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INTEGRABILITY OF POISSON BRACKETS

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Abstract

We discuss the integration of Poisson brackets, motivated by our recent solution to the integrability problem for general Lie brackets. We give the precise obstructions to integrating Poisson manifolds, describing the integration as a symplectic quotient, in the spirit of the Poisson sigma-model of Cattaneo and Felder. For regular Poisson manifolds we express the obstructions in terms of variations of symplectic areas, improving on results of Alcalde Cuesta and Hector. We apply our results (and our point of view) to decide about the existence of complete symplectic realizations, to the integrability of submanifolds of Poisson manifolds, and to the study of dual pairs, Morita equivalence and reduction.

Introduction

A **Poisson bracket** on a manifold M is a Lie bracket $\{\cdot,\cdot\}$ on the space $C^\infty(M)$ of smooth functions on M, satisfying the derivation property

$$\{fg,h\} = f\{g,h\} + g\{f,h\}, \quad f,g,h \in C^\infty(M).$$

The integrability problem for Poisson brackets can be loosely stated as:

- Is there a Lie group integrating this Lie algebra?

Stated as such, this problem is beyond our current state of knowledge, as is illustrated by the well-known *flux conjecture* in symplectic geometry [**26**] (see also Milnor's remarks about infinite dimensional groups in [**28**]). However, such problems become more tractable if instead of infinite dimensional Lie groups one brings finite dimensional objects known as *Lie groupoids* into the picture.

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A Poisson bracket on a manifold M gives rise to a Lie bracket $[\cdot,\cdot]$ on the space $\Omega^1(M)$ of 1-forms on M. This bracket is uniquely determined by the following two requirements:

(i) For exact 1-forms it coincides with the Poisson bracket:

$$[df, dg] = d\{f, g\}, \quad f, g \in C^\infty(M).$$

(ii) It satisfies the Leibniz identity:

$$[\alpha, f\beta] = f[\alpha, \beta] + \#\beta(f)\alpha, \quad \alpha, \beta \in \Omega^1(M),\ f \in C^\infty(M). \tag{1}$$

Here $\# : T^*M \to TM$ denotes contraction by the Poisson 2-tensor $\Pi \in \Gamma(\Lambda^2 TM)$ which is associated to the Poisson bracket by $\Pi(df, dg) = \{f, g\}$. An explicit formula for this bracket is

$$[\alpha, \beta] = \mathcal{L}_{\#\alpha}\beta - \mathcal{L}_{\#\beta}\alpha - d\Pi(\alpha, \beta), \quad \alpha, \beta \in \Omega^1(M). \tag{2}$$

The triple $(T^*M, [\cdot,\cdot], \#)$ is an example of a *Lie algebroid*. Lie algebroids can be thought as "infinite dimensional Lie algebras of geometric type", or "generalized tangent bundles". In general, a **Lie algebroid** is a vector bundle $A \to M$ together with a Lie bracket $[\cdot,\cdot]$ on the space of sections $\Gamma(A)$ and a bundle map $\# : A \to TM$, giving rise to a Lie algebra morphism $\# : \Gamma(A) \to \mathfrak{X}(M)$, such that the analogue of Leibniz's identity (1) is satisfied for all $\alpha, \beta \in \Gamma(A)$.

The global counterpart to Lie algebroids are *Lie groupoids*. A Lie groupoid consists of arrows (transformations) between different objects (points), which can be (smoothly) multiplied provided they match. More formally, a groupoid is a small category where every morphism is an isomorphism. A **Lie groupoid** is a groupoid in the differentiable category: it consists of a manifold Σ (the arrows), together with two submersions $\mathbf{s}, \mathbf{t} : \Sigma \to M$ (the source and the target maps) onto the base manifold M (the objects), an embedding $M \to \Sigma$, $x \mapsto 1_x$ (the unit section), and a smooth map $\Sigma \times_M \Sigma \to \Sigma$, $(g, h) \mapsto gh$ (the multiplication) defined on the space of pairs (g, h) with $\mathbf{s}(g) = \mathbf{t}(h)$. We will follow the conventions of [**4**].

To every Lie groupoid there is associated a Lie algebroid (see [**4**]). The converse is not true, and the precise obstructions to the integration of Lie algebroids to Lie groupoids were determined in [**10**]. Uniqueness, up to isomorphism, can be obtained by requiring the $\mathbf{s}$-fibers to be simply connected (i.e., connected with vanishing fundamental group) and, given Σ, one can construct a $\mathbf{s}$-simply connected groupoid $\widetilde{\Sigma}$ by taking universal coverings of the $\mathbf{s}$-fibers (see [**29**, **10**]). The integrability problem for Poisson manifolds can then be restated as:

- Is there a Lie groupoid integrating T^*M?

Note that the integrability of T^*M really amounts to the integrability of the brackets $[\cdot,\cdot]$ on $\Omega^1(M)$.

The integration of Poisson manifolds is, of course, a particular case of the integrability problem for general Lie algebroids. As already mentioned above, a complete solution for the latter was presented in [**10**]. However, it is important to understand this special case. On the one hand, the integration of Poisson manifolds is richer due to the presence of symplectic geometry on the leaves of the characteristic foliation, which gives rise to many properties that are not present in the general integrability problem. On the other hand, the integrability problem for Poisson brackets is relevant, for example, to symplectic reduction [**12**], to Poisson topology [**18**], to various quantization schemes [**5**, **24**, **35**], and to many other problems. As an example of the richer geometry presented in the special case of Poisson manifolds, we shall see that a Lie groupoid integrating a Poisson manifold has a natural symplectic structure, a well-known fact going back to the earlier works of Weinstein *et al.*

Recall that a **symplectic groupoid** is a Lie groupoid Σ together with a symplectic form, such that the graph of the multiplication is Lagrangian. This apparently mild condition is actually quite strong. It induces a natural Poisson structure on M, satisfying the following properties (see [**8**]):

(a) $\mathbf{s}$ is Poisson and $\mathbf{t}$ is anti-Poisson.[1]
(b) Both $\mathbf{s}$ and $\mathbf{t}$ are complete maps.
(c) The $\mathbf{s}$-fibers and the $\mathbf{t}$-fibers are symplectic orthogonal.
(d) M, viewed as the unit section, is a Lagrangian submanifold of Σ.
(e) The Lie algebroid of Σ is canonically isomorphic to T^*M.

The inverse problem (reconstructing Σ) is the symplectic version of the integrability problem:

- Is there a symplectic groupoid integrating M?

Again, it is not hard to see that if a Poisson manifold admits an integrating symplectic groupoid, then it admits a unique $\mathbf{s}$-simply connected one. In this case we say that M is an integrable Poisson manifold, and

[1] A *Poisson map* is a map $\phi : M \to N$ between two Poisson manifolds that preserves the Poisson brackets. A Poisson map is called *complete* if, whenever X_h is a complete Hamiltonian vector field on N, then $X_{\phi^* h}$ is also a complete Hamiltonian vector fields on M.

the **s**-simply connected groupoid $\Sigma = \Sigma(M)$ integrating M will be called the symplectic groupoid of M.

By property (e) above, if a Poisson manifold is integrable, then its associated algebroid T^*M is also integrable. The converse is also true as was shown by Mackenzie and Xu in [**25**]:

Theorem 1. *If M is a Poisson manifold such that T^*M is an integrable Lie algebroid then M is an integrable Poisson manifold.*

Many known results like this have a different and simpler proof in our unified approach to the integrability problem to be presented below. Let us give a short overview of our solution to the integrability problem and, at the same time, an outline of the content of the paper.

In [**10**], for *any* Lie algebroid A we have constructed a topological groupoid $\mathcal{G}(A)$, called the Weinstein groupoid of A which is a fundamental invariant of the Lie algebroid. Moreover, this groupoid has a compatible differentiable structure (i.e., is a Lie groupoid) if and only if A is an integrable Lie algebroid. In the case of Poisson manifolds, where $A = T^*M$, we will denote $\mathcal{G}(T^*M)$ by $\Sigma(M)$. One should think of $\Sigma(M)$ as the homotopy (or fundamental) groupoid of the Poisson manifold, and in fact it can be described, as we shall explain in Section 1 below, as a quotient

$$\Sigma(M) = \text{cotangent paths/cotangent homotopies}.$$

The obvious similarity to the ordinary homotopy group is related with the following basic *philosophical principle*:

> *In analogy with the role played by the tangent bundle TM of manifolds, Lie algebroids can be thought of as the "tangent bundles of* (*possibly singular*) *geometric structures".*

Accordingly, many constructions/results in differential geometry that can be carried out in terms of just the tangent bundle make sense for general Lie algebroids. However, sometimes this is not entirely obvious, and the Lie algebroid framework can reveal totally new patterns. For instance, in contrast with the fact that the fundamental groupoid of a manifold is always smooth, $G(A)$ may fail to be smooth. The main result of [**10**] gives the precise obstructions for a differentiable structure to exist in $\mathcal{G}(A)$, and expresses them in terms of so-called *monodromy groups.* For the special case of Poisson manifolds, we shall describe them in Section 2 below.

There are two special properties of T^*M that distinguish this case from the general case. First, the anchor and the bracket are both induced from the Poisson tensor and so are intimately related. Second, one has the duality between T^*M and TM. As we shall see in Section 3, these lead to a description of $\Sigma(M)$ as a symplectic quotient, a fact first noted in [**5**]. Then the symplectic form on $\Sigma(M)$, makes the Weinstein groupoid into a symplectic Lie groupoid, hence reproving Mackenzie-Xu's result mentioned above.

Our main result, whose proof is given in Section 4 below, can be stated as follows:

Theorem 2. *For a Poisson manifold M, the following are equivalent:*

(i) *M is integrable by a symplectic Lie groupoid.*
(ii) *The algebroid T^*M is integrable.*
(iii) *The Weinstein groupoid $\Sigma(M)$ is a smooth manifold.*
(iv) *The monodromy groups $\mathcal{N}_x$, with $x \in M$, are locally uniformly discrete.*

The last condition of the theorem makes no reference to algebroids or groupoids, and gives an integrability criteria which is computable in explicit examples. Moreover, the monodromy groups $\mathcal{N}_x$ (which are just some additive subgroups of the co-normal vector space $\nu_x^*(L)$ to the symplectic leaf L through x) are invariants of Poisson manifolds which are interesting on their own. For regular Poisson manifolds, as we discuss in Section 5 (improving on results of Alcalde Cuesta-Hector [**1**]), the monodromy can be expressed in terms of variations of symplectic areas of spheres along transverse directions to the symplectic leaves. In Section 6, we present many examples of integrable and non-integrable Poisson manifolds.

A well-known result of Karasev and Weinstein states that *any* Poisson manifold M has a *symplectic realization*, i.e, there exists a symplectic manifold S and a surjective Poisson submersion $\mu : S \to M$. The existence of symplectic realizations with μ *complete* has been an open problem, which can be thought as yet another instance of the integrability problem for Poisson manifolds:

- Is there a complete symplectic realization of M?

In Section 7 below we solve this problem (and sketch a different proof of the result of Karasev and Weinstein):

Theorem 3. *A Poisson manifold admits a complete symplectic realization if and only if it is integrable.*

This establishes the equivalence of all the different notions of integrability. In the last two sections of the paper we give some more applications of our integrability results.

In Section 8 we consider the Poisson brackets induced on a submanifold of a Poisson manifold. We clarify and improve the results of Xu [**38**] and Vaisman [**31**] on induced Poisson structures. We also discuss, for an integrable Poisson manifold, when is the induced Poisson bracket on a submanifold integrable.

In Section 9 we discuss Morita equivalence of Poisson manifolds, both for integrable and non-integrable Poisson manifolds. We observe that the original definition due to Xu only makes sense for integrable Poisson manifolds. For non-integrable Poisson manifolds we consider a weak notion of Morita equivalence, and we prove that many invariants of Poisson manifolds, relevant to the integrability problem, are *weak* Morita invariant.

Remark 1 (Hausdorff Issues)**.** The reader will notice that we are forced to allow non-Hausdorff manifolds in our paper. There are at least two simple reasons for this. First of all, a bundle of Lie algebras $\mathfrak{g}$ over a manifold B may not integrate to a bundle G of Lie groups over B (that is, the Lie algebra of the fiber Lie group G_b coincides with $\mathfrak{g}_b$, for all $b \in B$), if we require G to be Hausdorff. However, there exists always at least one which is a (possibly non-Hausdorff) manifold (see [**14**]). Secondly, there are simple examples of foliations whose graph, although a manifold, may be non-Hausdorff.

We mention here the objects of this paper which allow non-Hausdorff manifolds:

(a) Lie groupoids (Sections 3 and 4) may be non-Hausdorff manifolds. However, the base space (denoted M here), as well as the **s**-fibers and **t**-fibers, are always assumed to be Hausdorff.
(b) For a symplectic realization $\pi : S \to M$ (Section 7), S is allowed to be non-Hausdorff. However, the leaves of the foliation $\mathcal{F}(\pi)$ by fibers of π and of the symplectic orthogonal foliation $\mathcal{F}(\pi)^{\perp}$, are all Hausdorff.

In particular, the Poisson manifolds we study are always assumed to be Hausdorff. All manifolds are assumed to be 2nd countable.

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1. Cotangent paths and their homotopy

In the sequel M will denote a Poisson manifold. We let $\# : T^*M \to TM$ be the bundle map determined by contraction with the Poisson tensor, and we let $[\cdot,\cdot]$ be the Lie the bracket on 1-forms, which was defined in the introduction. We also denote by $\pi : T^*M \to M$ the canonical projection.

Definition 1. A **cotangent path** in M is a path $a : I \to T^*M$, where $I = [0, 1]$, satisfying

$$\frac{d}{dt}\pi(a(t)) = \#a(t).$$

Given a cotangent path $a : I \to T^*M$ and a vector field X on M we define, following [**19**], the path integral:

$$\int_a X \equiv \int_0^1 \langle a(t), X(\gamma(t))\rangle dt,$$

where $\gamma(t) = \pi(a(t)$ is the base path of a. If $X = X_h$ is a Hamiltonian vector field for some Hamiltonian function $h \in C^\infty(M)$, we see that

$$\int_a X_h = h(\gamma(1)) - h(\gamma(0)).$$

So for a Hamiltonian vector field the integral only depends on the end-points of the base of the cotangent path.

From a differential-geometric viewpoint, as was first explained in [**17**], cotangent paths are precisely the paths along which parallel transport can be performed whenever a contravariant connection has be chosen. Let us briefly recall how this works.

First of all, a **contravariant connection** ∇ on a vector bundle E over M is a bilinear map

$$\Omega^1(M) \times \Gamma(E) \to \Gamma(E),\ (\alpha, s) \mapsto \nabla_\alpha s,$$

such that for all $f \in C^\infty(M)$, $\alpha \in \Omega^1(M)$, $s \in \Gamma(E)$, we have:

(a) $\nabla_{f\alpha} s = f\nabla_\alpha s$.
(b) $\nabla_\alpha(fs) = f\nabla_\alpha s + \#\alpha(f)s$.

This is the analogue, in the Poisson category, of the usual connections.

The value of $\nabla_\alpha s$ at $x \in M$ only depends on the value of α at x and the values of s along the integral curve of $\#\alpha$ through x. Also, just like the usual covariant derivative, given a cotangent path a and path s in E above $\pi \circ a$, there is a well-defined contravariant derivative $\nabla_a s$, which is a path in E above $\pi \circ a$. This leads to the notion of **parallel transport** along a cotangent path a: it is the map $\tau_a : E_{\pi(a(0))} \to E_{\pi(a(1))}$ which takes $u \in E_{\pi(a(0))}$ to $s(1) \in E_{\pi(a(1))}$, where $s : I \to E$ is the solution of the differential equation $\nabla_a s = 0$, with initial condition $s(0) = u$. All this is described in detail in [**17**].

We will be mostly interested in contravariant connections on T^*M. A covariant connection ∇ on TM induces, apart from the dual covariant connection on T^*M, still denoted by ∇, a contravariant connection $\overline{\nabla}$ on TM defined by

$$\overline{\nabla}_\alpha X = \nabla_X \#\alpha + [\#\alpha, X],$$

and a dual contravariant connection on T^*M, also denoted by $\overline{\nabla}$, and given by:

$$\overline{\nabla}_\alpha \beta = \nabla_{\#\beta}\alpha + [\alpha, \beta].$$

These connections satisfy $\overline{\nabla}\# = \#\overline{\nabla}$ and they form a **basic connection** in the sense of [**17**]. The connection $\overline{\nabla}$ has **contravariant torsion** given by

$$T_\nabla(\omega, \eta) = \overline{\nabla}_\omega \eta - \overline{\nabla}_\eta \omega - [\omega, \eta].$$

Fix a covariant connection ∇ on TM. Given a family $a_\epsilon = a_\epsilon(t)$ of cotangent paths with the property that the base paths $\gamma_\epsilon(t) = \pi(a_\epsilon(t))$ have fixed end points, we consider the differential equation[2]

$$\partial_t b - \partial_\epsilon a = T_\nabla(a, b). \tag{3}$$

This equation has a unique solution $b = b(\epsilon, t)$ (running in T^*M) with initial condition $b(\epsilon, 0) = 0$, and we define the **variation** of a_ϵ to be the cotangent path:

$$\mathrm{var}(a_\epsilon) = b(\epsilon, 1).$$

Definition 2. A **cotangent homotopy** is a family $a_\epsilon = a_\epsilon(t)$ of cotangent paths with the property that the base paths $\gamma_\epsilon(t)$ have fixed end points and $\mathrm{var}(a_\epsilon) = 0$. Two cotangent paths a_0 and a_1 are said to be homotopic, and we write $a_0 \sim a_1$ if there is a cotangent homotopy joining them.

[2]Given a covariant connection ∇ on a bundle $E \to M$, a path $\gamma(t)$ in M and a path $u(t)$ in E over γ, we use the notation $\partial_t u \equiv \nabla_\gamma u$.

For an alternative definition of cotangent homotopy which avoids the use of connections we refer to [**10**].

Cotangent homotopy defines an equivalence relation on the set of cotangent paths and enjoys many properties similar to the ones enjoyed by the usual notion of homotopy. For example, since the analogue of closed 1-forms in this calculus are the Poisson vector fields, the following corresponds to the homotopy invariance of integrals of closed 1-forms along paths:

Proposition 1. *Let a_0 and a_1 be cotangent paths. If $a_0 \sim a_1$ then, for any Poisson vector field X on M, we have*

$$\int_{a_0} X = \int_{a_1} X.$$

Proof. Let $a = a(\epsilon, t)$ and $b = b(\epsilon, t)$ be as above, and set $I = \langle a, X \rangle$, $J = \langle b, X \rangle$. Fixing some connection ∇, for any $\omega \in \Gamma(T^*M)$, we have

$$\frac{d}{dt}\langle \omega(\gamma(t)), X(\gamma(t))\rangle = \left\langle \nabla_{\frac{d\gamma}{dt}}\omega, X(\gamma(t))\right\rangle + \left\langle \omega(\gamma(t)), \nabla_{\frac{d\gamma}{dt}} X\right\rangle .$$

A straightforward computation using the defining equation (3) and expression (2) for the Lie bracket, shows that

$$\frac{dI}{d\epsilon} - \frac{dJ}{dt} = \mathcal{L}_X \Pi(a, b).$$

Integrating first with respect to t, using $b(\epsilon, 0) = 0$, and then integrating with respect to ϵ, we find that

$$\int_{a_1} X - \int_{a_0} X = \int_{\mathrm{var}(a_\epsilon)} X + \int_0^1 \int_0^1 \mathcal{L}_X \Pi(a, b)\, dt d\epsilon.$$

for any vector field X, and any family $\{a_\epsilon\}$. For cotangent homotopies and Poisson vector fields this gives $\int_{a_0} X = \int_{a_1} X$. q.e.d.

Remark 2. In [**19**], Ginzburg gives a simple example (cf. Example 3.4) showing that the integral is not invariant under homotopy of the base path. Referring to the (non-)invariance of the integral under this "naive homotopy" he states: "this example shows that the above naive definition of homotopy is not a *correct* extension of this notion to the Poisson category". The proposition above shows that cotangent homotopy gives the desired extension.

Another instance of behavior similar to the usual notion of homotopy is obtained by looking at flat contravariant connections. For these connections we have the following contravariant version of a classical result in differential geometry:

Proposition 2. *Given a flat contravariant connection, parallel transport along cotangent paths is invariant under cotangent homotopy.*

The proof is similar to the proof of the previous proposition (see Proposition 1.6 in [**10**]), and will be omitted. Next we give the contravariant analogue of the fundamental group.

Definition 3. Let $x \in M$ be a point in the Poisson manifold M. The **isotropy group** $\Sigma(M, x)$ is the set of equivalence classes of cotangent paths whose base paths in M start and end at x. The group structure is defined by concatenation of paths. We also define the **restricted isotropy group** $\Sigma^0(M, x)$ by considering only cotangent paths whose base path is a contractible loop.

The groups $\Sigma(M, x)$ play a fundamental role in Poisson geometry. For instance, one can show that linear holonomy of Poisson manifolds (defined along cotangent paths, cf. [**17**]), only depends on homotopy classes, hence factors through the isotropy group. Also, if $\mu : S \to M$ is a complete symplectic realization, then there is a natural action of $\Sigma(M, x)$ on $\mu^{-1}(x)$, and, if the "reduction" $\mu^{-1}(x)/\Sigma(M, x)$ is smooth, then it carries a natural symplectic structure. More details and more examples will be given in the later sections. Notice also that, by Proposition 1, integration along closed cotangent paths gives a group homomorphism

$$\int : \Sigma(M, x) \to H^1_\Pi(M)^*,$$

where $H^1_\Pi(M)$ denotes the 1st Poisson cohomology group of M, i.e., the set of Poisson vector fields modulo the Hamiltonian vector fields.

Though one may view the groups $\Sigma(M, x)$ as *analogues* of the fundamental groups of manifolds, there is also a strong analogy with the construction of simply connected Lie groups integrating Lie algebras. In the remaining part of this section, we elaborate on these two analogies.

Notice that the two groups we have just defined do relate to the fundamental groups of the symplectic leaf L through x. We have a short exact sequence

$$1 \longrightarrow \Sigma^0(M, x) \longrightarrow \Sigma(M, x) \longrightarrow \pi_1(L, x) \longrightarrow 1, \tag{4}$$

where the first map is just the inclusion, while the second one associates to a cotangent path its underlying base path. In the Poisson context however, there are several new features which are not present in the case of classical fundamental groups. For example, one can assert that $\Sigma(M,x)$ and $\Sigma(M,y)$ are isomorphic only when x and y lie in the same symplectic leaf of M. Also, the groups $\Sigma(M,x)$ are not to be considered as discrete groups; this brings us to our second analogy, for which we need to recall the reconstruction of simply connected Lie groups in terms of paths in their Lie algebras, due to Duistermaat and Kolk [**15**].

Let $\widetilde{G}$ be a simply connected Lie group with Lie algebra $\mathfrak{g}$. We can identify elements $g \in \widetilde{G}$ with homotopy classes of paths $g(t)$ starting at $g(0) = e$ and ending at $g(1) = g$. Using derivatives and right translations, we obtain a bijective correspondence between paths $g(t)$ in the Lie group $\widetilde{G}$ starting at e, and paths $a(t)$ in the Lie algebra $\mathfrak{g}$ with free end points. Explicitly:

$$a(t) = \frac{d}{ds} g(s) g(t)^{-1} \Big|_{s=t} . \tag{5}$$

This bijection shows that there is a natural notion of homotopy of paths in $\mathfrak{g}$, which corresponds to the homotopy of paths in $\widetilde{G}$ (with fixed end points): if a_ϵ is a family of paths in $\mathfrak{g}$, the only thing we have to require is that the induced family g_ϵ of paths in $\widetilde{G}$ has fixed end points $g_\epsilon(1)$. If we let $\mathrm{var}(a_\epsilon) = \frac{d}{d\epsilon} g_\epsilon(1)$ one can show that $\mathrm{var}(a_\epsilon)$ can be defined directly in terms of the family a_ϵ, i.e., using data in $\mathfrak{g}$ and with no reference to the Lie group $\widetilde{G}$. This is in complete analogy with the variation for cotangent paths given above, with the extra simplification coming from the fact that here no connection is involved: $\mathrm{var}(a_\epsilon) = b(\epsilon, 1)$ where $b = b(\epsilon, t)$ is the solution of the differential equation

$$\frac{db}{dt} - \frac{da}{d\epsilon} = [b, a], \qquad b(\epsilon, 0) = 0.$$

The upshot is that the resulting group $G(\mathfrak{g})$ of homotopy classes of paths in $\mathfrak{g}$ can be defined directly in terms of the Lie algebra $\mathfrak{g}$, and hence makes sense even if we don't assume the existence of $\widetilde{G}$. Moreover, since the space of paths in $\mathfrak{g}$ (say of class C^1) carries a natural structure of Banach manifold, $G(\mathfrak{g})$ has a natural quotient topology, and at most one interesting smooth structure (the one for which the quotient map is a submersion). The fact that a smooth structure always exists, is far from trivial (see [**15**]). This gives a proof of Lie's third theorem asserting the existence of a simply-connected Lie group $G(\mathfrak{g})$ integrating $\mathfrak{g}$, and at the same time a proof of the fact that any simply-connected Lie group $\widetilde{G}$ must be isomorphic to $G(\mathfrak{g})$.

Let us look now at the isotropy groups $\Sigma(M,x)$. Since the space of cotangent paths carries a natural structure of a Banach manifold (for which the underlying topology is the C^1-topology), $\Sigma(M,x)$ has a natural quotient topology, and there is no ambiguity when looking for the smooth structure on $\Sigma(M,x)$ (and similarly for $\Sigma^0(M,x)$). However, unlike $G(\mathfrak{g})$, the isotropy group $\Sigma(M,x)$ is not always a Lie group (see Lemma 1 and Corollary 1).

Let us recall from our discussion in [**10**] why $\Sigma(M,x)$ may fail to be a Lie group (see, also, the informal discussion in [**30**]). First, observe that the Lie bracket on 1-forms induces a Lie bracket on the kernel of the map $\#_x : T^*M \to TM$. By skew-symmetry, this kernel coincides with the co-normal bundle to the leaf L through x. One calls $\nu_x^*(L) = \mathrm{Ker}(\#_x)$ the **isotropy Lie algebra** at x. Now, the simply connected Lie group $G(\nu_x^*(L))$ integrating the isotropy Lie algebra and the restricted isotropy Lie group $\Sigma^0(M,x)$ are both quotients of the space of paths in $\nu_x^*(L)$. Also, two paths in $\nu_x^*(L)$ which are homotopic as Lie algebra paths, are homotopic as cotangent paths. However, the converse is not true, since a cotangent homotopy may force one to go away from x. In other words, $\Sigma^0(M,x)$ is (topologically) a quotient of $G(\nu_x^*(L))$.

The following result summarizes some results from [**10**]:

Lemma 1. *Let M be a Poisson manifold, $x \in M$, and denote by L the symplectic leaf through x. The following are equivalent:*

(i) $\Sigma(M,x)$ *is Hausdorff.*
(ii) $\Sigma(M,x)$ *is a Lie group.*
(iii) $\Sigma^0(M,x)$ *is Hausdorff.*
(iv) $\Sigma^0(M,x)$ *is a Lie group.*

In this case, $\Sigma(M,x)$ has Lie algebra the isotropy Lie algebra $\nu_x^*(L)$, its connected component of the identity is $\Sigma^0(M,x)$, and the group of its connected components $\pi_0(\Sigma(M,x))$ is isomorphic to $\pi_1(L,x)$.

One can get a better grasp on the notions of homotopy and variation introduced above when T^*M is integrable to a symplectic groupoid Σ. Then, as in our discussion of Lie algebra paths, there is a bijection between cotangent paths and paths $g(t)$ in Σ which stay in an $\mathbf{s}$-fiber, and which start at an identity element. The variation of families a_ϵ of cotangent paths corresponds then to the derivatives with respect to ϵ of the end points of the associated paths in Σ. Hence, a_ϵ is a cotangent homotopy precisely when the corresponding family of paths in Σ is a homotopy with fixed end points.

Remark 3. Let us briefly explain the perspective one gains by using the language of algebroids/groupoids, as well as its unifying role. First of all, a cotangent path can be identified with a Lie algebroid morphism $TI \to T^*M$. Similarly, two cotangent paths a_0 and a_1 are homotopic if and only if there exists a Lie algebroid homomorphism $adt + bd\epsilon : T(I \times I) \to T^*M$, which covers a (ordinary) homotopy with fix end-points between the base paths $\pi(a_i(t))$, and which restricts to $a_i(t)$ on the boundaries. Therefore, all these constructions readily extend to any Lie algebroid. For example, in the case $A = TM$ one recovers the usual notions of path and homotopy of paths, and the fundamental group(oid). For a Lie algebra $\mathfrak{g}$, viewed as a Lie algebroid over a point, one recovers the construction of Duistermaat and Kolk of the simply connected Lie group $G(\mathfrak{g})$.[3]

We point out that we have given above the precise meaning of "cotangent homotopy with fixed end-points". For example, a family of cotangent paths $a_\varepsilon(t)$, $\varepsilon \in [0, 1]$, such that $a_\varepsilon(0) = a_\varepsilon(1) = 0$, in general, will *not be* a cotagent homotopy. In fact, such a definition would quickly lead to erroneous statements: for example, integration of Poisson vector fields over cotagent paths would not be invariant under cotangent homotopy. Similar remarks apply for Lie algebroids. We refer the reader to [**10**] for details on these constructions in the general Lie algebroid framework.

2. The monodromy groups of Poisson manifolds

In this section we give various descriptions of the monodromy groups of a Poisson manifold M at a point $x \in M$. We shall see that these are certain additive subgroups

$$\mathcal{N}_x \subset \nu^*(L)_x,$$

where L is the symplectic leaf through x, and $\nu^*_x(L)$ is the co-normal space to L at the point x. Recall that $\nu^*_x(L)$ carries a Lie algebra structure induced from the Lie bracket on 1-forms. The monodromy group actually sits inside the center of this Lie algebra.

We start with the shortest possible description of monodromy:

[3]Of course, for Lie groups, one uses pointwise multiplication of paths instead of concatenation, as in any first course on fundamental groups.

Description 1. The monodromy group $\mathcal{N}_x$ is the set of vectors $v \in Z(\nu_x^*(L))$, with the property that the constant cotangent path $a(t) = v$ is cotangent homotopic to the zero cotangent path.

To understand why this group shows up when looking at smoothness issues, we consider a slightly larger group: $\widetilde{\mathcal{N}}_x \subset G(\nu_x^*(L))$ consists of those elements represented by Lie algebra paths which, as cotangent paths, are homotopic to the zero path. First of all, by its very definition, there exists an exact sequence

$$0 \longrightarrow \widetilde{\mathcal{N}}_x \longrightarrow G(\nu_x^*(L)) \longrightarrow \Sigma^0(M,x) \longrightarrow 0, \tag{6}$$

so that

$$\Sigma^0(M,x) = G(\nu_x^*(L))/\widetilde{\mathcal{N}}_x.$$

Next, the group $\widetilde{\mathcal{N}}_x$ almost coincides with $\mathcal{N}_x$. The only difference comes from the fact that, although $\widetilde{\mathcal{N}}_x$ does lie in the center $Z(G(\nu_x^*(L)))$, it is only the connected component $Z^0(G(\nu_x^*(L)))$ of the center that naturally identifies (via the exponential) with the center of $\nu_x^*(L)$. It follows that

$$\mathcal{N}_x = \widetilde{\mathcal{N}}_x \cap Z^0(G(\nu_x^*(L))).$$

In particular, let us point out the following (see also Lemma 1)

Corollary 1. *The isotropy group $\Sigma(M,x)$ is a Lie group if and only if the monodromy group $\mathcal{N}_x$ is discrete.*

The next description not only explains the nature of the monodromy, but it is also suitable for computations.

Description 2. Consider a linear splitting $\sigma : TL \to T_L^*M$ of the short exact sequence

$$0 \longrightarrow \nu^*(L) \longrightarrow T_L^*M \xrightarrow{\#} TL \longrightarrow 0\ . \tag{7}$$

Assume that σ can be chosen so that its curvature

$$\Omega_\sigma(X,Y) \equiv \sigma([X,Y]) - [\sigma(X),\sigma(Y)]$$

is $Z(\nu^*(L))$-valued. Then

$$\mathcal{N}_x = \left\{ \int_\gamma \Omega_\sigma : [\gamma] \in \pi_2(L,x) \right\} \subset \nu^*(L)_x.$$

The integrals involved in this last description should be viewed as integrals of forms with coefficients in flat vector bundles:[4] the Bott connection gives $\nu(L)$ (hence also $\nu^*(L)$, and $Z(\nu^*(L))$) a natural flat vector bundle structure over L. We point out that the assumption that σ is center-valued is only needed to simplify the outcome. It is satisfied in many examples (e.g., whenever x is a regular point), but not always (see Example 6.4 in Section 6). In general, the formula

$$\partial([\gamma]) = \int_0^1 \Omega_\sigma\left(\frac{d\gamma}{dt}, \frac{d\gamma}{d\epsilon}\right) d\epsilon \tag{8}$$

defines a group homomorphism

$$\partial : \pi_2(L, x) \to G(\nu^*(L)_x),$$

and

$$\widetilde{\mathcal{N}}_x = \{\partial(\gamma) : [\gamma] \in \pi_2(L, x)\} \subset G(\nu^*(L)_x).$$

In the center-valued case, any center-valued Lie algebra path can be identified (i.e., defines the same Lie group element) with its integral, viewed as a central Lie group element, and we obtain

$$\partial([\gamma]) = \int_0^1 \int_0^1 \Omega_\sigma\left(\frac{d\gamma}{dt}, \frac{d\gamma}{d\epsilon}\right) d\epsilon dt = \int_\gamma \Omega_\sigma,$$

which is the description of the $\mathcal{N}_x$ given above.

Let us give a more conceptual explanation for the homomorphism ∂, which also provides the relation between the previous two descriptions. The "philosophical idea" on Lie algebroids mentioned in the introduction, and the remark that the homotopy long exact sequence of a (smooth) fibration can be carried out at the level of tangent bundles (see the construction of ∂ below), suggests a construction of such sequences for algebroids.

Description 3. Viewing (7) as analogous to a fibration, there is a long exact sequence

$$\cdots \longrightarrow \pi_2(L, x) \xrightarrow{\partial} G(\nu^*(L)_x) \xrightarrow{j} \Sigma(M, x) \longrightarrow \pi_1(L, x),$$

where $\partial : \pi_2(L, x) \to G(\nu^*(L)_x)$ is the map given above, and j is the composition of the projection from $G(\nu^*(L)_x)$ onto $\Sigma^0(M, x)$ with the inclusion.

[4]Note that forms with values in flat bundles can be integrated over simply connected cycles.

One can construct ∂ in complete analogy with the construction of the boundary map for fibrations: given a 2-loop $\gamma : I \times I \to L$ based at x, one lifts its differential $d\gamma$ to a morphism $adt + bd\epsilon : T(I \times I) \to A$. Since γ on the boundary of $I \times I$ takes the constant value x, the restriction of this morphism to the boundary, gives the path $a(1, t)$ in the Lie algebra $\nu^*(L)_x$, hence an element in the Lie group $G(\nu^*(L)_x)$. If we choose a splitting σ, we can use it to lift $d\gamma$, and the computation gives precisely the expression (8) for ∂.

3. The symplectic groupoid $\Sigma(M)$

In this section we describe the Weinstein groupoid $\Sigma(M)$ associated to a Poisson manifold. Note that, though $\Sigma(M)$ is just a special case (with $A = T^*M$) of the general construction of the Weinstein groupoid of a Lie algebroid A [**10**], the extra structure coming from the Poisson bracket and the cotangent bundle translates into some special properties of $\Sigma(M)$. In particular, we explain the relation with the work of Cattaneo and Felder [**5**] on the Poisson-sigma model (the Hamiltonian formulation).

We denote by $P(T^*M)$ the space of cotangent paths in M which are of class C^1, and we denote by $\sim$ the cotangent homotopy equivalence defined using cotangent homotopies a_ϵ in $P(T^*M)$ which are of class C^2 in ϵ. The Weinstein groupoid is defined as the quotient[5]

$$\Sigma(M) = P(T^*M)/\sim . \tag{9}$$

For the groupoid structure, the source and target maps $\mathbf{s}, \mathbf{t} : \Sigma(M) \to M$ take the equivalence class of a cotangent path to the end-points of the base path, while the multiplication is given by concatenation (see [**10**] for details). We shall see that this groupoid gives new insights into the global properties of a Poisson manifold.

As a simple example, let us consider the integration of Poisson vector fields along cotangent paths, discussed in Section 1 above. First, invariance under cotangent homotopy (see Proposition 1) shows that we can view integration as a map which associates to each Poisson vector field X a map

$$\int X : \Sigma(M) \to \mathbb{R}, \qquad g = [a] \mapsto \int_a X.$$

[5]In [**10**], this groupoid was denoted $\mathcal{G}(T^*M)$.

Second, additivity with respect to the concatenation of paths

$$\int_{g_0 g_1} X = \int_{g_0} X + \int_{g_1} X,$$

can be viewed as a cocycle condition for $\int X$: if $\Sigma(M)$ is a smooth manifold, this means that $\int X$ defines a **differentiable 1-cocycle** on $\Sigma(M)$. Third, and last, for an Hamiltonian vector field we can write:

$$\int_g X_h = h(\mathbf{t}(g)) - h(\mathbf{s}(g)),$$

which just says that $\int X_h$ is a **differentiable coboundary**. Hence, denoting by $H^1_{\mathrm{dif}}(\Sigma(M))$ the first differentiable cohomology group of $\Sigma(M)$, integration defines a map $H^1_\Pi(M) \to H^1_{\mathrm{dif}}(\Sigma(M))$.

On the other hand, there is a Van Est map (see [**9**]) which takes a differentiable cocycle into a Poisson cohomology class, and which is defined in all degrees. This map is known to be an isomorphism in degree one. One can think of the Van Est map as differentiation of differentiable cocycles. A simple check shows that:

Proposition 3. *The map $\int : H^1_\Pi(M) \to H^1_{\mathrm{dif}}(\Sigma(M))$ is the inverse of the Van Est map in degree one.*

The reader will notice that all this discussion extends to any Lie algebroid with appropriate changes.

On $\Sigma(M)$ we consider always the quotient topology, so that it becomes a topological groupoid. As in the case of the isotropy groups $\Sigma(M, x)$, there is no ambiguity when looking at the smoothness of $\Sigma(M)$: the smooth structure, if exists, will be the unique one for which the quotient map $P(T^*M) \to \Sigma(M)$ is a submersion. It is also convenient to consider the larger space $\widetilde{P}(T^*M)$ of all C^1-curves $a : I \to T^*M$, with base path $\gamma = \pi \circ a$ of class C^2. It has an obvious structure of a Banach manifold, and $P(T^*M)$ is a (Banach) submanifold of $\widetilde{P}(T^*M)$ (cf. Lemma 4.6 in [**10**]). We need an explicit description of the tangent spaces to these Banach manifolds, and a more geometric description of the equivalence relation defined on them by cotangent homotopies.

The tangent space $T_a\widetilde{P}(T^*M)$ consists of curves $U : I \to TT^*M$ such that $U(t) \in T_{a(t)}T^*M$, i.e., vector fields along a. Using a connection ∇ on TM and the associated contravariant connections $\overline{\nabla}$ (see Section 1), such a curve can be viewed as a pair (u, ϕ) consisting of curves $u : I \to T^*M$ and $\phi : I \to TM$ over γ (namely, the vertical and horizontal component of U). The subspace $T_aP(T^*M) \subset T_a\widetilde{P}(T^*M)$ consists (see [**10**], Section 5.2) of those pairs $U = (u, \phi)$ with the property that

$$\# u = \overline{\nabla}_a \phi.$$

Now let us describe the equivalence relation defined by cotangent homotopies in terms of a Lie algebra action. The Lie algebra consists of time-dependent 1-forms, vanishing at the end-points

$$P_0\Omega^1(M) = \left\{\eta_t \in \Omega^1(M), t \in I : \eta_0 = \eta_1 = 0,\ \eta_t \text{ of class } C^1 \text{ in } t\right\}$$

with Lie bracket the bracket on 1-forms with time varying as a parameter. The Lie algebra $P_0\Omega^1(M)$ acts on $\widetilde{P}(T^*M)$ and this action satisfies the following properties:

(a) It is tangent to $P(T^*M)$.
(b) The orbits in $P(T^*M)$ define a finite codimension foliation.
(c) Two cotangent paths are homotopic if and only if they belong to the same orbit.

This is proved in [**10**], and here we shall give only the definition of the action. This means we will define a Lie algebra map

$$P_0\Omega^1(M) \to \mathfrak{X}(\widetilde{P}(T^*M)), \quad \eta \mapsto X_\eta.$$

So, given a time-dependent 1-form $\eta \in P_0\Omega^1(M)$ and a path $a \in \widetilde{P}(T^*M)$ with underlying path $\gamma : I \to M$, we have to describe a path $X_{\eta,a}$ in T^*M above γ. Let us first assume that a is a cotangent path. To describe $X_{\eta,a}$, we specify the components (u, ϕ) with respect to a connection ∇:

$$u = \overline{\nabla}_a b, \quad \phi = \# b.$$

This does not depend on the connection, and, by the discussion above, it does define a vector tangent to $P(T^*M)$. We check that this can also be written as:

$$X_{\eta,a}(t) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \phi_\eta^{\epsilon,0} a(t) + \frac{d\eta_t}{dt}(\gamma(t))$$

where $\phi_\eta^{s,t}$ stands for the flow of the time dependent one form η.[6] Now, this formula make sense for *any* path $a : I \to T^*M$ and this is our definition of the action.

What we have said so far makes sense for any Lie algebroid. Let us now take advantage of the fact that we have $A = T^*M$, the cotangent bundle. We have a natural identification $\widetilde{P}(T^*M) \simeq T^*P(M)$, where $P(M)$ denotes the Banach space of paths $\gamma : I \to M$ of class C^2. Hence, $\widetilde{P}(T^*M)$ carries a natural symplectic structure. Moreover, the set of cotangent paths $P(T^*M)$ is the level set $J^{-1}(0)$ of the map $J : \widetilde{P}(T^*M) \to P_0\Omega^1(M)^*$ given by:

$$\langle J(a), \eta\rangle = \int_0^1 \langle \frac{d}{dt}\pi(a(t)) - \#a(t), \eta(t, \gamma(t))\rangle dt.$$

and one has the following result due to Cattaneo and Felder (see [**5**]):

Theorem 4. *The Lie algebra action of $P_0\Omega^1(M)$ on $\widetilde{P}(T^*M)$ is Hamiltonian, with equivariant moment map $J : \widetilde{P}(T^*M) \to P_0\Omega^1(M)^*$.*

Hence the Weinstein groupoid can be described alternatively as a Marsden-Weinstein reduction:

$$\Sigma(M) \simeq \widetilde{P}(T^*M)//P_0\Omega(M). \tag{10}$$

The two alternate descriptions (9) and (10) for the Weinstein groupoid of a Poisson manifold give the precise relationship between our integrability approach introduced in [**10**], which is valid for any Lie algebroid, and the approach of Cattaneo and Felder in [**5**].

Corollary 2. *If $\Sigma(M)$ is smooth, then it admits a symplectic form which turns $\Sigma(M)$ into a symplectic groupoid.*

Proof. We sketch a proof of Theorem 4. From what we saw above, we only need to check the compatibility of the symplectic form with the product, i.e., that the graph of multiplication is a Lagrangian submanifold. This can be restated in a slightly simpler form using the following proposition which will also be useful later.

---

[6]Given a time-dependent one-form η on M, its flow is a map $\phi_\eta^{t,s}(x) : T_x^*(M) \to T^*_{\Phi^{t,s}_{\#\alpha}}(M)$. It is uniquely determined by the conditions $\phi_\eta^{t,s}\phi_\eta^{s,u} = \phi_\eta^{t,u}$, $\phi_\eta^{t,t}$ =id, and the differential equation

$$\left.\frac{d}{dt}\right|_{t=s} (\phi_\eta^{t,s})^*\beta = [\eta^s, \beta],$$

where $(\phi_\eta^{t,s})^*(\beta)(x) = \phi_\eta^{s,t}\beta(\phi_{\#\eta}^{t,s}(x))$, with $\phi_{\#\eta}^{t,s}$ the flow of the vector field $\#\eta$.

Proposition 4. *Let $\mathcal{G}$ be a Lie groupoid, and let $\omega \in \Omega^2(\mathcal{G})$ be a symplectic form. The following statements are equivalent:*

(i) *The graph of multiplication $\gamma_m = \{(g, h, g\dot{h}) \in \mathcal{G} \times \mathcal{G} \times \mathcal{G} : (g, h) \in \mathcal{G}^{(2)}\}$ is a Lagrangian submanifold of $\mathcal{G} \times \mathcal{G} \times \bar{\mathcal{G}}$.*

(ii) *The relation $m^*\omega = \pi_1^*\omega + \pi_2^*\omega$ holds.*

Here, we denote by $m : \mathcal{G}^{(2)} \to \mathcal{G}$ the multiplication in $\mathcal{G}$ and by $\pi_1, \pi_2 : \mathcal{G}^{(2)} \to \mathcal{G}$ the projections to the first and second factors.

Using this Proposition we prove the Corollary above. First note that we have the following explicit formula for the symplectic form $\widetilde{\omega}$ in $\widetilde{P}(T^*M)$:

$$\widetilde{\omega}_a(U_1, U_2) = \int_0^1 \omega_{can}(U_1(t), U_2(t))dt,$$

for all $U_1, U_2 \in T_a\widetilde{P}(T^*M)$, where ω_{can} is the canonical symplectic form on T^*M. The additivity of the integral shows that that condition (ii) holds at the level of $\widetilde{P}(T^*M)$, hence it must hold also on the reduced space $\Sigma(M)$. q.e.d.

Proof of Proposition 4. We claim that the graph of multiplication γ_m satisfies the following two properties:

(a) γ_m is isotropic iff $m^*\omega = \pi_1^*\omega + \pi_2^*\omega$.

(b) γ_m is isotropic implies that M is also isotropic.

Assuming these claims, it should be clear that (i) implies (ii). Conversely, if (ii) holds, then by (a) we have that γ_m is an isotropic submanifold. Since $\dim \gamma_m = \dim \mathcal{G}^{(2)} = 2 \dim \mathcal{G} - \dim M$, we see that γ_m is Lagrangian provided that $\dim M = \frac{1}{2} \dim \mathcal{G}$. Since γ_m is isotropic, we know already that $2 \dim \mathcal{G} - \dim M \leq \frac{3}{2} \dim \mathcal{G}$, and so we have

$$\dim M \geq \frac{1}{2} \dim \mathcal{G}.$$

But, by (b), M is also an isotropic submanifold of $\mathcal{G}$ so that

$$\dim M \leq \frac{1}{2} \dim \mathcal{G}.$$

and equality must indeed hold.

To prove the claims let us denote by $\Omega = \omega \oplus \omega \oplus (-\omega)$ the symplectic form on $\mathcal{G} \times \mathcal{G} \times \bar{\mathcal{G}}$, and let $\gamma : \mathcal{G}^{(2)} \to \mathcal{G} \times \mathcal{G} \times \mathcal{G}$ be the embedding:

$$(g, h) \mapsto (g, h, g \cdot h).$$

Obviously, the graph γ_m is an isotropic submanifold of $\mathcal{G} \times \mathcal{G} \times \bar{\mathcal{G}}$ if and only if $\gamma^*\Omega = 0$, and this condition is clearly equivalent to

$$m^*\omega = \pi_1^*\omega + \pi_2^*\omega.$$

This shows that (a) holds.

In order to show that (b) also holds we observe that M, viewed as the identity section of $\mathcal{G}$, can be expressed as

$$M = \{x \in \mathcal{G} : \exists g \in \mathcal{G}, x \cdot g = g\}.$$

Hence we have $M = \pi(\Delta \cap \gamma_m)$ where $\pi : \mathcal{G} \times \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is projection onto the first fact and

$$\Delta \equiv \{(h, g, g) \in \mathcal{G} \times \mathcal{G} \times \mathcal{G}\}.$$

Now, it is easy to check that $\Delta \subset \mathcal{G} \times \mathcal{G} \times \bar{\mathcal{G}}$ is a coisotropic submanifold, $\pi : \Delta \to \mathcal{G}$ is projection along its characteristic foliation, and Δ and γ_m have clean intersection. By a standard argument in symplectic geometry (see [**26**], Lemma 5.34), γ_m being isotropic implies that the projection $M = \pi(\Delta \cap \gamma_m)$ is also isotropic. q.e.d.

4. The smoothness of $\Sigma(M)$

We are now in position to give a proof of our main result concerning the smoothness of $\Sigma(M)$. The discussion here depends heavily on our general integrability criteria for Lie algebroids [**10**].

Theorem 5. *For a Poisson manifold M the following statements are equivalent:*

(i) *M is integrable by a symplectic Lie groupoid.*
(ii) *The algebroid T^*M is integrable.*
(iii) *The Weinstein groupoid $\Sigma(M)$ is a smooth manifold.*
(iv) *The exponential map of $\Sigma(M)$ associated to a (or any) connection is injective around the zero section.*
(v) *The monodromy groups $\mathcal{N}_x$, where $x \in M$, are locally uniformly discrete.*

In this case $\Sigma(M)$ is the unique s-simply connected, symplectic groupoid which integrates M.

Before proceeding to the proof, we clarify conditions (iv) and (v).

The exponential map referred to in condition (iv) is defined as follows. Given a contravariant connection ∇, one can define the notion of cotangent geodesic as in the classical case, by the equation $\nabla_a a = 0$

(see [**17**]). Then, for $a_0 \in T^*M$ sufficiently close to zero, the geodesic starting at a_0 is defined up to the time $t = 1$, hence defines an element in $P(T^*M)$. This correspondence defines a map $\widetilde{\exp}_\nabla : T^*M \to P(T^*M)$, and, the map induced to the quotient, $\exp_\nabla : T^*M \to \Sigma(M)$, is called the **exponential map** associated to ∇. As in the classical case, the map is only defined on a neighborhood of the zero section.

Condition (v) should be viewed as the main "integrability criterion". The results of Section 2 show that it is suitable for computation in concrete examples. In order to measure the discreteness of the monodromy groups we define a function $r_{\mathcal{N}} : M \to [0, +\infty]$ by

$$r_{\mathcal{N}}(y) = d(\mathcal{N}_y - \{0_y\}, \{0_y\}),$$

where the distance is induced by some norm on T^*M. By convention, $r_{\mathcal{N}}(y) = +\infty$ if $\mathcal{N}_y = \{0_y\}$. Clearly, $\mathcal{N}_y$ is discrete if and only if $r_{\mathcal{N}}(y) \neq 0$. By uniform discreteness of $\mathcal{N}$ around y we mean that

$$\liminf_{y \to x} r_{\mathcal{N}}(y) \neq 0.$$

Proof of Theorem 5. If Σ is a symplectic groupoid integrating M, then its associated algebroid is isomorphic to T^*M, and this shows that (i) implies (ii). To see that (ii) implies (iii), we assume that Σ is a Lie groupoid integrating T^*M. Taking the universal covers of the **s**-fibers of Σ together, we get a new groupoid $\widetilde{\Sigma}$ which still integrates T^*M, and which is **s**-simply connected. More precisely, $\widetilde{\Sigma}$ is the collection of paths lying on the **s**-fibers, which start at the identity section, where we identify any two which are homotopic by homotopies with fixed end points (see [**10**] for details). It follows from Remark 3 that $\widetilde{\Sigma}$ is naturally identified with the space of cotangent paths, modulo homotopy, i.e., with $\Sigma(M)$. Since the quotient Σ of $\widetilde{\Sigma}$ is smooth, so is $\widetilde{\Sigma} \cong \Sigma(M)$. That (iii) implies (i) is the content of the previous section.

For any contravariant connection ∇ the exponential $\widetilde{\exp}_\nabla : T^*M \to P(T^*M)$ defines a transversal to the foliation on $P^*(TM)$. In particular, if the associated leaf space $\Sigma(M)$ is smooth, the differential of $\exp_\nabla$ at zero elements is the invertible, and this shows that (iii) implies (iv). Conversely, if (iv) holds, then one uses $\exp_\nabla$ to define the smooth structure on $\Sigma(M)$ around the identity elements, and then, using right translations, one extends the smooth structure to the entire topological groupoid $\Sigma(M)$ (cf. [**10**]). This explains why (iv) implies (iii).

Finally, the equivalence with (v) is the main theorem in [**10**], applied to the special case of the cotangent Lie algebroid of a Poisson manifold. q.e.d.

5. The regular case

A Poisson manifold is called regular if the Poisson tensor has constant rank. In the regular case, both $\mathcal{F} = \operatorname{Im}\#$ and $\nu^*(\mathcal{F}) = \ker\#$ are vector bundles over M, and the isotropy Lie algebras $\nu^*(\mathcal{F})_x$ are all abelian. This amounts to several simplifications and leads to a new geometric interpretation of the integrability criteria.

Let us consider, for instance, the long exact sequence of Section 2. In the regular case it becomes

$$\cdots \longrightarrow \pi_2(L,x) \xrightarrow{\partial} \nu^*(L)_x \longrightarrow \Sigma(M,x) \longrightarrow \pi_1(L,x)\ .$$

The monodromy groups can be described (or defined) as the image of ∂, which, in turn, is given by integration of a canonical cohomology 2-class $[\Omega_\sigma] \in H^2(L, \nu_x^*(L))$, and this class can be computed explicitly by using a section σ of $\# : T_L^*M \to TL$. In this section we will give a different geometric interpretation of monodromy based on the symplectic geometry of the leaves of M. We state the main results first, and delay all proofs until the end of the section.

Fix a point x in a Poisson manifold M, let L be the symplectic leaf through x, and consider a 2-sphere $\gamma : S^2 \to L$, which maps the north pole p_N to x. The symplectic area of γ is given, as usual, by

$$A_\omega(\gamma) = \int_{S^2} \gamma^*\omega,$$

where ω is the symplectic 2-form on the leaf L.

By a **deformation** of γ we mean a family $\gamma_t : S^2 \to M$ of 2-spheres parameterized by $t \in (-\varepsilon, \varepsilon)$, starting at $\gamma_0 = \gamma$, and such that for each fixed t the sphere γ_t has image lying entirely in a symplectic leaf. The **transversal variation** of γ_t (at $t = 0$) is the class of the tangent vector

$$\operatorname{var}_\nu(\gamma_t) \equiv \left[\frac{d}{dt}\gamma_t(p_N)\Big|_{t=0}\right] \in \nu_x(\mathcal{F}).$$

We shall see below that the quantity

$$\frac{d}{dt}A_\omega(\gamma_t)\Big|_{t=0}$$

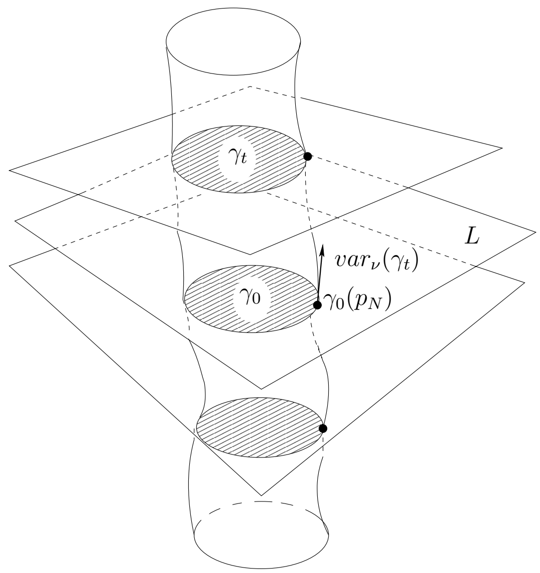

Figure 1. Symplectic spheres.

only depends on the homotopy class of γ and on $\mathrm{var}_\nu(\gamma_t)$. Finally, the formula

$$\langle A'_\omega(\gamma), \mathrm{var}_\nu(\gamma_t)\rangle = \frac{d}{dt}A_\omega(\gamma_t)\Big|_{t=0},$$

applied to different deformations of γ, gives a well-defined element

$$A'_\omega(\gamma) \in \nu^*_x(L).$$

The promised result is:

Proposition 5. *For any regular manifold M, the function $r_{\mathcal{N}}$ is lower semi-continuous, and*

$$\mathcal{N}_x = \{A'_\omega(\sigma) : \sigma \in \pi_2(L, x)\}.$$

Next, we form the (set theoretical) bundle of "variations of symplectic areas" (this already appears in [**1**] and in earlier work of Dazord, and it has been used in [**39**] in the case where the symplectic foliation is simple):

$$\mathcal{A}^{'}(M) = \bigcup \mathcal{N}_x \subset \nu^*(\mathcal{F}).$$

Similarly, the groups $\mathcal{S}_x(M) = \nu_x^*(L)/\mathcal{N}_x$ fit together into a (set theoretical, again) bundle of groups,

$$\mathcal{S}(M) = \nu^*(\mathcal{F})/\mathcal{A}^{'}(M) = \bigcup_{x \in M} \nu^*(\mathcal{F})_x/\mathcal{N}_x,$$

which will be called the **structure groupoid** of M. Note that the groups $\mathcal{S}_x(M)$ are closely related to the isotropy groups $\Sigma(M,x)$ introduced in Section 1. More precisely, it follows from Section 2 that:

$$\mathcal{S}_x(M) = \Sigma^0(M,x),$$

and we have a short exact sequence

$$1 \longrightarrow \mathcal{S}_x(M) \longrightarrow \Sigma(M,x) \longrightarrow \pi_1(L) \longrightarrow 1.$$

In the sequel we will only be interested on the structures on $\mathcal{S}(M)$ coming from $\nu^*(\mathcal{F})$ via the projection $\nu^*(\mathcal{F}) \to \mathcal{S}(M)$, and on structures on $\mathcal{A}^{'}(M)$ as a subspace of $\nu^*(\mathcal{F})$. In particular, both $\mathcal{A}^{'}(M)$ and $\mathcal{S}(M)$ are topological groupoids, and there is no ambiguity when asking for *the* smooth structure on $\mathcal{A}^{'}(M)$ and on $\mathcal{S}(M)$, respectively.

Theorem 6. *For a regular Poisson manifold M, the following are equivalent:*

(i) *M is integrable.*
(ii) *$\Sigma(M)$ is a Lie groupoid.*
(iii) *$\mathcal{S}(M)$ is a Lie groupoid.*
(iv) *$\mathcal{A}^{'}(M)$ is a Lie groupoid.*

Moreover, in this case, $\mathcal{A}^{'}(M)$ is étale (i.e., its source is a local diffeomorphism), and there is an exact sequence of Lie groupoids

$$1 \longrightarrow \mathcal{S}(M) \longrightarrow \Sigma(M) \xrightarrow{\Phi} \pi_1(\mathcal{F}) \longrightarrow 1\ .$$

Here $\pi_1(\mathcal{F})$ stands for the **homotopy groupoid** of $\mathcal{F}$. This object is well-known in foliation theory (sometimes under the name of monodromy groupoid), and it is one of the simplest examples of a Weinstein groupoid (namely the one associated to the Lie algebroid $T\mathcal{F}$). The arrows between $x, y \in M$ are the leafwise homotopy classes of paths (paths tangent to $\mathcal{F}$) with end-points in x and y, so that there are arrows only between points lying in the same leaf.

In the short exact sequence above, $\mathcal{S}(M)$ and $\pi_1(\mathcal{F})$ are (in principle) easily computable from the Poisson geometry of M, and the sequence will give in fact a pretty good indication to what the integration of M should be. On the other hand, this sequence shows that the symplectic groupoid of M is an extension of the monodromy groupoid $\pi_1(\mathcal{F})$ by the structure groupoid $\mathcal{S}(M)$. This suggests a different strategy to integrate a regular Poisson manifold: assuming $\mathcal{S}(M)$ to be a Lie groupoid one looks for an extension of $\pi_1(\mathcal{F})$ by $\mathcal{S}(M)$. In [**1**] it is shown that this strategy works under some additional technical assumptions, and Theorem 6 is actually an improvement over the results of [**1**].

We now turn to the proofs of the results we have stated so far. Recall that the normal bundle ν (hence also any associated tensor bundle) has a natural flat $\mathcal{F}$-connection $\nabla : \Gamma(T\mathcal{F}) \times \Gamma(\nu) \to \Gamma(\nu)$, given by

$$\nabla_X \overline{Y} = \overline{[X, Y]}.$$

In terms of the Lie algebroid $T\mathcal{F}$, ∇ is a flat Lie algebroid connection (see [**16**]) giving a Lie algebroid **representation** of $T\mathcal{F}$ on the vector bundle ν (and also on any associated tensor bundle). Hence the foliated forms with coefficients in ν, denoted $\Omega^\bullet(\mathcal{F};\nu) = \Gamma(\wedge^\bullet T^*\mathcal{F} \otimes \nu)$, carry a foliated de Rham operator

$$\begin{aligned} &d_{\mathcal{F}}\omega(X_1, \dots, X_{p+1}) \\ &= \sum_i (-1)^i \nabla_{X_i}(\omega(X_1, \dots, \hat{X}_i, \dots, X_{p+1})) \\ &\quad + \sum_{i<j} (-1)^{i+j-1} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{p+1}), \end{aligned}$$

for which the associated cohomology, called **foliated cohomology** with coefficients in ν, is denoted $H^*(\mathcal{F};\nu)$. Similarly one can talk about cohomology with coefficients in any tensorial bundle associated to ν. The special case of trivial coefficients $\mathbb{R}$ will be denoted $H^*(\mathcal{F})$.

For instance, the Poisson tensor in M determines a foliated 2-form $\omega \in \Omega^2(\mathcal{F})$, which is just another way of looking at the symplectic forms on the leaves. Therefore, we have a foliated cohomology class in the second foliated cohomology group:

$$[\omega] \in H^2(\mathcal{F}).$$

On the other hand, we saw above in "Description 2" of the monodromy groups, that a splitting σ determines a foliated 2-form $\Omega_\sigma \in \Omega^2(\mathcal{F};\nu^*)$, with coefficients in the co-normal bundle, and that the corresponding foliated cohomology class

$$[\Omega] \in H^2(\mathcal{F}; \nu^*)$$

does not depend on the choice of splitting.

These two classes are related in a very simple way. In fact, there is a map

$$d_\nu : H^2(\mathcal{F}) \longrightarrow H^2(\mathcal{F}; \nu^*),$$

which can be described as follows. We start with a class $[\theta] \in H^2(\mathcal{F})$, represented by a foliated 2-form θ. As with any foliated form, we have $\theta = \widetilde{\theta}|_{T\mathcal{F}}$ for some 2-form $\widetilde{\theta} \in \Omega^2(M)$. Since $d\widetilde{\theta}|_{\mathcal{F}} = 0$, it follows that the map $\Gamma(\wedge^2\mathcal{F}) \to \Gamma(\nu^*)$ defined by

$$(X, Y) \mapsto d\widetilde{\theta}(X, Y, -),$$

gives a closed foliated 2-form with coefficients in ν^*. It is easily seen that its cohomology class does not depend on the choice of $\widetilde{\theta}$, and this defines d_ν. Now the formula $\omega(X_f, X) = X(f)$, immediately implies that

$$d_\nu([\omega]) = [\Omega].$$

We emphasize that the operator d_ν is well-known in foliation theory and is part, together with $d_{\mathcal{F}}$, of the spectral sequence of a foliation. The advantage of this point of view comes from the fact that the construction of d_ν is functorial with respect to foliated maps (i.e., maps between foliated spaces which map leaves into leaves).

From this perspective, a deformation γ_t of 2-spheres is a foliated map $S^2 \times I \to M$, where in $S^2 \times I$ we consider the foliation $\mathcal{F}_0$ whose leaves are the spheres $S^2 \times \{t\}$. From $H^2(S^2) \simeq \mathbb{R}$, we get

$$H^2(\mathcal{F}_0) \simeq C^\infty(I) = \Omega^0(I)$$
$$H^2(\mathcal{F}_0; \nu^*) \simeq C^\infty(I)dt = \Omega^1(I),$$

where the isomorphisms are obtained by integrating over S^2. Hence, d_ν for $\mathcal{F}_0$ becomes the de Rham differential $d : \Omega^0(I) \to \Omega^1(I)$. Now, the functoriality of d_ν with respect to γ_t when applied to ω gives

$$\frac{d}{dt}\int_{S^2} \gamma_t^*\omega = \left\langle \int_{\gamma_t} \Omega, \frac{d}{dt}\gamma_t(p_N) \right\rangle.$$

This proves the last part of the proposition (and also the properties of the variation of the symplectic area, stated at the beginning of this section).

Next, we recall (see [**1**]) that the second homotopy groups of the leaves fit into a smooth étale groupoid (actually, a smooth bundle of discrete groups)

$$\pi_2(\mathcal{F}) = \bigcup_{x\in M} \pi_2(L_x, x).$$

Also, according to "Description 3" in Section 2, we should view ∂ as part of a long exact sequence (but now of groupoids rather than groups):

$$\cdots \longrightarrow \pi_2(\mathcal{F}) \xrightarrow{\partial} \nu^*(\mathcal{F}) \longrightarrow \Sigma(M) \xrightarrow{\Phi} \pi_1(\mathcal{F}) \ .$$

This clearly implies the exactness of the sequence of Theorem 6. Now, the smoothness of ∂, together with the fact that $\pi_2(\mathcal{F})$ is étale, imply the following property of the monodromy groups: for any $a \in \mathcal{N}_x$, there is a smooth local section α of $\nu^*(\mathcal{F})$ defined on some open U containing x such that $\alpha(x) = a$, and $\alpha(y) \in \mathcal{N}_y$ for all $y \in U$. This immediately implies that $r_{\mathcal{N}}$ is lower semi-continuous.

Let us turn now to the equivalence of the various statements in Theorem 6. We know from the general Lie algebroid case that (i) implies (ii). The fact that (ii) implies (iii) follows from the exact sequence above. To prove that (iii) implies (iv), we just have to observe that in the short exact sequence

$$1 \longrightarrow \mathcal{A}'(M) \longrightarrow \nu^*(\mathcal{F}) \longrightarrow \mathcal{S}(M) \longrightarrow 1 \ ,$$

the projection $\nu^*(\mathcal{F}) \to \mathcal{S}(M)$ will be a submersion if $\mathcal{S}(M)$ is smooth. We are left with showing that (iv) implies (i), and for that we will check the integrability conditions of Theorem 5. First of all, (iv) implies that $\mathcal{N}_x$ are closed subgroups of $\nu^*_x(\mathcal{F})$. But $\mathcal{N}_x = \partial(\pi_2(L_x, x))$ are at most countable (because second homotopy groups of manifolds are so), hence $\mathcal{N}_x$ must be discrete. This shows that $\mathcal{A}'(M)$ must be étale, which, in turn, implies that $\liminf_{y\to x} r_{\mathcal{N}}(y) \neq 0$. If not, we can find a sequence $a_n \in \mathcal{N}_{x_n}$ of nonzero elements, with $x_n \to x$, $a_n \to 0_x$. This is impossible since we can find a neighborhood of 0_x which only contains zero elements. This concludes our proof.

Remark 4. The last part of this proof can be restated as an interesting property of any integrable, regular, Poisson manifold: every transverse deformation $\mathrm{var}(\gamma_t)$ of a sphere γ_0 such that $\partial([\gamma_0]) = 0$ must be a trivial deformation, i.e., $\partial([\gamma_t]) = 0$ for all small enough t. By the results of this section, we can rewrite this as:

$$A'_\omega(\gamma_0) = 0 \quad \Longrightarrow \quad A'_\omega(\gamma_t) = 0, \ \forall t.$$

This property plays an important role in Alcalde Cuesta and Hector approach to integrability of regular Poisson manifolds, who state it as "every symplectic vanishing deformation is trivial" (see [**1**]).

6. Examples

In this section we present several examples of integrable and non-integrable Poisson manifolds. First, we give some immediate consequences of Theorem 5 and of the main properties of the monodromy groups.

Corollary 3. *If all the monodromy groups $\mathcal{N}_x$ of the Poisson manifold vanish, then M is integrable. This happens for instance in either of the following cases:*

(i) *For any $x \in M$, the symplectic leaf L through x has finite second homotopy groups.*
(ii) *The isotropy Lie algebra $\nu_x^*(L)$ is semi-simple.*

Not every Poisson manifold is locally integrable (see the example in Section 6.2 below). However, from the local form of regular foliations we deduce:[7]

Corollary 4 (Lie)**.** *If $x \in M$ is regular, then M is locally integrable around x, i.e., there exists a neighborhood of x in M which is integrable.*

After these simple criteria for integrability, we now turn to the examples.

6.1. Poisson manifolds of dimension 2.

The lowest dimension one can have nontrivial Poisson manifolds is 2. However, it follows immediately from Theorem 5 that in dimension 2 all Poisson manifolds are integrable.

Corollary 5. *Any 2-dimensional Poisson manifold is integrable.*

In particular, we recover the main positive integrability result of [**5**] that states that any Poisson structure on $\mathbb{R}^2$ is integrable.

Corollary 5 can be partially generalized to higher dimensions in the following sense: any $2n$-dimensional Poisson manifold whose Poisson tensor has rank $2n$ on a dense, open set, is integrable. This, in turn, is a consequence of a general result due to Debord [**13**] (see also [**10**], Corollary 5.9) that states that a Lie algebroid with almost injective anchor is integrable. The proof is more involved.

[7] As pointed out by Alan Weinstein in [**37**], this corollary should be attributed to Lie, who found the normal coordinates near a regular point and used them to prove a version of "Lie's third theorem".

6.2. A non-integrable Poisson manifold. Already in dimension 3 there are examples of non-integrable Poisson manifolds. We consider $M = \mathbb{R}^3$ with the Poisson bracket:

$$\{f, g\} = \det \begin{pmatrix} x & y & z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix}.$$

Identifying $\mathbb{R}^3$ with $\mathfrak{su}(2)^*$, this is just the Kirillov-Poisson bracket (see also the next example). We choose any smooth function $a = a(R)$ on M, which depends only on the radius R, and which is strictly positive for $R > 0$. We multiply the previous brackets by a, and we denote by M_a the resulting Poisson manifold. The bracket on $\Omega^1(M_a)$ is computed using the Leibniz identity and we get

$$\begin{aligned} [dx^2, dx^3] &= a dx^1 + b x^1 R \bar{n}, \\ [dx^3, dx^1] &= a dx^2 + b x^2 R \bar{n}, \\ [dx^1, dx^2] &= a dx^3 + b x^3 R \bar{n}, \end{aligned}$$

where $\bar{n} = \frac{1}{R} \sum_i x^i dx^i$ and $b(R) = a'(R)/R$. The bundle map $\# : T^* M_a \to T M_a$ is just

$$\#(dx^i) = a \bar{v}^i, \qquad i = 1, 2, 3$$

where $\bar{v}^i$ is the infinitesimal generator of a rotation about the i-axis:

$$\bar{v}^1 = x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3}, \quad \bar{v}^2 = x^1 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^1}, \quad \bar{v}^3 = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}.$$

The leaves of the symplectic foliation of M_a are the spheres $S^2_R \subset \mathbb{R}^3$ centered at the origin, and the origin is the only singular point. To compute the function $r_{\mathcal{N}}$, using the obvious metric on $T^* M_a$, we restrict to a leaf S^2_R with $R > 0$, and we use "Description 2" of Section 2. As splitting of $\#$ we choose the map defined by

$$\sigma(\bar{v}^i) = \frac{1}{a} \left(dx^i - \frac{x^i}{R} \bar{n} \right),$$

with curvature the center-valued 2-form

$$\Omega_\sigma = \frac{R a' - a}{a^2 R^3} \omega \bar{n},$$

where $\omega = x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2$. Since $\int_{S^2_R} \omega = 4\pi R^3$ it follows that

$$\mathcal{N}_{(x,y,z)} \simeq 4\pi \frac{Ra' - a}{a^2} \mathbb{Z}\bar{n} \subset \mathbb{R}\bar{n}.$$

The canonical generator of $\pi_2(S^2_R)$ defines *the* symplectic area of S^2_R, which is easily computable:

$$A_a(R) = 4\pi \frac{R}{a(R)}.$$

We recover in this way the relationship between the monodromy and the variation of the symplectic area (Proposition 5). Also,

$$r_{\mathcal{N}}(x,y,z) = \begin{cases} +\infty & \text{if } R = 0 \ \text{ or } A'_a(R) = 0, \\ \\ A'_a(R) & \text{otherwise,} \end{cases}$$

so the monodromy might vary in a nontrivial fashion, even for nearby regular leaves. Our computation also gives the isotropy groups

$$\Sigma(M_a,(x,y,z)) \cong \begin{cases} \mathbb{R}^3 & \text{if } R = 0,\ a(0) = 0, \\ \mathrm{SU}(2) & \text{if } R = 0,\ a(0) \neq 0, \\ \mathbb{R} & \text{if } R \neq 0,\ A'_a(R) = 0, \\ S^1 & \text{if } R \neq 0,\ A'_a(R) \neq 0. \end{cases}$$

We leave it to the reader the task of making the description of the groupoid $\Sigma(M_a)$ more explicit (see, also, [**2**, Sec. 6], where this example is further discussed).

6.3. Heisenberg-Poisson manifolds. The integrability of the Heisenberg-Poisson manifolds was discussed in [**34**]. We recall that the Heisenberg-Poisson manifold $M(S)$ associated to a symplectic manifold S, is the manifold $S \times \mathbb{R}$ with the Poisson structure given by

$$\{f, g\} \equiv t\{f_t, g_t\}_S,$$

where t stands for the real parameter, and f_t denotes the function on S obtained from f by fixing the value of t. We can now easily recover the main result of [**34**]:

Corollary 6. *For a symplectic manifold S, the following are equivalent:*

(i) *The Poisson-Heisenberg manifold $M(S)$ is integrable.*
(ii) *$\widetilde{S}$ is pre-quantizable.*

We recall that condition (ii) is usually stated as follows: when we pull back the symplectic form ω on S to a 2-form $\widetilde{\omega}$ on the covering space $\widetilde{S}$, the group of periods

$$\left\{ \int_{\gamma} \widetilde{\omega} : \gamma \in H_2(\widetilde{S}, \mathbb{Z}) \right\} \subset \mathbb{R}$$

is a multiple of $\mathbb{Z}$. Note that this group coincides with the **group of spherical periods** of ω

$$\mathcal{P}(\omega) = \left\{ \int_{\gamma} \omega : \gamma \in \pi_2(S) \right\},$$

so that (ii) says that $\mathcal{P}(\omega) \subset \mathbb{R}$ is a multiple of $\mathbb{Z}$.

Proof of Corollary 6. We have to compute the monodromy groups. The singular symplectic leaves are the points in $S \times \{0\}$ and they clearly have vanishing monodromy groups. The regular symplectic leaves are the submanifolds $S \times \{t\}$, where $t \neq 0$, with symplectic form ω/t. The most straightforward way to compute the monodromy groups is to invoke Proposition 5 of Section 5 (as an instructive exercise, the reader may also try to use “Description 2”of Section 2). We immediately get

$$\mathcal{N}_{(x,t)} = \frac{1}{t} \mathcal{P}(\omega) \subset \mathbb{R},$$

and the result follows. q.e.d.

Note that one can be quite explicit in describing $\Sigma(M(S))$. Straightforward computations show that it consists of two types of arrows:

(a) arrows which start and end at $(x, 0)$, which form a group isomorphic to the additive subgroup of $T^*_{x,0}M(S)$;

(b) arrows inside the symplectic leaves $S \times \{t\}$, $t \neq 0$, which consist of equivalence classes of pairs (γ, v), where γ is a path in S and $v \in \mathbb{R}$. Two such pairs (γ_i, v_i) are equivalent if and only if there is a homotopy $\gamma(\varepsilon, s)$ (with fixed end points) between the γ_i's, such that $v_1 - v_0 = \frac{1}{t} \int \gamma^* \omega$.

This explains how the general strategy of putting a smooth structure on $\Sigma(M)$ (see the end of Section 3) is related with the blowing-up techniques used in [**34**].

Remark 5. Condition (v) of Theorem 5 splits into two conditions:

(a) Each monodromy group $\mathcal{N}_x$ is discrete, i.e., $r_{\mathcal{N}}(x) > 0$.

(b) The monodromy groups are uniformly discrete, i.e.,

$$\liminf_{y \to x} r_{\mathcal{N}}(y) > 0.$$

If we choose a symplectic manifold which does not satisfy the prequantization condition, the Poisson-Heisenberg manifold of Section 6.3 gives a non-integrable Poisson manifold that violates condition (a). On the other hand, the example of Section 6.2 produces non-integrable Poisson manifolds in which condition (a) is satisfied, but condition (b) is not. Hence the two conditions are independent.

6.4. Linear Poisson structures. Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$. If we consider the Poisson manifold $M = \mathfrak{g}^*$ with the Kirillov-Kostant Poisson bracket, and let $A = T^*\mathfrak{g}^*$, we always obtain an integrable Lie algebroid: the source-simply connected Lie groupoid integrating $\mathfrak{g}^*$ is

$$\Sigma(\mathfrak{g}^*) = G \times \mathfrak{g}^*$$

with source and target maps

$$\mathbf{s}(g, \xi) = \xi, \qquad \mathbf{t}(g, \xi) = \mathrm{ad}^*\, g \cdot \xi,$$

and with multiplication $(g_1, \xi_1) \cdot (g_2 \cdot \xi_2) = (g_1 g_2, \xi_2)$, wherever defined. In spite of the fact that linear Poisson structures are always integrable, the symplectic geometry of their leaves varies in a nontrivial fashion, and their monodromy reflects this behavior.

For a specific example take $M = \mathfrak{su}^*(3)$. The symplectic leaves (i.e., the coadjoint orbits) are isospectral sets, and so we can understand them by looking at their point of intersection with the diagonal matrices with imaginary eigenvalues. There are orbits of dimension 6 (distinct eigenvalues), dimension 4 (two equal eigenvalues) and the origin (all eigenvalues equal). Let us take for example the (singular) orbit L through

$$x = \begin{pmatrix} i\lambda & 0 & 0 \\ 0 & i\lambda & 0 \\ 0 & 0 & -2i\lambda \end{pmatrix}.$$

Then we find its isotropy subalgebra to be

$$\mathfrak{g}_x = \left\{ \left(\begin{array}{c|c} X & 0 \\ \hline 0 & \text{- tr X} \end{array} \right) : \ X \in \mathfrak{u}(2) \right\}$$

and so we see that the simply connected Lie group integrating the Lie algebra $\mathfrak{g}_x$ is $G(\mathfrak{g}_x) = \mathbb{R} \times \mathrm{SU}(2)$. We can also compute the isotropy groups

$$\begin{aligned}\Sigma(\mathfrak{g}^*, x) &= \{g \in \mathrm{SU}(3) : \mathrm{ad}^* g \cdot x = x\} \\ &= \left\{ \left(\begin{array}{c|c} g & 0 \\ \hline 0 & \det g^{-1} \end{array} \right) : \; g \in \mathrm{U}(2) \right\}.\end{aligned}$$

We conclude that the orbit L is diffeomorphic to $\mathrm{SU}(3)/\mathrm{U}(2) = \mathbb{C}\mathrm{P}(2)$. In fact, one can show that it is symplectomorphic to $\mathbb{C}\mathrm{P}(2)$ with a multiple of its standard symplectic structure (see [**17**], Example 3.4.5). Also, we see that the long exact sequence

$$\cdots \to \pi_2(\mathbb{C}\mathrm{P}(2), x) \xrightarrow{\partial} G(\mathfrak{g}_x) \to \Sigma(\mathfrak{g}^*, x) \to \pi_1(\mathbb{C}\mathrm{P}(2), x),$$

reduces to:

$$\cdots \to \mathbb{Z} \xrightarrow{\partial} \mathbb{R} \times \mathrm{SU}(2) \xrightarrow{\rho} \mathrm{U}(2) \to \{1\},$$

where $\rho(\theta, A) = e^{i\theta} A$. We conclude that $\partial n = (\pi n, (-1)^n I)$, so that ∂ takes values in the center $Z(\mathbb{R} \times \mathrm{SU}(2)) = \mathbb{R} \times \{\pm I\}$, and

$$\widetilde{\mathcal{N}}_x = \mathrm{Im}\,\partial \simeq \mathbb{Z}, \qquad \mathcal{N}_x \simeq 2\mathbb{Z}.$$

This provides the example promised in Description 2: since the last two groups are distinct, there can be no splitting with center-valued curvature. Another argument is that such a splitting would define a flat connection on the co-normal bundle $\nu^*(L) = \mathrm{Ker}\,\#|_L$, and since $L = \mathbb{C}\mathrm{P}(2)$ is 1-connected, it would follow that the co-normal bundle would be a trivial bundle. This is not possible. In fact, the total Stiefel-Whitney class of $\mathbb{C}\mathrm{P}(2)$ is nontrivial, and so is the Stiefel-Whitney class of the normal bundle, when we embed $\mathbb{C}\mathrm{P}(2)$ in any Euclidean space. Hence, the co-normal bundle cannot be trivial.

7. Symplectic realizations

Recall (see [**37**]) that a **symplectic realization** of a Poisson manifold M is a surjective Poisson submersion $\mu : S \to M$, with connected fibers, from a symplectic manifold S onto M.[8] The symplectic manifold S comes equipped with a pair of foliations, in duality with respect to the symplectic structure: the one given by the fibers of μ, which will be denoted by $\mathcal{F}(\mu)$, and its symplectic orthogonal $\mathcal{F}(\mu)^{\perp}$. In terms of their tangent bundles (vectors tangent to the leaves) $\mathcal{F}(\mu)$ is the kernel of the differential of μ, while $\mathcal{F}(\mu)^{\perp}$ is spanned by the Hamiltonian vectors $X_{\mu^* f}$ ($f \in C^{\infty}(M)$). As explained in Remark 1, S may be non-Hausdorff, but we do require the leaves of $\mathcal{F}(\mu)$ and of $\mathcal{F}(\mu)^{\perp}$ to be Hausdorff. In particular, it makes sense to talk about the completeness of the vector fields $X_{\mu^* f}$. The symplectic realization is called **complete** if, for any complete Hamiltonian vector field X_f on M, the vector field $X_{\mu^*(f)}$ is complete.

The existence of symplectic realizations is guaranteed by the following basic result:

Theorem 7 (Karasev [**23**], Weinstein [**8**])**.** *Any Poisson manifold M admits a Hausdorff symplectic realization $\phi : S \to M$.*

Proof. Let M be a Poisson manifold. The Lie algebroid T^*M integrates to a local Lie groupoid $\Sigma_{\mathrm{loc}}(M)$. The proof of this result in [**5**] (cf. Corollary 5.1) shows that one can exhibit this local Lie groupoid as the quotient

$$\Sigma_{\mathrm{loc}}(M) = O/\sim$$

where O is a open set in the space of cotangent paths $P(T^*M)$. Hence, by a local version of the construction of Section 3, one can perform a Marsden-Weinstein reduction on a open set of $\widetilde{P}(T^*M)$ to obtain the structure of a local symplectic groupoid on $\Sigma_{\mathrm{loc}}(M)$. The source map (also the target map) $\mathbf{s} : \Sigma_{\mathrm{loc}}(M) \to M$ gives a symplectic realization. q.e.d.

The question of existence of *complete* symplectic realizations is the main topic of this section, and our main result is the following:

Theorem 8. *A Poisson manifold admits a complete symplectic realization if and only if it is integrable.*

The fact that integrability is somehow related to (and implies) the existence of complete symplectic realizations is well-known: if M is

[8]The reader should be aware that in the literature one often does not require ϕ to be a submersion or to be surjective, while the “symplectic realizations” defined above are also called “full symplectic realizations”.

integrable, the target map $\mathbf{s} : \Sigma(M) \to M$ gives the desired realization. Completeness follows from the remark that a path $\gamma : I \to M$ is an integral curve of X_f if and only if it is of the form $\gamma(t) = \mathbf{s}(\widetilde{\gamma}(t))$ where $\widetilde{\gamma} : I \to S$ is an integral curve of $X_{f\circ\mathbf{s}}$. Since this vector field is left invariant, its flow is left invariant, and the usual argument for Lie groups extends to the groupoid case to show that $X_{f\circ\mathbf{s}}$ is complete, provided X_f is complete.

The converse implication, which may look surprising at first sight, appears to be more difficult since there is no obvious way of constructing an integrating groupoid out of an arbitrary symplectic realization S (note that S may be quite different from $\Sigma(M)$). However, we can now take advantage of the fact that we always have the groupoid $\Sigma(M)$ around, and the only issue is to prove its smoothness. This is proved by pulling everything back to S, which one can think of as a desingularization of M, where the problem greatly simplifies.

Remark 6. We shall see that a complete symplectic realization of M can be thought of as a faithful representation of the Lie algebroid T^*M. We can use this representation to integrate T^*M, in a similar way as one does to integrate a Lie algebra to a Lie group, with the help of a faithful representation. Unlike the Lie algebra case, when such a representation always exists (Ado's theorem), in our case we have to assume the existence of such a representation, i.e., of the complete symplectic realization $\phi : S \to M$.

Remark 7. Note that, in light of Theorem 5 (v), the result above can be stated with no reference to integrability or groupoids (and this suggests that a different approach might be possible).

Proof of Theorem 8. We let $\mu : S \to M$ be a complete symplectic realization of M, and we first assume that S is Hausdorff. We split the proof into several steps.

Step 1. *The Lie algebroid T^*M acts on S.*

The map $df \mapsto X_{\mu^* f}$ defines a bundle map $\rho : \mu^* T^*M \to TS$, which can also be described as the composition of the dual of the differential of μ with the anchor map $T^*S \to TS$ induced by the symplectic form on S. It induces a Lie algebra homomorphism $\Omega^1(M) \to \mathfrak{X}(S)$, and

$$(d\mu)_s \circ \rho_y = \#_{\mu(y)}, \quad \text{for all } y \in S. \tag{11}$$

One should view ρ as an infinitesimal action of T^*M on S (see also below). Alternatively, we can also think of ρ as the horizontal lift of a flat, non-linear, contravariant connection (see [**17**]).

Step 2. *The action of* T^*M *on* S *integrates to an action of* $\Sigma(M)$.

Notice that it is for this reason that we need μ to be complete (compare with the integrability of infinitesimal actions of Lie algebras on manifolds). Given a cotangent path $a : I \to T^*M$, with base path γ starting at $x_0 \in M$, and given $y \in S$, the horizontal curve over a starting at y_0 is the solution $u : I \to S$ of the initial value problem:

$$\frac{d}{dt}u(t) = \rho_{u(t)}(a(t)), \quad u(0) = y_0.$$

Let us check that this equation has a unique solution, defined on the entire unit interval. We choose a time-dependent, compactly supported, one-form α with the property that $\alpha(t, \gamma(t)) = a(t)$, and we consider the induced time-dependent vector field on S, $X(t, y) = \rho_y(\alpha(t, \mu(y)))$. A solution of the equation above is an integral curve of X with initial condition y_0, hence uniqueness. Conversely, if u is an integral curve of X, (11) implies that $\mu \circ u$ is an integral curve of $\#\alpha$ starting at x_0, hence it must be γ. It follows that $\frac{d}{dt}u(t) = \rho_{u(t)}(\alpha(t, \gamma(t))) = \rho_{u(t)}(a(t))$. Finally, u is defined on the entire I because the completeness of μ implies that $\rho(\alpha)$ is complete whenever α is compactly supported.

Next we need to understand how the horizontal lifts depend on the cotangent path:

Lemma 2. *Let* a_0 *and* a_1 *be cotangent paths which are cotangent homotopic. Then their horizontal lifts* u_0 *and* u_1 *are homotopic paths relative to the end-points.*

Proof of Lemma 2. Let us fix the initial point, and consider a family a_ϵ of cotangent paths, so that $\gamma_\epsilon = \mu \circ a_\epsilon$ all start at x_0 and end at the same point. We obtain a corresponding family u_ϵ of paths in S as above, all starting at y_0, and all staying in the leaf of $\mathcal{F}(\mu)^\perp$ through y_0. Moreover,

$$\rho(\mathrm{var}(a_\epsilon)) = \frac{d}{d\epsilon}u_\epsilon(1), \tag{12}$$

for all ϵ. Of course, this is related to the very definition of $\mathrm{var}(a_\epsilon)$: by formula (2) in Proposition 1.3 of [**10**], we have

$$\mathrm{var}(a_\epsilon) = \int_0^1 \phi_{\alpha_\epsilon}^{t,s} \frac{d\alpha_\epsilon}{d\epsilon}(s, \gamma_\epsilon(s))ds, \tag{13}$$

where α_ϵ are time dependent 1-forms such that $\alpha_\epsilon(t, \gamma_\epsilon(t)) = a_\epsilon(t)$. Here $\phi^{t,s}_{\alpha_\epsilon}$ denotes the flows of α_ϵ (see the footnote in Section 3), which are covered by the flows of the vector field $X_\epsilon = \rho(\alpha_\epsilon)$, and so we find

$$\rho(\mathrm{var}(a_\epsilon)) = \int_0^1 \phi^{t,s}_{X_\epsilon} \frac{dX_\epsilon}{d\epsilon}(s, u_\epsilon(s))ds.$$

The left side of this equation is just the variation of parameters formula for flows of time-dependent vector fields, so (12) holds.

In particular, if a_ϵ is a cotangent homotopy we have $\mathrm{var}(a_\epsilon) = 0$ and so the end-points $u_\epsilon(1)$ are fixed. q.e.d.

Hence, for all $g = [a] \in \Sigma(M)$ with $\mathbf{s}(g) = x_0$ and $\mathbf{t}(g) = x_1$, given $y_0 \in S$ in the fiber above x_0, there is a well-defined element $u(1) \in S$ in the fiber above x_1, and we set $g \cdot y_0 \equiv u(1)$. In other words we have an induced action of $\Sigma(M)$ on S (the operation $g \cdot y$ does satisfy the usual axioms of an action).

Let us denote by $\mu^*\Sigma(M)$ the space of pairs (g, y), with $\mu(y) = \mathbf{s}(g)$. Then $\mu^*\Sigma(M)$ is a groupoid over S, called the **action groupoid** associated with the action of $\Sigma(M)$ on S: the source and target maps are given by $\mathbf{s}(g, y) = y$ and $\mathbf{t}(g, y) = g \cdot y$, while the multiplication is given by

$$(g, y) \cdot (h, z) \equiv (g \cdot h, z), \quad \text{whenever } y = h \cdot z.$$

Step 3. *The action groupoid of* $\Sigma(M)$ *on* S *is isomorphic to the homotopy groupoid of* $\mathcal{F}(\mu)^\perp$.

Note that Equation (12) above actually shows that a_ϵ is a cotangent homotopy if and only if u_ϵ is a homotopy with fixed end points. Recall that, similar to $\Sigma(M)$, one has a homotopy groupoid $G(\mathcal{F})$ associated to any regular foliation $\mathcal{F}$: it consists of homotopy classes of paths with fixed end-points, staying in a single leaf. Now, since ρ is actually an isomorphism from μ^*T^*M to the symplectic orthogonal foliation $\mathcal{F}(\mu)^\perp$, (12) shows that

$$\mu^*\Sigma(M) \cong G(\mathcal{F}(\mu)^\perp) \tag{14}$$

an isomorphism of topological groupoids.

Step 4. $\Sigma(M)$ *is a Lie groupoid.*

Since the homotopy groupoid $G(\mathcal{F})$ of any regular foliation is always smooth, the action groupoid $\mu^*\Sigma(M)$ is also smooth and we want to conclude from this that $\Sigma(M)$ must be smooth too. To prove it, we will use the exponential map $\exp_\nabla : T^*M \to \Sigma(M)$ with respect to a connection ∇ on T^*M, and verify condition (iv) of Theorem 5.

Consider the induced map $\mu^*T^*M \to \mu^*\Sigma(M)$, denoted $\mu^*\exp_\nabla$, and also its composition $F : \mu^*T^*M \to G(\mathcal{F}(\mu)^\perp)$ with the homeomorphism (14). Then F associates to a pair (v, y) ($y \in S$, $v \in T^*_{\mu(y)}M$) the homotopy class of the path $t \mapsto \exp_\nabla(tv)y$. This map is a local diffeomorphism at the zero section. Indeed, after the identification $\mu^*T^*M \cong \mathcal{F}(\mu)^\perp$, it is just the exponential map of $\mathcal{F}(\mu)^\perp$ with respect to the pullback connection $\mu^*\nabla$. Alternatively, its differential at a point $(0, y) \in \mu^*T^*M$ is an isomorphism since it fits into a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & T^*M \arrow[r] \arrow[d, "\rho_y"'] & T_{(0,y)}\mu^*T^*M \arrow[r] \arrow[d, "dF"'] & T_yS \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\
0 \arrow[r] & \mathcal{F}(\mu)^\perp \arrow[r] & T_{1_y}G(\mathcal{F}(\mu)^\perp) \arrow[r, "d\mathbf{s}"'] & T_yS \arrow[r] & 0.
\end{tikzcd}$$

It then follows that $\mu^*\exp_\nabla$ is locally injective around the zero section, which immediately implies the similar property for $\exp_\nabla$. Hence, by (iv) of Theorem 5, it follows that M must be integrable.

When S is not necessarily Hausdorff, basically the same proof applies. All the complete vector fields we have used were actually tangent to the leaves of $\mathcal{F}(\mu)^\perp$, which are Hausdorff. Of course, we also need to know that the construction of the homotopy groupoid $G(\mathcal{F})$ works well (i.e., is a smooth manifold) for any regular foliation $\mathcal{F}$ (on a possibly non-Hausdorff manifold) with Hausdorff leaves, but this works exactly as in the Hausdorff case. q.e.d.

The previous arguments become even more natural when using the language of algebroids. Recall that an **action of an algebroid** A over M on a map $\mu : S \to M$ consists of a bundle map $\rho : \mu^*A \to TS$ with the property that it induces a Lie algebra homomorphism $\Gamma(A) \to \mathfrak{X}(S)$ and satisfies:

$$(d\mu)_s \circ \rho_s = \#_{\mu(s)}, \qquad \text{for all } s \in S.$$

The action is called **complete** if $\rho(\alpha)$ is a complete vector field whenever $\alpha \in \Gamma(A)$ has compact support.

A Lie algebroid action ρ determines a Lie algebroid structure on the pullback μ^*A, called the **action Lie algebroid** of ρ: the anchor is simply ρ, while the bracket is uniquely determined by the Leibniz rule and $[\mu^*\alpha, \mu^*\beta] = \mu^*[\alpha, \beta]$. Similarly, if a groupoid G acts on $\mu : S \to M$, there is an induced groupoid μ^*G over S, called the **action Lie groupoid**: its arrows consist of pairs $(g, y) \in G \times S$ with the property that $\mu(y) = \mathbf{s}(g)$. The source of (g, y) is y, its target is $g \cdot y$, and the multiplication (product) $(g, y) \cdot (h, z)$, defined when $y = h \cdot z$, equals to (gh, z).

The same arguments as in the previous proof show that:

Corollary 7. *Let A be a Lie algebroid. Then:*

(a) *A complete action of A on $\mu : S \to M$ determines a (topological) action of the Weinstein groupoid $G(A)$ on S, and $\mu^* G(A) \cong G(\mu^* A)$, as groupoids. Moreover, for any connection ∇ on A, the pullback to S of the associated exponential map $\exp_\nabla\ A \to G(A)$, identifies with the exponential $\exp_{\mu^*\nabla} : \mu^* A \to G(\mu^* A)$ with respect to the pullback connection $\mu^*\nabla$.*

(b) *If A admits a complete action on $\mu : S \to M$ such that the action Lie algebroid $\mu^* A$ is integrable, then A is integrable.*

The proof above consisted in observing that any symplectic realization $\mu : S \to M$ comes equipped with an action ρ_μ of T^*M, which is complete if and only if μ is complete. Moreover, $df \mapsto X_{\mu^* f}$ defines a Lie algebroid isomorphism between $\mu^* T^* M$ and $T\mathcal{F}(\mu)^\perp$. Since the latter is integrable, T^*M must also be integrable.

Let us point out several consequences.

Corollary 8. *Any complete symplectic realization $\mu : S \to M$ comes equipped with a (smooth) locally free action of $\Sigma(M)$ on S. The action is free if and only if the leaves of the symplectic orthogonal foliation $\mathcal{F}(\mu)^\perp$ are simply connected.*

Proof. That the action is locally free, but not necessarily free, comes from the fact that $\mu^*\Sigma(M)$ is isomorphic only to the homotopy groupoid $G(\mathcal{F}(\mu)^\perp)$. Freeness corresponds to the case where $G(\mathcal{F}(\mu)^\perp)$ is a subset of $S \times S$, or, equivalently, to the simply connectedness of the leaves. q.e.d.

In [**27**], Mikami and Weinstein have explained that "symplectic reduction" can be performed in the general context of actions of symplectic groupoids. With our "infinitesimal point of view", we find that all is needed is a complete symplectic realization:

In particular, we find that "symplectic reduction" can be performed in the general context of complete symplectic realizations (compare with the similar result of Mikami and Weinstein [**27**], where one starts with an action of the symplectic groupoid):

Corollary 9. *For any complete symplectic realization $\mu : S \to M$:*

(i) *There is a natural action of the isotropy group $\Sigma(M,x)$ on the fiber $\mu^{-1}(x)$.*
(ii) *If $\mu^{-1}(x)/\Sigma(M,x)$ is smooth, then it carries a natural symplectic structure.*
(iii) *Every cotangent path a, starting at x and ending at y, induces a bijection $\mu^{-1}(x)/\Sigma(M,x) \to \mu^{-1}(y)/\Sigma(M,y)$. It only depends on the homotopy class of a, and, in the smooth case, it is a symplectic diffeomorphism.*

Proof. The reduced symplectic structure comes from the fact that, since the action of $\Sigma(M)$ on S is locally free, the tangent space of $\mu^{-1}(x)/\Sigma(M,x)$ at some $y \in \mu^{-1}(x)$ is given by

$$T_y\mathcal{F}(\mu)_y^{\perp}/T_y\mathcal{F}(\mu) \cap T_y\mathcal{F}(\mu)^{\perp}.$$

Then linear symplectic reduction shows that we have a nondegenerated bilinear form induced by the symplectic form on S. On the other hand, the action of the cotangent paths on the symplectic quotients $\mu^{-1}(x)/\Sigma(M,x)$ is just the one induced from the action of $\Sigma(M)$ on S. q.e.d.

Corollary 10. *Let $\mu : S \to M$ be a complete symplectic realization, and assume that the orbit space $S/\Sigma(M)$ is smooth. Then it carries a natural Poisson structure, whose symplectic leaves can be identified with the symplectic manifolds $\mu^{-1}(x)/\Sigma(M,x)$.*

These results show that the map $\mu : S \to M$ can be seen as a **moment map** for the groupoid action of $\Sigma(M)$ on S. In fact, it is easy to see that the graph of the action $\{(g, y, g \cdot y) : \mathbf{s}(g) = \mu(y)\}$ is a Lagrangian submanifold of $\Sigma(M) \times S \times \overline{S}$, so the groupoid action of $\Sigma(M)$ on S is symplectic in the sense of Mikami and Weinstein [**27**]. In fact, the corollaries above also follow from this observation hence the general results of [**27**], since we know now that $\Sigma(M)$ is smooth (see also the recent preprint [**33**]).

Notice that a complete symplectic realization gives rise to a two leg diagram

$$\begin{tikzcd} & S \arrow[dl, "\mu"'] \arrow[dr, "\pi"] & \\ M & & S/\Sigma(M) \end{tikzcd}$$

where the quotient $S/\Sigma(M)$ and the map π are Poisson under some favorable circumstances (e.g., when $S/\Sigma(M)$ is smooth and S is a principal $\Sigma(M)$-bundle over $S/\Sigma(M)$). Moreover, if the fibers of μ are simply connected, the symplectic groupoid of this new Poisson manifold is just the gauge groupoid $S \times_{\Sigma(M)} S$. Note also that if one repeats this construction on $S/\Sigma(M)$ and the symplectic realization given by π, then one gets back M.

Example 1. A well-known instance of this is provided by a complete symplectic realization $J : S \to \mathfrak{g}^*$ of the dual of some Lie algebra. In this case we have $\Sigma(\mathfrak{g}^*) = \mathfrak{g}^* \times G$, where G is a simply connected Lie group with Lie algebra $\mathfrak{g}$, and $\Sigma(\mathfrak{g}^*, \nu) = G_\nu$ is the isotropy group of $\nu \in \mathfrak{g}^*$ for the coadjoint action (see Section 6.2). Then we obtain a free action of G on S which is Hamiltonian with momentum map J, and the symplectic structures on $J^{-1}(\nu)/\Sigma(\mathfrak{g}^*, \nu)$ are the well-known Marsden-Weinstein symplectic quotients. They form the leaves of the Poisson manifold $S/\Sigma(\mathfrak{g}^*) = S/G$, provided this quotient is smooth. We will came back to dual pairs later in our discussion of Morita equivalence (see Section 9).

Corollaries 8 through 10 improve and explain the results of Mikami and Weinstein [**27**, **33**]. We can summarize this section by saying that we can view a symplectic realization of a Poisson manifold M as a momentum map for an action of the groupoid $\Sigma(M)$ on S. Being the target of a momentum map, M can be thought of as the dual of the Lie algebra(oid) of its Weinstein group(oid). This fits well with Alan Weinstein's remark (see [**4**], p. 46) that "it is tempting to think of any symplectic manifold S as the dual of the Lie algebra of $\pi_1(S)$".

8. Induced Poisson structures

In this section we discuss submanifolds which have a canonical induced Poisson structure.

Let M be a manifold with a smooth foliation $\mathcal{F}$, which may be singular (for singular foliations, see e.g., Vaisman's book [**32**]). By a **smooth family of symplectic forms** on the leaves we mean a family of symplectic forms $\left\{\omega_L \in \Omega^2(L) : L \in \mathcal{F}\right\}$ such that for every smooth function $f \in C^\infty(M)$ the Hamiltonian vector field X_f defined by

$$i_{X_f}\omega_L = d(f|_L), \quad \forall L \in \mathcal{F}$$

is a smooth vector field in M.

If $(M, \{\cdot,\cdot\})$ is a Poisson manifold, then the symplectic foliation with the induced symplectic forms on the leaves gives a smooth foliation with a smooth family of symplectic forms.

Conversely, let M be a manifold with a smooth foliation $\mathcal{F}$, furnished with a smooth family of symplectic forms on the leaves. Then, we have a Poisson bracket on M defined by the formula

$$\{f, g\} \equiv X_f(g),$$

for which the associated symplectic foliation is precisely $\mathcal{F}$. Hence, a Poisson structure can be defined by its symplectic foliation instead of the Poisson bracket (see also [**32**], Theorem 2.14). This motivates the following definition:

Definition 4. Let M be a Poisson manifold. A submanifold $N \subset M$ is called a **Poisson-Dirac submanifold** if N is a Poisson manifold and:

(i) The symplectic foliation of N is $N \cap \mathcal{F} = \{L \cap N : L \in \mathcal{F}\}$.
(ii) For every leaf $L \in \mathcal{F}$, $L \cap N$ is a symplectic submanifold of L.

Let us clarify this definition. First, the intersections $N \cap L$ need not be connected, so in (i) we really mean that:

(ia) N intersects L cleanly, so that $N \cap L$ is a submanifold of N and L and $T(N \cap L) = TN \cap TL$.
(ib) The symplectic leaves of N are the connected components of the intersections $N \cap L$.

Also, condition (ii) means that the symplectic forms on $L \cap N$ are the pullbacks $i^*\omega_L$, where $i : N \cap L \hookrightarrow L$ is the inclusion. Then we must have

$$TN \cap \#(TN^0) = \{0\}\,, \tag{15}$$

since the left-hand side is the kernel of the pullback $i^*\omega_L$.

By the remarks above, a submanifold N has at most one Poisson structure satisfying these properties, and this Poisson structure is completely determined by the Poisson structure of M. Poisson submanifolds are obvious examples of Poisson-Dirac submanifolds, and we will see many other examples later.

In order to discuss the problem of integrability of Poisson-Dirac submanifolds, it will be convenient to recall a few facts about Dirac structures. At the same time, this will allow us to justify our usage of the term *Poisson-Dirac submanifold*.

8.1. Poisson-Dirac subspaces. Recall (see [**7**]) that a **linear Dirac structure** on a vector space V is a subspace $L \subset V \oplus V^*$ which is maximally isotropic with respect to the canonical symmetric pair $\langle \cdot, \cdot \rangle$ given by:

$$\langle (v,\xi), (w,\eta) \rangle = \frac{1}{2}(\xi(w) + \eta(v)).$$

We remark that linear Dirac structures can always be restricted to subspaces: if $W \subset V$ is a subspace, then on W one has the induced Dirac structure

$$L_W = \{(v, \xi|_W) : (v,\xi) \in L, v \in W\}.$$

Now a Poisson vector space (V, Π) (a vector space V with a bivector $\Pi \in \wedge^2 V$) is the same as a linear Dirac structure on V, with the property that the projection $L \to V^*$ is bijective. Namely, a bivector Π is completely determined by its graph $L^\Pi \subset V \oplus V^*$, where

$$L^\Pi \equiv \{(\#\xi, \xi) : \xi \in V^*\}.$$

Hence, given a subspace $W \subset V$ one has the induced Dirac structure

$$L^\Pi_W = \{(\#\xi, \xi|_W) : \xi \in V^*, \#\xi \in W\}.$$

However, in general, the projection $L^\Pi_W \to W^*$ will not be bijective so this Dirac structure is not defined by some bivector in W. Noticing that the kernel of this projection is precisely $W \cap \#(W^0)$ we introduce the following definition.

Definition 5. Let (V, Π) be a Poisson vector space. A subspace $W \subset V$ is called a **Poisson-Dirac subspace** if

$$W \cap \#(W^0) = \{0\}. \tag{16}$$

Therefore, a Poisson-Dirac subspace W of a Poisson vector space (V, Π) has a natural induced bivector Π_W. Let us set

$$A_W \equiv (\#(W^0))^0 = \{\xi \in V^* : \#\xi \in W\}.$$

The bivector Π_W is given by

$$\Pi_W(\xi, \eta) = \Pi(\widetilde{\xi}, \widetilde{\eta}), \tag{17}$$

where $\widetilde{\xi}$ and $\widetilde{\eta}$ are extensions to V of ξ and η, at least one of which lies in A_W. On the other hand, condition (16) can also be written as:

$$W^0 + A_W = T^*M.$$

Also, if (16) holds, we see that

$$W^0 \cap A_W = W^0 \cap \operatorname{Ker} \#,$$

so we have a short exact exact sequence

$$0 \longrightarrow W^0 \cap \operatorname{Ker} \# \longrightarrow A_W \longrightarrow W^* \longrightarrow 0 .$$

The space A_W has the role of relating V^* and W^*, and shows that Π_W is obtained by linear presymplectic reduction. Indeed, A_W is a subspace of the (presymplectic) vector space V^*, while W^* is a quotient of A_W with the quotient map just the restriction to W.

We shall call a **Dirac projection** of W any projection $p : V \to W$ such that $p|_{\#(W^0)} = 0$. Notice that $p : V \to W$ is a Dirac projection if and only if $p^* : W^* \to V^*$ is a splitting of the short exact sequence above. The role of a Dirac projection $p : V \to W$ is to replace the choice of extensions in the description (17) of Π_W:

$$\Pi_W(\xi, \eta) = \Pi(p^*\xi, p^*\eta).$$

Any projection $p : V \to W$ determines a splitting $V = W \oplus E_p$, where $E_p = (1-p)V$, and Dirac projections correspond to Π-orthogonal complements of W in V (that is, $\Pi(\xi, \eta) = 0$ for all $\xi \in W^0$, $\eta \in E_p^0$). This orthogonality condition shows that Π decomposes as

$$\Pi = \Pi_W + \Pi_{E_p},$$

for a unique $\Pi_{E_p} \in \wedge^2 E_p$.

We define the **rank** of a Poisson-Dirac subspace $W \subset V$ to be the number

$$\operatorname{rank} W = \dim(W^0 \cap \operatorname{Ker} \#).$$

This number $r = \operatorname{rank} W$ determines the dimensions of the other spaces involved. For instance, $\dim A_W = \dim W + r$ and $\dim V = \dim(W + \#(W^0)) + r$. We find that

$$\operatorname{codim} W - \operatorname{rank} W = \operatorname{rank} \Pi - \operatorname{rank} \Pi_N,$$

so that $0 \leq \operatorname{rank} W \leq \operatorname{codim} W$.

Example 2. A **cosymplectic subspace** of a Poisson vector space (V, Π) is a subspace $W \subset V$ such that $V = W + \#(W^0)$. Hence, a cosymplectic subspace is the same thing as a Poisson-Dirac subspace of minimal rank (i.e., $\operatorname{rank} W = 0$). Since this is the same as satisfying the transversality condition $W + \operatorname{Im}(\#) = V$, a cosymplectic subspace is a nonzero Poisson-Dirac subspace which has a unique Dirac projection.

At the other extreme, we can consider Poisson-Dirac subspaces $W \subset V$ of maximal rank (i.e., $\operatorname{rank} W = \operatorname{codim} W$). Then we must have $W^0 = \ker \#$ or, equivalently, $\operatorname{Im} \#_W = \operatorname{Im} \#$. This just means that W is a **Poisson subspace**.

Finally, note that a Poisson vector space (V, Π) is completely determined by the linear symplectic space (S, ω), where $S = \operatorname{Im} \# \subset V$ and

$$\omega(\#\xi, \#\eta) = \Pi(\xi, \eta).$$

This is the linear counterpart of the remark made at the beginning of the section, since the leaves of Π are just the linear symplectic affine spaces obtained from (S, ω) by translation.

Now for any Poisson-Dirac subspace $W \subset V$, we have

$$\operatorname{Im} \#_W = W \cap \operatorname{Im} \#.$$

Hence, the linear symplectic space (S_W, ω_W) that corresponds to W, is such that $S_W \hookrightarrow S$ and $\omega_W = i^*\omega$, with i the inclusion. Conversely, the canonical splitting

$$\operatorname{Im} \# = \operatorname{Im} \#_W \oplus \#(W^0),$$

shows that if $W \cap \operatorname{Im}(\#)$ is a symplectic subspace of $\operatorname{Im}(\#)$ then W is Poisson-Dirac.

Therefore, if we view a Poisson vector space (V, Π) as a linear symplectic space (S, ω), the Poisson-Dirac subspaces of V correspond to the symplectic subspaces of S.

8.2. Poisson-Dirac submanifolds. We now consider the non-linear case. Let (M, Π) be a Poisson manifold, and consider a submanifold $N \subset M$ which is "pointwise Poisson-Dirac", i.e., T_xN is a Poisson-Dirac subspace of T_xM (with respect to $\Pi_x \in \wedge^2 T_xM$) for all $x \in N$:

$$T_xN \cap \#(T_xN^0) = \{0\}.$$

From the linear case it follows that there is an induced two-tensor on N, denoted Π_N, but there is nothing to ensure us that it is smooth.

Example 3. Let $M = \mathbb{C}^3$ with complex coordinates (x, y, z). We consider the (regular) foliation of $\mathbb{C}^3$ by complex lines defined by

$$dy = 0, \quad dz - y dx = 0.$$

The leaves of this foliation are symplectic submanifolds of $\mathbb{C}^3$ with the canonical symplectic form. Hence, we have a Poisson structure on $M = \mathbb{C}^3$ with this symplectic foliation, and we denote it by Π. Now consider the submanifold $N = \{(x, y, z) : z = 0\} \subset M$. A complex line in the foliation intersects N in a point (if $y \neq 0$) or in a complex line (if $y = 0$), which are symplectic submanifolds. This shows, that

$$TN \cap \#(TN^0) = \{0\},$$

so N is pointwise Poisson-Dirac, and Π_N is not smooth, for its image is a non-smooth distribution. (Notice that it is nonetheless an integrable distribution!)

If Π_N is smooth, the next proposition shows that Π_N satisfies automatically the integrability condition $[\Pi_N, \Pi_N] = 0$ (i.e., the induced almost Dirac structure on N is integrable):

Proposition 6. *Let N be a submanifold of the Poisson manifold M, such that:*

(i) *N is pointwise Poisson-Dirac, i.e., $TN \cap \#(TN^0) = \{0\}$.*
(ii) *The induced tensor Π_N is smooth.*

Then Π_N is a Poisson tensor on N.

Proof. Fix any $x \in N$. We claim that $[\Pi_N, \Pi_N]_x = 0$. By the linear theory, we can choose a splitting $T_N M = TN \oplus E$ where E is some vector bundle over N, such that

$$\Pi = \Pi_N + \Pi_E,$$

where $(\Pi_E)_x \in \wedge^2 E_x$, and $(E_x)^0$ and $(T_x N)^0$ are Π_x-orthogonal complements.[9] Since $[\Pi, \Pi] = 0$, it follows that

$$\begin{aligned} [\Pi_N, \Pi_N]_x &= [\Pi_N + \Pi, \Pi_N - \Pi]_x, \\ &= -[2\Pi_N + \Pi_E, \Pi_E]_x. \end{aligned}$$

The left-hand side of this expression lies in $\wedge^3 T_x N \subset \wedge^2 T_x M \wedge T_x N$, while the right-hand side lies in $\wedge^2 T_x M \wedge E_x$. Hence they must both be zero. q.e.d.

This proposition has the following corollary which justifies our usage of the term **Poisson-Dirac submanifold** in Definition 4.

Corollary 11. *A submanifold $N \subset M$ of a Poisson manifold is a Poisson-Dirac submanifold iff it is pointwise Poisson-Dirac and the induced tensor is smooth.*

[9]Notice that for $y \neq x$, in general, we will have $(\Pi_E)_y = (\Pi - \Pi_N)_y \notin \wedge^2 E_y$, but that is irrelevant for the argument.

Proof. If $N \subset M$ is a Poisson-Dirac submanifold then it is pointwise Poisson-Dirac. Also, the induced tensor coincides with the Poisson tensor on N so it is smooth.

Conversely, if $N \subset M$ is pointwise Poisson-Dirac and the induced tensor is smooth, then by the proposition Π_N is a Poisson tensor, so N is a Poisson manifold. By the linear theory, the smooth, integrable, distribution defined by Π_N on N (namely $\mathrm{Im}(\#_N)$) is just $TN \cap \mathrm{Im}(\#)$. Hence, the symplectic foliation of N is $N \cap \mathcal{F}$, where $\mathcal{F}$ is the symplectic foliation of M, and the leaves of N are symplectic submanifolds of the leaves of M. Therefore, N is a Poisson-Dirac submanifold. q.e.d.

A **Dirac projection** of N is a smooth bundle map $p : T_N M \to TN$ with the property that $p_x : T_x M \to T_x N$ is a linear Dirac projection for each $x \in N$. In other words, $\{p_x\}$ is a family of Dirac projections depending smoothly on x. The following proposition shows that Poisson-Dirac submanifolds which admit a Dirac projection are the objects which Vaisman calls in [**31**] *quasi-Dirac submanifolds*.

Proposition 7. *Let M be a Poisson manifold and $N \subset M$ a submanifold. The following statements are equivalent:*

(i) *N is a Poisson-Dirac submanifold admitting a Dirac projection $p : T_N M \to TN$.*
(ii) *There exists a bundle E such that $T_N M = TN \oplus E$ and $\#(E^0) \subset TN$.*

Proof. To show that (i) $\Rightarrow$ (ii), we just observe that if $p : T_N M \to TN$ is a Dirac projection then $E = \mathrm{Ker}\, p$ is a subbundle such that $T_N M = TN \oplus E$ and $\#(E^0) \subset TN$.

For the converse, we observe (as in the linear case) that (ii) gives a decomposition

$$\Pi = \Pi_N + \Pi_E$$

with $\Pi_N \in \Gamma(\wedge^2 TN)$ and $\Pi_E \in \Gamma(\wedge^2 E)$, where now both Π_N and Π_E must be smooth bivector fields. By Corollary 11, N is a Poisson-Dirac submanifold. q.e.d.

Remark 8. If, around each point of N, we have local Dirac projections we can use a partition of unity to glue them into a global Dirac projection. In fact, notice that a convex combination of linear Dirac projections is a linear Dirac projection. Therefore, the existence of a Dirac projection for a given Poisson-Dirac submanifold is a local issue.

Corollary 12. *Let N be a submanifold of Poisson manifold which is pointwise Poisson-Dirac and for which the Poisson-Dirac subspaces $T_xN \subset T_xM$ have constant rank. Then N is a Poisson-Dirac submanifold admitting a Dirac projection.*

Proof. The assumptions imply the existence of a Dirac projection: if $T_xN \subset T_xM$ have constant rank then $\#(TN^0)$ has constant rank, and so we can choose a bundle F such that $T_NM = TN \oplus \#(TN^0) \oplus F$. Then projection onto TN is a Dirac projection. q.e.d.

Example 4. Let N be a submanifold of a Poisson manifold M. If M is a cosymplectic submanifold (i.e., each T_xN is a cosymplectic subspace of T_xM), then N is a Poisson-Dirac submanifold. Similarly, if N is a Poisson submanifold (i.e., each T_xN is a Poisson subspace of T_xM), then N is also a Poisson-Dirac submanifold.

For a general Poisson-Dirac submanifold $N \subset M$, the rank of the linear Poisson-Dirac subspaces $T_xN \subset T_xM$ will vary. Let us call this number the **rank of the Poisson-Dirac submanifold** at x, and denote it by $\operatorname{rank}_x N$. By the linear theory, we have:

$$\operatorname{codim} N - \operatorname{rank}_x N = \operatorname{rank} \Pi_x - \operatorname{rank}(\Pi_N)_x.$$

Obviously, cosymplectic and Poisson submanifolds are examples of Poisson-Dirac submanifolds of constant rank. Here is a simple example of a Poisson-Dirac submanifold with nonconstant rank.

Example 5. Let $M = \mathbb{R}^3$ with coordinates (x, y, z) and the following Poisson bracket:

$$\{x, y\} = f(z), \quad \{x, z\} = \{y, z\} = 0,$$

where $f \in C^\infty(\mathbb{R})$ is such that $f(z) = 0$ for $z \leq 0$ and $f(z) > 0$ for $z > 0$. Then one checks easily that $N = \left\{(0, 0, z) \in \mathbb{R}^3\right\}$ is a Poisson-Dirac submanifold. This Poisson-Dirac submanifold has rank two for $z \leq 0$, and has rank zero for $z > 0$.

Example 6. In the previous example, a Dirac projection still exists. Now, let $M = \mathbb{R}^4$ with coordinates (x, y, z, w) and Poisson bracket defined by:

$$\{x, y\} = x^2, \quad \{z, w\} = z.$$

Then one checks easily that $N = \left\{(x, y, z, w) \in \mathbb{R}^4 : y = 0, x = z^2\right\}$ is a Poisson-Dirac submanifold. Let us use coordinates (z, w) for N. The induced Poisson bracket on N is simply

$$\{z, w\}_N = z.$$

and N is a Poisson-Dirac submanifold of rank 2 at points $(0, w)$ and of rank 0 elsewhere. Hence, for $z \neq 0$ there is a unique Dirac projection. The subbundle $E \subset T_N M$, whose existence is asserted by Proposition 7, is given by

$$E_{(z,w)} = \left\{(-z^4 a_1, z^4 a_2, 0, -2z^2 a_2) : a_1, a_2 \in \mathbb{R}\right\}, \quad (z \neq 0).$$

This bundle can be extended over points with $z = 0$, where it will have fiber

$$E_{(0,w)} = \{(a_1, 0, 0, a_2) : a_1, a_2 \in \mathbb{R}\}.$$

Note, however, that at these points $TN \cap E \neq 0$, so in a neighborhood of any point $(0, w)$ there are no Dirac projections.

Let us restrict now to constant rank Poisson-Dirac submanifolds. Then we have the vector bundle:

$$A_N(M) \equiv \{\xi \in T_N^* M : \#\xi \in TN\},$$

and this has in fact a Lie algebroid structure over N, with bracket induced from the bracket on T^*M. On the other hand, we also have the vector bundle

$$\mathfrak{g}_N(M) \equiv TN^0 \cap \operatorname{Ker} \#,$$

which is in fact a bundle of Lie algebras over N. Hence, we see that one has a short exact sequence of Lie algebroids:

$$0 \longrightarrow \mathfrak{g}_N(M) \longrightarrow A_N(M) \longrightarrow T^*N \longrightarrow 0. \tag{18}$$

The Lie algebroid $A_N(M)$ determines a subgroupoid $\Sigma_N(M)$ of the Weinstein groupoid $\Sigma(M)$, which consists of equivalence classes $[a] \in \Sigma(M)$ that can be represented by a cotangent path a with $\#a \in TN$. The short exact sequence above then gives a groupoid homomorphism $\Sigma_N(M) \to \Sigma(N)$ with kernel a bundle of Lie groups $\mathcal{G}(\mathfrak{g}_N)$. We shall show now that, in the integrable case, the diagram

$$\begin{tikzcd} \Sigma_N(M) \arrow[r, hook] \arrow[dr] & \Sigma(M) \\ & \Sigma(N) \end{tikzcd}$$

corresponds in fact to symplectic reduction.

Proposition 8. *Let N be a constant rank Poisson-Dirac submanifold of a Poisson manifold M. If M is integrable then $\Sigma_N(M)$ is a Lie subgroupoid of $\Sigma(M)$. Moreover, $\Sigma_N(M) \subset \Sigma(M)$ is a presymplectic submanifold with rank equal to $2\dim N$ and its characteristic foliation has leaves the orbits of the equivalence relation on $\Sigma_N(M)$ defined by the bundle of Lie groups $\mathcal{G}(\mathfrak{g}_N)$.*

Proof. Subalgebroids of an integrable Lie algebroid integrate to Lie subgroupoids of the corresponding Weinstein groupoids. Hence, the only thing to check is the statement about the characteristic foliation. By invariance of the symplectic form $\omega \in \Sigma(M)$, it is enough to check that the kernel of $i^*\omega$, where $i : \Sigma_N(M) \hookrightarrow \Sigma(M)$ is the inclusion, coincides with $(\mathfrak{g}_N)_x$ at points $x \in M$.

Now observe that if $x \in M$, we have an isomorphism of symplectic vector spaces

$$T_x\Sigma(M) \simeq T_xM \oplus T_x^*M$$

where on the right the symplectic form is the the one defined by

$$\omega_x((v_1,\xi_1),(v_2,\xi_2)) = \xi_1(v_2) - \xi_2(v_1) + \Pi(\xi_1,\xi_2)$$

(the identity section gives a splitting of the differential of the source map). Under this isomorphism, we have

$$T_x\Sigma_N(M) \simeq T_xN \oplus A_N(M)_x,$$

and we check that

$$\omega_x((v_1,\xi_1),(v_2,\xi_2)) = 0, \quad \forall v_2 \in T_xN, \xi_2 \in A_N(M)_x \qquad \Longleftrightarrow \quad v_1 = 0,\ \xi_1 \in \mathfrak{g}_x,$$

so the result follows. q.e.d.

We conclude that if both N and M are integrable, the symplectic form on $\Sigma_N(M)$ is obtained by symplectic reduction from the symplectic form on $\Sigma(M)$: we first pullback to $\Sigma_N(M)$, to obtain a presymplectic form, and then we project to $\Sigma(N)$ along the characteristic foliation.

Example 7. For a cosymplectic submanifold, we have $\operatorname{rank} N = 0$ so $\Sigma_N(M)$ is a symplectic subgroupoid of $\Sigma(M)$ of dimension $2\dim N$ and we have $\Sigma(N) = \Sigma_N(M)$ (but not the converse; see below). On the other hand, for a Poisson submanifold we have $\operatorname{codim} N = \operatorname{rank} N$ and this happens precisely when $\Sigma_N(M)$ is a coisotropic submanifold of $\Sigma(M)$.

Notice that M being integrable guarantees that A_N is integrable. Although the foliation of the Lie algebroid A_N coincides with the symplectic foliation of N, this is not enough to guarantee that N is integrable, since the isotropy Lie algebras of A_N and T^*N are distinct. Also, $\mathfrak{g}_N(M)$ being a bundle of Lie algebras is always integrable.

In general, a Poisson-Dirac submanifold will not have constant rank. However, we can still define $\Sigma_N(M)$ which will be a non-smooth subgroupoid of $\Sigma(M)$. We will still have a diagram as above relating $\Sigma(M)$, $\Sigma(N)$ and $\Sigma_N(M)$. So, morally, Poisson-Dirac submanifolds are the base manifolds of *presymplectic groupoids*, with Poisson submanifolds corresponding to base manifolds of coisotropic subgroupoids.

By a **presymplectic groupoid** we mean a Lie groupoid $\mathcal{G}$ with a closed 2-form $\omega \in \Omega^2(\mathcal{G})$ such that:

(i) ω is compatible with the product: $m^*\omega = \pi_1^*\omega + \pi_2^*\omega$.
(ii) ω has a characteristic foliation defined by a normal Lie subgroupoid $\mathcal{H} \subset \mathcal{G}$.

Property (i) is similar to the corresponding property for symplectic groupoids (see Proposition 4). Now, to explain (ii), recall that a normal subgroupoid $\mathcal{H} \subset \mathcal{G}$ is a wide subgroupoid such that that:

$$\forall h \in \mathcal{H}_x, g \in \mathcal{G} \text{ with } s(g) = x \implies ghg^{-1} \in \mathcal{H}.$$

Notice that this condition only involves the isotropy groups of $\mathcal{H}$. A normal subgroupoid $\mathcal{H} \subset \mathcal{G}$ defines an equivalence relation in $\mathcal{G}$ and an equivalence relation in the base N:

- For $g_1, g_2 \in \mathcal{G}$, $g_1 \sim g_2$ if and only if there exist $h, h' \in \mathcal{H}$ such that $hg_1h' = g_2$.
- For $x, y \in M$, $x \sim y$ if and only if there exists $h \in \mathcal{H}$ with $s(h) = x$ and $t(h) = y$;

So condition (ii) means that the leaves of the characteristic foliation are the $\mathcal{H}$ orbits in $\mathcal{G}$.

Presymplectic groupoids are interesting objects which we believe deserve more attention. Let us list some of the properties we hope they will satisfy: Let $\mathcal{G}$ be a presymplectic groupoid over N with characteristic foliation defined by $\mathcal{H} \subset \mathcal{G}$. Then:

(a) The base manifold N has a Dirac structure with null foliation the foliation defined by $\mathcal{H}$.
(b) Conversely, every Dirac structure has a (Weinstein) presymplectic groupoid.

(c) If the leaf space $N/\mathcal{H}$ is smooth it is a Poisson manifold. Its symplectic groupoid is $\mathcal{G}/\mathcal{H}$ provided it is smooth.
(d) The groupoid $\mathcal{G}'$ obtained from $\mathcal{G}$ by factoring out only the isotropy of $\mathcal{H}$, has Lie algebroid $L_N \subset TN \oplus T^*N$ (the Dirac subbundle), provided it is smooth.
(e) By a groupoid-equivariant coisotropic embedding theorem, $\mathcal{G}$ should embed into a symplectic groupoid Σ, whose germ is unique up to isomorphism. The base M of Σ is a Poisson submanifold and the Dirac structure on $N \subset M$ is the one induced from M.

Remark 9. The definition given here of a presymplectic groupoid, does not guarantee that to each (integrable) Dirac structure there will be a unique presymplectic groupoid integrating it. To obtain uniqueness, one must require a non-degeneracy condition which can be expressed by saying that $\ker \omega$ intersects the fibers of $\mathbf{s}$ and $\mathbf{t}$ transversely. After the submission of this work, the papers [**2**, **6**] have appeared. There one can find a serious discussion of presymplectic groupoids, multiplicative forms and their main properties.

8.3. Lie-Dirac submanifolds. We consider now Poisson-Dirac submanifolds which admit a Dirac projection compatible with the Lie brackets on one-forms induced by the Poisson structures.

Definition 6. A submanifold $N \subset M$ of a Poisson manifold is called a **Lie-Dirac submanifold** if it admits a Dirac projection $p : T_N M \to TN$ such that $p^* : T^*N \to T^*M$ preserves Lie brackets (i.e., is a Lie algebroid map). Such projections will be called **Lie-Dirac projections**.

Lie-Dirac submanifolds were first introduced by Xu in [**38**], under the name "Dirac submanifolds". Note that while the property of being a Poisson-Dirac submanifold is a local property, the property of being a Lie-Dirac submanifold is a global property. The obstructions will be studied below.

Let us first give alternative characterizations of these submanifolds. The proof is immediate and is left to the reader.

Proposition 9. *Let M be a Poisson manifold and $N \subset M$ a submanifold. The following statements are equivalent:*
(i) *N is a Lie-Dirac submanifold.*
(ii) *There exists a Dirac projection $p : T_N M \to TN$ such that*
$$p_*([\Pi, X]) = [\Pi_N, p(X)]$$
for every $X \in \mathfrak{X}(M)$.

(iii) *There exists a bundle* E *such that* $T_N M = TN \oplus E$ *and* $E^0 \subset T^*M$ *is a Lie subalgebroid.*
(iv) *There exists a bundle* E *such that* $T_N M = TN \oplus E$ *and* E *is a coisotropic submanifold of* TM.

Note that for property (iv) one considers on TM the tangent Poisson structure.

Example 8. Any cosymplectic submanifold is a Lie-Dirac submanifold. However, Poisson-Dirac submanifolds of constant rank, Poisson submanifolds, or even symplectic leaves, may fail to be Lie-Dirac submanifolds.

A Poisson-Dirac submanifold $N \subset M$ of constant rank is Lie-Dirac if and only if the sequence of Lie algebroids (18) has a Lie algebroid splitting. Since this splitting integrates to a homomorphism of the corresponding Weinstein groupoids, we can complete the diagram above to a commutative diagram

$$\begin{tikzcd} \Sigma_N(M) \arrow[r, hook] \arrow[dr] & \Sigma(M) \\ & \Sigma(N) \arrow[u] \end{tikzcd}$$

where the map going up is an embedding.

Now note that even if N does not have constant rank, the Lie-Dirac projection induces a groupoid homomorphism $\Sigma(N) \to \Sigma(M)$ and we have the following theorem, which is a slight improvement of Xu's results.

Theorem 9. *Let* M *be an integrable Poisson manifold. Then any Lie-Dirac submanifold* $N \subset M$ *is integrable, and* $\Sigma(N)$ *is a symplectic subgroupoid of* $\Sigma(M)$. *More precisely, any Lie-Dirac projection* p *induces a groupoid embedding of* $\Sigma(N)$ *into* $\Sigma(M)$, *which is a symplectomorphism onto a symplectic subgroupoid of* $\Sigma(M)$. *Conversely, any such embedding is of this type.*

Proof. The first part of the theorem is clear. So assume that $i : \Sigma' \hookrightarrow \Sigma(M)$ is a symplectic groupoid embedding of a symplectic groupoid Σ' over N. We can identify $T^*N \simeq T_N\Sigma'/TN$ and $T^*M \simeq T_M\Sigma/TM$. Hence, we obtain a Lie algebroid map $i_* : T^*N \to T^*M$ whose image is a Lie subalgebroid $A \subset T^*M$ which is transversal to $(TN)^0$. It is clear that $E = A^0$ satisfies (iii) of Proposition 9, so N is a Lie-Dirac submanifold and the embedding i can be identified with the embedding $\Sigma(N) \to \Sigma(M)$. q.e.d.

Therefore, Lie-Dirac submanifolds of M are the base manifolds of symplectic subgroupoids of $\Sigma(M)$. It should be noted that for a given Lie-Dirac submanifold $N \subset M$ of a Poisson manifold there can be *distinct* connected symplectic subgroupoids of $\Sigma(M)$ over N. In other words, the Poisson manifold N does not determine uniquely the subgroupoid of $\Sigma(M)$ (the image of the embedding). This is, of course, because there can be several distinct Lie-Dirac projections. Here is a very simple example.

Example 9. A manifold M with the zero bracket integrates to the symplectic groupoid $\Sigma = T^*M$, where ω is the canonical symplectic structure, $\mathbf{s} = \mathbf{t}$ is the projection to M, and the group operation is addition in the fibers. If $N \subset M$ is any submanifold, then a vector subbundle $E \subset T^*M$ over N, with $\operatorname{rank} E = \dim N$, is a symplectic subgroupoid $\Sigma' \subset \Sigma$ if and only if $E \cap (TN)^\perp = \{0\}$. In this case, N is obviously a Poisson submanifold of M, but there are many symplectic subgroupoids with the same base N.

As was already remarked by Xu in [**38**], not every Poisson submanifold is a Lie-Dirac submanifold. Here we give the obstruction and show that it is related to the monodromy of Section 2. So let $i : N \hookrightarrow M$ be a Poisson submanifold, giving rise to the short exact sequence of Lie algebroids

$$0 \longrightarrow \nu^*(N) \longrightarrow T^*_N M \xrightarrow{i^*} T^*N \longrightarrow 0. \tag{19}$$

Choosing a Dirac projection $p : T_N M \to TN$ is the same as choosing a splitting $p^* : T^*N \to T^*_N M$ of this exact sequence. This choice gives rise to a contravariant connection $\nabla : \Omega^1(N) \times \Gamma(\nu^*(N)) \to \Gamma(\nu^*(N))$ defined by:

$$\nabla_\alpha \beta = [p^*(\alpha), \beta].$$

If the Lie algebra bundle $\nu^*(N)$ is abelian (this is the case, for example, if N consists only of regular points), this connection is independent of the splitting, and it is flat:

$$R_\nabla(\alpha, \beta) \equiv \nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha - \nabla_{[\alpha,\beta]} = 0.$$

In other words, $\nu^*(N)$ is canonically a flat Poisson vector bundle over N. Then the curvature 2-form

$$\Omega(\alpha,\beta)=p^*([\alpha,\beta])-[p^*(\alpha),p^*(\beta)],$$

defines a Poisson cohomology 2-class $[\Omega]\in H^2_\Pi(N,\nu^*(N))$ which is the obstruction for N to be a Lie-Dirac submanifold. For example, for a regular leaf L of a Poisson manifold the Poisson cohomology $H^2_\Pi(L,\nu^*(L))$ coincides with the cohomology $H^2(L,\nu^*(L))$ and we conclude that:

Corollary 13. *Let L be a regular leaf of a Poisson manifold. Then L is a Lie-Dirac submanifold if and only if the canonical cohomology class $[\Omega]\in H^2(L,\nu^*(L))$ vanishes. If L is simply connected, then L is a Lie-Dirac submanifold if and only if its monodromy group vanishes.*

Example 10. Let us return to the Poisson brackets on $\mathfrak{su}^*(2)$ of Section 6.2. The origin is always a Lie-Dirac submanifold. The spheres S^2_R, for $R>0$, are regular, compact and simply connected leaves, and it follows that S^2_R is a Lie-Dirac submanifold if and only if the variation of the symplectic area $A'(R)$ vanishes.

Recall now (see e.g., [**32**]) that the **characteristic form class** of the regular Poisson manifold M is the relative cohomology class $[\xi]=[d\omega]\in H^3_{\mathrm{rel}}(M,\mathcal{F})$, where ω is the foliated symplectic form. The characteristic form class is obviously the obstruction to the existence of a closed 2-form on M which pulls back to the symplectic 2-form on each leaf. Equivalently, by the coisotropic embedding theorem of Gotay [**20**], it is the obstruction for the existence of a leafwise symplectic embedding of M. From the remarks in Section 5, we have a commutative diagram relating the different classes associated with a regular Poisson manifold

$$\begin{tikzcd}
{[\omega]\in H^2(\mathcal{F})} \arrow[d, "\delta"'] \arrow[dr, "d_\nu"] & \\
{[\xi]\in H^3_{\mathrm{rel}}(M,\mathcal{F})} \arrow[r] & {H^2(\mathcal{F},\nu^*)\ni[\Omega]}
\end{tikzcd}$$

where δ is defined from the long exact sequence of the pair $(M,\mathcal{F})$:

$$\cdots\longrightarrow H^k_{\mathrm{rel}}(M,\mathcal{F})\longrightarrow H^k(M)\longrightarrow$$
$$H^k(\mathcal{F})\xrightarrow{\delta}H^{k+1}_{\mathrm{rel}}(M,\mathcal{F})\longrightarrow\cdots.$$

In particular we obtain:

Corollary 14. *Let M be a regular Poisson manifold. Then:*

(i) *If M admits a leafwise symplectic embedding then every leaf of M is a Lie-Dirac submanifold.*
(ii) *If every leaf of M is a Lie-Dirac submanifold then M is integrable.*

Note that the reverse implications in general do not hold.

9. Morita equivalence

In this section we discuss Morita equivalence of Poisson manifolds. This notion, introduced by P. Xu [**39**], originates on Weinstein's dual pairs as a global form of Lie's dual function groups. Intuitively, Morita equivalence of Poisson manifolds means isomorphism from the point of view of transversal Poisson geometry (in particular, it induces Poisson diffeomorphisms between the transversal Poisson structures).

9.1. Integrable Poisson manifolds. Let us start by recalling Xu's definition:

Definition 7. Two Poisson manifolds M_1 and M_2 are **Morita equivalent** if there exists a two leg diagram

$$\begin{tikzcd} & S \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ M_1 & & \overline{M}_2 \end{tikzcd}$$

where S is a symplectic manifold and the maps π_1 and π_2 satisfy:

(a) Each π_i is a complete Poisson map.
(b) Each π_i is a surjective submersion.
(c) π_1 and π_2 have symplectic orthogonal, simply connected, fibers.

From our section on symplectic realizations we see that, in a Morita equivalence, both Poisson manifolds have to be integrable. Also, if M is integrable, then the associated symplectic groupoid $\Sigma(M)$ defines a Morita equivalence of M with itself (π_i the source/the target). Hence:

Proposition 10. *For a Poisson manifold M, the following are equivalent:*

(i) *M is Morita equivalent to another Poisson manifold.*
(ii) *M is Morita equivalent to itself.*
(iii) *M is integrable.*

Moreover, if S defines an equivalence between M_1 and M_2, we also see that S comes equipped with free (symplectic) actions of $\Sigma(M_1)$ and $\Sigma(M_2)$, and the associated orbit spaces are M_1, and M_2, respectively:

$$\begin{tikzcd}
\Sigma(M_1) \arrow[d, shift left] \arrow[d, shift right] & S \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \Sigma(M_2) \arrow[d, shift left] \arrow[d, shift right] \\
M_1 & & M_2.
\end{tikzcd}$$

Or, in the terminology of [**39**, **40**], we recover Xu's result that two integrable Poisson manifolds are Morita equivalent if and only if their symplectic groupoids $\Sigma(M_1)$ and $\Sigma(M_2)$ are symplectic Morita equivalent. Now we can proceed as in the ring-theoretic version of Morita equivalence and compose such equivalences: if S_i defines an equivalence between M_i and M_{i+1}, $i \in \{1, 2\}$, then $S_1 \otimes_{\Sigma(M_2)} S_2$, that is, the quotient of $S_1 \times_{M_2} S_2$ by the diagonal action of $\Sigma(M_2)$, is an equivalence between M_1 and M_3. In particular:

Corollary 15. *On the class of integrable Poisson manifolds, Morita equivalence is an equivalence relation.*

9.2. Weak Morita Equivalence. For general Poisson manifolds, there are two important questions one should address:

- Find a satisfactory notion of Morita equivalence which works well also for non-integrable Poisson manifolds.
- Construct "Morita invariants", i.e., invariants which allow us to distinguish non-Morita equivalent Poisson manifolds and eventually classify Morita equivalence classes.

The notion of weak Morita equivalence of Poisson manifolds is very similar to that of Morita equivalence of foliations (discussed in [**9**, **18**]) which, in turn, is an infinitesimal version of Morita equivalence of holonomy groupoids (see [**21**]). More precisely, given a submersion $\phi : Q \to M$ and a (regular) foliation $\mathcal{F}$ on M, the pullback foliation on Q, denoted $\phi^\star\mathcal{F}$, is the foliation whose leaves are the connected components of $\phi^{-1}(L)$, with L leaf of $\mathcal{F}$. In terms of the associated involutive subbundles, $\phi^\star\mathcal{F}$ consists of vectors X tangent to Q with the property that $d\phi(X) \in \mathcal{F}$, and is in general different from the pullback vector bundle $\phi^*\mathcal{F}$ (note the difference in the notation). We will say that two (regular) foliations $\mathcal{F}_i$ on M_i ($i \in \{1, 2\}$) are Morita equivalent if there exists a manifold Q, and submersions π_i from Q onto M_i with simply connected fibers:

$$M_1 \xleftarrow{\pi_1} Q \xrightarrow{\pi_2} M_2,$$

and such that $\pi_1^\star \mathcal{F} = \pi_2^\star \mathcal{F}$.

For Poisson manifolds we proceed similarly. First of all, for a submersion $\phi : Q \to M$ from a manifold Q into a Poisson manifold M, we form the bundle $\phi^\star T^*M$ over Q:

$$\phi^\star T^*M = \{(\alpha, X) \in \phi^* T^*M \times TQ : \#\alpha = \phi_* X\}.$$

This is a Lie algebroid over Q with the anchor $(\alpha, X) \mapsto X$, and with the Lie bracket

$$[(f\phi^*\alpha, X), (g\phi^*\beta, Y)] = (fg\phi^*[\alpha, \beta] + X(g)\phi^*\beta - Y(f)\phi^*\alpha, [X, Y]),$$

where $f, g \in C^\infty(Q)$, $X, Y \in \mathfrak{X}(Q)$, $\alpha, \beta \in \Omega^1(M)$, and $\phi^*\alpha$ is the induced section of ϕ^*T^*M. The idea is that this should represent the pullback of the Poisson structure to Q. The point is that pull backs of Poisson structures generally only make sense as Dirac structures, and $\phi^\star T^*M$ is precisely this pullback Dirac structure, viewed as a Lie algebroid.

Definition 8. Let M_1 and M_2 be two Poisson manifolds. We say that they are weakly Morita equivalent if there exists a manifold Q together with two submersions onto M_1 and M_2 with simply connected fibers

$$M_1 \xleftarrow{\pi_1} Q \xrightarrow{\pi_2} M_2,$$

such that the pullback algebroids (equivalentely, Dirac structures) on P are isomorphic:

$$\pi_1^\star T^*M_1 \simeq \pi_2^\star T^*M_2.$$

The reader will notice that Dirac structures furnish a natural setting fot Morita equivalence. We refer to [**3**] for details on this approach. Also, the previous notion of Morita equivalence originates in [**9**] (see Theorem 2 there and the comment preceeding it) and it coincides with the equivalence used in [**18**].

Proposition 11. (i) *Morita equivalence implies weak Morita equivalence.*
(ii) *Weak Morita equivalence is an equivalence relation on the class of all Poisson manifolds.*

Proof. Let S be as in the definition of Morita equivalence. We denote by $\rho_i : \pi^* T^* M_i \to TS$ the induced actions on S. There are bundle isomorphisms

$$\pi_1^\star T^* M_1 \simeq \pi_1^* T^* M_1 \oplus \pi_2^* T^* M_2 \simeq \pi_2^\star T^* M_2,$$

which, for sections α of $\pi_1^* T^* M_1$ and β of $\pi_2^* T^* M_2$ which are pullbacks of one forms on M_1 and M_2, are given by

$$(\alpha, \rho_1(\alpha) + \rho_2(\beta)) \longleftarrow\!\shortmid\ (\alpha, \beta) \longmapsto (\beta, \rho_1(\alpha) + \rho_2(\beta)).$$

A careful computation shows that this is actually a Lie algebroid isomorphism. Assume now that P defines another weak Morita equivalence between M_2 and M_3. We then form the fibered product $R = Q \times_{M_2} P$ and denote by q and p, the projections into Q, and P, respectively:

$$\begin{tikzcd}
 & & R \arrow[dl, "q"'] \arrow[dr, "p"] & & \\
 & Q \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & P \arrow[dl, "\pi_3"'] \arrow[dr, "\pi_4"] & \\
M_1 & & M_2 & & M_3.
\end{tikzcd}$$

Then R, together with $\pi_1 q$ and $\pi_4 p$ defines a weak Morita equivalence. This follows from the functoriality of the operation $\phi^\star$ on Lie algebroids. q.e.d.

We now proceed with the description of various (weak) Morita invariants. Due to the similarity with the notion of Morita equivalence of foliations, it is not surprising that:

Proposition 12. *Let M_1 and M_2 be weakly Morita equivalent Poisson manifolds with symplectic foliations $\mathcal{F}_1$ and $\mathcal{F}_2$. Then any weak Morita equivalence Q between them induces a homeomorphism*

$$\phi_Q : M_1/\mathcal{F}_1 \simeq M_2/\mathcal{F}_2,$$

and the fundamental groups of corresponding leaves are isomorphic

$$L_1 \xleftrightarrow{\phi_Q} L_2 \Longrightarrow \pi(L_1) \cong \pi(L_2).$$

Moreover, if one of the two leaf spaces is a smooth manifold, then so is the other one, and the map above is a diffeomorphism.

Proof. It is easy to see that given a submersion $\phi : Q \to M$ onto a Poisson manifold M, the leaves of $\phi^{\star}T^{*}M$ are of the form $\phi^{-1}(L)$, where L is a leaf of M. Hence, given a Morita equivalence $M_1 \xleftarrow{\pi_1} Q \xrightarrow{\pi_2} M_2$, the map $L \mapsto \pi_2(\pi_1^{-1}(L))$ is a bijective correspondence between the leaves of M_1 and the leaves of M_2. This bijection makes the following diagram commute:

$$\begin{tikzcd} & Q \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ M_1 \arrow[d] & & M_2 \arrow[d] \\ M_1/\mathcal{F}_1 \arrow[rr, leftrightarrow] & & M_2/\mathcal{F}_2. \end{tikzcd}$$

The statements on the leaf spaces follow from the fact that the maps going down are quotient maps. On the other hand, the statement about the fundamental groups of corresponding leaves, follows from the fact that any submersion with simply connected fibers induces isomorphisms in the first homotopy groups (cf. the Appendix in [**18**]). q.e.d.

Our next result shows that a large number of geometric invariants are weak Morita invariants, including the monodromy groups we have introduced before. In this respect, note that the higher homotopy groups of corresponding leaves may not be isomorphic, so one may think of the monodromy groups (which come from the second homotopy groups of the leaves), as the next level in the hierarchy.

Theorem 10. *The monodromy groups, isotropy Lie algebras and groups, first Poisson cohomology groups and integration along cotangent paths, are all weak Morita invariants. More precisely, let Q be a weak Morita equivalence between M_1 and M_2, and let $x \in M_1$, $y \in M_2$ be points whose leaves correspond. Then the following are isomorphic:*

(i) *The monodromy groups $\mathcal{N}_x(M_1)$ and $\mathcal{N}_y(M_2)$;*
(ii) *the isotropy Lie algebras $\mathfrak{g}_x$, and $\mathfrak{g}_y$;*
(iii) *the isotropy groups $\Sigma(M_1, x)$ and $\Sigma(M_2, x)$;*
(iv) *the reduced isotropy groups $\Sigma^0(M_1, x)$ and $\Sigma^0(M_2, x)$;*
(v) *the Poisson cohomology groups $H^1_\Pi(M_1)$ and $H^1_\Pi(M_2)$, and the two maps*

$$\Sigma(M_i, x_i) \times H^1_\Pi(M_i) \longrightarrow \mathbb{R}, ([a], X) \mapsto \int_a X \quad (i = 1, 2).$$

Proof. The idea is the same as in the previous proof: given a Lie algebroid A over M, and a submersion $\phi : Q \to M$ with simply connected fibers, we prove that all these groups for $\phi^\star A$ are isomorphic to the ones of A. We use here that all we have said about T^*M and $\Sigma(M)$ works for general Lie algebroids (cf. [**10**]).

We have already remarked that the leaves of $\phi^\star A$ are of type $\phi^{-1}L$, with L leaf of A. Since the fibers are simply connected, the maps $\pi_2(\phi^{-1}L) \to \pi_2(L)$ are surjective [**18**].

Next, the kernel of the anchor of $\phi^\star A$ is clearly isomorphic to the kernel of the anchor of A, i.e., the isotropy Lie algebras $\mathfrak{g}_q(\phi^\star A)$ and $\mathfrak{g}_{\phi(q)}(A)$ are isomorphic. For the other groups, due to the canonical sequences (4), (6) (and their obvious versions for Lie algebroids), and the fact that ϕ induces isomorphism in the first homotopy groups, it is enough to prove the statement for the reduced isotropy group $\Sigma^0(M,x)$ (and its Lie algebroid versions, denoted $G^0(A,x)$). Both groups $G(\phi^\star A, q)$ and $G(A,x)$ ($x = \phi(q)$) are quotients of the same space (of paths in the isotropy Lie algebra), and one only has to check that the equivalence relation (homotopy) is the same. For this one uses that, when working in the smooth category, submersions do behave like Serre fibrations (cf. the Appendix in [**18**]). Alternatively, one can prove all these isomorphisms at once, computing the groupoid $G(\phi^\star A)$ of $\phi^\star A$ (see [**10**] for notations):

First of all, a path for $\phi^\star A$ is a pair $(a, \widetilde{\gamma})$, where A is an A-path, and $\widetilde{\gamma}$ is a path in Q over the base path of a. However, since:

(a) For any two paths $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ in Q covering the same path γ in M, there is a homotopy $\widetilde{\gamma}_\epsilon$ between them, covering γ (this follows since the fibers are simply connected, see the Appendix in [**18**]).
(b) If $\widetilde{\gamma}_\epsilon$ is as above, and a is an A-path with base path γ, then $(a, \widetilde{\gamma}_\epsilon)$ is a $\phi^\star A$-homotopy.

It follows that, when looking at homotopy classes of $\phi^\star A$-paths $(a, \widetilde{\gamma})$, it is only a and the end points of $\widetilde{\gamma}$ that matter. Then, working with triples (p, a, q), where a is an A-path, and $\phi(p) = \gamma(0)$, $\phi(q) = \gamma(1)$, and writing down what $\phi^\star A$-homotopy means, ones gets $G(\phi^\star A) = Q \times_M G(A) \times_M Q$ consists of triples (p, g, q) with $\phi(p) = \mathbf{s}(g)$, $\phi(q) = \mathbf{t}(g)$, with the multiplication $(q, h, r)(p, g, q) = (p, hg, r)$. This immediately imply the rest.

Finally, the statement about Poisson cohomology follows from the fact (see [**9**], Theorem 2) that if $\phi : Q \to M$ is a surjective submersion with simply-connected fibers then A and $\phi^\star A$ have isomorphic first cohomology groups. q.e.d.

Example 11. Let us look at some examples:

1. The fundamental group $\pi_1(S)$ is a complete Morita invariant in the symplectic case.
2. The isotropy Lie algebra at zero is a complete invariant in the case of linear structures (we actually see that $\mathfrak{g}^*$ and $\mathfrak{h}^*$ are weakly Morita equivalent around the origin if and only if and only if $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic Lie algebras).
3. Consider the sphere S^2 with its standard symplectic structure (area form) and form the Poisson-Heisenberg manifold $M(S^2)$ (see the example in 6.3). This Poisson manifold is not Morita equivalent to any of the Poisson manifolds M_a considered in the example of Section 6.2, just because they have different leaf spaces.
4. The Poisson manifolds M_a, for different a's, all have the same leaf spaces. However, one can find a's for which the monodromy groups vary differently, hence the associated Poisson structures are not weakly Morita equivalent (this will actually force one of the manifolds to be non-integrable, but this can be avoided by going to leaf spaces of higher dimensions).

We say that a Poisson manifold M has simple symplectic foliation if $\mathcal{F}$ is regular, and the leaf space $B = M/\mathcal{F}$ is smooth. Equivalently, one has a submersion $\pi : M \to B$ with connected symplectic fibers, and M has the induced Poisson structure. Note that in this case there are canonical identifications $\nu_x \cong T_{\pi(x)}B$, and the monodromy groups $\mathcal{N}_x$ define a bundle of subgroups

$$\mathcal{N}(M) \subset T^*B.$$

We know that the continuity of $\mathcal{N}(M)$ is related to the integrability of M. It is not difficult to see that the class of such Poisson manifolds is closed under weak Morita equivalence. From the previous proofs we see that:

Corollary 16. *For Poisson manifolds M with simple symplectic foliation, the pair $(B, \mathcal{N}(M))$ is a weak Morita invariant. More precisely, a weak Morita equivalence Q between two such M_1 and M_2 induces a diffeomorphism $\phi : B_1 \to B_2$ with $(d\phi)_x^*(\mathcal{N}(M_1)) = \mathcal{N}(M_1)$.*

Specializing to product foliations, Xu proved a similar result for ordinary Morita equivalence, along with a converse (see also the next proposition).

Remark 10. It is important that the isomorphism between the monodromy is induced by the differential. For instance, consider the Poisson manifolds $M_a^{'} = M_a \setminus \{0\}$. They all have the same leaf space, and one can choose a and b such that the corresponding Poisson manifolds have the same (i.e., isomorphic) monodromy, without being Morita equivalent.

The following proposition suggests that, in some cases, the leaf space, the homotopy groups of the leaves, and the monodromy groups form a *complete* set of weak Morita invariants:

Proposition 13. *Let M be a Poisson manifold with simple symplectic foliation $\mathcal{F}$ and compact simply connected leaves. If $\mathcal{N}(M) = 0$, then M with the induced Poisson structure is weakly Morita equivalent to $B = M/\mathcal{F}$ with the zero Poisson structure.*

Proof. Let $\sigma : \mathcal{F} \to T^*M$ be a splitting of the anchor, and let $\Omega \in \Omega^2(\mathcal{F}; \nu^*)$ be its curvature. Since each leaf L is simply connected, the condition on the monodromy shows that $\Omega|_L$ is exact. From the Reeb stability theorem each L has a neigborhood of type $T \times L$, hence Ω is exact in a neigborhood L. By choosing a partition of unity supported on such opens neigborhoods, and which is constant on each leaf (e.g., the pullback by $\pi : M \to B$ of an open cover of the base), we see that Ω is exact. Hence σ has a global splitting compatible with the brackets. This shows that $\Sigma(M) = \pi(\mathcal{F}) \times_M \nu^*$, which is clearly Morita equivalent to T^*B (as a bundle of abelian Lie groups over B). q.e.d.

Remark 11. Our aim here was not only to describe Morita invariants, but also to point out that all the algebraic invariants we know are weak Morita invariants. Nevertheless, although this notion does behave well for all Poisson manifolds, it is not our intention to present it as the "satisfactory" notion of Morita equivalence that we asked at the beginning of this section, for it does not take the symplectic/Poisson picture fully into account. Let us point out two other possible notions of Morita equivalence for non-integrable Poisson manifolds:

Symplectic Morita Equivalence: We mimic Xu's definition (see Definition 7) but we remove the completeness assumption. Then we obtain a symmetric relation which is also reflexive (because of the existence of a local symplectic groupoid integrating any M). It is however not clear that this relation is transitive, so we should take the equivalence relation it generates.

Algebraic Morita Equivalence: For M_1 and M_2 to be algebraic Morita equivalent we require the existence of a Lie algebroid bi-module for T^*M_1 and T^*M_2: there are commuting free actions ρ_1 and ρ_2 of T^*M_1 and T^*M_2 on Q, with moment maps $M_1 \xleftarrow{\pi_1} Q \xrightarrow{\pi_2} M_2$, where each π_i is a surjective submersion with simply connected fibers, such that the orbits of ρ_1 (respectively, ρ_2) are the fibers of π_2 (respectively, π_1).

It is not hard to see that one has the following chain of implications:

$$\text{symplectic Morita} \implies \text{algebraic Morita} \implies \text{weak Morita}$$

In our opinion, it is an important open question to show that these relations coincide or else to find new Morita invariants which are not weak Morita invariants.

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