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Received August 20, 1975

(1059)

Integrability of seminorms, the 0-1 law and the affine kernel for product measures

by

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Abstract. Let (X_n) be a sequence of random variables taking values in a measurable linear space E , and let q be a quasi-convex subadditive function on E^∞ . The first part of the paper deals with the problem of finding conditions, which assures that $E(e^{\varepsilon M})$ is finite for some positive ε , where $M = \sup q(X_1, \dots, X_n, 0, 0, \dots)$.

In the second and third part of the paper we take $E = \mathbf{R}^n$ and we show that if X_n has no mass points, then every linear subspace of \mathbf{R}^∞ has probability 0 or 1. Finally, we study the affine kernel of (X_n) , i.e. the intersection of all affine subspaces of probability 1, and we give an analytic expression for this.

1. Introduction. If μ is a Gaussian measure on a locally convex space, there are three main results which have proved to be useful.

The first is the result of Fernique stating that, if φ is a measurable a.e. finite seminorm, then $E(\exp(\varepsilon\varphi^2)) < \infty$ where E denotes expectation with respect to μ . A similar result has been proved by C. Borel ([1]) for certain other classes of measures. In Section 2 of this paper we shall prove some results in this direction when μ is a product measure on $(\prod_1^\infty E_n, \otimes_1^n B_n)$ and (E_n, B_n) is a measurable linear space. Here we define a measurable linear space, (E, B) , to be a linear space E equipped with a σ -algebra B satisfying

$$(1.1) \quad (x, y) \rightarrow x + y \text{ is measurable: } (E \times E, B \otimes B) \rightarrow (E, B),$$

$$(1.2) \quad (\lambda, y) \rightarrow \lambda y \text{ is measurable: } (E \times \mathbf{R}, B \otimes B(\mathbf{R})) \rightarrow (E, B).$$

The methods and the results of that section are closely related to the results of Marcus and Jain in [9] and to the results in [2] and [3].

The second result is the 0-1 law by Kallianpur in [5]. C. Borel has in [1] shown that the same result holds for certain other classes of measure. In Section 3 we show that, if μ is a product measure on \mathbf{R}^∞ with non-atomic factors, then $\mu(A) = 0$ or 1 for all μ -measurable affine subsets.

The third result on Gaussian measures, which has proved to be a very powerful tool, is the reproducing kernel Hilbert space which, in case μ

has mean 0, can be characterized as the intersection of all μ -measurable linear subsets of measure 1. In Section 4 we shall show that, if μ is a product measure on \mathbf{R}^∞ with non-degenerated factors, then \mathbf{R}_0^∞ (the set of sequences with at most finitely many non-zero coordinates) is contained in the intersection of all μ -measurable linear sets with measure 1. We shall also give an analytic expression for this intersection under some restrictions on μ .

2. Integrability of quasi-convex functions. Let (E_n, \mathbf{B}_n) be a measurable linear space, and (X_n) a sequence of independent random vectors so that X_n takes its values in E_n for all $n \geq 1$. Let φ be a map from E_∞ into $\mathbf{R}_+ = [0, \infty]$, where

$$E_\infty = \prod_{n=1}^{\infty} E_n \quad \text{and} \quad \mathbf{B}_\infty = \bigotimes_{n=1}^{\infty} \mathbf{B}_n$$

and suppose that

$$(2.1) \quad \varphi \text{ is } \mathbf{B}_\infty\text{-measurable,}$$

$$(2.2) \quad \varphi\left(\frac{1}{2}x + \frac{1}{2}y\right) \leq \max\{\varphi(x), \varphi(y)\} \quad \forall x, y \in E_\infty,$$

$$(2.3) \quad \varphi(x+y) \leq \varphi(x) + \varphi(y) \quad \forall x, y \in E_\infty,$$

that is, φ is a measurable quasi-convex subadditive function on E_∞ . We shall say that φ is *symmetric* if

$$(2.4) \quad \varphi(x) = \varphi(-x) \quad \forall x.$$

Let us define the random variables

$$N_j = \varphi(0, \dots, 0, X_j, 0, \dots), \quad N = \sup_j N_j,$$

$$M_j = \varphi(X_1, \dots, X_j, 0, \dots), \quad M = \sup_j M_j;$$

we shall consider their tail probabilities

$$G_j(t) = \mathbf{P}(N_j > t), \quad G(t) = \mathbf{P}(N > t),$$

$$F_j(t) = \mathbf{P}(M_j > t), \quad F(t) = \mathbf{P}(M > t).$$

We shall use these notations throughout this section.

LEMMA 2.1. *Suppose that X_n is symmetric for all $n \geq 1$; then we have*

$$(2.1.1) \quad \mathbf{P}(\max_{1 \leq j \leq n} M_j > t) \leq 2\mathbf{P}(M_n > t) \quad \forall t \geq 0,$$

$$(2.1.2) \quad \mathbf{P}(M > t) \leq 2 \liminf_{n \rightarrow \infty} \mathbf{P}(M_n > t) \quad \forall t \geq 0.$$

Proof. The proof is fairly standard and actually only requires quasi-convexity of φ . Let

$$T = \inf\{1 \leq j \leq n \mid M_j > t\} \quad (\inf(\emptyset) = \infty),$$

$$Y_n = (X_1, \dots, X_n, 0, \dots) \quad \text{for } n \geq 1,$$

$$Y_{nj} = (X_1, \dots, X_j, -X_{j+1}, \dots, -X_n, 0, \dots) \quad \text{for } n \geq j \geq 0;$$

then $Y_j = \frac{1}{2}Y_n + \frac{1}{2}Y_{nj}$ and so

$$M_j = \varphi(Y_j) \leq \max\{\varphi(Y_n), \varphi(Y_{nj})\}.$$

If $T = j$, then $M_j > t$, and so either $\varphi(Y_n) > t$ or $\varphi(Y_{nj}) > t$, however, Y_n and Y_{nj} are equidistributed and so

$$\begin{aligned} \mathbf{P}(T = j) &\leq \mathbf{P}(T = j, \varphi(Y_n) > t) + \mathbf{P}(T = j, \varphi(Y_{nj}) > t) \\ &= 2\mathbf{P}(T = j, M_n > t). \end{aligned}$$

Hence we have

$$\mathbf{P}(\max_{1 \leq j \leq n} M_j > t) = \mathbf{P}(T \leq n) \leq 2\mathbf{P}(M_n > t).$$

Now let $a > \liminf_{n \rightarrow \infty} F_n(t)$; then there exist integers $n_1 < n_2 < \dots$ so that $F_{n_j}(t) \leq a$ for all $j \geq 1$. From the first part of the lemma we know that

$$\mathbf{P}(\max_{1 \leq v \leq n_j} M_v > t) \leq 2a \quad \forall j \geq 1$$

and since $\max_{1 \leq v \leq n_j} M_v$ increases to M , we have $\mathbf{P}(M > t) \leq 2a$, and (2.1.2) follows.

LEMMA 2.2. *Let (E, \mathbf{B}) be measurable linear space and ψ a \mathbf{B} -measurable function from E into \mathbf{R}_+ , so that ψ is symmetric quasi-convex and subadditive. If Z is an E -valued random vector and $\mathbf{P}(\psi(Z) \leq a) \geq \frac{1}{2}$ for some $a \geq 0$, then*

$$(2.2.1) \quad \mathbf{P}(\psi(Z^*) > 2t) \leq 2\mathbf{P}(\psi(Z) > t) \quad \forall t \geq 0,$$

$$(2.2.2) \quad \mathbf{P}(\psi(Z) > t+a) \leq 2\mathbf{P}(\psi(Z^*) > t) \quad \forall t \geq 0,$$

where Z^* is a symmetrization of Z .

Proof. (2.2.1) is obvious. If μ is the distribution law of Z , then

$$\mathbf{P}(\psi(Z^*) > t) = \int_{\mathbf{R}_E} \mathbf{P}(\psi(Z-x) > t) \mu(dx) \geq \int_A \mathbf{P}(\psi(Z-x) > t) \mu(dx),$$

where $A = \{x \in E \mid \psi(x) \leq a\}$. Then $\mu(A) \geq \frac{1}{2}$ and so there exists $x_i \in A$ for all $i \geq 0$ so that

$$\mathbf{P}(\psi(Z-x_i) > t) \leq 2\mathbf{P}(\psi(Z^*) > t)$$

and since $\psi(Z) \leq \psi(Z-x_i) + a$, (2.2.2) follows.

THEOREM 2.3. Suppose that X_n is symmetric for all $n \geq 1$; then we have

$$(2.3.1) \quad F(2t+s) \leq 2G(s) + 4F(t)^2 \quad \forall t, s \geq 0.$$

If we drop the assumption about symmetry of X_n , but assume that φ is symmetric, then we have

$$(2.3.2) \quad F(4t+s+a) \geq 8G(s) + 32F(t)^2 \quad \forall s, t \geq 0,$$

where $a \geq 0$ is determined by $\mathbf{P}(M \leq a) \geq \frac{1}{2}$.

Proof. Suppose first that (X_n) is symmetric and define

$$T = \inf\{n \geq 1 \mid M_n > t\};$$

then $M_k > 2t+s$ implies that $T \leq k$ and so

$$F_k(2t+s) = \sum_{j=1}^k \mathbf{P}(T = j, M_k > 2t+s).$$

Now let $Z_j = (0, \dots, 0, X_j, 0, \dots)$ and $Y_j = (X, \dots, X_j, 0, \dots)$; then we have

$$Y_k = (Y_k - Y_j) + Y_{j-1} + Z_j$$

and so

$$M_k \leq \varphi(Y_k - Y_j) + M_{j-1} + N.$$

Hence, if $T = j$, $N \leq s$ and $M_k > 2t+s$, then $M_{j-1} \leq t$, and $\varphi(Y_k - Y_j) > t$, and so we have

$$\begin{aligned} \mathbf{P}(T = j, M_k > 2t+s) &\leq \mathbf{P}(T = j, N > s) + \mathbf{P}(T = j, \varphi(Y_k - Y_j) > t) \\ &= \mathbf{P}(T = j, N > s) + \mathbf{P}(T = j) \mathbf{P}(\varphi(Y_k - Y_j) > t) \end{aligned}$$

since $\{T = j\}$ and $Y_k - Y_j$ are independent. Now we observe that by Lemma 2.1 we have

$$\mathbf{P}(\varphi(Y_k - Y_j) > t) \leq 2\mathbf{P}(\varphi(Y_k) > t) \leq 2\mathbf{P}(M > t).$$

Hence we find

$$F_k(2t+s) \leq \sum_{j=1}^k \mathbf{P}(T = j, N > s) + 2 \sum_{j=1}^k \mathbf{P}(T = j) \mathbf{P}(M > t) \leq G(s) + 2F(t)^2$$

and from Lemma 2.1, we find

$$F(2t+s) \leq 2 \liminf_{k \rightarrow \infty} F_k(2t+s) \leq 2G(s) + 4F(t)^2$$

which proves (2.3.1).

Now suppose that φ is symmetric, and let (X_n^*) be a symmetrization. Define M^* and N^* in the obvious way and let F^* , respectively, G^* be their tail probabilities; then by Lemma 2.2 and (2.3.1) we find

$$F(4t+2s+a) \leq 2F^*(4t+2s) \leq 4G^*(2s) + 8F^*(2t)^2 \leq 8G(s) + 32F(t)^2.$$

THEOREM 2.4. If (X_n) is symmetric, $M_n < \infty$ a.s. for $n \geq 1$, and $\{M_n\}$ stochastically bounded, then $M < \infty$ a.s.

If φ is symmetric, $M_n < \infty$ a.s. for all $n \geq 1$, and $\{M_n\}$ is stochastically bounded, then $M < \infty$ a.s.

Proof. The first case is an immediate consequence of Lemma 2.1. So suppose that φ is symmetric, and let (X_n^*) be a symmetrization of (X_n) . Let M_n^* and M^* be defined in the obvious way. Then (2.2.2) shows that $\{M_n^*\}$ is stochastically bounded, and so $M^* < \infty$ a.s. Hence we find

$$1 = \mathbf{P}(M^* < \infty) = \int_{E_\infty} \mathbf{P}(\varphi_0(X-w) < \infty) \mu(dw),$$

where μ is the distribution of X , and

$$\varphi_0(x) = \sup_n \varphi(x_1, \dots, x_n, 0, \dots) \quad \forall x = (x_j) \in E_\infty.$$

So for some $x \in E_\infty$ we have $\varphi_0(X-x) < \infty$ a.s. Let $y_n = (x_1, \dots, x_n, 0, \dots)$ and $Y_n = (X_1, \dots, X_n, 0, \dots)$, then we have

$$\varphi(y_n) \leq \varphi(Y_n - y_n) + \varphi(Y_n) \leq \varphi_0(X-x) + M_n.$$

Now we choose $a \geq 0$ so that $\mathbf{P}(M_n \leq a) > \frac{1}{2}$ and $\mathbf{P}(\varphi_0(X-x) \leq a) > \frac{1}{2}$, then for each $n \geq 1$ we have

$$\{M_n \leq a\} \cap \{\varphi_0(X-x) \leq a\} \neq \emptyset$$

and so $\varphi(y_n) \leq 2a$ for all $n \geq 1$. That is

$$M = \sup_n \varphi(Y_n) \leq \varphi_0(X-x) + 2a < \infty \text{ a.s.}$$

and the theorem is proved.

THEOREM 2.5. Suppose that $M < \infty$ a.s., and let g be an increasing function from $[t_0, \infty)$ into \mathbf{R}_+ so that

$$(2.5.1) \quad \lim_{t \rightarrow \infty} g(t) = \infty,$$

$$(2.5.2) \quad G(t) \leq K \exp(-g(t)) \quad \forall t \geq t_0,$$

for some constants $K > 0$ and $\varepsilon > 0$. Let $c > 1$ and put $\lambda = (\log c)/(\log 2)$, and $\nu = \lambda^{-1}$.

(A) If g satisfies:

$$(2.5.3) \quad \lim_{t \rightarrow \infty} (g(ct) - 2g(t)) = -\infty,$$

then there exists $K_1 > 0$, $\delta > 0$ and $t_1 \geq t_0$ so that

$$(2.5.4) \quad F(t) \leq K_1 \exp(-g(\delta t)) \quad \forall t \geq t_1 \quad \text{if } c > 2,$$

$$(2.5.5) \quad F(t) \leq K_1 \exp(-g(\delta t^c)) \quad \forall t \geq t_1 \quad \text{if } c < 2,$$

$$(2.5.6) \quad F(t) \leq K_1 \exp\left(-g\left(\frac{\delta t}{\log t}\right)\right) \quad \forall t \geq t_1 \quad \text{if } c = 2.$$

(B) If g satisfies

$$(2.5.7) \quad g(ct) \geq 2g(t) \quad \forall t \geq t_1,$$

then there exist $K_2 > 0$, $\delta > 0$ and $t_1 \geq t_0$ so that

$$(2.5.8) \quad F(t) \leq K_2 \exp(-\delta t) \quad \forall t \geq t_1 \quad \text{if } c < 2,$$

$$(2.5.9) \quad F(t) \leq K_2 \exp(-\delta t^c) \quad \forall t \geq t_1 \quad \text{if } c > 2,$$

$$(2.5.10) \quad F(t) \leq K_2 \exp\left(-\frac{\delta t}{\log t}\right) \quad \forall t \geq t_1 \quad \text{if } c = 2.$$

(C) If g satisfies

$$(2.5.11) \quad \int_{t_0}^{\infty} g(t)^{-1} dt < \infty,$$

then there exist $K_3 > 0$ and $\delta > 0$ so that

$$(2.5.12) \quad F(t) \leq K_3 \exp(-\delta t) \quad \forall t \geq t_1.$$

Proof. Let (X_n^*) be a symmetrization of (X_n) and let M^* and N^* be defined in the obvious way. Let $F^*(t) = P(M^* > t)$ and $G^*(t) = P(N^* > t)$, then we have

$$F^*(2t+s) \leq 2G^*(s) + 4F^*(t)^2 \quad \forall t, s \geq 0,$$

$$G^*(2s) \leq 2G(s),$$

$$F(t+a) \leq 2F^*(t),$$

where $a \geq 0$ is determined by $P(M \leq a) \geq \frac{1}{2}$. Now let

$$F_0(t) = 8F^*(t-a), \quad G_0(t) = 32G(2t),$$

then we have

$$(i) \quad F_0(2t+2s+3a) \leq \frac{1}{2}G_0(s) + \frac{1}{2}F_0(t)^2 \leq \max\{G_0(s), F_0(t)^2\},$$

$$(ii) \quad F(t) \leq 8F_0(t),$$

$$(iii) \quad G_0(t) \leq \exp(-g_0(t)) \quad \forall t \geq t_0 \text{ where } g_0(t) = g\left(\frac{1}{2}et\right) - \log(32K).$$

Now suppose that σ_n ($n \geq 0$) and τ_0 is given; then we define

$$\tau_n = 2^n \tau_0 + \sum_{j=0}^{n-1} 2^{n-j-1} (2\sigma_j + 3a) = 2\tau_{n-1} + 2\sigma_{n-1} + 3a.$$

Then we have

$$(iv) \quad \tau_n \leq K_1 2^n + 2^n \sum_{j=0}^{n-1} 2^{-j} \sigma_j \quad \forall n \geq 0,$$

$$(v) \quad F_0(\tau_n) \leq \max\{G_0(\sigma_{n-1}), G_0(\sigma_{n-2})^2, \dots, G_0(\sigma_0)^{2^{n-1}}, F_0(\tau_0)^{2^n}\}.$$

Case (A), $c \neq 2$. Now we choose $\sigma_0 \geq t_0$ from (2.5.3) so that

$$g(ct) \leq 2g(t) - 3\log(32K) \quad \forall t \geq 2e^{-1}\sigma_0,$$

then $g_0(ct) \leq 2g_0(t)$ for all $t \geq \sigma_0$. Then we put $\sigma_n = c^n \sigma_0$ for $n \geq 0$ and choose $\tau_0 \geq \sigma_0$, so that $\log F(\tau_0) \leq -\frac{1}{2}g_0(\sigma_0)$. Now since $2^{-j}g_0(\sigma_j)$ decreases in j we have

$$G_0(\sigma_j)^{2^{n-j-1}} \leq \exp(-2^{n-j-1}g_0(\sigma_j)) \leq \exp(-g_0(\sigma_{n-1}))$$

for all $0 \leq j \leq n-1$, and since $g_0(\sigma_{n-1}) \leq 2^{n-1}g_0(\sigma_0)$, we have

$$F_0(\tau_0)^{2^n} \leq \exp(-2^{n-1}g_0(\sigma_0)) \leq \exp(-g_0(\sigma_{n-1})).$$

So by (v) we find

$$F_0(\tau_n) \leq \exp(-g_0(c^{n-1}\sigma_0)) \quad \forall n \geq 0.$$

And from (iv) we find

$$\tau_n \leq 2^n K_1 + 2^n \sigma_0 \sum_{j=0}^{n-1} (\frac{1}{2}c)^j \leq K_2 d^n \quad \forall n \geq 0,$$

where $d = \max\{2, c\}$.

Now let $t_1 = K_2$, and $t \geq t_1$; then there exists an integer $n \geq 0$ so that $K_2 d^n \leq t \leq K_2 d^{n+1}$, and so

$$F_0(t) \leq F_0(\tau_n) \leq \exp(-g_0(c^{n-1}\sigma_0)).$$

However, $c = d^\gamma$, where $\gamma = \min\{1, \lambda\}$, and so

$$c^{n-1}\sigma_0 = c^{-2}\sigma_0 d^{\gamma(n+1)} \geq \alpha t^\gamma$$

where $\alpha = K_2^{-\gamma} c^{-2} \sigma_0 > 0$, and so

$$F(t) \leq 8F_0(t) \leq 8 \exp(-g_0(\alpha t^\gamma)) = 256 \cdot K \exp(-g(\delta t^\gamma))$$

where $\delta = \frac{1}{2}\epsilon\alpha$. This proves case (A) $c \neq 2$.

Case (A), $c = 2$. σ_0 and τ_0 is chosen as before and we find that

$$F_0(\tau_n) \leq \exp(-g_0(\sigma_{n-1})).$$

However, in this case we have

$$\tau_n \leq 2^n K_1 + n\sigma_0 2^n \leq n 2^n K_3 \quad \forall n \geq 1.$$

Now let $t_1 = 2K_3$, and let $t \geq t_1$; then there exists $n \geq 1$ so that $n 2^n K_3 \leq t \leq (n+1) 2^{n+1} K_3$, then

$$F_0(t) \leq F_0(\tau_n) \leq \exp(-g_0(2^{n-1}\sigma_0)).$$

Now we observe that

$$\frac{\beta t}{\log t} \leq \frac{\beta(n+1)2^{n+1}K_3}{(n+1) + \log(n+1) + \log K_3} \leq \beta 2^{n+1}K_3 = 2^{n-1}\sigma_0$$

if $\beta = K_3^{-1}2^{-2}\sigma_0$. Hence,

$$F(t) \leq 8F_0(t) \leq 8 \exp\left(-g_0\left(\frac{\beta t}{\log t}\right)\right) = 256 \cdot K \exp\left(-g\left(\frac{\delta t}{\log t}\right)\right),$$

where $\delta = \frac{1}{2}\varepsilon\beta$.

Case (B). In this case we choose $\sigma_n = c^n t_0$, and τ_0 so that $\log F(\tau_0) \leq \frac{1}{2}g_0(t_0)$. Now $g_0(ct) \geq 2g_0(t)$ for all $t \geq t_0$ by (2.5.7) and the definition of g_0 , hence we have that $2^{-j}g_0(\sigma_j)$ increases with j , and so

$$G_0(\sigma_j)^{2^{n-j-2}} \leq \exp(-2^{n-j-1}g_0(\sigma_0)) \leq \exp(-2^{n-1}g_0(t_0)),$$

$$F_0(\tau_0)^{2^n} \leq \exp(-2^{n-1}g_0(t_0)).$$

From (v) we find that

$$F_0(\tau_n) \leq \exp(-2^{n-1}g_0(t_0))$$

and as in case (A) we find

$$\tau_n \leq \begin{cases} 2^n K_4, & \text{if } c < 2, \\ n 2^n K_5, & \text{if } c = 2, \\ c^n K_6, & \text{if } c > 2. \end{cases}$$

Case (B) now follows by an argument similar to the proof of case (A).

Case (C). Let σ_n be defined by

$$\sigma_n = \inf\{t \geq t_0 \mid g_0(t) \geq 2^n g_0(t_0)\} \quad \forall n \geq 0;$$

then $g_0(\sigma_n) = 2^n g_0(t_0)$, and we have

$$G_0(\sigma_j)^{2^{n-j-1}} \leq \exp(-2^{n-j-1}g_0(\sigma_j)) = \exp(-2^{n-1}g_0(t_0)).$$

So if τ_0 is chosen so that $\log F(\tau_0) \leq -\frac{1}{2}g_0(t_0)$, then we have by (v)

$$F_0(\tau_n) \leq \exp(-2^{n-1}g_0(t_0)).$$

Now let $b_j = \sigma_j g_0(\sigma_j)^{-1} = k \sigma_j 2^{-j}$, then we have

$$a_j = \int_{\sigma_{j-1}}^{\sigma_j} g_0(t)^{-1} dt \geq (\sigma_j - \sigma_{j-1}) g_0(\sigma_j)^{-1} = b_j - \frac{1}{2} b_{j-1},$$

$$b \leq \frac{1}{2} b_{j-1} + a_j \leq \left(\frac{1}{2}\right)^j b_0 + \sum_{i=1}^j a_i \left(\frac{1}{2}\right)^{j-i}.$$

So using (2.5.11) we find that

$$\begin{aligned} \tau_n &\leq K_1 2^n + 2^n \sum_{j=0}^{n-1} 2^{-j} \sigma_j \leq K_1 2^n + 2^n \left(\sigma_0 + \sum_{j=1}^{\infty} k^{-1} b_j\right) \\ &\leq K_1 2^n + 2^n \left(\sigma_0 + b_0 + \sum_{j=1}^{\infty} \sum_{i=1}^j k^{-1} a_i \left(\frac{1}{2}\right)^{j-i}\right) \\ &= 2^n \left(K_1 + b_0 + \sigma_0 + k^{-1} \sum_{i=1}^{\infty} a_i\right) \\ &\leq 2^n \left(K_1 + b_0 + \sigma_0 + k^{-1} \int_{t_0}^{\infty} g_0(t)^{-1} dt\right) = K_7 2^n \end{aligned}$$

from which (2.5.11) follows as above.

EXAMPLES. (a) $G(t) \leq K_1 f(t)^{-1}$, where f satisfies

$$(2.4) \quad f \text{ is increasing on } [t_0, \infty) \text{ and } \lim_{t \rightarrow \infty} f(t) = \infty.$$

$$(2.5) \quad f(2t) \leq M f(t) \quad \forall t \geq t_0, \quad \text{for some } M > 0,$$

then $g(t) = \log f(t)$ satisfies (2.5.3) for all $c > 1$, and so we find

$$(2.6) \quad F(t) \leq K_2 f(t)^{-1}.$$

Note that $f(t) = t^p$ satisfies (2.4) and (2.5) for all $p > 0$.

A simple computation, using integration by parts, shows that if f satisfies (2.4) and (2.5), then we have

$$(2.7) \quad \mathbf{E}f(N) < \infty \text{ implies } \mathbf{E}f(M) < \infty.$$

(b) $G(t) \leq K_1 \exp(-\varepsilon_1 t^p)$, where $p > 0$, then $g(t) = t^p$ satisfies (2.5.7) for $c = 2^{1/p}$, hence we have

$$(2.8) \quad F(t) \leq K_2 \exp(-\varepsilon_2 t) \quad \text{if } p > 1,$$

$$(2.9) \quad F(t) \leq K_2 \exp(-\varepsilon_2 t^p) \quad \text{if } p < 1,$$

$$(2.10) \quad F(t) \leq K_2 \exp(-\varepsilon_2 t(\log t)^{-1}) \quad \text{if } p = 1.$$

(c) $G(t) \leq K_1 \exp(-\varepsilon_1 t(\log t)^p)$ for some $p > 1$, then $g(t) = t(\log t)^p$ satisfies (2.5.11), and so we have

$$(2.11) \quad F(t) \leq K_2 \exp(-\varepsilon_2 t).$$

3. The zero-one law for product measures. Let μ_n be a probability measure on (R, B) , and let

$$R^\infty = \prod_{n=1}^{\infty} R, \quad B^\infty = \otimes_{n=1}^{\infty} B, \quad \mu = \prod_{n=1}^{\infty} \mu_n.$$

A set $A \subseteq R^\infty$ is called a tail event if A satisfies

(3.1) If $x \in \mathbf{R}^\infty$ and $x_j = y_j$ for all $j \geq n$, for some $y \in A$ and some $n \geq 1$, then $X \in A$.

If \mathbf{R}_0^∞ is the set of $x = (x_j) \in \mathbf{R}^\infty$ with almost finitely many non-zero coordinates, then it is easily checked that (3.1) is equivalent to

$$(3.2) \quad \mathbf{R}_0^\infty + A \subseteq A.$$

It is well known that $\mu(A)$ is either 0 or 1 if A is a μ -measurable tail event. We shall in this section show that the 0-1 law also holds for linear subspaces of \mathbf{R}^∞ .

THEOREM 3.1. *Suppose that μ_n is non-atomic for all $n \geq 1$, and let L be a μ -measurable linear subset of \mathbf{R}^∞ . Then either $\mu(L) = 0$ or $\mu(L) = 1$ and in the latter case we have $\mathbf{R}_0^\infty \subseteq L$.*

Proof. Since μ is a Radon measure on \mathbf{R}^∞ , we can find a σ -compact linear subset $L_0 \subseteq L$, so that $\mu(L_0) = \mu(L)$, so we may as well assume that L itself is σ -compact.

Now suppose that $\mu(L) > 0$, and that $\mathbf{R}_0^\infty \not\subseteq L$. Then for some $n \geq 1$ we have $e_n \notin L$, where e_n is the n th unit vector in \mathbf{R}^∞ . Let

$$F = L \cap \mathbf{R}_0^\infty$$

and let G be a linear complement to F in \mathbf{R}_0^∞ , so that $e_n \in G$. Then G is σ -compact (note that all linear subsets of \mathbf{R}_0^∞ are σ -compact), and so $L_1 = L + G$ is σ -compact. Moreover, we have

$$L_1 \supseteq F + G = \mathbf{R}_0^\infty, \quad \mu(L_1) \geq \mu(L) > 0.$$

Since L_1 is linear, we find that L_1 is a tail event, and so $\mu(L_1) = 1$. Now we observe that $L \cap G = (0)$, and so

$$(x, y) \mapsto x + y$$

is a continuous bijection from $L \times G$ to $L + G$, hence, by Theorem III.8.4 in [4], we see that there exist linear Borel maps p and q so that

$$p: L + G \rightarrow L, \quad q: L + G \rightarrow G,$$

$$x = p(x) + q(x) \quad \forall x \in L_1.$$

Let π_n be the projection of \mathbf{R}^∞ onto the n th coordinate, and put $p_0 = \pi_n \circ p$ and $q_0 = \pi_n \circ q$. Then p_0 and q_0 are Borel measurable linear functionals which are defined μ -a.e., and we have

$$\begin{aligned} q_0(x) &= 0 & \text{for } x \in L, \\ q_0(e_n) &= 1, & p_0(e_n) = 0. \end{aligned}$$

Hence we have

$$p_0(x) = p_0(x_1, \dots, x_{n-1}, 0, x_{n+1}, \dots) \quad \forall x = (x_j) \in L_1$$

so p_0 and π_n are independent and $p_0 + q_0 = \pi_n$ a.s. Hence if ν is the distribution law of p_0 , we have

$$\begin{aligned} \mu(L) &\leq \mu(q_0 = 0) = \mu(p_0 = \pi_n) \\ &= \int_{-\infty}^{\infty} \mu(\pi_n = x) \nu(dx) = \int_{-\infty}^{\infty} \mu_n(\{x\}) \nu(dx) = 0 \end{aligned}$$

since μ_n is non-atomic. So we have derived a contradiction, and we must have that $\mathbf{R}_0^\infty \subseteq L$ whenever $\mu(L) > 0$. However, since L is linear, we have that $\mathbf{R}_0^\infty \subseteq L$ implies that L is a tail event and so $\mu(L) = 1$.

4. The affine kernel of a product measure. Let $\mu = \prod_{n=1}^{\infty} \mu_n$ be a product probability measure on \mathbf{R}^∞ , then the affine kernel of μ , $A(\mu)$, is defined by

$$A(\mu) = \bigcap \{A \mid A \text{ } \mu\text{-measurable affine set with } \mu(A) = 1\}.$$

The measurable affine subsets of \mathbf{R}^∞ with full measure are often too badly behaved, and we shall mainly restrict ourselves to consider only those affine sets, $A \subseteq \mathbf{R}^\infty$ which satisfy

$$(4.1) \quad \forall \varepsilon > 0 \exists K \text{ compact convex, so that } K \subseteq A \text{ and } \mu(K) > 1 - \varepsilon.$$

An affine space satisfying (4.1) will be called a μ -Lusin affine space, and we define the Lusin affine kernel of μ , $A_L(\mu)$, by

$$A_L(\mu) = \bigcap \{A \mid A \text{ is a } \mu\text{-Lusin affine space}\}.$$

Let $\Lambda(\mu)$ denote the set of μ -measurable linear functions, $f: \mathbf{R}^\infty \rightarrow \mathbf{R}$. Again these functions are often too badly behaved, and we shall call $f \in \Lambda(\mu)$ a μ -Lusin functional, if f satisfies

$$(4.2) \quad \forall \varepsilon > 0 \exists K \text{ compact convex, so that } \mu(K) > 1 - \varepsilon \text{ and } f|_K \text{ is continuous.}$$

And we denote the set of μ -Lusin functionals by $\Lambda_L(\mu)$. The space $\Lambda(\mu)$ and $\Lambda_L(\mu)$ may be considered as subspaces of $L^0(\mu)$, where $L^0(\mu)$ is the space of all μ -measurable functions, X , equipped with the metric

$$\|X\|_\mu = \int_{\mathbf{R}^\infty} \varphi(|x|) d\mu$$

where $\varphi(x) = x/(1+x)$. It is well known (see Theorem 1 in [10]) that

$$(4.3) \quad \Lambda(\mu) \text{ and } \Lambda_L(\mu) \text{ are closed subspaces of } L^0(\mu) \text{ and } \Lambda_L(\mu) = \text{cl}(\mathbf{R}_0^\infty) \text{ (closure in } L^0(\mu)), \text{ where } \mathbf{R}_0^\infty \text{ is identified with the dual of } \mathbf{R}^\infty.$$

The main results of this section state under certain conditions on μ that every $f \in \mathcal{A}_L(\mu)$ admit a representation of the form

$$f(x) = \sum_{j=1}^{\infty} a_j x_j,$$

where $a = (a_j)$ belongs to the set:

$$C(\mu) = \left\{ (a_j) \mid \sum_{j=1}^{\infty} a_j x_j \text{ converges } \mu\text{-a.e.} \right\}$$

and that the Lusin affine kernel $\mathcal{A}_L(\mu)$, is equal to the sequence dual, $C^*(\mu)$, of $C(\mu)$, that is

$$C^*(\mu) = \left\{ (b_j) \in \mathbf{R}^{\infty} \mid \sum_{j=1}^{\infty} a_j b_j \text{ converges } \forall a \in C(\mu) \right\}.$$

The condition, which assures the validity of these two results, is that $0 \in \mathcal{A}_L(\mu)$, and we shall say that μ is centered at 0, if $0 \in \mathcal{A}_L(\mu)$.

If $a \in C(\mu)$, then we define

$$|a|_{\mu} = \sup_{\mathbf{R}^{\infty}} \int \varphi \left(\left| \sum_{j=1}^n a_j x_j \right| \right) \mu(dx).$$

A slight modification of the proof of Theorem 5.2 in [2] shows that, if μ_n is non-degenerated for all $n \geq 1$, then

$$(4.4) \quad (C(\mu), |\cdot|_{\mu}) \text{ is a Fréchet space (in general non-locally convex).}$$

THEOREM 4.1. *If μ_n is non-degenerated for all $n \geq 1$, and A is a μ -measurable affine set with $\mu(A) = 1$, then $A + \mathbf{R}_0^{\infty} \subseteq A$.*

Proof. Let μ^* be the symmetrization of μ . Then

$$\mu^*(B) = \int_{\mathbf{R}^{\infty}} \mu(B+x) \mu(dx) \quad \forall B \in \mathbf{B}^{\infty},$$

and, in particular, we have

$$\mu^*(A - x_0) = \int_{\mathbf{R}^{\infty}} \mu(A - x_0 + x) \mu(dx) = \int_A \mu(A - x_0 + x) \mu(dx).$$

So if $x_0 \in A$, then $A - x_0 + x = A$ for all $x \in A$, and so

$$\mu^*(A - x_0) = 1 \quad \forall x_0 \in A.$$

Let $L = A - x_0$, where $x_0 \in A$, then L is a linear set and $\mu^*(L) = 1$. Let Σ be the set of linear transformations, $\sigma: \mathbf{R}^{\infty} \rightarrow \mathbf{R}^{\infty}$, of the form

$$\sigma x = (\varepsilon_n x_n) \quad \text{for } x = (x_n) \in \mathbf{R}^{\infty}$$

where $\varepsilon_n = \pm 1$ and $\varepsilon_n = 1$ for all $n \geq N$ for some $N \geq 1$. Then Σ is a countable group and

$$\mu^*(\sigma(B)) = \mu^*(B) \quad \forall B \in \mathbf{B}^{\infty} \quad \forall \sigma \in \Sigma.$$

So if $L_0 = \bigcap_{\sigma \in \Sigma} \sigma(L)$, then $\mu^*(L_0) = 1$. Let $n \geq 1$, now since μ_n is non-degenerated we can find $y \in L_0$ so that $y_n \neq 0$. Now let σ be defined by

$$\sigma x = (x_1, \dots, x_{n-1}, -x_n, x_{n+1}, \dots);$$

then $\sigma \in \Sigma$ and since $\sigma(L_0) = L_0$, we have that $\sigma y \in L_0$. Moreover, since L_0 is linear, we have

$$e_n = (2y_n)^{-1}(y - \sigma y) \in L_0,$$

where e_n is the n th unit vector. Hence we have

$$\mathbf{R}_0^{\infty} \subseteq L_0 \subseteq L = A - x_0 \quad \forall x_0 \in A$$

from which we find that $\mathbf{R}_0^{\infty} + A \subseteq A$.

THEOREM 4.2. *Suppose that μ_n is non-degenerated for all $n \geq 1$ and let μ^* be the symmetrization of μ , that is*

$$\mu^*(B) = \int_{\mathbf{R}^{\infty}} \mu(B+x) \mu(dx) \quad \forall B \in \mathbf{B}^{\infty}.$$

Then μ is centered at 0 if and only if $C(\mu) = C(\mu^)$.*

Proof. Suppose that μ is centered at 0, and let $a \in C(\mu^*)$. Let

$$L = \left\{ (x_j) \in \mathbf{R}^{\infty} \mid \sum_{j=1}^{\infty} a_j x_j \text{ converges} \right\};$$

then L is a linear Borel subset of \mathbf{R}^{∞} with $\mu^*(L) = 1$, that is

$$1 = \mu^*(L) = \int_{\mathbf{R}^{\infty}} \mu(L+x) \mu(dx)$$

and so $\mu(M) = 1$, where $M = \{x \mid \mu(L+x) = 1\}$. Now let $b = (b_j) \in M$, then

$$L + b = \left\{ (x_j) \in \mathbf{R}^{\infty} \mid \sum_{j=1}^{\infty} a_j (x_j - b_j) \text{ converges} \right\},$$

$$\mu(L + b) = 1.$$

Let $\varepsilon > 0$ be given, then by Egoroff's theorem there exists $B \in \mathbf{B}^{\infty}$ and $n_1 < n_2 < \dots$ so that

$$\left| \sum_{j=n+1}^{n+m} a_j (x_j - b_j) \right| \leq 2^{-k} \quad \forall n \geq n_k \quad \forall m, k \geq 1 \quad \forall x \in B,$$

$$\mu(B) > 1 - \varepsilon.$$

Now let

$$C = \bigcap_{k=1}^{\infty} \bigcap_{n=n_k}^{\infty} \bigcap_{m=1}^{\infty} \{x \mid \sum_{j=n+1}^{n+m} a_j(x_j - b_j) \mid \leq 2^{-k}\};$$

then $B \subseteq C \subseteq L+b$, and C is convex and closed. Now we choose a compact set $D \subseteq B$ so that $\mu(D) > 1 - \varepsilon$, and put K equal to the closed convex hull of D . Then $D \subseteq K \subseteq C \subseteq L+b$, since C is convex and closed, and $\mu(K) > 1 - \varepsilon$. This shows that $L+b$ is a μ -Lusin affine space and so by assumption $0 \in L+b$ for all $b \in M$, or $M \subseteq L$. Hence $\mu(M) = \mu(L) = 1$, and $a \in C(\mu)$ which proves that $C(\mu^*) \subseteq C(\mu)$, the other inclusion is trivial.

Now suppose that $C(\mu^*) = C(\mu)$, then $|\cdot|_{\mu}$ and $|\cdot|_{\mu^*}$ are Fréchet metrics on $C(\mu)$, so that

$$|a|_{\mu^*} \leq 2 |a|_{\mu} \quad \forall a \in C(\mu).$$

Hence the two metrics are equivalent, and so we have

(i) $\forall \varepsilon > 0 \exists \delta > 0$ so that $|a|_{\mu^*} \leq \delta$ implies $|a|_{\mu} \leq \varepsilon$.

Now we consider the linear operator, $T: C(\mu) \rightarrow L^0(\mu)$, defined by

$$Ta = \sum_{j=1}^{\infty} a_j x_j,$$

then $\|Ta\|_{\mu} \leq |a|_{\mu}$, so T is continuous. Let $\varepsilon > 0$ be given and choose $\delta > 0$ according to (i), if $a \in C(\mu)$ and $\|2Ta\|_{\mu} < \delta$, then by Theorem 2.6 in [3], we have

$$\int_{\mathbf{R}^{\infty}} \varphi \left(\left| \sum_{j=1}^n a_j x_j \right| \right) \mu^*(dx) \leq \frac{1}{2} \|2Ta\|_{\mu^*} \leq \|2Ta\|_{\mu} \leq \delta$$

for all $n \geq 1$. Hence $|a|_{\mu^*} \leq \delta$ and so $|a|_{\mu} \leq \varepsilon$, that is $\|2Ta\|_{\mu} \leq \delta$ implies that $|a|_{\mu} \leq \varepsilon$. This shows that T is an isomorphism and $T(C(\mu))$ is a closed subspace of $L^0(\mu)$. So by (4.3) we have

(ii) $A_L(\mu) \subseteq T(C(\mu))$.

Let A be a μ -Lusin affine space, and suppose that $0 \notin A$, then there exists a linear functional, f , so that $f(x) = 1$ for $x \in A$. Obviously, we have that $f \in A_L(\mu)$, and so by (ii) we have $f = Ta$ for some $a \in C(\mu)$. That is

$$\sum_{j=1}^{\infty} a_j x_j = 1 \quad \mu\text{-a.e.}$$

and so $a \neq 0$. Let $n \geq 1$ be chosen, so that $a_n \neq 0$, then the random variables (defined on $(\mathbf{R}^{\infty}, \mathbf{B}^{\infty}, \mu)$)

$$X_n(x) = a_n x_n, \quad Y_n = \sum_{j \neq n} a_j x_j,$$

are independent and $X_n + Y_n = 1$ a.s., but this implies that X_n is degenerated contrary to the assumption that μ_n is non-degenerated. Hence $0 \in A$ and μ is centered at 0.

THEOREM 4.3. *Suppose that μ is centered at 0, and μ_n is non-degenerated for all $n \geq 1$. Then there exists a linear functional, $\lambda: A(\mu) \rightarrow \mathbf{R}$, so that*

$$(4.2.1) \quad \lambda(f) = \lim_{n \rightarrow \infty} f(0, \dots, 0, x_n, x_{n+1}, \dots) \quad \text{for a.a. } x \in \mathbf{R}^{\infty};$$

$$(4.2.2) \quad (f(e_j)) \in C(\mu) \quad \text{and} \quad f(x) = \lambda(f) + \sum_{j=1}^{\infty} x_j f(e_j) \quad \text{a.e.};$$

$$(4.2.3) \quad \lambda(f) = 0 \quad \forall f \in A_L(\mu);$$

$$(4.2.4) \quad \lambda = 0 \quad \text{if } \mu_n \text{ is symmetric for all } n \geq 1.$$

Proof. Let $f \in A(\mu)$ and define $a_j = f(e_j)$ and

$$X_j(x) = a_j x_j,$$

$$Y_n(x) = f(0, \dots, 0, x_{n+1}, x_{n+2}, \dots),$$

$$Z_n(x) = \sum_{j=1}^n x_j f(e_j) = \sum_{j=1}^n X_j(x) = f(x) - Y_n(x),$$

for $x = (x_j) \in \mathbf{R}^{\infty}$. Then X_1, \dots, X_n, Y_n are independent as random variables on $(\mathbf{R}^{\infty}, \mathbf{B}^{\infty}, \mu)$ so if φ, φ_n and ψ_n denote the characteristic functions of f, X_n and Y_n , we have

$$\varphi(t) = \psi_n(t) \prod_{j=1}^n \varphi_j(t)$$

and so

$$|\varphi(t)| \leq \lim_{n \rightarrow \infty} \prod_{j=1}^n |\varphi_j(t)|.$$

Hence by Corollary 2, p. 251, and Theorem b, p. 250 in [7], we find that $(f(e_j)) = a \in C(\mu^*)$, and so by Theorem 4.2 we have $a \in C(\mu)$. That is,

$$\lambda_0(f) = \lim_{n \rightarrow \infty} Y_n$$

exists a.e., and $\lambda_0(f)$ is measurable with respect to the tail σ -algebra, that is $\lambda_0(f)$ is constant a.s. Hence there exists $\lambda(f) \in \mathbf{R}$ so that

$$\lambda(f) = \lim_{n \rightarrow \infty} f(0, \dots, 0, x_n, x_{n+1}, \dots) \quad \text{a.e.}$$

and, clearly, λ is linear on $A(\mu)$, and we have

$$f(x) = \lambda(f) + \sum_{j=1}^{\infty} x_j f(e_j) \quad \text{a.e.}$$

Now suppose that $f \in A_L(\mu)$, and let

$$L = \left\{ x \mid \sum_{j=1}^{\infty} a_j f(e_j) \text{ converges} \right\},$$

$$L_0 = \left\{ x \in L \mid f(x) - \sum_{j=1}^{\infty} a_j f(e_j) = \lambda(f) \right\};$$

then an argument similar to the first part of the proof of Theorem 4.2 shows that L_0 is a μ -Lusin affine space and so by assumption we have $0 \in L$. That is $\lambda(f) = 0$.

Now suppose that μ_n is symmetric for all $n > 1$, and let $f \in A(\mu)$. If we define

$$g(x) = f(x) - \sum_{j=1}^{\infty} a_j f(e_j),$$

then g and $-g$ has the same distribution law, and $g = \lambda(f)$ a.e. Hence $\lambda(f) = 0$.

THEOREM 4.4. *Suppose that μ is centered at 0, and μ_n is non-degenerated for all $n \geq 1$. Then we have that $C^*(\mu) = A_L(\mu)$.*

Proof. Let $b \in C^*(\mu)$ and let A be a μ -Lusin affine space. Then A is linear, since μ is centered at 0, and we can find compact convex symmetric sets, K_n , so that

$$K_n \subseteq A, \quad K_n + K_n \subseteq K_{n+1}, \quad \mu(K_n) > 1 - 2^{-n}.$$

Let $L = \bigcup_{n=1}^{\infty} K_n$; then L is a linear space $L \subseteq A$ and $\mu(L) = 1$. Now let

$$F = \{ f \mid f \text{ linear: } L \rightarrow \mathbf{R}, f|_{K_n} \text{ is continuous } \forall n \},$$

$$\|f\|_n = \sup_{x \in K_n} |f(x)| \quad \forall f \in F,$$

$$\varrho(f, g) = \sum_{n=1}^{\infty} 2^{-n} \|f - g\|_n \quad \forall f, g \in F;$$

then it is easily seen that (F, ϱ) is a locally convex Fréchet space. Moreover, it is clear that the ϱ -topology is stronger than the $\sigma(F, L)$ -topology (note that (F, L) is a dual pair).

If $y \notin K_n$, then there exists $f \in (\mathbf{R}^{\infty})'$, so that $f(y) > 1$ and $|f(x)| \leq 1$ for all $x \in K_n$, that is $f \in K_n^0$, where K_n^0 is the polar of K_n in F . Since $f(y) > 1$, this shows that $y \notin K_n^{00}$, where K_n^{00} is the polar of K_n^0 in L . Hence $K_n^{00} \subseteq K_n$, and since the converse inclusion is obvious, we have $K_n = K_n^{00}$. So K_n is $\sigma(L, F)$ -compact and convex.

Since the ϱ -topology is the topology of uniform convergence on each K_n , we find that the ϱ -topology is weaker than the Mackey topology: $\tau(F, L)$. Hence we have

$$(i) \quad (F, \varrho)' = L.$$

Now let $f \in F$ and define

$$S_n f = \sum_{j=1}^n b_j f(e_j);$$

then S_n is a linear functional on F , and S_n is continuous since $e_j \in L$ for all $j \geq 1$ by Theorem 4.1. Moreover, we have

$$\lim_{n \rightarrow \infty} S_n f = S f = \sum_{j=1}^{\infty} b_j f(e_j)$$

since $(f(e_j)) \in C(\mu)$ by Theorem 4.3 and $b \in C^*(\mu)$. But (F, ϱ) is a Fréchet space and so $S \in F'$. So by (i) there exist $x \in L$ with

$$\sum_{j=1}^{\infty} b_j f(e_j) = f(x) \quad \forall f \in F.$$

Now the functional $f_j(x) = x_j$ for $x \in L$ belongs to F and so $b_j = x_j$ for all $j \geq 1$. Hence $b = x \in L \subseteq A$ and so $C^*(\mu) \subseteq A_L(\mu)$.

Now suppose that $b \notin C^*(\mu)$, then there exists $a \in C(\mu)$ so that $\sum_{j=1}^{\infty} a_j b_j$ diverges. Let

$$L = \left\{ x \in \mathbf{R}^{\infty} \mid \sum_{j=1}^{\infty} a_j x_j \text{ converges} \right\};$$

then as in the proof of Theorem 4.2 one finds that L is a μ -Lusin affine space, and since $b \notin L$, we have $b \notin A_L(\mu)$, and so $C^*(\mu) \not\subseteq A_L(\mu)$.

THEOREM 4.5. *If μ_n is symmetric for all $n \geq 1$, then μ is centered at 0, and actually we have $0 \in A(\mu)$.*

Proof. Let A be a μ -measurable affine set, then $\mu(A) = \mu(-A)$, so if $\mu(A) = 1$, then $A \cap (-A) \neq \emptyset$ and so $0 \in A$.

LEMMA 4.6. *Let μ_n^* be the symmetrization of μ_n ; then the following five statements are equivalent:*

$$(4.6.1) \quad \exists a > 0, \text{ so that } \sup_n \sup_{x \in \mathbf{R}} \mu_n^*([x-a, x+a]) < 1;$$

$$(4.6.2) \quad \exists a > 0, \exists m_n \text{ a median for } \mu_n, \text{ so that } \sup_n \mu_n^*([m_n - a, m_n + a]) < 1;$$

$$(4.6.3) \quad \exists a > 0, \text{ so that } \sup_n \mu_n^*([-a, a]) < 1;$$

$$(4.6.4) \quad \delta_0 \text{ is not a weak limit point of } \{\mu_n^*\};$$

$$(4.6.5) \quad \inf_n \int_{-\infty}^{\infty} \frac{|x|}{1+|x|} \mu_n^*(dx) > 0.$$

Remark. If $\{\mu_n\}$ satisfies one of the five statements above, we shall say that μ is *totally non-degenerated*. Note that μ is totally non-degenerated if and only if μ^* is totally non-degenerated. Condition (4.6.3) appears in [8] as condition (4.2).

Proof. From Theorem A, p. 245 in [7], we have

$$\mu_n(x \mid |x - m_n| > a) \leq 2\mu_n^*(x \mid |x| > a) \leq 4\mu_n(x \mid |x - b| > \frac{1}{2}a)$$

if m_n is a median of μ_n . Hence it follows that (4.6.1), (4.6.2) and (4.6.3) are equivalent. It is obvious that (4.6.3) implies (4.6.4), and (4.6.4) implies (4.6.5).

Now suppose that (4.6.5) holds, then we have

$$b = \inf_n \int_{-\infty}^{\infty} \frac{|x|}{1+|x|} \mu_n^*(dx) > 0$$

and if $a = b(2-b)^{-1}$, then $a > 0$ and

$$b \leq \mu_n^*(x \mid |x| > a) + \frac{a}{1+a} = \frac{1}{2}b + \mu_n^*(x \mid |x| > a)$$

and so $\mu_n^*([-a, a]) \leq 1 - \frac{1}{2}b$. Hence (4.6.5) implies (4.6.3) and the lemma is proved.

LEMMA 4.7. Suppose that μ_n satisfies

$$(4.7.1) \quad \int_{-\infty}^{\infty} x \mu_n(dx) = 0 \quad \forall n,$$

$$(4.7.2) \quad \inf_n \int_{-\infty}^{\infty} |x| \mu_n(dx) > 0,$$

$$(4.7.3) \quad \sup_n \int_{-\infty}^{\infty} |x|^p \mu_n(dx) < \infty,$$

for some $p > 1$, then μ is totally non-degenerated.

Proof. Let $1 < q < \infty$ be taken so that $1/p + 1/q = 1$, then a slight modification of the proof of Inequality II, p. 6, in [6], shows that

(i) $\mathbf{P}(X \geq \lambda \mathbf{E}X) \geq (1 - \lambda)^q (\mathbf{E}X)^q (\mathbf{E}X^p)^{-q/p}$ whenever X is a non-negative random variable with finite p th moment and $0 \leq \lambda \leq 1$.

Let X_n be a random variable with distribution law μ_n , and let $X_n^* = X_n' - X_n''$ be a symmetrization of X_n . Then by Theorem 2.6 in [3], we have

$$\begin{aligned} \mathbf{E}|X_n| &\leq \mathbf{E}|X_n^*|, \\ \mathbf{E}|X_n^*|^p &\leq 2^p \mathbf{E}|X_n|. \end{aligned}$$

Now let $a = \inf_n \mathbf{E}|X_n|$ and $A = \sup_n \mathbf{E}|X_n|^p$, then $a > 0$ and $A < \infty$ by (4.7.2) and (4.7.3), hence by (i) we have

$$\begin{aligned} \mathbf{P}(|X_n^*| > \frac{1}{2}a) &\geq \mathbf{P}(|X_n^*| \geq \frac{1}{2} \mathbf{E}|X_n^*|) \\ &\geq 2^{-q} (\mathbf{E}|X_n^*|)^q (\mathbf{E}|X_n^*|^p)^{-q/p} \geq 2^{-2q} a^q A^{-q/p} \end{aligned}$$

and so (4.6.3) is satisfied.

THEOREM 4.8. If μ is totally non-degenerated, then $C(\mu) \subseteq l^2$ (here l^2 is the space of square-summable sequences).

Proof. We have $C(\mu) \subseteq C(\mu^*)$, where μ^* is a symmetrization of μ . Let $\varepsilon > 0$ be chosen so that $\mu_n^*([- \varepsilon, \varepsilon]) \leq 1 - \varepsilon$. Now we can find Borel functions f_n from $I = [0, 1]$ into \mathbf{R} so that

$$\lambda(f_n^{-1}(A)) = \mu_n^*(A) \quad \forall n \geq 1 \quad \forall A \in \mathbf{B}(\mathbf{R}),$$

where λ is the Lebesgue measure on I . Let $\Omega = I^\infty$ and $\mathbf{P} = \prod_1^\infty \lambda$, and let

$$X_n(\omega) = f_n(\omega_n) \quad \text{for } \omega = (\omega_n) \in \Omega.$$

Then X_1, X_2, \dots are independent random variables and X_n has distribution law μ_n^* . Since we have that

$$\lambda(|f_n| > \varepsilon) \geq \varepsilon$$

we can find $a_n \geq \varepsilon$ so that

$$\lambda(|f_n| > a_n) \leq \varepsilon \leq \lambda(|f_n| \geq a_n)$$

and so there exist a Borel set A_n so that $\lambda(A_n) = \varepsilon$ and

$$\{|f_n| > a_n\} \subseteq A_n \subseteq \{|f_n| \geq a_n\}.$$

Now let

$$Y_n(\omega) = \begin{cases} 1 & \text{if } \omega_n \in A_n, \\ 0 & \text{if } \omega_n \notin A_n; \end{cases}$$

then Y_1, Y_2, \dots are independent identically distributed random variables, so that

$$\mathbf{P}(Y_n = 1) = \varepsilon, \quad \mathbf{P}(Y_n = 0) = 1 - \varepsilon.$$

Moreover, we have $0 \leq Y_n \leq a_n^{-1} |X_n| \leq \varepsilon^{-1} |X_n|$.

Now let $b = (b_j) \in C(\mu^*)$; then by Proposition 2.8 in [3] we have that

$$\sum_{j=1}^{\infty} b_j^2 X_j(\omega)^2 < \infty \text{ a.s.}$$

and so

$$\sum_{j=1}^{\infty} b_j^2 Y_j^2 < \infty \text{ a.s.}$$

Since $b_j X_j \rightarrow 0$ a.s. and μ is totally non-degenerated, we must have that $\{b_j\}$ is bounded. Hence

$$\sup_j |b_j^2 Y_j|^2 \leq \sup_j |b_j|^2 < \infty.$$

So from Corollary 3.3 in [3] we find that

$$\mathbf{E} \left(\sum_{j=1}^{\infty} b_j^2 Y_j^2 \right) = \varepsilon \sum_{j=1}^{\infty} b_j^2 < \infty$$

and so $b \in \ell^2$. Hence $C(\mu) \subseteq C(\mu^*) \subseteq \ell^2$.

THEOREM 4.9. *Suppose that μ satisfies*

$$(4.9.1) \quad \int_{-\infty}^{\infty} x \mu_n(dx) = 0 \quad \forall n \geq 1,$$

$$(4.9.2) \quad \sup_n \int_{-\infty}^{\infty} x^2 \mu_n(dx) < \infty;$$

then $\ell^2 \subseteq C(\mu)$ and $A_L(\mu) \subseteq \ell^2$.

Proof. Let $a = (a_j) \in \ell^2$, then

$$\mathbf{E} \left| \sum_{j=n+1}^{n+m} a_j X_j \right|^2 \leq \sup_j \mathbf{E} |X_j|^2 \sum_{j=n+1}^{n+m} |a_j|^2$$

if X_1, X_2, \dots are independent and X_j has distribution law μ_j . Hence $\sum_{j=1}^{\infty} a_j X_j$ converges in L^2 and so $a \in C(\mu)$. That is $\ell^2 \subseteq C(\mu)$.

It is easily seen that $A_L(\mu) \subseteq C^*(\mu)$, and since $\ell^2 \subseteq C(\mu)$ we have $C^*(\mu) \subseteq \ell^2$, and so the theorem is proved.

THEOREM 4.10. *Suppose that μ satisfies*

$$(4.10.1) \quad \int_{-\infty}^{\infty} x \mu_n(dx) = 0 \quad \forall n,$$

$$(4.10.2) \quad \sup_n \int_{-\infty}^{\infty} x^2 \mu_n(dx) < \infty,$$

$$(4.10.3) \quad \mu \text{ is totally non-degenerated,}$$

then μ is centered at 0 and we have

$$(4.10.4) \quad \Lambda(\mu) \subseteq \ell^2(\mu),$$

$$(4.10.5) \quad C(\mu) = C(\mu^*) = C^*(\mu) = A_L(\mu) = \ell^2.$$

Remark. Note that under (4.10.1) and (4.10.2) we have that

$$(4.10.3)^* \quad \inf_{-\infty}^{\infty} \int |x| \mu_n(dx) > 0$$

implies (4.10.3) by Lemma 4.7.

Proof. By Theorem 4.8 and Theorem 4.10, we have $C(\mu) = C(\mu^*) = \ell^2$ and so μ is centered at 0. Moreover, it is clear that

$$\sum_{j=1}^{\infty} a_j x_j \in L^2(\mu) \quad \forall a = (a_j) \in \ell^2.$$

Hence by Theorem 4.3 we have that $\Lambda(\mu) \subseteq L^2(\mu)$, and from Theorem 4.4 it follows that $A_L(\mu) = C^*(\mu) = \ell^2$.

THEOREM 4.11. *Let g be a non-negative function on \mathbf{R}_+ and $p \geq 1$. If we have*

$$(4.11.1) \quad \int_{-\infty}^{\infty} x \mu_n(dx) = 0 \quad \forall n,$$

$$(4.11.2) \quad R_n(ts) \leq g(t) R_n(s) \quad \forall t, s \geq a \quad \forall n \geq 1,$$

$$(4.11.3) \quad \mu \text{ is totally non-degenerated,}$$

$$(4.11.4) \quad \int_{-\infty}^{\infty} t^{p-1} g(t) dt < \infty,$$

where $R_n(t) = \mu_n(x \in \mathbf{R} \mid |x| > t)$ and a is a positive number, then μ is centered at 0, and we have

$$(4.11.5) \quad \Lambda(\mu) \subseteq L^p(\mu).$$

Remark. From Lemma 4.7 it follows easily (4.11.1), (4.11.2), (4.11.4) and

$$(4.11.3)^* \quad \inf_{-\infty}^{\infty} \int |x| \mu_j(dx) > 0$$

implies (4.11.3), whenever $p > 1$.

Proof. Let X_1, X_2, \dots be independent random variables with distributions μ_1, μ_2, \dots . If $a \in C(\mu)$, then $\sum_{j=1}^{\infty} a_j X_j$ converges a.s., and so $\sup_j |a_j X_j| < \infty$ a.s. Hence by Lemma 3.1 in [9] we have that

$$\sum_{j=1}^{\infty} R_j(a_j^{-1} t_0) < \infty$$

for some $t_0 > 0$, moreover, since $a \in \mathcal{L}^2$ by Theorem 4.8, we have that $b = \inf_j \{t_0 a_j^{-1}\} > 0$, and we may assume that t_0 is taken so large that $t_0 a_j^{-1} \geq a$ for all $j \geq 1$. Then we have

$$P(\sup_j |a_j X_j| > t) \leq \sum_{j=1}^{\infty} R_j(t a_j^{-1}) \leq g(t t_0^{-1}) \sum_{j=1}^{\infty} R_j(t_0 a_j^{-1})$$

for $t \geq a t_0$, and so $\sup_j |a_j X_j| \in L^p$ by (4.11.4). Hence from Corollary 3.3 in [3] we have that

$$(i) \sum_{j=1}^{\infty} a_j w_j \in L^p(\mu) \quad \forall a = (a_j) \in C(\mu).$$

Now we may define

$$|a|_p = \sup \left\{ \int \left| \sum_{j=1}^n a_j w_j \right|^p \mu(dx) \right\}^{1/p} \quad \forall a \in C(\mu),$$

$$T a = \sum_{j=1}^{\infty} a_j w_j \quad \forall a \in C(\mu).$$

Then by Theorem 2.6 in [3] we have

$$(ii) \|T a\|_p = \left(\int \left| \sum_{j=1}^{\infty} a_j w_j \right|^p \mu(dx) \right)^{1/p} = |a|_p \quad \forall a \in C(\mu).$$

Moreover, a slight modification of the proof of Theorem 5.2 in [2] shows that

$$(iii) (C(\mu), |\cdot|_p) \text{ is a Banach space.}$$

Since the identity map, $I: (C(\mu), |\cdot|_p) \rightarrow (C(\mu), |\cdot|_{\mu})$, obviously is continuous, we have that $|\cdot|_p$ and $|\cdot|_{\mu}$ give the same topology on $C(\mu)$ by the closed graph theorem. Hence by (ii) we find that T is an isomorphism of $(C(\mu), |\cdot|_{\mu})$ onto a closed subspace of $L^p(\mu)$. Then $T(C(\mu))$ is certainly also closed in $L^p(\mu)$ and by (4.3) we have

$$(iv) \mathcal{L}_{\mathcal{X}}(\mu) \subseteq T(C(\mu)).$$

Now let A be a μ -Lusin affine space. If $0 \notin A$, then there exists a linear functional, f , so that $f(x) = 1$ for all $x \in A$. Then $f \in \mathcal{L}_{\mathcal{X}}(\mu)$ and so we can find $a \in C(\mu)$, such that

$$\sum_{j=1}^{\infty} a_j w_j = 1 \text{ a.e.}$$

But the sum converges in $L^p(\mu)$ and $p \geq 1$, and so

$$1 = \int d\mu = \sum_{j=1}^{\infty} a_j \int_{-\infty}^{\infty} x \mu_j(dx) = 0.$$

Hence the assumption of $0 \notin A$ leads to a contradiction, and so we must have that μ is centered at 0. Finally, (4.11.5) follows from (iv) and Theorem 4.3.

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Received November 24, 1975

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