Integrability of the Frobenius algebra-valued Kadomtsev-Petviashvili hierarchy

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We introduce a Frobenius algebra-valued Kadomtsev-Petviashvili (KP) hierarchy and show the existence of Frobenius algebra-valued τ-function for this hierarchy. In addition, we construct its Hamiltonian structures by using the Adler-Dickey-Gelfand method. As a byproduct of these constructions, we show that the coupled KP hierarchy, defined by Casati and Ortenzi [J. Geom. Phys. **56**, 418-449 (2006)], has at least n-"basic" different local bi-Hamiltonian structures. Finally, via the construction of the second Hamiltonian structures, we obtain some local matrix, or Frobenius algebra-valued, generalizations of classical W-algebras. © *2015 AIP Publishing LLC.*
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I. INTRODUCTION

The Kadomtsev-Petviashvili (KP) hierarchy is defined by the set of equations

$$\frac{\partial}{\partial t_r}L = [B_r, L], \quad r = 1, 2, \ldots, \tag{1.1}$$

where $L = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \cdots$ is a pseudo-differential operator with coefficients $u_1, u_2, \ldots$ being smooth functions of infinitely many variables $\mathbf{t} = (t_1, t_2, \ldots)$, with $t_1 = x$ and $B_r = L_+^r$ is the pure differential part of the operator L^r and $\partial = \frac{\partial}{\partial x}$.

A fundamental result, due to Sato, is the existence of a τ-function for the KP hierarchy (see the survey[5]). Another fundamental property of this hierarchy is that it has two compatible local Hamiltonian structures. The first structure was suggested by Watanabe,[22] the second by Dickey[6] and, shortly after that, Radul[18] proved that not only one pair of structures can be built but infinitely many. Essentially, the construction was a slight modification of the Adler-Gelfand-Dickey (AGD) method for the nth-Korteweg-deVries (KdV) (GD$_n$) hierarchy in Refs. 1 and 12. We refer to Ref. 8 for a more detailed description.

Recently, there are several types of noncommutative generalizations of the KP hierarchy (see, for example, Ref. 14 and the references therein). Most of them do not preserve the above two fundamental properties. For example, the matrix KP[3] has two compatible Hamiltonian structures via the AGD method, on utilizing the matrix trace map, but the second is nonlocal. Furthermore, there is no τ-function for this hierarchy.

In this paper, we study certain properties of a Frobenius algebra-valued KP hierarchy. The first motivation stems from the work of Casati and Ortenzi.[4] With the use of vertex operator representations of polynomial Lie algebras, they obtained a class of coupled KP hierarchy formulated as a "coupled Hirota bilinear equation." Shortly afterwards, van de Leur in Ref. 15, starting from these bilinear equations, recovered the corresponding wave functions and Lax equations with a $\mathcal{Z}_n$-valued Lax operator L, where $\mathcal{Z}_n = \mathbb{C}[\Lambda]/(\Lambda^n)$ is the maximal commutative subalgebra of $gl(n, \mathbb{R})$ and

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$\Lambda = (\delta_{i,j+1}) \in gl(m, \mathbb{R})$. A natural problem then arises as to how to construct Hamiltonian structures for these coupled KP hierarchies. The main problem in using the AGD method is that the bilinear form constructed using the usual matrix trace $\langle A, B \rangle = \text{trace}\,(AB)$ for $A, B \in \mathcal{Z}_n$ is degenerate. In order to solve this, one of the current authors[24] introduced a somewhat strange looking trace-type map $\text{tr}_n : gl(n, \mathbb{R}) \longrightarrow \mathbb{R}$ defined by

$$
\text{tr}_n(A) = \text{trace} \left[\begin{pmatrix} \dfrac{1}{n} & \dfrac{1}{n-1} & \cdots & 1 \\ 0 & \dfrac{1}{n} & \ddots & \dfrac{1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \dfrac{1}{n} \end{pmatrix} A \right]. \tag{1.2}
$$

We remark that this trace-type map is not symmetric on $gl(n, \mathbb{R})$ but when restricted to the subalgebra $\mathcal{Z}_n$, is nondegenerate and symmetric.

Our second motivation is due to the following crucial observation. Let $\mathbf{1_n}$ be the identity matrix and $\circ$ the matrix multiplication, then $\{\mathcal{Z}_n, \text{tr}_n, \mathbf{1_n}, \circ\}$ is a Frobenius algebra. This observation motivates us to study the $\mathcal{A}$-valued KP hierarchy via $\mathcal{A}$-valued Lax operators, where $\mathcal{A}$ is a Frobenius algebra.

This paper is organized as follows. In Section II, we will show the existence of the $\mathcal{A}$-valued τ-function for the $\mathcal{A}$-KP hierarchy. In Section III, we will construct Hamiltonian structures of the $\mathcal{A}$-valued KP hierarchy. In Section IV, we will list some similar results for the $\mathcal{A}$-valued dispersionless KP hierarchy. Section V is devoted to various conclusions and a discussion of some open problems.

II. THE FROBENIUS ALGEBRA-VALUED KP HIERARCHY AND ITS τ-FUNCTION

In this section, we will introduce an $\mathcal{A}$-valued KP hierarchy via $\mathcal{A}$-valued Lax operators and show the existence of an $\mathcal{A}$-valued τ-function for the $\mathcal{A}$-KP hierarchy.

A. Frobenius algebra

We begin with the definition of a Frobenius algebra.[9]

Definition 2.1. A Frobenius algebra $\{\mathcal{A}, \circ, e, \omega\}$ over $\mathbb{R}$ satisfies the following conditions:

(i) $\circ : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a commutative, associative algebra with unity e;
(ii) $\omega \in \mathcal{A}^{\star}$ defines a non-degenerate inner product $\langle a, b \rangle = \omega(a \circ b)$, which is often called a trace form (or Frobenius form).

Example 2.2 (Ref. 21). Let $\mathcal{A}$ be a two-dimensional commutative and associative algebra with a basis $e = e_1, e_2$ satisfying

$$
e_1 \circ e_1 = e_1, \quad e_1 \circ e_2 = e_2, \quad e_2 \circ e_2 = \varepsilon e_1 + \mu e_2, \quad \varepsilon, \mu \in \mathbb{R}, \tag{2.1}
$$

then $\mathcal{Z}_{2,k}^{\varepsilon,\mu} := \{\mathcal{A}, \circ, e, \omega_k\}$, $k = 1, 2$ are Frobenius algebras, where

$$
\omega_k(a) = a_k + a_2(1 - \delta_{k,2})\delta_{\varepsilon,0}, \quad k = 1, 2, \tag{2.2}
$$

for $a = a_1 e_1 + a_2 e_2 \in \mathcal{A}$.

Example 2.3 (Ref. 21). Let $\mathcal{A}$ be an n-dimensional nonsemisimple commutative associative algebra $\mathcal{Z}_n$ over $\mathbb{R}$ with a unity e and a basis $e_1 = e, \ldots, e_n$ satisfying

$$
e_i \circ e_j = \begin{cases} e_{i+j-1}, & i+j \le n+1, \\ 0, & i+j = n+2. \end{cases} \tag{2.3}
$$

Taking $\Lambda = (\delta_{i,j+1}) \in gl(m, \mathbb{R})$, one obtains a matrix representation of $\mathcal{A}$ as

$$e_j \mapsto \Lambda^{j-1}, \quad j = 1, \ldots, n.$$

Similarly, for any $a = \sum_{k=1}^{n} a_k e_k \in \mathcal{A}$, *we introduce n trace-type forms, called "basic" trace-type forms, as follows:*

$$\omega_{k-1}(a) = a_k + a_n(1 - \delta_{k,n}), \quad k = 1, \ldots, n. \tag{2.4}$$

Every trace map ω_k *induces a nondegenerate symmetric bilinear form on* $\mathcal{A}$ *given by*

$$\langle a, b \rangle_k := \omega_k(a \circ b), \quad a, b \in \mathcal{A}, \quad k = 0, \ldots, n-1. \tag{2.5}$$

Thus, all of $\{\mathcal{A}, \circ, e, \omega_{k-1}\}$ *are nonsemisimple Frobenius algebras, denoted by* $\mathcal{Z}_{n,k-1}$ *for* $k = 1, \ldots, n$. *We remark that the trace-type map* tr_n *in (1.2) is exactly a linear combination of n "basic" trace-type forms as*

$$\mathrm{tr}_n := \sum_{s=0}^{n-1} \omega_s - (n-1)\omega_{n-1}.$$

Unless otherwise stated, we assume that $\{\mathcal{A}, \circ, e := \mathbf{1_n}, \omega := \mathrm{tr}\}$ is an n-dimensional Frobenius algebra over $\mathbb{R}$ with the basis $e_1 = \mathbf{1_n}, e_2 \cdots, e_n$.

B. The $\mathcal{A}$-valued KP hierarchy

Let

$$L = \mathbf{1_n}\partial + U_1\partial^{-1} + U_2\partial^{-2} + \cdots, \quad \partial = \frac{\partial}{\partial x} \tag{2.6}$$

be an $\mathcal{A}$-valued pseudo-differential operator (ΨDO) with coefficients $U_1, U_2, \ldots$ being smooth $\mathcal{A}$-valued functions of an infinite many variables $t = (t_1, t_2, \ldots)$ and $t_1 = x$.

Definition 2.4. The $\mathcal{A}$-KP hierarchy is the set of equations

$$\frac{\partial L}{\partial t_r} = [B_r, L] := B_r \circ L - L \circ B_r, \quad B_r = L_+^r, \quad r = 1, 2, \ldots, \tag{2.7}$$

where B_r is the pure differential part of the operator $L^r = \underbrace{L \circ \cdots \circ L}_{r \text{ terms}}.$

Generally, by imposing the constraint $(L^m)_- = 0$, $\mathcal{A}$-KP hierarchy (2.7) reduces to the $\mathcal{A}$-GD$_m$ hierarchy. The $\mathcal{A}$-KP hierarchy is equivalent to

$$\frac{\partial B_l}{\partial t_r} - \frac{\partial B_r}{\partial t_l} + [B_l, B_r] = 0. \tag{2.8}$$

Consider a case ($r = 2$, $l = 3$), then system (2.8) becomes

$$U_{1,t_2} = U_{1,xx} + 2U_{2,x}, \quad 2U_{1,t_3} = 2U_{1,xxx} + 3U_{2,xx} + 3U_{2,t_2} + 6U_1 \circ U_{1,x}. \tag{2.9}$$

If we eliminate U_2 in (2.9) and rename $t_2 = y$, $t_3 = t$ and $\mathcal{U} = U_1$, we obtain

$$(4\mathcal{U}_t - 12\mathcal{U} \circ \mathcal{U}_x - \mathcal{U}_{xxx})_x - 3\mathcal{U}_{yy} = 0. \tag{2.10}$$

All this follows, the scalar case verbatim. But as the following example shows, when written in terms of a specific basis, this structure is broken and the underlying Frobenius algebra is hidden.

Example 2.5. Suppose that $\mathcal{A}$ is the $\mathcal{Z}_{2,2}^{\varepsilon,\mu}$ algebra and $\mathcal{U} = ve_1 + we_2$. Then, system (2.10) in component form is

$$\begin{cases} (4v_t - 12vv_x - v_{xxx} - 12\varepsilon\, ww_x)_x - 3v_{yy} = 0, \\ (4w_t - 12(vw)_x - w_{xxx} - 12\mu\, ww_x)_x - 3w_{yy} = 0. \end{cases} \tag{2.11}$$

When $\varepsilon = \mu = 0$, system (2.11) reduces to the coupled KP equation (e.g., Refs. 4 and 24). Furthermore, if $v_y = w_y = 0$, the coupled KP equation reduces to the coupled KdV equation[11,16,13]

$$\begin{cases} 4v_t - 12vv_x - v_{xxx} = 0, \\ 4w_t - 12(vw)_x - w_{xxx} = 0. \end{cases} \tag{2.12}$$

Thus, certain multicomponent examples that have appeared in the literature are best viewed as a single $\mathcal{A}$-valued equation: writing them in terms of basis-dependent component fields obscures the underlying algebraic structure.

C. The τ-function

Let us represent the $\mathcal{A}$-valued Lax operator L in (2.6) in a dressing form

$$L = \Phi^{-1} \circ \mathbf{1_n}\partial \circ \Phi, \quad \Phi = \sum_{i=0}^{\infty} W_i \partial^{-i}, \text{with } W_0 = \mathbf{1_n}, \tag{2.13}$$

where the $\mathcal{A}$-valued dressing operator Φ is determined up to a multiplication on the right by $\mathbf{1_n} + \sum_{k=1}^{\infty} C_k \partial^{-k}$, with arbitrary constant elements $C_k \in \mathcal{A}$. Then, using (2.7), we obtain

$$\partial_r \Phi = -L_-^r \circ \Phi, \qquad \partial_r = \frac{\partial}{\partial t_r}. \tag{2.14}$$

For simplicity, let $\xi(t, z) = \sum_{k=1}^{\infty} t_k z^k$ and $\widehat{W}(t, z) = \sum_{i=0}^{\infty} W_i z^i$, where $z \in \mathbb{C}$ is a parameter. The wave function of $\mathcal{A}$-KP hierarchy (2.7) is defined by the $\mathcal{A}$-valued function

$$W(t, z) := \Phi e^{\xi(t,z)} = \widehat{W}(t, z)e^{\xi(t,z)}. \tag{2.15}$$

Similarly, the adjoint wave function is given by

$$\widetilde{W}(t, z) := (\Phi^{-1})^* e^{-\xi(t,z)} = \widehat{\widetilde{W}}(t, z)e^{\xi(t,z)}. \tag{2.16}$$

Lemma 2.6. The following identities hold:

$$(1) \quad \mathrm{res}_z\big(\partial_1^{i_1} \cdots \partial_k^{i_k} W(t, z)\big) \circ \widetilde{W}(t, z) = 0, \quad i_j \in \mathbb{Z}_{\geq 0}, \tag{2.17}$$

$$(2) \quad \widehat{W}(t, z)^{-1} = G(z)[\widehat{\widetilde{W}}(t, z)], \tag{2.18}$$

$$(3) \quad \partial \ln \widehat{W}(t, z) = W_1(t) - G(z)[W_1(t)], \tag{2.19}$$

where $G(z)$ is a shift operator defined by

$$G(z)[f(t; z, s)] = f(t_1 - \frac{1}{z}, t_2 - \frac{1}{2z^2}, \dots; z, s). \tag{2.20}$$

Identity (2.17) is called the $\mathcal{A}$-valued bilinear identity.

Proof. (1) By definitions,

$$L^r \circ W(t, z) = (\Phi \circ \partial \Phi^{-1})^r \circ \Phi e^{\xi(t,z)} = z^r W(t, z). \tag{2.21}$$

Using (2.14) and (2.18), we have

$$\partial_r W(t, z) = \partial_r(\Phi e^{\xi(t,z)}) = (\partial_r \Phi)e^{\xi(t,z)} + \Phi(\partial_r e^{\xi(t,z)}) = L_+^r W(t, z). \tag{2.22}$$

With (2.19), it suffices to consider only the case when all i_j for $j > 1$ vanish. Then,

$$\mathrm{res}_z\big(\partial^i W(t, z)\big) \circ \widetilde{W}(t, z) = \mathrm{res}_z\big(\partial^i \Phi e^{xz}\big) \circ \big((\Phi^{-1})^* e^{-xz}\big) = \mathrm{res}_\partial \partial^i \Phi \circ \Phi^{-1} = 0.$$

In the second step, we use a simple formula $\mathrm{res}_z(Pe^{xz}) \circ (Qe^{-xz}) = \mathrm{res}_\partial P \circ Q^*$, where P and Q are two $\mathcal{A}$-valued ΨDOs.

(2) Bilinear identity (2.17) implies

$$\mathrm{res}_z W(t,z) \circ G(\zeta)[\widetilde{W}(t,z)] = 0. \tag{2.23}$$

Using (2.23) and the identity $e^{\sum_{k=1}^{\infty} \frac{z^k}{k\zeta^k}} = (1 - \frac{z}{\zeta})^{-1}$, we obtain

$$0 = \mathrm{res}_z \widehat{W}(t,z) \circ G(\zeta)[\widehat{\widetilde{W}}(t,z)](1 - \frac{z}{\zeta})^{-1} = \zeta\big(\widehat{W}(t,\zeta) \circ G(\zeta)[\widehat{\widetilde{W}}(t,z)] - \mathbf{1_n}\big)$$

which yields identity (2.18).

(3) Similarly, from bilinear identity (2.17), we have

$$\mathrm{res}_z \partial W(t,z) \circ G(\zeta)[\widetilde{W}(t,z)] = 0.$$

So using (2.18), we get

$$
\begin{aligned}
0 &= \mathrm{res}_z \partial\widehat{W}(t,z) \circ G(\zeta)[\widehat{\widetilde{W}}(t,z)](1 - \frac{z}{\zeta})^{-1} \\
&= \big(\partial\widehat{W}(t,\zeta) + \zeta\widehat{W}(t,\zeta)\big) \circ G(\zeta)[\widehat{\widetilde{W}}(t,\zeta)] - \mathbf{1_n}\zeta - W_1(t) + G(\zeta)[W_1(t)] \\
&= \partial\widehat{W}(t,\zeta) \circ \widehat{W}(t,\zeta)^{-1} - W_1(t) + G(\zeta)[W_1(t)]
\end{aligned}
$$

which implies identity (2.19).   □

We are now in a position to state the main theorem in this section, which can be regarded as an $\mathcal{A}$-valued counterpart of Sato's theorem[5,7,8,19] for the scalar KP hierarchy.

Theorem 2.7. *There is an $\mathcal{A}$-valued function $\tau = \tau(t)$ such that*

$$\widehat{W}(t,z) = G(z)[\tau(t)] \circ \tau(t)^{-1}. \tag{2.24}$$

The $\mathcal{A}$-valued τ-function is determined up to a multiplication by $C_0 \circ e^{\sum_{k=1}^{\infty} C_k t_k}$, with arbitrary constant elements $C_k \in \mathcal{A}, k \in \mathbb{N}$ and arbitrary invertible constant element $C_0 \in \mathcal{A}$.

Proof. With bilinear identity (2.17), we get

$$\mathrm{res}_z W(t,z) \circ G(\zeta_1)[G(\zeta_2)[\widetilde{W}(t,z)]] = 0$$

and

$$\mathrm{res}_z \widehat{W}(t,z) \circ G(\zeta_1)[G(\zeta_2)[\widehat{\widetilde{W}}(t,z)]](1 - \frac{z}{\zeta_1})^{-1}(1 - \frac{z}{\zeta_2})^{-1} = 0. \tag{2.25}$$

It follows from (2.25) that

$$\widehat{W}(t,\zeta_1) \circ G(\zeta_2)[G(\zeta_1)[\widehat{\widetilde{W}}(t,\zeta_1)]] = \widehat{W}(t,\zeta_2) \circ G(\zeta_1)[G(\zeta_2)[\widehat{\widetilde{W}}(t,\zeta_2)]],$$

which becomes, using (2.18),

$$\widehat{W}(t,\zeta_1) \circ G(\zeta_2)[\widehat{W}(t,\zeta_1)^{-1}] = \widehat{W}(t,\zeta_2) \circ G(\zeta_1)[\widehat{W}(t,\zeta_2)^{-1}]. \tag{2.26}$$

Letting $\mu(t,z) = \ln\widehat{W}(t,z)$ and taking into account (2.26), we have

$$\mu(t,\zeta_1) - G(\zeta_2)[\mu(t,\zeta_1)] = \mu(t,\zeta_2) - G(\zeta_1)[\mu(t,\zeta_2)]. \tag{2.27}$$

For simplicity, we denote

$$N(z) := \frac{\partial}{\partial z} - \sum_{k=1}^{\infty} z^{-k-1}\partial_k, \quad B_i := \mathrm{res}_z z^i N(z)\mu(t,z).$$

Applying the operator to (2.27) after renaming $\zeta_1 = z$ and $\zeta_2 = \zeta$, we get

$$N(z)\mu(t,z) - G(\zeta)[N(z)\mu(t,z)] = -\sum_{k=1}^{\infty} z^{-k-1}\partial_k\mu(t,\zeta). \tag{2.28}$$

Multiplying by z^i on both sides of (2.28) and taking the residues res_z, we obtain

$$B_i = G(\zeta)[B_i] - \partial_i\mu(t,z) \tag{2.29}$$

and furthermore,

$$\partial_j B_i - \partial_i B_j = G(\zeta)[\partial_j B_i - \partial_i B_j] \tag{2.30}$$

which yields $\partial_j B_i - \partial_i B_j = const \in \mathcal{A}$. The left side of (2.30) is a differential polynomial in $W_i(t)$ without constant terms, we thus have $\partial_j B_i = \partial_i B_j$. So, there is an $\mathcal{A}$-valued function $\tau = \tau(t)$ such that $B_i = \partial_i \ln \tau$. By using (2.29), we get

$$\partial_i \mu(t, z) = \partial_i (G(\zeta)[\ln \tau] - \ln \tau)$$

which yields (2.24). The rest of the theorem is obvious. □

Corollary 2.8. For any $i \in \mathbb{N}$, the following identity holds:

$$\operatorname{res} L^i = \frac{\partial}{\partial t_i}(\tau_x \circ \tau^{-1}). \tag{2.31}$$

Proof. Equating the residue on both sides of (2.14), we have

$$\operatorname{res} L^i = -\partial_i W_1(t). \tag{2.32}$$

Observe that $\operatorname{res}_z z^i N(z) \mu(t, z) = B_i = \partial_i \ln \tau$ and $\mu(t, z) = \ln \widehat{W}(t, z)$, then we get

$$\frac{\partial}{\partial t_i}(\tau_x \circ \tau^{-1}) = \partial \partial_i \ln \tau = \operatorname{res}_z z^i N(z) \partial \ln \widehat{W}(t, z) \qquad \text{using (2.19)}$$

$$= \operatorname{res}_z z^i N(z)(W_1(t) - G(z)[W_1(t)]) = \operatorname{res}_z z^i N(z) W_1(t)$$

$$= \operatorname{res}_z z^i \sum_{k=1}^{\infty} z^{-k-1} \partial_k W_1(t) = -\partial_i W_1(t).$$

Taking into account (2.32), we obtain desired formula (2.31). □

Example 2.9. Let $A \in \mathcal{A}$ be a constant element, then

$$\tau = \mathbf{1}_n + \exp(2Ax + 2A^3 t)$$

is an $\mathcal{A}$-valued τ-function of the $\mathcal{A}$-valued KdV equation $4\mathcal{U}_t - 12\mathcal{U} \circ \mathcal{U}_x - \mathcal{U}_{xxx} = 0$.

Taking $\mathcal{A}$ to be the Frobenius algebra $\mathcal{Z}_2$, the $\mathcal{A}$-valued KdV equation is exactly coupled KdV equations (2.12). By choosing $A = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \in \mathcal{A}$, we then have

$$\tau = \begin{pmatrix} 1 + \exp(2ax + 2a^3 t) & 0 \\ (2bx + 2b^3 t)\exp(2ax + 2a^3 t) & 1 + \exp(2ax + 2a^3 t) \end{pmatrix} := \begin{pmatrix} \tau_0 & 0 \\ \tau_1 & \tau_0 \end{pmatrix}$$

and

$$\begin{pmatrix} v & 0 \\ w & v \end{pmatrix} = \mathcal{U} = \frac{\partial}{\partial x}(\tau_x \tau^{-1}) = \begin{pmatrix} (\log \tau_0)_{xx} & 0 \\ \left(\dfrac{\tau_1}{\tau_0}\right)_{xx} & (\log \tau_0)_{xx} \end{pmatrix}.$$

Thus, we obtain a solution of couple KdV equation (2.12) given by

$$v = (\log \tau_0)_{xx}, \quad w = \left(\frac{\tau_1}{\tau_0}\right)_{xx}. \tag{2.33}$$

We remark that variable transformation (2.33) has been used to derive the coupled KdV equation from the Hirota equation in Ref. 4. The form of this (i.e., Equation (2.33)) may thus be traced back to the nilpotent elements that appear in the Frobenius algebra $\mathcal{A}$.

III. HAMILTONIAN STRUCTURES OF THE $\mathcal{A}$-VALUED KP HIERARCHY

In this section, we will use the AGD-scheme (e.g., Refs. 1, 12, and 8) to construct Hamiltonian structures of the $\mathcal{A}$-KP hierarchy. For the clarity, let $P = \sum_i P_i \partial^i$ be an $\mathcal{A}$-valued ΨDO, in what follows, we denote P_+ the pure differential part of the operator P and

$$P_- = P - P_+, \quad \operatorname{res}(P) = P_{-1}, \quad P^* = \sum_i (-1)^i \partial^i P_i.$$

Lemma 3.1. Suppose A and B are two $\mathcal{A}$-valued ΨDOs, then

$$\operatorname{tr} \int \operatorname{res} A \circ B \, dx = \operatorname{tr} \int \operatorname{res} B \circ A \, dx. \tag{3.1}$$

Proof. We first show that

$$\operatorname{res} [A, B] = \frac{\partial h(x, \mathbf{t})}{\partial x}, \tag{3.2}$$

where $h(x, \mathbf{t})$ is a certain $\mathcal{A}$-valued function. By linearity, it is sufficient to prove (3.1) for any two $\mathcal{A}$-valued monomials $A = A_i \partial^i$, $B = B_j \partial^j$. If $i, j \geq 0$ or $i + j < 1$, then $\operatorname{res} [A, B] = 0$ and so, $h = 0$. We thus only need consider the case $i \geq 0$, $j < 0$ and $i + j \geq 1$. A direct computation gives

$$\operatorname{res} [A, B] = C_i^{i+j+1} \left(A_i \circ B_j^{(i+j+1)} + (-1)^{i+j} B_j \circ A_i^{(i+j+1)} \right)$$

$$= \frac{\partial}{\partial x} \left(C_i^{i+j+1} \sum_{s=0}^{i+j} (-1)^s A_i^{(s)} \circ B_j^{i+j-s} \right) := \frac{\partial}{\partial x} h.$$

Obviously, h is $\mathcal{A}$-valued. Furthermore, taking the trace form tr on both sides of (3.2), we obtain $\operatorname{tr} \operatorname{res} [A, B] = \operatorname{tr} \frac{\partial h}{\partial x}$. With this, identity (3.1) follows immediately. □

A. Case $U_0 \neq 0$, i.e., $V_{m-1} \neq 0$

Let $L = \mathbf{1_n} \partial + U_0 + U_1 \partial^{-1} + U_2 \partial^{-2} + \cdots$ be an $\mathcal{A}$-valued ΨDO with an additional term U_0. Denoting

$$\mathcal{L} := L^m = \mathbf{1_n} \partial^m + V_{m-1} \partial^{m-1} + V_{m-2} \partial^{m-2} + \cdots, \quad V_i = \sum_{q=1}^n v_{[i]q} e_q. \tag{3.3}$$

In the following, our Hamiltonian structures will be established in terms of the "dynamical coordinates" $\{v_{[i]q}\}$.

We denote by $\mathfrak{D}$ the differential algebra of polynomials in formal symbols $\left\{ v_{[i]q}^{(j)} \right\}$, where $v_{[i]q}^{(j)} = \frac{\partial^j v_{[i]q}}{\partial x^j}$ for $q = 1, \ldots, n$ and $j = 0, 1, \ldots$. We consider a subalgebra $\mathcal{D}$ of $\mathfrak{D}$ with the element of the form $\operatorname{tr} F(V)$, where $F(V)$ is an $\mathcal{A}$-valued differential polynomial with respect to its arguments V_i. We denote the space of functionals by

$$\widetilde{\mathcal{D}} = \left\{ \tilde{f} = \int \operatorname{tr} F(V) dx \, \middle| \, \operatorname{tr} F(V) \in \mathcal{D} \right\}.$$

The variational derivative with respect to an algebra-valued field has been discussed in Ref. 17. In the present context, for $V = \sum_{q=1}^n v_q e_q$, the variational derivative $\frac{\delta F}{\delta V}$ is defined by

$$\tilde{f}(v + \delta v) - \tilde{f}(v) = \int \operatorname{tr} \left(\frac{\delta F}{\delta V} \circ \delta V + o(\delta V) \right) dx = \int \sum_{q=1}^n \left(\frac{\delta f}{\delta v_q} \delta v_q + o(\delta v) \right) dx, \tag{3.4}$$

where $f(v) = \operatorname{tr} F(V)$, $\delta V = \sum_{q=1}^n \delta v_q e_q \in \mathcal{A}$, and $\frac{\delta f}{\delta v_q} = \sum_{j=0}^\infty (-\partial)^j \frac{\partial f}{\partial v_q^{(j)}}$. Without confusion, we use the notation $\frac{\delta f}{\delta V}$ instead of $\frac{\delta F}{\delta V}$.

Suppose $\mathbf{a} = (a_{m-1}, a_{m-2}, \ldots)$ with elements

$$a_i = \sum_{q=1}^n a_{[i]q} e_q \in \mathcal{A}, \quad i = m - 1, m - 2, \ldots.$$

We define a vector field associated to **a** by the formula

$$\partial_{\mathbf{a}} = \sum_{i=-\infty}^{m-1} \sum_{j=0}^{\infty} \sum_{q=1}^{n} a_{[i]q}^{(j)} \frac{\partial}{\partial v_{[i]q}^{(j)}}. \tag{3.5}$$

Obviously, $\partial_{\mathbf{a}}$ and ∂ commute, i.e.,

$$\partial \partial_{\mathbf{a}} f = \partial_{\mathbf{a}} \partial f, \quad \text{for} \quad f \in \mathcal{D}. \tag{3.6}$$

The set of all vector fields $\partial_{\mathbf{a}}$ will be denoted by $\mathcal{V}$, which is a Lie algebra with respect to the commutator $[\partial_{\mathbf{a}}, \partial_{\mathbf{b}}] = \partial_{\partial_{\mathbf{a}}\mathbf{b} - \partial_{\mathbf{b}}\mathbf{a}}$. Let Ω^1 be the dual space of $\mathcal{V}$ consisting of formal $\mathcal{A}$-valued integral operators

$$X = \sum_{i=-\infty}^{m-1} \partial^{-i-1} X_i, \quad X_i \in \mathcal{A}$$

with the pairing

$$\langle \partial_{\mathbf{a}}, X \rangle = \langle \mathbf{a}, X \rangle = \operatorname{tr} \int \operatorname{res}(\mathbf{a} \circ X) dx. \tag{3.7}$$

With the use of formulae (3.4) and (3.6), the action of $\mathcal{V}$ on $\mathcal{D}$ can be transferred to $\widetilde{\mathcal{D}}$,

$$\partial_{\mathbf{a}} \tilde{f} = \partial_{\mathbf{a}} \int f \, dx = \int \partial_{\mathbf{a}} f \, dx = \sum_{i=-\infty}^{m-1} \sum_{q=1}^{n} \int \frac{\delta f}{\delta v_{[i]q}} a_{[i]q} dx = \operatorname{tr} \int \sum_{i=-\infty}^{m-1} a_i \circ \frac{\delta f}{\delta V_i} \, dx.$$

If we set

$$\frac{\delta f}{\delta \mathcal{L}} = \sum_{i=-\infty}^{m-1} \partial^{-i-1} \frac{\delta f}{\delta V_i} \tag{3.8}$$

and identify the vector $\mathbf{a} = (a_{n-1}, a_{n-2}, \ldots)$ with the $\mathcal{A}$-valued ΨDO $\mathbf{a} = \sum_{i=-\infty}^{m-1} a_i \partial^i$, we then have

$$\partial_{\mathbf{a}} \tilde{f} = \operatorname{tr} \int \operatorname{res}\left(\mathbf{a} \circ \frac{\delta f}{\delta \mathcal{L}}\right) dx, \tag{3.9}$$

which follows

$$\left\langle \partial_{\mathbf{a}}, \frac{\delta f}{\delta \mathcal{L}} \right\rangle = \partial_{\mathbf{a}} \tilde{f} = \langle \partial_{\mathbf{a}}, d\tilde{f} \rangle, \quad d\tilde{f} = \frac{\delta f}{\delta \mathcal{L}} \in \Omega^1. \tag{3.10}$$

Lemma 3.2. The mapping $\mathcal{H} : \Omega^1 \to \mathcal{V}$ defined by $\mathcal{H}(X) = \partial_{A^{(z)}(X)}$ is a Hamiltonian mapping,[28] where

$$A^{(z)}(X) = (\widetilde{\mathcal{L}} \circ X)_+ \circ \widetilde{\mathcal{L}} - \widetilde{\mathcal{L}} \circ (X \circ \widetilde{\mathcal{L}})_+ \tag{3.11}$$

and $\widetilde{\mathcal{L}} = \mathcal{L} - z$ and z is an arbitrary parameter.

Proof. When the Frobenius algebra $\mathcal{A}$ is taken to be $\mathbb{R}$, this mapping is the famous Adler mapping which is a Hamiltonian mapping. For a general commutative Frobenius algebra, its trace form is nondegenerate and symmetric. We thus follow the same ideas as used in Ref. 6 to obtain the proof by replacing the scalar operators by $\mathcal{A}$-valued operators.   □

We rewrite $A^{(z)}(X)$ in (3.11) as

$$A^{(z)}(X) = H^{m(0)}(X) + z \, H^{m(\infty)}(X),$$

that is to say,

$$H^{m(0)}(X) = (\mathcal{L} \circ X)_+ \circ \mathcal{L} - \mathcal{L} \circ (X \circ \mathcal{L})_+, \quad H^{m(\infty)}(X) = [\mathcal{L}_-, X_+]_- - [\mathcal{L}_+, X_-]_+. \tag{3.12}$$

By using Lemma 3.2, $H^{m(0)}$ and $H^{m(\infty)}$ are Hamiltonian mappings. We thus get two compatible Poisson brackets of the $\mathcal{A}$-KP hierarchy associated with $\mathcal{L} := L^m$ are given by

$$\{\tilde{f}, \tilde{g}\}^{m(\infty)} = \mathrm{tr} \int \mathrm{res}\, H^{m(\infty)} \left(\frac{\delta f}{\delta \mathcal{L}} \right) \circ \frac{\delta g}{\delta \mathcal{L}}\, dx \tag{3.13}$$

$$= \mathrm{tr} \int \mathrm{res} \left(\left[\mathcal{L}_-, \left(\frac{\delta f}{\delta \mathcal{L}} \right)_+ \right]_- - \left[\mathcal{L}_+, \left(\frac{\delta f}{\delta \mathcal{L}} \right)_- \right]_+ \right) \circ \frac{\delta g}{\delta \mathcal{L}}\, dx$$

and

$$\{\tilde{f}, \tilde{g}\}^{m(0)} = \mathrm{tr} \int \mathrm{res}\, H^{m(0)} \left(\frac{\delta f}{\delta \mathcal{L}} \right) \circ \frac{\delta g}{\delta \mathcal{L}}\, dx \tag{3.14}$$

$$= \mathrm{tr} \int \mathrm{res} \left(\left(\mathcal{L} \circ \frac{\delta f}{\delta \mathcal{L}} \right)_+ \circ \mathcal{L} - \mathcal{L} \circ \left(\frac{\delta f}{\delta \mathcal{L}} \circ \mathcal{L} \right)_+ \right) \circ \frac{\delta g}{\delta \mathcal{L}}\, dx,$$

where $\tilde{f}$, $\tilde{g}$ are two functionals. Furthermore, we have the following.

Theorem 3.3. *The $\mathcal{A}$-KP hierarchy $\frac{\partial L}{\partial t_r} = [B_r, L]$ admits a bi-Hamiltonian representation given by*

$$\frac{\partial \mathcal{L}}{\partial t_r} = H^{m(0)} \left(\frac{\delta h_r}{\delta \mathcal{L}} \right) = H^{m(\infty)} \left(\frac{\delta g_r}{\delta \mathcal{L}} \right), \tag{3.15}$$

with the Hamiltonians

$$\tilde{h}_r = \frac{m}{r} \mathrm{tr} \int \mathrm{res}\, L^r\, dx \quad and \quad \tilde{g}_r = -\frac{m}{r + m} \mathrm{tr} \int \mathrm{res}\, L^{m+r}\, dx.$$

Proof. Observe that the $\mathcal{A}$-KP hierarchy $\frac{\partial L}{\partial t_r} = [B_r, L]$ is equivalent to $\frac{\partial \mathcal{L}}{\partial t_r} = [B_r, \mathcal{L}]$. By definition in (3.4), one obtains

$$\frac{\delta}{\delta \mathcal{L}} \mathrm{tr} \int \mathrm{res}\, L^r dx = \frac{r}{m} L^{r-m}.$$

With the help of (3.12), one thus gets

$$H^{m(0)}(\frac{\delta h_r}{\delta \mathcal{L}}) = H^{m(0)}(L^{r-m}) = [B_r, \mathcal{L}]$$

and

$$H^{m(\infty)}(\frac{\delta g_r}{\delta \mathcal{L}}) = -H^{m(\infty)}(L^r) = [\mathcal{L}_+, L_-^r]_+ - [\mathcal{L}_-, L_+^r]_-$$

$$= [\mathcal{L}, L_-^r]_+ - [\mathcal{L}, L_+^r]_- = [B_r, \mathcal{L}],$$

which yields this theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

B. Case $U_0 = 0$, i.e., $V_{m-1} = 0$

If we restrict to $V_{m-1} = 0$, it is easy to check that the first Hamiltonian structure automatically reduces to this submanifold, but the second one is reducible if and only if

$$\mathrm{res}\, [\mathcal{L}, \frac{\delta f}{\delta \mathcal{L}}] = 0 \tag{3.16}$$

which is equivalent to the condition

$$X_{m-1} = \frac{1}{m} \sum_{i=-\infty}^{m-2} \left(\binom{-i-1}{m-i} X_i^{(m-i-1)} + \sum_{j=i+1}^{m-1} \binom{-i-1}{j-i} (X_i \circ V_j)^{(j-i-1)} \right), \tag{3.17}$$

where $X_i = \frac{\delta f}{\delta V_i} \in \mathcal{A}$. We denote the corresponding reduced brackets by $\{\,,\,\}^{m(\infty)}$ and $\{\,,\,\}_D^{m(0)}$.

Corollary 3.4. The coupled KP hierarchy defined in Ref. 4 has at least n "basic" different local bi-Hamiltonian structures.

Proof. As explained in the Introduction, the coupled KP hierarchy defined in Refs. 4 and 15 is exactly the $\mathcal{Z}_n$-KP hierarchy. According to Example 2.3, the algebra $\mathcal{Z}_n$ has at least n-"basic"

different ways to be realized as the Frobenius algebra. With this, the corollary follows immediately from Theorem 3.3.   □

Definition 3.5. In terms of the basis $\{v_{[i]q}\}$, the second Poisson bracket $\{\ ,\ \}^{m(0)}$ for L^m in (3.3) and the reduced bracket $\{\ ,\ \}_D^{m(0)}$ for L^m with the constraint $V_{m-1} = 0$ will provide two kinds of local W-type algebras, we call them the $W_{\mathcal{AKP}}^{(n,m)}$-algebra and the $W_\infty^{(n,m)}$-algebra, respectively. Under the reduction $L_-^m = 0$, the corresponding algebras are called the $W_{\mathcal{AGD}}^{(n,m)}$-algebra and the $W_{(n,m)}$-algebra, respectively.

With the use of (3.3) and (3.17), one knows that all of them are **local matrix** generalizations of W-algebras. To conclude this section, two examples will be given to illustrate our construction.

Example 3.6. Consider the $\mathcal{A}$-KdV hierarchy with the Lax operator $L^2 = \mathbf{1_n}\partial^2 + V$, i.e., $L_-^2 = 0$. We denote $X = \partial^{-2}X_1 + \partial^{-1}X_0$ and $Y = \partial^{-2}Y_1 + \partial^{-1}Y_0$. Condition (3.16) becomes $X_1 = \frac{1}{2}X_0'$, then we have

$$H^{2(\infty)} = [X, L^2]_+ = -2X_0'$$

and

$$H^{2(0)}(X) = (L^2 \circ X)_+ \circ L^2 - L^2 \circ (X \circ L^2)_+ = 2V \circ X_0' + X_0 \circ V' + \frac{1}{2}X_0'''.$$

Thus, two compatible Poisson brackets of the $\mathcal{A}$-KdV hierarchy[25] are given by

$$\{\tilde{f}, \tilde{g}\}^{2(\infty)} = 2\,\mathrm{tr} \int \frac{\delta f}{\delta V} \circ \frac{\partial}{\partial x} \frac{\delta g}{\delta V}dx$$

and

$$\{\tilde{f}, \tilde{g}\}_D^{2(0)} = -\frac{1}{2}\,\mathrm{tr} \int \frac{\delta f}{\delta V} \circ \left(\mathbf{1_n}\frac{\partial^3}{\partial x^3} + 2V\frac{\partial}{\partial x} + 2\frac{\partial}{\partial x}V\right) \circ \frac{\delta g}{\delta V}dx.$$

In particular, if one chooses the algebra $\mathcal{A}$ to be the algebra $\mathcal{Z}_2$ defined in Example 2.3, one obtains the $\mathcal{Z}_2$-KdV equation for $V = ve_1 + we_2$ given by

$$4v_t - 12vv_x - v_{xxx} = 0, \quad 4w_t - 12(vw)_x - w_{xxx} = 0. \tag{3.18}$$

According to Corollary 3.4, system (3.18) can be written as

$$\begin{pmatrix} v \\ w \end{pmatrix}_t = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}\begin{pmatrix} \frac{\delta H_2}{\delta v} \\ \frac{\delta H_2}{\delta w} \end{pmatrix} = \begin{pmatrix} 0 & J_0 \\ J_0 & J_1 \end{pmatrix}\begin{pmatrix} \frac{\delta H_1}{\delta v} \\ \frac{\delta H_1}{\delta w} \end{pmatrix}$$

with Hamiltonians

$$H_1 = \int_{\mathbb{S}^1} vw\,dx, \quad H_2 = \int_{\mathbb{S}^1} (\frac{3}{2}v^2w + \frac{1}{4}vw_{xx})dx$$

and

$$\begin{pmatrix} v \\ w \end{pmatrix}_t = \begin{pmatrix} 0 & \partial \\ \partial & -\partial \end{pmatrix}\begin{pmatrix} \frac{\delta \widetilde{H}_2}{\delta v} \\ \frac{\delta \widetilde{H}_2}{\delta w} \end{pmatrix} = \begin{pmatrix} 0 & J_0 \\ J_0 & J_1 - J_0 \end{pmatrix}\begin{pmatrix} \frac{\delta \widetilde{H}_1}{\delta v} \\ \frac{\delta \widetilde{H}_1}{\delta w} \end{pmatrix}$$

with Hamiltonians

$$\widetilde{H}_1 = \int_{\mathbb{S}^1} (\frac{1}{2}v^2 + vw)dx, \quad \widetilde{H}_2 = \int_{\mathbb{S}^1} (\frac{3}{2}v^2w + \frac{1}{4}vw_{xx} + \frac{1}{2}v^3 + \frac{1}{8}vv_{xx})dx,$$

where $J_0 = \frac{1}{4}\partial^3 + v\partial + \partial v$ and $J_1 = w\partial + \partial w$.

Example 3.7 (The $\mathcal{A}$-Boussinesq hierarchy). In this case, we have $\mathcal{L} = I_m\partial^3 + V_1\partial + V_0$. Let us take $\tilde{f}, \tilde{g} \in \widetilde{\mathcal{D}}$ and denote

$$X_j = \frac{\delta f}{\delta V_j}, \quad Y_j = \frac{\delta g}{\delta V_j}, \quad j = 0, 1.$$

Using condition ([3.17](#)), we have

$$\frac{\delta f}{\delta \mathcal{L}} = \partial^{-3} X_2 + \partial^{-2} X_1 + \partial^{-1} X_0, \quad \frac{\delta g}{\delta \mathcal{L}} = \partial^{-3} Y_2 + \partial^{-2} Y_1 + \partial^{-1} Y_0,$$

where $X_2 = X_1' - \frac{1}{3} X_0'' - \frac{1}{3} X_0 V_1$ and $Y_2 = Y_1' - \frac{1}{3} Y_0'' - \frac{1}{3} Y_0 V_1$.

A direct calculation gives two Poisson brackets of the $\mathcal{A}$-Boussinesq hierarchy

$$\{\tilde{f}, \tilde{g}\}^{3(\infty)} = 3 \, \mathrm{tr} \int (X_1 Y_0' + X_0 Y_1') dx$$

and

$$\begin{aligned}
\{\tilde{f}, \tilde{g}\}_D^{3(0)} = {}& \mathrm{tr} \int \left(\frac{2}{3} X_0 Y_0^{(5)} - X_0 Y_1^{(4)} + X_1 Y_0^{(4)} - 2 X_1 Y_1^{(3)} \right) dx \\
& + \mathrm{tr} \int \left(\frac{1}{3} X_0 Y_0' - \frac{1}{3} X_0' Y_0 \right) V_1^2 \, dx \\
& + \mathrm{tr} \int \left(\frac{2}{3} X_0 Y_0^{(3)} - \frac{2}{3} X_0^{(3)} Y_0 + X_1'' Y_0 - X_0 Y_1'' + X_1' Y_1 - X_1 Y_1' \right) V_1 \, dx \\
& + \mathrm{tr} \int \left(X_0 Y_0'' - X_0'' Y_0 + 2 X_1' Y_0 - X_1 Y_0' + X_0' Y_1 - 2 X_0 Y_1' \right) V_0 \, dx.
\end{aligned}$$

More specifically, by analogy to the classical W-algebra in Refs. [2] and [10], we set

$$W_2 = V_1, \quad W_3 = V_0 - \frac{1}{2} V_1',$$

then for any two $\mathcal{A}$-valued test functions F and G, we have

$$\left\{ \mathrm{tr} \int F W_2 dx \, , \, \mathrm{tr} \int G W_2 dx \right\}_D^{3(0)} = \mathrm{tr} \int \left(2 F^{(3)} + \mathbf{2} W_2 F' + W_2' F \right) G \, dx,$$

$$\left\{ \mathrm{tr} \int F W_2 dx \, , \, \mathrm{tr} \int G W_3 dx \right\}_D^{3(0)} = \mathrm{tr} \int \left(\mathbf{3} W_3 F' + W_3' F \right) G \, dx,$$

and

$$\begin{aligned}
\left\{ \mathrm{tr} \int F W_3 dx \, , \, \mathrm{tr} \int G W_3 dx \right\}_D^{3(0)} = {}& \frac{1}{6} \, \mathrm{tr} \int \left((2 \, F G' - 2 \, F' G) W_2^2 \right. \\
& \left. + \, F G^{(5)} \right) \, dx + \frac{1}{12} \, \mathrm{tr} \int \left(2 \, F G^{(3)} - 2 \, F^{(3)} G + 3 F'' G' - 3 F' G'' \right) W_2 dx.
\end{aligned}$$

We thus confirm that $W_{\mathbf{k}}$ for $\mathbf{k} = 2, 3$ are spin-$\mathbf{k}$ conformally primary $\mathcal{A}$-valued fields. But notice that the equation $\mathrm{tr}\, F W_2^2 = (\mathrm{tr}\, F W_2)^2$ has no $\mathcal{A}$-valued non-zero solution, which means the classical W_3-algebra is not a subalgebra of the $W_{(n,3)}$-algebra for $\dim \mathcal{A} = n > 1$.

IV. THE DISPERSIONLESS $\mathcal{A}$-KP HIERARCHY

Because of the similarities in the theories of dispersionless and dispersive KP equations (see Refs. [20] and [23]), we list here the analogous results for the $\mathcal{A}$-dKP hierarchy without proofs. We will use the following notation in this part. For an $\mathcal{A}$-valued Laurent series of the form $A = \sum_i A_i p^i$, we denote by A_+ the polynomial part of the Laurent series A and $A_- = A - A_+$, $\mathrm{res}\,(A) = a_{-1}$. Let

$$L = \mathbf{1_n} p + U_1 p^{-1} + U_2 p^{-2} + \cdots \tag{4.1}$$

be an $\mathcal{A}$-valued Laurent series.

Definition 4.1. The $\mathcal{A}$-dKP hierarchy is the set of equations of motion

$$\frac{\partial L}{\partial t_r} = \{L_+^r, L\}, \tag{4.2}$$

where $\{\,,\,\}$ is defined by $\{A, B\} = \frac{\partial A}{\partial p} \circ \frac{\partial B}{\partial x} - \frac{\partial A}{\partial x} \circ \frac{\partial B}{\partial p}$.

Let us assume that L^m, $m \in \mathbb{N}$, is of the form

$$\mathcal{L} := L^m = \mathbf{1_n}p^m + V_{m-1}p^{m-1} + \cdots. \tag{4.3}$$

Taking a dispersionless limit of Hamiltonian structures for the $\mathcal{A}$-KP hierarchy, we get the first and the second Poisson brackets of the $\mathcal{A}$-dKP hierarchy associated with $\mathcal{L}$ in (4.3) as follows:

$$\{\tilde{f}, \tilde{g}\}^{m(\infty)} = \mathrm{tr} \int \mathrm{res} \left(\left\{ \mathcal{L}_-, (\frac{\delta f}{\delta \mathcal{L}})_+ \right\}_- - \left\{ \mathcal{L}_+, (\frac{\delta f}{\delta \mathcal{L}})_- \right\}_+ \right) \circ \frac{\delta g}{\delta \mathcal{L}} \, dx \tag{4.4}$$

and

$$\{\tilde{f}, \tilde{g}\}^{m(0)} = \mathrm{tr} \int \mathrm{res} \left((\mathcal{L} \circ \frac{\delta f}{\delta \mathcal{L}})_+ \circ \mathcal{L} - \mathcal{L} \circ (\frac{\delta f}{\delta \mathcal{L}} \circ \mathcal{L})_+ \right) \circ \frac{\delta g}{\delta \mathcal{L}} \, dx, \tag{4.5}$$

where $\tilde{f}$, $\tilde{g} \in \tilde{\mathcal{D}}$ are two functionals. The variational derivative $\frac{\delta f}{\delta \mathcal{L}}$ is given by

$$\frac{\delta f}{\delta \mathcal{L}} = \sum_{i=-\infty}^{m-1} \frac{\delta f}{\delta V_i} p^{-i-1}, \tag{4.6}$$

where $\frac{\delta f}{\delta V_i}$ is defined in (3.4). When we restrict these to the submanifold $V_{m-1} = 0$, the first Hamiltonian structure automatically reduces to this submanifold, but the second one is reducible if and only if

$$\mathrm{res} \left\{ \mathcal{L}, \frac{\delta f}{\delta \mathcal{L}} \right\} = 0. \tag{4.7}$$

Similarly, in terms of the basis $\{v_{[i]q}\}$, the second Poisson bracket $\{\ ,\ \}^{m(0)}$ for L^m in (4.3) and the reduced bracket $\{\ ,\ \}_D^{m(0)}$ for L^m with the constraint $V_{m-1} = 0$ will provide two kinds of local w-type algebras.

V. CONCLUSIONS

In summary, we have introduced the Frobenius algebra-valued KP hierarchy and studied the existence of τ-functions and Hamiltonian structures. Regarding scalar fields as components of a more basic $\mathcal{A}$-valued field is a more elegant approach: it is not basis dependent and it automatically stresses the algebraic properties more clearly. Other properties can then be traced back, for example, to the freedom in the definition of the Frobenius form. Via the properties of the second Hamiltonian structures, we have obtained some local matrix generalizations of W-algebras. An interesting byproduct is that the coupled KP hierarchy in Ref. 4 has at least n-"basic" different local bi-Hamiltonian structures. The methods in the paper may clearly be applied to other theories of a similar type which have an underlying Lax equation, for example, Toda-hierarchies and reductions of these theories.[26,27]

In a separate paper, $\mathcal{A}$-valued Frobenius manifolds, topological quantum field theories, and bi-Hamiltonian structures are constructed.[21] These constructions are different in character to those in this paper: they are developed without any use of Lax equations, relying on a "lifting" construction from scalar to algebra-valued fields. There will, clearly, be an overlap, with the theory of $\mathcal{A}$-valued KdV and dKdV equations being the most obvious example.

ACKNOWLEDGMENTS

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[27] K-L. Tian and D. Zuo, "Free field realizations of some local matrix W-algebras" (unpublished).

[28] A skew mapping $\mathcal{H} : \Omega^1 \to \mathcal{V}$ is said to be Hamiltonian if (1) $\mathcal{H}\Omega^1 \subset \mathcal{V}$ is a Lie subalgebra and (2) the 2-form ω defined by $\omega(\mathcal{H}(X), \mathcal{H}(Y)) = \langle \mathcal{H}(X), Y \rangle$ is closed.