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INTEGRABLE ALMOST TANGENT STRUCTURES

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Suppose that Γ is a given pseudogroup of local diffeomorphisms $f: R^n \to R^n$. A maximal atlas of charts of a manifold M whose changes of coordinates belong to Γ determines a Γ *structure* on M. These particular charts of M are said to be *adapted* for the Γ structure.

The set Γ_G of all local diffeomorphisms f whose derivatives Df have values in some given Lie subgroup G of $GL(R^n)$ is an important example of a pseudogroup. A Γ_G structure on M is called an *integrable G structure*. Examples of these are integrable almost complex structures and integrable almost tangent structures.

Γ structures on manifolds M and M_1 are said to be *isomorphic* if there exists a bijection

$$\phi: M_1 \to M$$

such that x is an adapted chart of M iff $x \circ \phi$ is an adapted chart of M_1.

Any complex manifold has a standard integrable almost complex structure. A well-known theorem states that any integrable almost complex structure on a manifold M is isomorphic to this standard structure on some complex manifold.

Any tangent manifold has a standard integrable almost tangent structure. But an integrable almost tangent structure on a manifold M is not necessarily isomorphic to this standard structure on some tangent manifold. In this paper we find necessary and sufficient conditions for the existence of such a tangent manifold.

1. Locally affine structures

Suppose given a Γ structure on a manifold M. An atlas of adapted charts of M whose changes of coordinates belong to a given subpseudogroup Γ' of Γ determines a *subordinate* Γ' structure on M. In general such a subordinate structure does not exist.

A *locally affine structure* on M is a pseudogroup structure with coordinate transformations of the type

$$z \to Az + b \ ,$$

where $A \in GL(R^n)$, $b \in R^n$. A manifold with such a structure carries a standard

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flat linear connection whose components with respect to the adapted charts are zero.

A subpseudogroup structure with coordinate transformations of the type

$$z \to Az$$

is a locally *centro-affine structure*.

A locally affine structure does not necessarily admit a subordinate locally centro-affine structure. In order to be able to state conditions for it to do so we recall [3] that a vector field V on a manifold M is *concurrent* with respect to a linear connection ∇ on M if

$$\nabla_X V = X$$

for any vector field X in M.

Theorem 1. *A locally affine structure on M has a subordinate locally centro-affine structure iff M admits a global vector field concurrent with respect to its flat connetion.*

Proof. Suppose that M has a subordinate locally centro-affine structure. For each chart y adapted for this structure we define the local vector field

$$y^j \partial / \partial y^j \ .$$

These local fields agree on the intersection of their domains and so define a global vector field V on M. This is concurrent with respect to the flat connection ∇ since, if $X = a^i \partial / \partial y^i$,

$$\nabla_X V = a^i \nabla_{\partial / \partial y^i} (y^j \partial / \partial y^j) = a^i \partial / \partial y^i = X \ .$$

Conversely, suppose that M carries a concurrent vector field V. For each chart y adapted for the locally affine structure let $V = v^i \partial / \partial y^i$. Then for any vector field $X = a^i \partial / \partial y^i$

$$\nabla_X V = a^i \frac{\partial v^j}{\partial y^i} \frac{\partial}{\partial y^j} \ .$$

Since this must be X it follows that

$$v = y + c$$

for some $c \in R^n$. Such functions v are therefore charts adapted for the locally affine structure and so, on any intersection of domains,

$$\bar{v} = Av + b \ ,$$

where A, b have values in $GL(R^n)$ and R^n respectively. But because V is a global vector field, it follows that $b = 0$. The charts v therefore define a locally centro-affine structure on M subordinate to the given locally affine structure.

q.e.d.

We shall say that a locally affine structure on a manifold M is *complete* if the flat connection on M is complete. Any complete locally affine structure on a connected manifold M determines a complete locally affine structure on its simply connected covering manifold M'. This structure on M' is isomorphic to the standard locally affine structure on R^n determined by its identity chart [2].

Theorem 2. *A locally affine structure on a connected manifold M which*

(i) *is complete,*

(ii) *admits a subordinate locally centro-affine structure,*

is isomorphic to the standard structure on R^n.

Proof. Theorem 1 shows that M carries a concurrent vector field V. This lifts to a concurrent vector field W on R^n. If y is the identity chart on R^n, an argument used in the previous proof shows that

$$W = (y^i + c^i)\frac{\partial}{\partial y^i}$$

for some $c^i \in R$, and so W has just one zero. This arises from a zero of the vector field V. But since W has only one zero, R^n must cover M just once.

2. Integrable almost tangent structures

A manifold M modelled on $R^n \oplus R^n$ carries a *foliation* (of dimension n) if if it has a pseudogroup structure whose coordinate transformations are local diffeomorphisms of the type

$$(z, w) \to (fz, g(z, w)) ,$$

where f is a local diffeomorphism in R^n. If (x, y) is an adapted chart at a point $m \in M$, the leaf F_m containing m is determined locally by $x = xm$ and it admits $y | F_m$ as a chart.

An *integrable almost tangent structure* [1] on M is a pseudogroup structure with coordinate transformations of the type

$$(z, w) \to (fz, (Df)_z w + bz) ,$$

where f is a local diffeomorphism in R^n, and the local function $b\colon R^n \to R^n$ is differentiable. An integrable almost tangent manifold carries an underlying foliation $\mathscr{F}$. A chart (x, y) which is adapted for the almost tangent structure is necessarily adapted for $\mathscr{F}$, and the charts $y | F_m$ determine a locally affine structure on the leaf F_m.

A subpseudogroup structure with coordinate transformations of the type

$$(z, w) \to (fz, (Df)_z w)$$

is a *nearly tangent structure.*

Theorem 3. *An integrable almost tangent structure on M has a subordinate nearly tangent structure iff M admits a global vector field which is tangent to the underlying foliation $\mathscr{F}$ and concurrent with respect to the locally affine structure on each leaf of $\mathscr{F}$.*

Proof. Suppose that M has a subordinate nearly tangent structure with charts (x, y). The local vector fields $y^i\partial/\partial y^i$ agree on the intersection of their domains since

$$y^i\frac{\partial}{\partial y^i} = y^i\left(\frac{\partial \bar{x}^j}{\partial y^i}\frac{\partial}{\partial \bar{x}^j} + \frac{\partial \bar{y}^j}{\partial y^i}\frac{\partial}{\partial \bar{y}^j}\right) = y^i\frac{\partial \bar{x}^j}{\partial x^i}\frac{\partial}{\partial \bar{y}^j} = \bar{y}^i\frac{\partial}{\partial \bar{y}^i},$$

and together they define a global vector field Λ on M. This vector field is tangent to the foliation $\mathscr{F}$, and is concurrent with respect to the locally affine structures on the leaves.

Conversely suppose that M carries such a vector field Λ. For each chart (x, y) adapted for the almost tangent structure let $\Lambda = v^i\partial/\partial y^i$, where v^i depend on x, y. Then for any vector field $X = a^i\partial/\partial y^i$ on a leaf

$$\nabla_X \Lambda = a^i\frac{\partial v^j}{\partial y^i}\frac{\partial}{\partial y^j} .$$

Since this must be X, it follows that

$$v = y + c(x)$$

for some local differentiable function $c\colon R^n \to R^n$. Consequently (x, v) are also charts of M adapted for the almost tangent structure and so on any intersection of their domains

$$\bar{x}^i = f^i(x) , \qquad \bar{v}^i = \frac{\partial \bar{x}^i}{\partial x^j}v^j + b^i(x) .$$

In terms of such a chart, $\Lambda = v^i\partial/\partial v^i$, and because Λ is a global vector field it follows that $b^i = 0$. The charts (x, v) therefore define a subordinate nearly tangent structure on M. q.e.d.

Suppose that M is a manifold with a nearly tangent structure, and that Λ is the associated vector field defined in Theorem 3.

Theorem 4. *The set of zeros of Λ can be given the structure of a regular submanifold M' of M of dimension n.*

Proof. Let M' be the set of zeros of Λ, and p a given point of M'. Choose a chart $\xi = (x, y)$ at p adapted for the nearly tangent structure and having range $U \times V \subset R^n \oplus R^n$, where U and V are open cubes in R^n. The intersection $M' \cap$ (domain ξ) is just the set of points for which $y = 0$.

If $y: M' \to M$ is the natural injection, then $x' = x \circ j$ is an injection $M' \to R^n$ with range U. It is therefore a chart for M'. If $\bar{x}'$ is another chart for M' at p obtained from $\bar{\xi} = (\bar{x}, \bar{y})$, then $\bar{x}' \circ x'^{-1} = \bar{x} \circ x^{-1}$, and this is a local diffeomorphism in R^n. As p varies over M', the charts x' obtained in this way define a manifold structure of dimension n on M'.

The representative for j in terms of the charts x', ξ is the function $z \to (z, 0)$. Consequently M' is a submanifold of M. The domain of x' is the intersection of the domain of ξ with M'. This implies that M' is a regular submanifold of M.

3. The main theorem

We shall say that an integrable almost tangent structure on a manifold M is complete if the locally affine structure on each leaf of the foliation $\mathscr{F}$ is complete.

Suppose that such a structure is given on M, and that it admits a subordinate nearly tangent structure. Each leaf then admits a subordinate locally centro-affine structure. According to Theorem 2, the locally affine structure on each leaf is isomorphic to the standard structure on R^n, and therefore admits a global adapted chart s.

The vector field Λ, introduced in Theorem 3, has just one zero in each leaf. We can therefore define the function $\pi: m \to p$, where p is the zero in the leaf F_m. This maps M onto M'.

Choose a point $p \in M'$, and let $j_p: F_p \to M$ be the natural injection. Let $\xi = (x, y)$ be a chart of M at p chosen as in the proof of Theorem 4. Then the chart $y \circ j_p$ is adapted for the locally centro-affine structure on F_p and on its domain

$$y \circ j_p = As + b$$

for some $A \in GL(R^n)$, $b \in R^n$. The function $As + b$ is a global chart for the manifold F_p. It can be shown to be adapted to the locally centro-affine structure on F_p. This extension of the chart $y \circ j_p$ can be carried out for each point p in the open set

$$W' = (x')^{-1}U$$

of M'. Consequently it defines a function Y on $\pi^{-1}(W')$ with values in R^n.

The function x is constant on the slices of ξ, and so it also can be extended to a function X on $\pi^{-1}(W')$ with values in R^n. This is constant on the leaves of $\mathscr{F}$.

Lemma. *The function $(X, Y): M \to R^n \oplus R^n$ is a chart adapted to the nearly tangent structure on M.*

Proof. In the first place we observe that (X, Y) is a local injection onto the open set $U \times R^n$.

We shall say that (X, Y) is *nearly tangent at a point m*, if it is defined on some neighborhood of m, and if its restriction to this neighborhood is a chart adapted to the nearly tangent structure on M. To prove the lemma it is sufficient to show that (X, Y) is nearly tangent at each point of its domain $\pi^{-1}(W')$.

Let p be a point of W', and consider the set S of points in F_p at which (X, Y) is nearly tangent. S is not empty because (X, Y) is nearly tangent at all points of the domain of the chart ξ. S is, of course, an open subset of F_p. We complete the proof of this lemma by showing that S is also a closed subset of F_p and will therefore coincide with F_p.

Choose a point m in the closure of S, and then choose a chart $\bar{\xi} = (\bar{x}, \bar{y})$ at m, with range $\bar{U} \times \bar{V}$ (where $\bar{U}, \bar{V}$ are open in R^n), adapted for the nearly tangent structure. The domain of $\bar{\xi}$ will meet S, and we choose a point q of the intersection. Since (X, Y) is nearly tangent at q, there is a neighborhood

$$W_1 = \bar{\xi}^{-1}(U_1 \times V_1)$$

of q, where U_1 and V_1 are cubes contained in $\bar{U}$ and $\bar{V}$, such that (X, Y) is defined on W_1 and its restriction to W_1 is a chart adapted to the nearly tangent structure. Consequently on W_1

$$X^i = f^i(\bar{x}) \ , \qquad Y^i = \frac{\partial X^i}{\partial \bar{x}^j}\bar{y}^j \ ,$$

where f is a local diffeomorphism in R^n with domain U_1.

The first relation holds on $\bar{W} = \bar{\xi}^{-1}(U_1 \times \bar{V})$. The second relation also holds on $\bar{W}$ because the functions Y and $\bar{y}$ induce the same locally centro-affine structure on each leaf of $\mathscr{F}$ which meets W_1. Consequently the function (X, Y) is defined on $\bar{W}$ and its restriction to $\bar{W}$ is a chart adapted to the nearly tangent structure on M. But $m \in \bar{W}$ and therefore $m \in S$. It follows that S is a closed subset of F_p. q.e.d.

Suppose that M' is any manifold modelled on R^n, and that TM' is its *tangent manifold*. Let $\pi' \colon TM' \to M'$ be the natural projection. Associated with any chart x' of M' with domain W' we have a standard chart (X', Y') of TM' with domain $(\pi')^{-1}W'$ defined by

$$X' \colon v \to x'm \ , \qquad Y' \colon v \to a \ ,$$

where $v = a^i(\partial/\partial x'^i)_m$. These charts define a nearly tangent structure on TM' whose underlying integrable almost tangent structure is complete.

Conversely, we have

Theorem 5. *An integrable almost tangent structure on a manifold M which*

(i) *is complete,*

(ii) *admits a subordinate nearly tangent structure,*
is isomorphic to the standard structure on a tangent manifold.

Proof. Let M' be the submanifold of M defined in Theorem 4. Choose any point $m \in M$ and a chart $\xi = (x, y)$ of M at $p = \pi m$ as in the proof of Theorem 4. The previous lemma shows that this can be extended to a chart (X, Y) at m.

Let x' be the chart of M' at p associated with ξ. The local function

$$m \rightarrow Y^i(m)(\partial / \partial x'^i)_p$$

is independent of the choice of ξ. Such functions determine a bijection ϕ between M and TM'. In terms of the chart (X, Y) of M and the standard chart (X', Y') of TM' associated with x', the representative of ϕ is the identity function on $U \times R^n$. Consequently ϕ is a diffeomorphism of M onto TM'.

Since it maps the atlas of adapted charts (X, Y) of M to the atlas of adapted charts (X', Y') of TM', ϕ is an isomorphism of the integrable almost tangent structures on these manifolds. q.e.d.

A manifold with an integrable almost tangent structure does not necessarily admit any subordinate nearly tangent structure.

To illustrate this, consider the circle S and its atlas $\mathscr{A}$ whose charts are restrictions of the global function $(\cos\alpha, \sin\alpha) \rightarrow \alpha$. Two such charts differ by a coordinate transformation given locally by

$$z \rightarrow z + c$$

for some $c \in R$, and so $\mathscr{A}$ determines a locally affine structure $\Gamma_{\mathscr{A}}$ on S. This structure is complete. Now consider the torus $T = S \times S$. The atlas of charts $x \times y$, where $x, y \in \mathscr{A}$, defines an integrable almost tangent structure Σ on T. The leaves of the foliation $\mathscr{F}$ are the circles $F_p = p \times S$. The locally affine structure on any leaf is isomorphic to $\Gamma_{\mathscr{A}}$ and so it is complete. Since T is compact, it cannot be a tangent manifold. It follows from Theorem 5 that the integrable almost tangent structure Σ cannot admit any subordinate nearly tangent structure.

References

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