We have checked by computer that with the potentials of Sec. III B the operators $\hat{\mathcal{H}}_l$ (2.9)–(2.13) commute for $n = 3, 4$, and expect this result to be true for any n.

Conjecture 4.1: The operators $\hat{\mathcal{H}}_l$ (2.9)–(2.13) with potentials given by (3.16)–(3.18) commute.

As regards the generalization of the system in Sec. III C, the situation seems less straightforward. Probably there again exist functions $U_{I,p}$ such that the operators $\hat{\mathcal{H}}_1,\ldots,\hat{\mathcal{H}}_n$ commute if the potentials v, w are given by (3.19), (3.20). Furthermore, it is expected that these functions $U_{I,p}$ have similar properties as those in Secs. II and III.

Conjecture 4.2: Let $\hat{\mathcal{H}}_l$ be of the form (2.9)–(2.12) with potentials v and w given by (3.19) and (3.20). Then there exist functions $U_{I,p}$ with $U_{I,0}=1$ such that $\hat{\mathcal{H}}_1,\ldots,\hat{\mathcal{H}}_n$ commute. Furthermore, the functions $U_{I,p}$ have the following properties:

    i. $U_{I,p}$ does not depend on x_i, $i \notin I$, and is invariant under permutations of x_i, $i \in I$;

    ii. $U_{I,p}$ is even, meromorphic, and doubly periodic in x_i (with primitive periods $2\omega_1$ and $2\omega_2$);

    iii. $U_{I,p}$ is holomorphic in $\hbar$ at $\hbar = 0$;

    iv. $U_{I,p}$ is real for x_i real and parameters given by (2.14) [with half-periods as in (3.10)].

Unfortunately we have not yet succeeded in producing general formulas for $U_{I,p}$, nor have we proved the existence of such functions by any other means. Having said this, let us continue by outlining some partial results that support Conjecture 4.2.

Generalization of the two-particle Hamiltonian $\hat{\mathcal{H}}_1$ in Sec. III C leads to a Hamiltonian of the form

$$\hat{\mathcal{H}}_1 = \sum_{\substack{1 \leq j \leq n \\ \varepsilon = \pm 1}} V_{\varepsilon j}\, e^{-\varepsilon \beta \hat{\theta}_j} + U, \tag{4.1}$$

with

$$V_{\varepsilon j} = w(\varepsilon x_j) \prod_{k \neq j} v(\varepsilon x_j + x_k) v(\varepsilon x_j - x_k), \tag{4.2}$$

$$U = \sum_{0 \leq r \leq 3} c_r \prod_{1 \leq j \leq n} \frac{\sigma_r(\mu - \gamma + x_j)}{\sigma_r(-\gamma + x_j)} \frac{\sigma_r(\mu - \gamma - x_j)}{\sigma_r(-\gamma - x_j)}, \tag{4.3}$$

where c_r is again given by (3.24). Furthermore, as generalization of the integral $\hat{\mathcal{H}}_2$ we have found an operator $\hat{\mathcal{H}}_n$ that commutes with the n-particle Hamiltonian (4.1) if Condition (3.21) holds:

$$\hat{\mathcal{H}}_n = \sum_{\substack{J \subset \{1,\ldots,n\} \\ \varepsilon_j = \pm 1,\, j \in J}} U_{J^c}\, W_{\varepsilon J}\, V_{\varepsilon J;\, J^c}\, e^{-\beta \hat{\theta}_{\varepsilon J}}, \tag{4.4}$$

with $W_{\varepsilon J}$ and $V_{\varepsilon J;\, K}$ given by (2.10), (2.11), and

$$U_K = \sum_{\varepsilon_k = \pm 1,\, k \in K} (-1)^{|K|} \prod_{k \in K} w(\varepsilon_k x_k) \prod_{\substack{k,k' \in K \\ k < k'}} v(\varepsilon_k x_k + \varepsilon_{k'} x_{k'}) v(-\varepsilon_k x_k - \varepsilon_{k'} x_{k'} - 2\gamma). \tag{4.5}$$

Although in the present case the combinatorics involved in computing the commutator is much more laborious, the commutativity of $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_n$ can be verified with the same techniques used already in Sec. III C (i.e., by an analysis of the residues at poles occurring in the coefficients of the commutator, and by invoking Lemma B.1).

Theorem 4.3: *The operators $\hat{\mathscr{H}}_1$ and $\hat{\mathscr{H}}_n$ (4.1)–(4.5), with potentials v, w given by (3.19), (3.20), commute if the coupling constants of the external field satisfy Condition (3.21).*

When all coupling constants of the external fields are zero (so $w=1$ and $U=0$), then a straightforward calculation shows that

$$\hat{\mathscr{H}}_n = (\hat{\mathscr{H}}_+ - \hat{\mathscr{H}}_-)^2 = \hat{\mathscr{H}}_+^2 + \hat{\mathscr{H}}_-^2 - \hat{\mathscr{H}}_+\,\hat{\mathscr{H}}_- - \hat{\mathscr{H}}_-\,\hat{\mathscr{H}}_+, \tag{4.6}$$

with

$$\hat{\mathscr{H}}_\epsilon = \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_n \in \{1,-1\} \\ \varepsilon_1\cdots\varepsilon_n = \epsilon}} \prod_{1\leqslant j < k \leqslant n} v(\varepsilon_j x_j + \varepsilon_k x_k) e^{-\beta(\varepsilon_1\hat{\theta}_1 + \cdots + \varepsilon_n\hat{\theta}_n)/2}, \quad \epsilon = \pm 1. \tag{4.7}$$

The operators $\hat{\mathscr{H}}_\pm$ are an elliptic generalization of certain trigonometric difference operators introduced by Macdonald that are associated with the half-spin weights of the root system D_n.[9,11] The commutativity of $\hat{\mathscr{H}}_+$, $\hat{\mathscr{H}}_-$, and $\hat{\mathscr{H}}_1$ amounts to functional equations for $v(z)$ that can again be verified with the techniques from Sec. III. Thus, in this special case we have three independent integrals, which implies the integrability of the model for $n=3$. (In fact, for $n=3$ integrability follows already from Ref. 5 because in that case our system coincides up to coordinate transformation with the four-particle RCM model in center of mass coordinates; indeed, the root systems D_3 and A_3 are isomorphic.) The structure of the integrals $\hat{\mathscr{H}}_\pm$ is not of the form anticipated in Conjecture 4.2. However, the combinations $(\hat{\mathscr{H}}_+\,\hat{\mathscr{H}}_- + \hat{\mathscr{H}}_-\,\hat{\mathscr{H}}_+)/2$ and $\hat{\mathscr{H}}_+^2 + \hat{\mathscr{H}}_-^2$ are of the correct form corresponding to $l = n-1$ and $l = n$, respectively.

Let us return to the situation with a nontrivial external field. Assuming parameters as in Eq. (2.14) and transforming $\hat{\mathscr{H}}_1$ to the Hermitian gauge leads to the operator

$$\hat{H}_1 = \sum_{\substack{1\leqslant j\leqslant n \\ \varepsilon = \pm 1}} V_{\varepsilon j}^{1/2}\, e^{-\varepsilon\beta\hat{\theta}_j}\, V_{-\varepsilon j}^{1/2} + U. \tag{4.8}$$

Expanding (4.8) in β entails the following generalization of corresponding formulas in Sec. III D:

$$\hat{H}_1(\beta) = \text{const} + \hat{H}_{1,0}\,\beta^2 + o(\beta^2), \tag{4.9}$$

with

$$\hat{H}_{1,0} = \sum_{1\leqslant j\leqslant n} \hat{\theta}_j^2 + g(g-\hbar)\sum_{1\leqslant j\neq k\leqslant n}(\wp(x_j+x_k)+\wp(x_j-x_k))$$

$$+ \sum_{\substack{1\leqslant j\leqslant n \\ 0\leqslant r\leqslant 3}} \tilde{g}_r(\tilde{g}_r-\hbar)\,\wp(\omega_r+x_j), \tag{4.10}$$

$$\text{const} = 2n + \frac{1}{g(g-\hbar)}\sum_{0\leqslant r\leqslant 3} g_r'(2g_r-\hbar). \tag{4.11}$$

Clearly $\hat{H}_{1,0}$ (4.10) amounts (up to a factor 2) to a quantization of the Inozemtsev Hamiltonian (1.3).

Remarks: i. Consider the operators $\hat{\mathscr{H}}_{l,\,\text{lead}}$, which consist of those terms in $\hat{\mathscr{H}}_l$ that are of highest order in the exponentials $\exp(-\beta\hat{\theta}_j)$:

$$\hat{\mathscr{H}}_{l,\,\text{lead}} = \sum_{\substack{J\subset\{1,\ldots,n\}\\|J|=l}} W_J\,V_{J;J^c}\,e^{-\beta\hat{\theta}_J}. \tag{4.12}$$

It is clear that if the operators $\hat{\mathscr{H}}_l$ (2.9) commute, then their leading parts $\hat{\mathscr{H}}_{l,\text{lead}}$ must also commute. A similarity (gauge) transformation, resembling the ones we have seen before, turns $\hat{\mathscr{H}}_{l,\,\text{lead}}$ into the integrals of the quantum RCM system:

$$\hat{H}_{l,\,\text{RCM}} = \Delta_{\text{RCM}}^{1/2}\,C^{-1}\hat{\mathscr{H}}_{l,\,\text{lead}}C\,\Delta_{RCM}^{-1/2}$$

$$= \Delta_{\text{RCM}}^{1/2}\left(\sum_{\substack{J\subset\{1,\ldots,n\}\\|J|=l}}\prod_{\substack{j\in J\\k\in J^c}} v(x_j-x_k)\,e^{-\beta\hat{\theta}_J}\right)\Delta_{\text{RCM}}^{-1/2}$$

$$= \sum_{\substack{J\subset\{1,\ldots,n\}\\|J|=l}}\prod_{\substack{j\in J\\k\in J^c}} v^{1/2}(x_j-x_k)\,e^{-\beta\hat{\theta}_J}\prod_{\substack{j\in J\\k\in J^c}} v^{1/2}(x_k-x_j), \tag{4.13}$$

where

$$\Delta_{\text{RCM}} = \prod_{1\leqslant j<k\leqslant n} d_v(x_j-x_k),\quad C = \prod_{1\leqslant j<k\leqslant n} c_v(x_j+x_k)\prod_{1\leqslant j\leqslant n} c_w(x_j), \tag{4.14}$$

with d_v, c_v, and c_w defined in Appendix A. The commutativity of $\hat{H}_{l,\,\text{RCM}}$ and thus also of $\hat{\mathscr{H}}_{l,\,\text{lead}}$ follows from Ref. 5. Notice that $\hat{\mathscr{H}}_{l,\,\text{lead}} = \lim_{R\to\infty}\exp(-l\beta\hbar R)\,\Lambda_R^{-1}\hat{\mathscr{H}}_l\,\Lambda_R$ with $\Lambda_R = \exp(-iR(x_1+\cdots+x_n))$, so the RCM system can be seen as a limiting case of the system studied here.

ii. It is not difficult to see that operators $\hat{\mathscr{H}}_l$ of the form (2.9)–(2.13) annihilate constant functions [for $l=1$ this is immediate from (2.1)]. This implies that, at least formally, the function $\Delta^{1/2}$ of Appendix A is a joint eigenfunction with eigenvalue zero of the corresponding Hermitian operators $\hat{H}_l$ (2.15) [we assume parameters according to (2.14)]. In the trigonometric case this eigenfunction corresponds to the ground state of $\hat{H}$ (2.1) (cf. Remark *iii.* of Sec. II with $\lambda = 0$). In other words: the system has a factorized (or Jastrow-like) wave function representing the ground state.

A similar picture holds for the systems with potentials of type I, II, and for the special system with elliptic potentials of Conjecture 4.1. However, in the latter cases it has not been demonstrated yet that $\hat{H}$ (2.1) is positive, and for type I or II potentials the wave function $\Delta^{1/2}$ does not correspond to a true bound state because it is not normalizable.

It is most unlikely that the ground state wave function of the Hamiltonian (4.8) with elliptic potentials (3.19), (3.20) factorizes for general parameters. Indeed, the function $\Delta^{1/2}$ is no longer an eigenfunction of $\hat{H}_1$ (4.8).

iii. Sending periods to infinity in the special system of Conjecture 4.1 leads to operators $\hat{\mathscr{H}}_l$ (2.9)–(2.13) with type I–III potentials corresponding to a special value for the parameter μ. Recall that in a similar transition from hyperbolic/trigonometric potentials to rational ones, a more general external field is obtained if first certain coupling constants are shifted over a half-period (Remark *ii.* of Sec. II). The same phenomenon occurs here; after introducing the appropriate shifts we obtain operators (2.9)–(2.13) with the following potentials. (For convenience we have chosen $s = 1$.)

$\omega_1\to\infty$, $\omega_2 = i\pi/(2\alpha)$ $(\mu_1\to\mu_1-\omega_1,\ \mu_3\to\mu_3+\omega_1)$

$$v(z) = 1/\text{sh}\,(\alpha z), \tag{4.15}$$

$$w(z) = \frac{\text{sh } \alpha(\mu_0+z)\text{sh } \alpha(\mu_1+z)\text{ch } \alpha(\mu_2+z)\text{ch } \alpha(\mu_3+z)}{\text{sh } (\alpha z)\text{ch } (\alpha z)} \frac{\text{sh } \alpha(\mu_0'+\gamma+z)}{\text{sh } \alpha(\gamma+z)} ; \quad (4.16)$$

$\omega_1 = \pi/(2\alpha), \quad \omega_2 \to i\infty \ (\mu_2 \to \mu_2 - \omega_2, \ \mu_3 \to \mu_3 + \omega_2)$

$$v(z) = \cot(\alpha z), \quad (4.17)$$

$$w(z) = \frac{\sin \alpha(\mu_0+z)\cos \alpha(\mu_1+z)\sin \alpha(\mu_2+z)\cos \alpha(\mu_3+z)}{\sin(\alpha z) \cos(\alpha z)}$$

$$\times \frac{\sin \alpha(\mu_0'+\gamma+z)\cos \alpha(\mu_1'+\gamma+z)}{\sin \alpha(\gamma+z) \cos \alpha(\gamma+z)} ; \quad (4.18)$$

$\omega_1 \to \infty, \quad \omega_2 \to i\infty \ (\mu_r \to \mu_r - \omega_r)$

$$v(z) = 1/z, \quad (4.19)$$

$$w(z) = \frac{(\mu_0+z)(\mu_1+z)(\mu_2+z)(\mu_3+z)(\mu_0'+\gamma+z)}{z(\gamma+z)} . \quad (4.20)$$

The above transitions can be verified with the aid of the product representation of the σ-function.[3] Using this representation one derives the asymptotics of the potentials v (3.16) and w (3.17). The potential w gives rise to a multiplicative constant converging to zero. Because the operator $\hat{\mathscr{H}}_l$ (2.9)–(2.13) is homogeneous in w, it is possible to collect such multiplicative constants into an overall factor. In order to get the above results, one should divide by this factor before sending the periods to infinity. This type of renormalization is to be compared with the division by α^{2l} in the before-mentioned transition II/III→I (see Remark *ii.*, Sec. II).

It should be stressed that the cases (4.15), (4.16), and (4.17), (4.18), are different in the sense that they are not connected via analytic continuation. Notice also that for the potentials (4.16) and (4.20) Condition (3.18) ($s=1$) may be omitted because the dependence on μ_1' has dropped out in the limit.

By studying similar limiting cases of the Hamiltonian (4.8) we have found,[14] besides further generalizations of the models in Sec. II, certain difference Toda chains with boundary conditions generalizing the (free-end) relativistic Toda system.[15] For $n=2$ the integrability of these novel models is an immediate consequence of Theorem 3.3.

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V. THE CLASSICAL SYSTEM: LIOUVILLE INTEGRABILITY

We conclude this paper with some results regarding the classical counterpart of our difference CM models. Let us first restrict to the case of type I–III potentials. The classical counterpart $\mathscr{H}_l$ of the operator $\hat{\mathscr{H}}_l$ (2.9)–(2.13) is obtained by substituting real variables θ_j for the partials $\hat{\theta}_j$ and setting $\gamma = 0$ [cf. Eq. (2.7)]. It is explained in Appendix C that the Poisson commutativity of $\mathscr{H}_1,\ldots,\mathscr{H}_n$ is an immediate consequence of the quantum integrability of the difference system. Thus, we arrive at the following corollary of Theorem 2.1:

Corollary 5.1 (Liouville integrability): The classical versions $\mathscr{H}_1,\ldots,\mathscr{H}_n$ of the commuting Hamiltonians in Theorem 2.1 are in involution (with respect to the Poisson bracket induced by the standard symplectic form $\omega = \Sigma_j \, dx_j \wedge d\theta_j$).

Similar substitutions in Eq. (2.15) lead to the classical version H_l of the Hermitian operator $\hat{H}_l$. The integrals $\mathscr{H}_l$ and H_l are connected by means of the canonical gauge transformation in Eqs. (A4), (A5) (Appendix A). The classical counterpart of Theorem 2.2 becomes:

Corollary 5.2 (transition to the BC_n-type CM system): The limits

$$H_{l,0} = \lim_{\beta \to 0} \beta^{-2l} H_l(\beta), \quad l = 1, \ldots, n, \tag{5.1}$$

exist and the resulting functions $H_{1,0}, \ldots, H_{n,0}$ Poisson commute.

For $l = 1$ we recover the classical BC_n-type CM Hamiltonian, which is given by the classical version of (2.19) (obtained by substituting $\hat{\theta}_j \mapsto \theta_j$ and $\hbar = 0$).

Next, we turn to the case of elliptic potentials. Up to a factor two the classical Hamiltonian associated with $\hat{H}_1$ (4.8) reads

$$H_1 = \sum_{1 \leqslant j \leqslant n} \prod_{0 \leqslant r \leqslant 3} f_{r,\mu_r}(x_j) f_{r,\mu_r'}(x_j) \prod_{k \neq j} f(x_j - x_k) f(x_j + x_k) \; \mathrm{ch}\beta\theta_j + \sum_{0 \leqslant r \leqslant 3} c_r \prod_{1 \leqslant j \leqslant n} f_{r,\mu}^2(x_j), \tag{5.2}$$

with

$$f(z) = \left(\frac{\sigma(\mu+z)\sigma(\mu-z)}{\sigma(z)\sigma(-z)} \right)^{1/2}, \quad f_{r,\nu}(z) = \left(\frac{\sigma_r(\nu+z)\sigma_r(\nu-z)}{\sigma_r(z)\sigma_r(-z)} \right)^{1/2}, \tag{5.3}$$

and

$$c_r = \sigma(\mu)^{-2} \prod_{0 \leqslant s \leqslant 3} \sigma_s(\mu_{\pi_r(s)}) \sigma_s(\mu'_{\pi_r(s)}). \tag{5.4}$$

For $\beta \to 0$ we now recover the Inozemtsev Hamiltonian (1.3) [cf. Expansion (4.9)–(4.11) with $\hat{\theta}_j \mapsto \theta_j$ and $\hbar = 0$].

In the case of two particles the Liouville integrability of H_1 (5.2) [with Condition (3.21)] is a consequence of Theorem 3.3 and Appendix C. The $n = 2$ specialization of the Hamiltonian (5.2) unifies various two-particle models for which the Liouville integrability was already demonstrated in Ref. 16. In particular, the situation that all parameters μ_r' are equal to zero (so $c_r = 0$) corresponds to a model studied there.

For some special values of the coupling constants H_1 (5.2)–(5.4) can be seen as a reduction of the RCM Hamiltonian (1.2). More precisely, by setting $x_{m+1-j} = -x_j$ and $\theta_{m+1-j} = -\theta_j$ in the m-particle version of H_{RCM} (1.2) one obtains (up to a factor 2)

$$H_{\mathrm{red}} = \sum_{1 \leqslant j \leqslant n} f(2x_j) \prod_{k \neq j} f(x_j + x_k) f(x_j - x_k) \; \mathrm{ch}\beta\theta_j \tag{5.5}$$

if $m = 2n$, and

$$H_{\mathrm{red}} = \sum_{1 \leqslant j \leqslant n} f(x_j) f(2x_j) \prod_{k \neq j} f(x_j + x_k) f(x_j - x_k) \; \mathrm{ch}\beta\theta_j \quad + 1/2 \prod_{1 \leqslant j \leqslant n} f^2(x_j) \tag{5.6}$$

if $m = 2n + 1$. [Because of Eq. (3.8), the function $f(z)$ used here is essentially the same as the one used in the introduction.] With the aid of the duplication formula for σ-functions (3.7) one verifies that H_1 (5.2) reduces to (5.5) for $\mu_r = \mu/2$, $\mu_r' = 0$, and to (5.6) if instead of taking μ_0' equal to zero one sets $\mu_0' = \mu$.

A complete set of (parity invariant) integrals for the (m-particle) RCM model is given by the classical version of $(\hat{H}_{l,\,\mathrm{RCM}} + \hat{H}_{m-l,\,\mathrm{RCM}} \hat{H}_{m,\,\mathrm{RCM}}^{-1})$ [cf. Eq. (4.13) with the convention $\hat{H}_{0,\,\mathrm{RCM}} \equiv 1$]:[5]

$$H_{l,\mathrm{rel}} \sim \sum_{\substack{J \subset \{1,\ldots,m\} \\ |J|=l}} \prod_{\substack{j \in J \\ k \in J^c}} f(x_j - x_k)\, \mathrm{ch}\beta\theta_J, \quad l=1,\ldots,m. \tag{5.7}$$

The Hamiltonian H_{RCM} (1.2) corresponds to $l=1$. If the initial particle-configuration is chosen invariant with respect to reflection of the particles in the origin, then the commuting $H_{l,\mathrm{rel}}$-flows preserve this symmetry. By computing the Hamiltonians that generate the reduced motion (i.e., setting as above $x_{m+1-j}=-x_j$ and $\theta_{m+1-j}=-\theta_j$), one finds a complete set of integrals for H_{red} (5.5), (5.6); the structure of the resulting integrals is compatible with Conjecture 4.2:

$$H_{l,\mathrm{red}} = \sum_{\substack{J \subset \{1,\ldots,n\},\, |J| \leqslant l \\ \varepsilon_j = \pm 1,\, j \in J}} U_{J^c,l-|J|}\, F_{\varepsilon J;J^c}\, \mathrm{ch}\beta\theta_{\varepsilon J}, \quad l=1,\ldots,n \tag{5.8}$$

with

$$F_{\varepsilon J;K} = \prod_{\substack{j,j' \in J \\ j<j'}} f^2(\varepsilon_j x_j + \varepsilon_{j'} x_{j'}) \prod_{\substack{j \in J \\ k \in K}} f(x_j+x_k) f(x_j-x_k)$$

$$\times \begin{cases} \displaystyle\prod_{j \in J} f(2x_j), & m=2n \\[2ex] \displaystyle\prod_{j \in J} f(2x_j) f(x_j), & m=2n+1 \end{cases} \tag{5.9}$$

and

$$U_{I,p} = \sum_{\substack{I' \subset I \\ |I'|=[p/2]}} \prod_{\substack{i' \in I' \\ i \in \Lambda\, I'}} f^2(x_{i'}+x_i) f^2(x_{i'}-x_i)$$

$$\times \begin{cases} 0 \ (p \text{ odd}), \quad 1 \ (p \text{ even}), & m=2n \\[2ex] \displaystyle\prod_{i \in \Lambda\, I'} f^2(x_i) \ (p \text{ odd}), \quad \prod_{i' \in I'} f^2(x_{i'}) \ (p \text{ even}), & m=2n+1 \end{cases} \tag{5.10}$$

(here $[p/2]$ denotes the integer part of $p/2$). Unfortunately, it does not seem straightforward to generalize the integrals $H_{l,\mathrm{red}}$ to the situation where the coupling constants of the external field are not related to μ.

Remarks: i. For type I/II potentials the above reduction of the classical relativistic Calogero–Moser system has been studied in Ref. 17 (Sec. 5B); it is shown that the action-angle transformation for the reduced system is obtained by restricting the action-angle map for the relativistic system.

ii. Recently it has been observed that similar reductions of the nonrelativistic CM system[1] may be viewed as real cases of more general 'duplication' procedures in the complex plane;[18] these duplication methods give rise to new types of integrable n-particle models on the line.

ACKNOWLEDGMENTS

APPENDIX A: A GAUGE TRANSFORMATION

If the functions $c_v(z)$, $c_w(z)$ ($\neq 0$) satisfy the difference equations

$$c_v(z+i\beta\hbar)=c_v(z)/v(z), \quad c_w(z+i\beta\hbar)=c_w(z)/w(z), \tag{A1}$$

then it is not difficult to see that conjugation with $\Delta^{1/2}$, where

$$\Delta = \prod_{1\leqslant j<k\leqslant n} d_v(x_j+x_k)d_v(x_j-x_k) \prod_{1\leqslant j\leqslant n} d_w(x_j), \tag{A2}$$

and

$$d_v(z)=1/(c_v(z)\,c_v(-z)), \quad d_w(z)=1/(c_w(z)\,c_w(-z)), \tag{A3}$$

transforms operators $\hat{\mathscr{H}}_l$ of the form (2.9) into operators $\hat{H}_l$ of the form (2.15). The transition $\mathscr{H}_l \mapsto H_l$ for the corresponding classical systems (Sec. V) boils down to a canonical gauge transformation:

$$x_j \mapsto x_j. \tag{A5}$$

$$\theta_j \mapsto \theta_j + \frac{1}{2\beta}\ln w(x_j) + \frac{1}{2\beta}\sum_{k\neq j}(\ln v(x_j+x_k)+\ln v(x_j-x_k))$$

$$-\frac{1}{2\beta}\ln w(-x_j) - \frac{1}{2\beta}\sum_{k\neq j}(\ln v(-x_j+x_k)+\ln v(-x_j-x_k)), \tag{A4}$$

It follows from classical results that the first-order difference equations (A1) have nontrivial solutions,[19] which are unique up to periodic multiplicative factors. In general, the solutions of (A1) may be badly singular. It turns out, however, that for all potentials of interest there exist solutions $c_v(z)$, $c_w(z)$ that are meromorphic in z.

The potentials $v(z)$ and $w(z)$ are of the form

$$v(z)=\frac{s(i\beta g+z)}{s(z)}, \quad w(z)=\prod_r \frac{s_r(i\beta g_r+z)\,s_r(i\beta g_r'+\gamma+z)}{s_r(z)\,s_r(\gamma+z)} \tag{A6}$$

with I: $s(z),s_0(z)=z$; II: $s(z),s_0(z)=\text{sh}(\alpha z)$, $s_1(z)=\text{ch}(\alpha z)$; III: $s(z),s_0(z)=\sin(\alpha z)$, $s_1(z)=\cos(\alpha z)$; IV: $s(z),s_0(z)=\sigma(z)$, $s_r(z)=\sigma_r(z)$ ($1\leqslant r\leqslant 3$). For convenience, $\hbar$ will be taken equal to one from now on. For $g,g_r\in\mathbb{N}$ it is now straightforward to write down meromorphic solutions of (A1):

$$c_v(z)=\prod_{0\leqslant k<g} s^{-1}(i\beta k+z), \tag{A7}$$

$$c_w(z)=\prod_r \prod_{0\leqslant k_r<g_r} s_r^{-1}(i\beta k_r+z) \prod_{0\leqslant k_r'<g_r'} s_r^{-1}(i\beta k_r'+\gamma+z). \tag{A8}$$

In the rational and the trigonometric case one can generalize (A7) and (A8) to (meromorphic) solutions valid for arbitrary $g,g_r\geqslant 0$ by rewriting in terms of gamma functions or q-shifted factorials, respectively (recall $\gamma=i\beta/2$):

I. Rational case

$$c_v(z) = \frac{\Gamma(z/i\beta)}{\Gamma(g+z/i\beta)} , \quad c_w(z) = \frac{\Gamma(z/i\beta)\Gamma(1/2+z/i\beta)}{\Gamma(g_0+z/i\beta)\Gamma(g_0'+1/2+z/i\beta)} ; \tag{A9}$$

III. Trigonometric case

$$c_v(z) = e^{i\alpha zg} \frac{(e^{-2\alpha\beta g}e^{2i\alpha z}; \ e^{-2\alpha\beta})_\infty}{(e^{2i\alpha z}; \ e^{-2\alpha\beta})_\infty} , \tag{A10}$$

$$c_w(z) = e^{i\alpha z(g_0+g_1+g_0'+g_1')}$$

$$\times \frac{(e^{-2\alpha\beta g_0}e^{2i\alpha z}, \ -e^{-2\alpha\beta g_1}e^{2i\alpha z}, \ e^{-2\alpha\beta(g_0'+1/2)}e^{2i\alpha z}, \ -e^{-2\alpha\beta(g_1'+1/2)}e^{2i\alpha z}; e^{-2\alpha\beta})_\infty}{(e^{4i\alpha z}; \ e^{-2\alpha\beta})_\infty} .$$

$$\tag{A11}$$

In the case of hyperbolic and elliptic potentials meromorphic solutions $c_v(z)$ valid for all $g \geq 0$ have been obtained by Ruijsenaars;[6,20] in the latter situations $c_v(z)$ is given by an integral formula (hyperbolic case) or in terms of a product expansion (elliptic case). Appropriate shifts over half-periods and over γ lead to the corresponding $c_w(z)$.

Remark: In the hyperbolic case the system is periodic in the parameters g , g_r, with period $2\pi/(\alpha\beta)$. Thus, if this period is not a rational number, then the parameter values for which c_v and c_w can be written in terms of the elementary hyperbolic functions (A7), (A8), form a dense subset.

APPENDIX B: A LEMMA

In this appendix it is shown that certain quasiperiodic functions are identically zero. The result was used in Sec. III to prove the integrability of the two-particle difference model with elliptic potentials. As usual $2\omega_r$, $r = 1,2,3$, denote the periods of the Weierstrass functions and $\eta_r = \zeta(\omega_r)$.

Lemma B.1: Let $K_\mu(z)$ be a complex function such that

i. $K_\mu(z)$ *is entire in both z and μ;*
ii. $K_\mu(z)$ *is quasiperiodic in z: $K_\mu(z+2\omega_r) = e^{2\eta_r p(\mu)} K_\mu(z)$, $r=1,2$, with $p(\mu)$ a certain polynomial of degree ≥ 1 in μ.*

Then $K_\mu(z)$ must be the zero function.

Proof: Consider

$$G(w) \equiv e^{-[\eta_1 p(\mu)/\pi i]\ln w} K_\mu\left(\frac{\omega_1}{\pi i}\ln w\right). \tag{B1}$$

Because of *i.* and *ii.*, $G(w)$ is holomorphic and univalent in w on $\mathbb{C}\backslash\{0\}$; therefore, $G(w)$ has a Laurent expansion around zero, which converges for all $w \neq 0$

$$G(w) = \sum_{n \in \mathbb{Z}} k_n(\mu) w^n. \tag{B2}$$

Substituting $w = e^{[\pi i/\omega_1]z}$ yields a series expansion for $K_\mu(z)$, which is valid for all $z \in \mathbb{C}$

$$K_\mu(z) = e^{[\eta_1 p(\mu)/\omega_1]z} \sum_{n \in \mathbf{Z}} k_n(\mu) e^{(n\pi i/\omega_1)z}.$$ (B3)

It follows from the quasiperiodicity relation *ii.* for $r = 2$ [using Legendre (3.3)] that the coefficients $k_n(\mu)$ satisfy the relation

$$k_n(\mu) = e^{(\pi i/\omega_1)(p(\mu) + 2n\omega_2)} k_n(\mu).$$

Thus, $k_n(\mu)$ must be zero.

$\square$

APPENDIX C: THE CLASSICAL LIMIT

In this appendix we consider n-particle difference Hamiltonians of the form (finite summation)

$$\hat{H} = \sum_p V_p(x, \hbar)\, e^{-\kappa_p \cdot \hat{\theta}}$$ (C1)

with $\kappa_p \in \mathbf{R}^n$ and $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_n)$. It will be assumed that the coefficients $V_p(x, \hbar)$ are holomorphic in $\hbar$ at $\hbar = 0$, and holomorphic in x on $\mathcal{U} + i\,\mathbf{R}^n$, where $\mathcal{U}$ is an open dense subset of $\mathbf{R}^n$ (these assumptions correspond to our applications). The classical Hamiltonian associated with $\hat{H}$ reads

$$H = \sum_p V_p(x,\ 0) e^{-\kappa_p \cdot \theta} \quad (\theta \in \mathbf{R}^n).$$ (C2)

Our aim is to show that the classical version of $[\hat{H}_1, \hat{H}_2]/i\hbar$ coincides with $\{H_1, H_2\}$, i.e., the assignment $\hat{H} \mapsto H$ is a Lie algebra homomorphism. ($[\cdot, \cdot]$ denotes the commutator product and $\{\cdot, \cdot\}$ is the Poisson bracket induced by the standard symplectic form $\omega = \Sigma_j\, dx_j \wedge d\theta_j$.)

Clearly, it is sufficient to consider monomials (bilinearity of the brackets):

$$\hat{H}_p = V_p(x, \hbar)\, e^{-\kappa_p \cdot \hat{\theta}}, \quad H_p = V_p(x, 0)\, e^{-\kappa_p \cdot \theta}, \quad p = 1, 2.$$ (C3)

The relevant brackets are of the form:

$$[\hat{H}_1, \hat{H}_2] = V_{[1,2]}(x, \hbar)\, e^{-(\kappa_1 + \kappa_2) \cdot \hat{\theta}},$$ (C4)

$$\{H_1, H_2\} = V_{\{1,2\}}(x)\, e^{-(\kappa_1 + \kappa_2) \cdot \theta}.$$ (C5)

Proposition C.1: One has

$$\lim_{\hbar \to 0} \frac{1}{i\hbar}\, V_{[1,2]}(x, \hbar) = V_{\{1,2\}}(x).$$ (C6)

Proof: Working out the commutator product/Poisson bracket of (C3) yields for the r.h.s. of (C4), (C5):

$$V_{[1,2]}(x, \hbar) = V_1(x, \hbar)\, V_2(x + i\hbar\kappa_1, \hbar) - V_2(x, \hbar)\, V_1(x + i\hbar\kappa_2, \hbar),$$ (C7)

$$V_{\{1,2\}}(x) = V_1(x, 0)\, (\kappa_1 \cdot \nabla V_2)(x, 0) - V_2(x, 0)\, (\kappa_2 \cdot \nabla V_1)(x, 0).$$ (C8)

Taylor expansion of (C7) around $\hbar = 0$ entails

$$V_{[1,2]}(x,\hbar) = i\hbar\, V_{\{1,2\}}(x) + O(\hbar^2).$$ (C9)

$\square$

Corollary C.2: One has

$$[\hat{H}_1, \hat{H}_2] = 0 \;\Rightarrow\; \{H_1, H_2\} = 0.$$ (C10)

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