INTEGRABILITY OF DISTRIBUTIONS IN *GCR*-LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KAEHLER MANIFOLDS

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Abstract. In present paper we establish conditions for the integrability of various distributions of *GCR*-lightlike submanifolds and obtain conditions for the distributions to define totally geodesic foliations in *GCR*-lightlike submanifolds.

1. Introduction

The study of geometry of Cauchy-Riemann (*CR*)-submanifolds in Kaehler manifolds was initiated by Bejancu [2], which include holomorphic and totally real submanifolds as special cases and has been further studied by Bejancu [3, 4], Chen [6], Duggal [7, 8], Yano and Kon [17, 18] and others.

The geometry of lightlike submanifolds was initiated by Kupeli and further developed by Bejancu and Duggal [9] and they also introduced the notion of *CR*-lightlike submanifolds of indefinite Kaehler manifolds. But this class of submanifolds exclude the complex and totally real submanifolds as subcases. Later on, Duggal and Sahin [11] introduced *SCR*-lightlike submanifolds of indefinite Kaehler manifolds. Since there was no inclusion relation between *CR* and *SCR* cases therefore Duggal and Sahin [12] introduced a new class called *GCR*-lightlike submanifolds of indefinite Kaehler manifolds which is an umbrella for all these types of submanifolds. In present paper we establish conditions for the integrability of various distributions of *GCR*-lightlike submanifolds and obtain conditions for the distributions to define totally geodesic foliations in *GCR*-lightlike submanifolds.

2. Lightlike submanifolds

We recall notations and fundamental equations for lightlike submanifolds, which are due to the book [9] by Duggal and Bejancu.

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m + n - 1$ and (M, g) be an

m-dimensional submanifold of $\bar{M}$ and g the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M, then M is called a lightlike submanifold of $\bar{M}$. For a degenerate metric g on M

$$(1) \qquad TM^{\perp} = \cup\{u \in T_x\bar{M} : \bar{g}(u,v) = 0, \forall v \in T_xM, x \in M\},$$

is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $\text{Rad}T_xM = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping

$$(2) \qquad \text{Rad}TM : x \in M \longrightarrow \text{Rad}T_xM,$$

defines a smooth distribution on M of rank $r > 0$, then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and $\text{Rad}TM$ is called the radical distribution on M.

Screen distribution $S(TM)$ is a semi-Riemannian complementary distribution of $\text{Rad}(TM)$ in TM, that is

$$(3) \qquad TM = \text{Rad}TM \perp S(TM)$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $\text{Rad}TM$ in $TM^{\perp}$. Let $\text{tr}(TM)$ and $\text{ltr}(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $\text{Rad}TM$ in $S(TM^{\perp})^{\perp}$ respectively. Then we have

$$(4) \qquad \text{tr}(TM) = \text{ltr}(TM) \perp S(TM^{\perp}),$$

$$(5) \quad T\bar{M}\mid_M = TM \oplus \text{tr}(TM) = (\text{Rad}TM \oplus \text{ltr}(TM)) \perp S(TM) \perp S(TM^{\perp}).$$

Let u be a local coordinate neighborhood of M and consider the local quasi-orthonormal fields of frames of $\bar{M}$ along M, on u as $\{\xi_1, \ldots, \xi_r, W_{r+1}, \ldots, W_n, N_1, \ldots, N_r, X_{r+1}, \ldots, X_m\}$, where $\{\xi_1, \ldots, \xi_r\}, \{N_1, \ldots, N_r\}$ are local lightlike bases of $\Gamma(\text{Rad}TM\mid_u)$, $\Gamma(\text{ltr}(TM)\mid_u)$ and $\{W_{r+1}, \ldots, W_n\}, \{X_{r+1}, \ldots, X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp})\mid_u)$ and $\Gamma(S(TM)\mid_u)$ respectively. For this quasi-orthonormal fields of frames, we have:

Theorem 2.1 ([9]). *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $\text{ltr}(TM)$ of $\text{Rad}TM$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(\text{ltr}(TM)\mid_u)$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}\mid_u$, where u is a coordinate neighborhood of M such that*

$$(6) \qquad \bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0 \ \ \text{for any} \ \ i, j \in \{1, 2, \ldots, r\},$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $\Gamma(\text{Rad}(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ then according to the decomposition (5), the Gauss and Weingarten formulas are given by

$$(7) \qquad \bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \forall \ X, Y \in \Gamma(TM),$$

$$(8) \qquad \bar{\nabla}_X U = -A_U X + \nabla_X^{\perp} U, \quad \forall \ X \in \Gamma(TM), U \in \Gamma(\text{tr}(TM)),$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla_X^\perp U\}$ belong to $\Gamma(TM)$ and $\Gamma(\text{tr}(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, A_U is a linear operator on M and known as shape operator.

According to (4), considering the projection morphisms L and S of $\text{tr}(TM)$ on $\text{ltr}(TM)$ and $S(TM^\perp)$, respectively, then (7) and (8) become

$$(9) \qquad \bar{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y),$$

$$(10) \qquad \bar{\nabla}_X U = -A_U X + D_X^l U + D_X^s U,$$

where we put $h^l(X,Y) = L(h(X,Y))$, $h^s(X,Y) = S(h(X,Y))$, $D_X^l U = L(\nabla_X^\perp U)$, $D_X^s U = S(\nabla_X^\perp U)$.

As h^l and h^s are $\Gamma(\text{ltr}(TM))$-valued and $\Gamma(S(TM^\perp))$-valued respectively, therefore they are called as the lightlike second fundamental form and the screen second fundamental form on M. In particular

$$(11) \qquad \bar{\nabla}_X N = -A_N X + \nabla_X^l N + D^s(X,N),$$

$$(12) \qquad \bar{\nabla}_X W = -A_W X + \nabla_X^s W + D^l(X,W),$$

where $X \in \Gamma(TM)$, $N \in \Gamma(\text{ltr}(TM))$ and $W \in \Gamma(S(TM^\perp))$. Using (9)-(12) we obtain

$$(13) \qquad \bar{g}(h^s(X,Y), W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y),$$

$$(14) \qquad \bar{g}(h^l(X,Y), \xi) + \bar{g}(Y, h^l(X,\xi)) + g(Y, \nabla_X \xi) = 0,$$

$$(15) \qquad \bar{g}(A_N X, N') + \bar{g}(N, A_{N'} X) = 0$$

for any $\xi \in \Gamma(\text{Rad}TM)$, $W \in \Gamma(S(TM^\perp))$ and $N, N' \in \Gamma(\text{ltr}(TM))$.

Let P be the projection morphism of TM on $S(TM)$ then using (3), we can induce some new geometric objects on the screen distribution $S(TM)$ on M as

$$(16) \qquad \nabla_X PY = \nabla_X^* PY + h^*(X, PY),$$

$$(17) \qquad \nabla_X \xi = -A_\xi^* X + \nabla_X^{*t} \xi$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\text{Rad}TM)$, where $\{\nabla_X^* PY, A_\xi^* X\}$ and $\{h^*(X, PY), \nabla_X^{*t} \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(\text{Rad}TM)$, respectively. ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $\text{Rad}TM$, respectively. h^* and A^* are $\Gamma(\text{Rad}TM)$-valued and $\Gamma(S(TM))$-valued bilinear forms and called as the second fundamental forms of distributions $S(TM)$ and $\text{Rad}TM$, respectively.

Using (9), (10), (16) and (17), we obtain

$$(18) \qquad \bar{g}(h^l(X, PY), \xi) = g(A_\xi^* X, PY),$$

$$(19) \qquad \bar{g}(h^*(X, PY), N) = \bar{g}(A_N X, PY)$$

for any $X, Y \in \Gamma(TM), \xi \in \Gamma(\text{Rad}(TM))$ and $N \in \Gamma(\text{ltr}(TM))$.

From the geometry of Riemannian submanifolds and non degenerate submanifolds, it is known that the induced connection ∇ on a non degenerate submanifold is a metric connection. Unfortunately, this is not true for a lightlike submanifold. Indeed, considering $\bar{\nabla}$ a metric connection then we have

$$(20) \qquad (\nabla_X g)(Y, Z) = \bar{g}(h^l(X, Y), Z) + \bar{g}(h^l(X, Z), Y)$$

for any $X, Y, Z \in \Gamma(TM)$. From [9] page 171, using the properties of linear connections we have

$$(21) \qquad (\nabla_X h^l)(Y, Z) = \nabla_X^l(h^l(Y, Z)) - h^l(\nabla_X Y, Z) - h^l(Y, \nabla_X Z),$$

$$(22) \qquad (\nabla_X h^s)(Y, Z) = \nabla_X^s(h^l(Y, Z)) - h^s(\nabla_X Y, Z) - h^s(Y, \nabla_X Z).$$

Barros and Romero [1] defined indefinite Kaehler manifolds as:

Definition 2.2. Let $(\bar{M}, \bar{J}, \bar{g})$ be an indefinite almost Hermitian manifold and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$. Then $\bar{M}$ is called an indefinite Kaehler manifold if $\bar{J}$ is parallel with respect to $\bar{\nabla}$, that is

$$(23) \qquad (\bar{\nabla}_X \bar{J})Y = 0, \quad \forall \quad X, Y \in \Gamma(T\bar{M}).$$

3. Generalized Cauchy-Riemann lightlike submanifolds

Definition 3.1 ([12])**.** Let $(M, g, S(TM))$ be a real lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g}, \bar{J})$ then M is called a generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied:

(A) There exist two subbundles D_1 and D_2 of $\text{Rad}(TM)$ such that

$$(24) \qquad \text{Rad}(TM) = D_1 \oplus D_2, \quad \bar{J}(D_1) = D_1, \quad \bar{J}(D_2) \subset S(TM).$$

(B) There exist two subbundles D_0 and D' of $S(TM)$ such that

$$(25) \qquad S(TM) = \{\bar{J}D_2 \oplus D'\} \perp D_0, \quad \bar{J}(D_0) = D_0, \quad \bar{J}(D') = L_1 \perp L_2,$$

where D_0 is a non degenerate distribution on M, L_1 and L_2 are vector subbundle of $\text{ltr}(TM)$ and $S(TM)^\perp$, respectively.

Then the tangent bundle TM of M is decomposed as

$$(26) \qquad TM = D \perp D', \quad D = \text{Rad}(TM) \oplus D_0 \oplus \bar{J}D_2.$$

M is called a proper GCR-lightlike submanifold if $D_1 \neq \{0\}, D_2 \neq \{0\}, D_0 \neq \{0\}$ and $L_2 \neq \{0\}$.

Let Q, P_1 and P_2 be the projections on $D, \bar{J}(L_1) = M_1 \subset D'$ and $\bar{J}(L_2) = M_2 \subset D'$, respectively. Then for any $X \in \Gamma(TM)$ we have

$$(27) \qquad X = QX + P_1 X + P_2 X,$$

applying $\bar{J}$ to (27) we obtain

$$(28) \qquad \bar{J}X = TX + wP_1 X + wP_2 X,$$

and we can write (28) as

$$(29) \qquad \bar{J}X = TX + wX,$$

where TX and wX are the tangential and transversal components of $\bar{J}X$, respectively.

Similarly

$$(30) \qquad \bar{J}V = BV + CV$$

for any $V \in \Gamma(\text{tr}(TM))$, where BV and CV are the sections of TM and $\text{tr}(TM)$, respectively.

Differentiating (28) and using (9)-(12) and (30) we have

$$(31) \quad D^s(X, wP_1Y) = -\nabla^s_X wP_2Y + wP_2\nabla_XY - h^s(X, TY) + Ch^s(X, Y),$$

$$(32) \quad D^l(X, wP_2Y) = -\nabla^l_X wP_1Y + wP_1\nabla_XY - h^l(X, TY) + Ch^l(X, Y).$$

Using Kaehlerian property of $\bar{\nabla}$ with (11) and (12), we have the following lemmas.

Lemma 3.2. *Let M be a GCR-lightlike submanifold of an indefinite Kaehlerian manifold $\bar{M}$. Then we have*

$$(33) \qquad (\nabla_X T)Y = A_{wY}X + Bh(X, Y)$$

and

$$(34) \qquad (\nabla^t_X w)Y = Ch(X, Y) - h(X, TY),$$

where $X, Y \in \Gamma(TM)$ and

$$(35) \qquad (\nabla_X T)Y = \nabla_X TY - T\nabla_X Y,$$

$$(36) \qquad (\nabla^t_X w)Y = \nabla^t_X wY - w\nabla_X Y.$$

Lemma 3.3. *Let M be a GCR-lightlike submanifold of an indefinite Kaehlerian manifold $\bar{M}$. Then we have*

$$(37) \qquad (\nabla_X B)V = A_{CV}X - TA_V X$$

and

$$(38) \qquad (\nabla^t_X C)V = -wA_V X - h(X, BV),$$

where $X \in \Gamma(TM)$, $V \in \Gamma(\text{tr}(TM))$ and

$$(39) \qquad (\nabla_X B)V = \nabla_X BV - B\nabla^t_X V,$$

$$(40) \qquad (\nabla^t_X C)V = \nabla^t_X CV - C\nabla^t_X V.$$

4. Integrability of the distributions

Theorem 4.1. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. If the distribution D is integrable, then the following assertions hold*

    (i) $\bar{g}(D^l(\bar{J}X, W), Y) = \bar{g}(D^l(X, W), \bar{J}Y) \Leftrightarrow \bar{g}(A_W \bar{J}X, Y) = \bar{g}(A_W X, \bar{J}Y)$.
    (ii) $\bar{g}(D^l(\bar{J}X, W), \xi) = \bar{g}(A_W X, \bar{J}\xi)$.
    (iii) $\bar{g}(D^l(X, W), \xi) = \bar{g}(A_W \bar{J}X, \bar{J}\xi)$ *for any* $X, Y \in \Gamma(D), \xi \in \Gamma(D_2)$ *and* $W \in \Gamma(S(TM^\perp))$.

Proof. Let the distribution D be integrable then for $X, Y \in \Gamma(D)$ and $W \in \Gamma(S(TM^\perp))$ we have

$$
\begin{aligned}
\bar{g}(D^l(\bar{J}X, W), Y) &= \bar{g}(\bar{\nabla}_{\bar{J}X} W + A_W \bar{J}X - \nabla^s_{\bar{J}X} W, Y) \\
&= -\bar{g}(W, \bar{\nabla}_{\bar{J}X} Y) + \bar{g}(A_W \bar{J}X, Y) \\
&= -\bar{g}(W, h(\bar{J}X, Y)) + \bar{g}(A_W \bar{J}X, Y) \\
&= -\bar{g}(W, h(X, \bar{J}Y)) + \bar{g}(A_W \bar{J}X, Y) \\
&= -\bar{g}(W, \bar{\nabla}_X \bar{J}Y) + \bar{g}(A_W \bar{J}X, Y) \\
&= \bar{g}(\bar{\nabla}_X W, \bar{J}Y) + \bar{g}(A_W \bar{J}X, Y) \\
&= -\bar{g}(A_W X, \bar{J}Y) + \bar{g}(D^l(X, W), \bar{J}Y) + \bar{g}(A_W \bar{J}X, Y),
\end{aligned}
$$
(41)

or

(42) $\bar{g}(D^l(\bar{J}X, W), Y) - \bar{g}(D^l(X, W), \bar{J}Y) = \bar{g}(A_W \bar{J}X, Y) - \bar{g}(A_W X, \bar{J}Y)$.

Therefore part (i) of the assertion follows. Substituting $Y = \xi$ and $Y = \bar{J}\xi$ in (42), for any $\xi \in \Gamma(D_2)$, we obtain (ii) and (iii), respectively. $\square$

Theorem 4.2. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then D' is integrable if and only if $\nabla_X \bar{J}Y = \nabla_Y \bar{J}X$ for any $X, Y \in \Gamma(D')$.*

Proof. For any $X, Y \in \Gamma(D')$ we have

(43) $$ h(X, \bar{J}Y) = \bar{\nabla}_X \bar{J}Y - \nabla_X \bar{J}Y $$

and

(44) $$ h(\bar{J}X, Y) = \bar{\nabla}_Y \bar{J}X - \nabla_Y \bar{J}X. $$

Subtracting (44) from (43), we obtain

$$
\begin{aligned}
h(X, \bar{J}Y) - h(\bar{J}X, Y) &= \bar{J}\bar{\nabla}_X Y - \bar{J}\bar{\nabla}_Y X - \nabla_X \bar{J}Y + \nabla_Y \bar{J}X \\
&= \bar{J}[X, Y] - \nabla_X \bar{J}Y + \nabla_Y \bar{J}X \\
&= T[X, Y] + w[X, Y] - \nabla_X \bar{J}Y + \nabla_Y \bar{J}X.
\end{aligned}
$$

Equating tangential parts of above equation, we obtain

(45) $$ T[X, Y] = \nabla_X \bar{J}Y - \nabla_Y \bar{J}X. $$

Hence from (45) the result follows. $\square$

Theorem 4.3. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then D_0 is integrable if and only if*

(i) $\bar{g}(h^*(X,Y),N) = \bar{g}(h^*(Y,X),N)$,

(ii) $\bar{g}(h^*(X,\bar{J}Y),N') = \bar{g}(h^*(Y,\bar{J}X),N')$,

(iii) $h^s(X,\bar{J}Y) = h^s(\bar{J}X,Y)$,

(iv) $\bar{g}(\nabla^*_X Y,\bar{J}\xi) = \bar{g}(\nabla^*_Y X,\bar{J}\xi)$ *for any X, $Y \in \Gamma(D_0)$, $N \in \Gamma(\mathrm{ltr}(TM))$, $N' \in \Gamma(L_1)$ and $\xi \in \Gamma(D_2)$.*

Proof. Using the definition of GCR-lightlike submanifold, D_0 is integrable if and only if

$$\bar{g}([X,Y],N) = \bar{g}([X,Y],\bar{J}N') = \bar{g}([X,Y],\bar{J}W) = \bar{g}([X,Y],\bar{J}\xi) = 0$$

for any $X,Y \in \Gamma(D_0)$, $N \in \Gamma(\mathrm{ltr}(TM))$, $N' \in \Gamma(L_1)$, $W \in \Gamma(L_2)$ and $\xi \in \Gamma(D_2)$.

Using (9) and (16) we have

(46)
$$\bar{g}([X,Y],N) = \bar{g}(h^*(X,Y),N) - \bar{g}(h^*(Y,X),N)$$

and

$$\begin{aligned}\bar{g}([X,Y],\bar{J}N') &= \bar{g}(\bar{\nabla}_X Y,\bar{J}N') - \bar{g}(\bar{\nabla}_Y X,\bar{J}N')\\ &= -\bar{g}(\bar{\nabla}_X \bar{J}Y,N') + \bar{g}(\bar{\nabla}_Y \bar{J}X,N'),\end{aligned}$$

using (16) we have

(47)
$$\bar{g}([X,Y],\bar{J}N') = -\bar{g}(h^*(X,\bar{J}Y),N') + \bar{g}(h^*(Y,\bar{J}X),N').$$

Again using (9) we obtain

$$\begin{aligned}\bar{g}([X,Y],\bar{J}W) &= \bar{g}(\bar{\nabla}_X Y,\bar{J}W) - \bar{g}(\bar{\nabla}_Y X,\bar{J}W)\\ &= -\bar{g}(\bar{\nabla}_X \bar{J}Y,W) + \bar{g}(\bar{\nabla}_Y \bar{J}X,W)\end{aligned}$$

(48)
$$\qquad\qquad = -\bar{g}(h^s(X,\bar{J}Y),W) + \bar{g}(h^s(\bar{J}X,Y),W).$$

Finally from (16) we obtain

(49)
$$\bar{g}([X,Y],\bar{J}\xi) = \bar{g}(\nabla^*_X Y,\bar{J}\xi) - \bar{g}(\nabla^*_Y X,\bar{J}\xi).$$

Thus from (46)-(49) the result follows. $\qquad\square$

Corollary 4.4. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then D_0 is integrable if and only if*

(i) $g(X,A_N Y) = g(Y,A_N X)$,

(ii) $g(\bar{J}X,A_{N'}Y) = g(\bar{J}Y,A_{N'}X)$,

(iii) $h^s(X,\bar{J}Y) = h^s(\bar{J}X,Y)$,

(iv) $\bar{g}(h^l(X,\bar{J}Y),\xi) = \bar{g}(h^l(Y,\bar{J}X),\xi)$ *for any $X,Y \in \Gamma(D_0)$, $N \in \Gamma(\mathrm{ltr}(TM))$, $N' \in \Gamma(L_1)$ and $\xi \in \Gamma(D_2)$.*

Theorem 4.5. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then $\mathrm{Rad}(TM)$ is integrable if and only if*

(i) $\bar{g}(h^l(\xi,\bar{J}\xi''),\xi') = \bar{g}(h^l(\xi',\bar{J}\xi''),\xi)$,

(ii) $\bar{g}(h^l(\xi,Z),\xi') = \bar{g}(h^l(\xi',Z),\xi)$,

(iii) $h^s(\xi', \bar{J}\xi) = h^s(\bar{J}\xi', \xi)$,
(iv) $\bar{g}(\xi, h^l(\xi', \bar{J}N)) = \bar{g}(\xi', h^l(\xi, \bar{J}N))$ *for any* $Z \in \Gamma(D_0)$, $\xi'' \in \Gamma(D_2)$, ξ, $\xi' \in \Gamma(\mathrm{Rad}(TM))$, $W \in \Gamma(L_2)$ *and* $N \in \Gamma(L_1)$.

Proof. Using the definition of GCR-lightlike submanifolds, $\mathrm{Rad}(TM)$ is integrable if and only if

$$\bar{g}([\xi, \xi'], Z) = \bar{g}([\xi, \xi'], \bar{J}N) = \bar{g}([\xi, \xi'], \bar{J}W) = \bar{g}([\xi, \xi'], \bar{J}\xi'') = 0$$

for any $Z \in \Gamma(D_0)$, $\xi'' \in \Gamma(D_2)$, ξ, $\xi' \in \Gamma(\mathrm{Rad}(TM))$, $W \in \Gamma(L_2)$ and $N \in \Gamma(L_1)$.

Using (9) and (16), we obtain

$$
\begin{aligned}
\bar{g}([\xi, \xi'], \bar{J}\xi'') &= \bar{g}(\bar{\nabla}_\xi \xi', \bar{J}\xi'') - \bar{g}(\bar{\nabla}_{\xi'}\xi, \bar{J}\xi'') \\
&= -\bar{g}(\xi', \bar{\nabla}_\xi \bar{J}\xi'') + \bar{g}(\xi, \bar{\nabla}_{\xi'}\bar{J}\xi'') \\
&= -\bar{g}(h^l(\xi, \bar{J}\xi''), \xi') + \bar{g}(h^l(\xi', \bar{J}\xi''), \xi)
\end{aligned}
$$
(50)

and using (18)

$$
\begin{aligned}
\bar{g}([\xi, \xi'], Z) &= \bar{g}(\nabla_\xi \xi', Z) - \bar{g}(\nabla_{\xi'}\xi, Z) \\
&= -\bar{g}(A^*_{\xi'}\xi, Z) + \bar{g}(A^*_\xi \xi', Z) \\
&= -\bar{g}(h^l(\xi, Z), \xi') + \bar{g}(h^l(\xi', Z), \xi)
\end{aligned}
$$
(51)

and

$$
\begin{aligned}
\bar{g}([\xi, \xi'], \bar{J}W) &= \bar{g}(\bar{\nabla}_\xi \xi', \bar{J}W) - \bar{g}(\bar{\nabla}_{\xi'}\xi, \bar{J}W) \\
&= -\bar{g}(W, \bar{\nabla}_\xi \bar{J}\xi') + \bar{g}(W, \bar{\nabla}_{\xi'}\bar{J}\xi) \\
&= -\bar{g}(h^s(\xi, \bar{J}\xi'), W) + \bar{g}(h^s(\xi', \bar{J}\xi), W)
\end{aligned}
$$
(52)

and

$$
\begin{aligned}
\bar{g}([\xi, \xi'], \bar{J}N) &= \bar{g}(\bar{\nabla}_\xi \xi', \bar{J}N) - \bar{g}(\bar{\nabla}_{\xi'}\xi, \bar{J}N) \\
&= -\bar{g}(\xi', \bar{\nabla}_\xi \bar{J}N) + \bar{g}(\xi, \bar{\nabla}_{\xi'}\bar{J}N) \\
&= -\bar{g}(\xi', h^l(\xi, \bar{J}N)) + \bar{g}(\xi, h^l(\xi', \bar{J}N)).
\end{aligned}
$$
(53)

Thus from (50)-(53), the result follows. $\qquad\square$

Corollary 4.6. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then $\mathrm{Rad}(TM)$ is integrable if and only if*

(i) $A^*_\xi \xi' \notin \Gamma(M_1)$,
(ii) $g(\nabla^*_{\xi'}\bar{J}Z, \bar{J}\xi) = g(\nabla^*_\xi \bar{J}Z, \bar{J}\xi')$,
(iii) $g(A_W \xi', \bar{J}\xi) = g(A_W \xi, \bar{J}\xi')$,
(iv) $g(A_N \xi', \bar{J}\xi) = g(A_N \xi, \bar{J}\xi')$ *for any* $Z \in \Gamma(D_0)$, ξ, $\xi' \in \Gamma(\mathrm{Rad}(TM))$ *and* $N \in \Gamma(\mathrm{ltr}(TM))$.

Theorem 4.7. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the distribution D_1 is integrable if and only if*

(i) $\nabla^{*t}_X \bar{J}Y - \nabla^{*t}_Y \bar{J}X \in \Gamma(D_1)$,
(ii) $A^*_{\bar{J}Y}X = A^*_{\bar{J}X}Y$,

(iii) $Bh(X, \bar{J}Y) = Bh(Y, \bar{J}X)$ *for any* $X, Y \in \Gamma(D_1)$.

Proof. Since $\bar{J}$ is the almost complex structure on M therefore for any $X, Y \in \Gamma(D_1)$ we have

$$\bar{\nabla}_X Y = -\bar{\nabla}_X \bar{J}^2 Y = -\bar{J}\bar{\nabla}_X \bar{J}Y,$$

using (9), we have

$$\nabla_X Y + h(X, Y) = -\bar{J}(\nabla_X \bar{J}Y + h(X, \bar{J}Y)).$$

Now, using (17), (29)-(30) and then equating the tangential components of the resulting equation both sides we obtain

$$(54) \qquad \nabla_X Y = T A^*_{\bar{J}Y} X - T \nabla^{*t}_X \bar{J}Y - Bh(X, \bar{J}Y),$$

replacing X by Y and then subtracting the resulting equation from (54) we obtain

(55)
$$[X, Y] = T(A^*_{\bar{J}Y} X - A^*_{\bar{J}X} Y) - T(\nabla^{*t}_X \bar{J}Y - \nabla^{*t}_Y \bar{J}X) - Bh(X, \bar{J}Y) + Bh(Y, \bar{J}X).$$

Hence from (55) the result follows.   □

Corollary 4.8. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then D_1 defines a totally geodesic foliation in M if and only if*

   (i) $\nabla^{*t}_X \bar{J}Y \in \Gamma(D_1)$,
   (ii) $A^*_{\bar{J}Y} X = 0$,
   (iii) $Bh(X, \bar{J}Y) = 0$ *for any* $X, Y \in \Gamma(D_1)$.

Theorem 4.9. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the distribution D_2 is integrable if and only if*

   (i) $\nabla^*_X \bar{J}Y - \nabla^*_Y \bar{J}X \in \Gamma(\bar{J}D_2)$,
   (ii) $h^*(X, \bar{J}Y) = h^*(Y, \bar{J}X)$,
   (iii) $Bh(X, \bar{J}Y) = Bh(Y, \bar{J}X)$ *for any* $X, Y \in \Gamma(D_2)$.

Proof. Since $\bar{J}$ is the almost complex structure on M therefore for any $X, Y \in \Gamma(D_2)$ we have

$$\nabla_X Y + h(X, Y) = -\bar{J}(\nabla_X \bar{J}Y + h(X, \bar{J}Y)),$$

using (16) and (29)-(30) and then equating tangential components of the resulting equation both sides we obtain

$$(56) \qquad \nabla_X Y = -T\nabla^*_X \bar{J}Y - Th^*(X, \bar{J}Y) - Bh(X, \bar{J}Y),$$

replacing X and Y and then subtracting the resulting equation from (56) we obtain

$$
\begin{aligned}
[X, Y] &= -T(\nabla^*_X \bar{J}Y - \nabla^*_Y \bar{J}X) - T(h^*(X, \bar{J}Y) - h^*(Y, \bar{J}X)) \\
(57) &\quad - Bh(X, \bar{J}Y) + Bh(Y, \bar{J}X).
\end{aligned}
$$

Hence from (57) the theorem follows.   □

Corollary 4.10. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the distribution D_2 defines a totally geodesic foliation in M if and only if*

(i) $\nabla_X^* \bar{J}Y \in \Gamma(\bar{J}D_2)$,
(ii) $h^*(X, \bar{J}Y) = 0$,
(iii) $Bh(X, \bar{J}Y) = 0$ *for any* $X, Y \in \Gamma(D_2)$.

Theorem 4.11. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then $\bar{J}D_2$ is integrable if and only if*

(i) $\bar{g}(\bar{J}\xi, A_N \bar{J}\xi') = \bar{g}(\bar{J}\xi', A_N \bar{J}\xi)$,
(ii) $h^s(\bar{J}\xi, \xi') = h^s(\bar{J}\xi', \xi)$,
(iii) $\bar{g}(h^l(\bar{J}\xi, \xi'), \xi'') = \bar{g}(h^l(\xi, \bar{J}\xi'), \xi'')$,
(iv) $\bar{g}(\bar{J}Z, A_\xi^* \bar{J}\xi') = \bar{g}(\bar{J}Z, A_{\xi'}^* \bar{J}\xi)$ *for any* $Z \in \Gamma(D_0)$, ξ, ξ', $\xi'' \in \Gamma(D_2)$, $W \in \Gamma(L_2)$ *and* $N \in \Gamma(\mathrm{ltr}(TM))$.

Proof. Using the definition of GCR-lightlike submanifolds, $\bar{J}D_2$ is integrable if and only if

$$\bar{g}([\bar{J}\xi', \bar{J}\xi], N) = \bar{g}([\bar{J}\xi', \bar{J}\xi], \bar{J}W) = \bar{g}([\bar{J}\xi', \bar{J}\xi], \bar{J}\xi'') = \bar{g}([\bar{J}\xi', \bar{J}\xi], Z) = 0$$

for any $Z \in \Gamma(D_0)$, ξ, ξ', $\xi'' \in \Gamma(D_2)$, $W \in \Gamma(L_2)$ and $N \in \Gamma(\mathrm{ltr}(TM))$. Using (11) we have

$$
\begin{aligned}
\bar{g}([\bar{J}\xi', \bar{J}\xi], N) &= \bar{g}(\bar{\nabla}_{\bar{J}\xi'} \bar{J}\xi, N) - \bar{g}(\bar{\nabla}_{\bar{J}\xi} \bar{J}\xi', N) \\
&= -\bar{g}(\bar{J}\xi, \bar{\nabla}_{\bar{J}\xi'} N) + \bar{g}(\bar{J}\xi', \bar{\nabla}_{\bar{J}\xi} N) \\
(58) \qquad &= \bar{g}(\bar{J}\xi, A_N \bar{J}\xi') - \bar{g}(\bar{J}\xi', A_N \bar{J}\xi)
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{g}([\bar{J}\xi', \bar{J}\xi], \bar{J}W) &= \bar{g}(\bar{\nabla}_{\bar{J}\xi'} \bar{J}\xi, \bar{J}W) - \bar{g}(\bar{\nabla}_{\bar{J}\xi} \bar{J}\xi', \bar{J}W) \\
&= \bar{g}(\bar{\nabla}_{\bar{J}\xi'} \xi, W) - \bar{g}(\bar{\nabla}_{\bar{J}\xi} \xi', W) \\
(59) \qquad &= \bar{g}(h^s(\bar{J}\xi', \xi), W) - \bar{g}(h^s(\bar{J}\xi, \xi'), W)
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{g}([\bar{J}\xi', \bar{J}\xi], \bar{J}\xi'') &= \bar{g}(\bar{\nabla}_{\bar{J}\xi'} \bar{J}\xi, \bar{J}\xi'') - \bar{g}(\bar{\nabla}_{\bar{J}\xi} \bar{J}\xi', \bar{J}\xi'') \\
&= \bar{g}(\bar{\nabla}_{\bar{J}\xi'} \xi, \xi'') - \bar{g}(\bar{\nabla}_{\bar{J}\xi} \xi', \xi'') \\
(60) \qquad &= \bar{g}(h^l(\bar{J}\xi', \xi), \xi'') - \bar{g}(h^l(\bar{J}\xi, \xi'), \xi'')
\end{aligned}
$$

and finally

$$
\begin{aligned}
\bar{g}([\bar{J}\xi', \bar{J}\xi], Z) &= \bar{g}(\bar{\nabla}_{\bar{J}\xi'} \bar{J}\xi, Z) - \bar{g}(\bar{\nabla}_{\bar{J}\xi} \bar{J}\xi', Z) \\
&= -\bar{g}(\bar{\nabla}_{\bar{J}\xi'} \xi, \bar{J}Z) + \bar{g}(\bar{\nabla}_{\bar{J}\xi} \xi', \bar{J}Z) \\
(61) \qquad &= \bar{g}(\bar{J}Z, A_\xi^* \bar{J}\xi') - \bar{g}(\bar{J}Z, A_{\xi'}^* \bar{J}\xi).
\end{aligned}
$$

Hence from (58)-(61) the result follows. $\qquad\qquad\square$

Lemma 4.12. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. If M is D-geodesic, then D defines a totally geodesic foliation in M.*

Proof. For $X, Y \in \Gamma(D)$, using (31) and (32), we obtain $w\nabla_X Y = h(X, TY) - Ch(X, Y)$, then using the hypothesis, we obtain $w\nabla_X Y = 0$. Thus the proof is complete. $\qquad\square$

Lemma 4.13. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. If T is parallel, then $Bh(X, Y) = 0$ for any $Y \in \Gamma(D)$.*

Proof. For any $X, Y \in \Gamma(TM)$, from (33) we have

$$(62) \qquad\qquad (\nabla_X T)Y = A_{wY} X + Bh(X, Y).$$

From the hypothesis of lemma, the proof is complete. $\qquad\square$

Theorem 4.14 ([12])**.** *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the distribution D defines a totally geodesic foliation in M if and only if $Bh(X, Y) = 0 \ \forall \ X, Y \in \Gamma(D)$.*

Theorem 4.15. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. If T is parallel, then the distribution D defines a totally geodesic foliation in M.*

Proof. Let T is parallel, then from Lemma 4.13, we obtain $Bh(X, Y) = 0$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$. Then using Theorem 4.14, D defines a totally geodesic foliation in M. $\qquad\square$

Theorem 4.16. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the distribution D' is parallel if and only if $A_{\bar{J}Y} X$ has no components in holomorphic distribution for any $X, Y \in \Gamma(D')$.*

Proof. From (33), we have

$$-T\nabla_X Y = A_{wY} X + Bh(X, Y)$$

for any $X, Y \in \Gamma(D')$. Hence the proof follows. $\qquad\square$

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