Integrable discrete PT symmetric model

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An exactly solvable discrete PT invariant nonlinear Schrödinger-like model is introduced. It is an integrable Hamiltonian system that exhibits a nontrivial nonlinear PT symmetry. A discrete one-soliton solution is constructed using a left-right Riemann-Hilbert formulation. It is shown that this pure soliton exhibits unique features such as power oscillations and singularity formation. The proposed model can be viewed as a discretization of a recently obtained integrable nonlocal nonlinear Schrödinger equation.

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I. INTRODUCTION

Discrete nonlinear models play a central role in many branches of sciences [1]. A prototypical example is the discrete nonlinear Schrödinger (DNLS) equation $i\dot{u}_n = u_{n+1} + u_{n-1} + |u_n|^2 u_n$ where dot stands for time derivative and u_n is a time-dependent function of the integer n. It is a valuable model describing many important physical and biological phenomena. For example, in the context of nonlinear optics it describes wave propagation in coupled waveguide arrays. In this regard the DNLS model was first proposed in Ref. [2] and used to find localized solitary waves. Subsequently such discrete nonlinear modes were experimentally observed [3,4]. The DNLS equation is also an effective model of certain biophysical systems [5], molecular crystals [6], and atomic chains [7]. It also explains recent observations related to PT symmetric arrays of linearly and/or nonlinearly coupled optical waveguides [8–14]. In this regard, several PT invariant DNLS-like models with gain and/or loss have been recently proposed [15–18].

In 1975–1976 Ablowitz and Ladik [19,20] showed that if one replaces the nonlinear term $|u_n|^2 u_n$ in the DNLS equation by $|u_n|^2(u_{n+1} + u_{n-1})/2$ then the resulting equation $i\dot{u}_n = u_{n+1} + u_{n-1} + |u_n|^2(u_{n+1} + u_{n-1})/2$ (IDNLS) is integrable: It is solvable via the inverse scattering transform, has soliton solutions, possesses an infinite number of conserved quantities, a Hamiltonian structure, as well as other interesting properties related to integrable systems.

In this paper, the following integrable nonlocal discrete nonlinear Schrödinger equation is introduced and investigated in detail

$$i\frac{dQ_n}{dt} = Q_{n+1} - 2Q_n + Q_{n-1} \pm Q_n Q^*_{-n}(Q_{n+1} + Q_{n-1}), \tag{1}$$

where $*$ denotes complex conjugation and Q_n is a complex function and n is an integer. Equation (1) admits a linear (Lax) pair formulation and possesses an infinite number of conservation laws, hence it is an integrable system. It is a discretization of a recently obtained integrable nonlocal nonlinear Schrödinger equation [21]. Corresponding to a rapidly decaying initial data one can exactly linearize Eq. (1) using the inverse scattering transform and obtain solutions including pure solitons. Below, some of the important properties of Eq. (1) are contrasted with the IDNLS model where the nonlocal nonlinear term $Q_n Q^*_{-n}$ is replaced by $|Q_n|^2$. Indeed we note that both equations share the symmetry that when $n \to -n$, $t \to -t$ and a complex conjugate is taken, then the equation remains invariant. Thus, the new nonlocal equation is PT symmetric [22], which, in the case of classical optics, amounts to the invariance of the so-called nonlinear self-induced potential cf. Ref. [23] $V_n = Q_n Q^*_{-n}$ under the combined action of parity and time reversal symmetry. Importantly, PT symmetry breaking in coupled waveguide arrays has been experimentally observed in classical optics [11–13].

II. LINEAR PAIR AND THE NONLOCAL DNLS EQUATION

We begin our analysis by considering the so-called Ablowitz-Ladik scattering problem [19]

$$v_{n+1} = \begin{pmatrix} z & Q_n \\ R_n & z^{-1} \end{pmatrix} v_n, \tag{2}$$

$$\frac{dv_n}{dt} = \begin{pmatrix} \mathsf{A}_n & \mathsf{B}_n \\ \mathsf{C}_n & \mathsf{D}_n \end{pmatrix} v_n, \tag{3}$$

where $v_n = (v_n^{(1)}, v_n^{(2)})^T$, Q_n and R_n vanish rapidly as $n \to \pm\infty$, z is a complex spectral parameter, $\mathsf{A}_n = iQ_nR_{n-1} - \frac{i}{2}(z - z^{-1})^2$, $\mathsf{B}_n = -i(zQ_n - z^{-1}Q_{n-1})$, $\mathsf{C}_n = i(z^{-1}R_n - zR_{n-1})$, and $\mathsf{D}_n = -iR_nQ_{n-1} + \frac{i}{2}(z - z^{-1})^2$. The discrete compatibility condition $\frac{d}{dt}v_{n+1} = (\frac{d}{dt}v_m)_{m=n+1}$ yields

$$i\frac{d}{dt}Q_n = \Delta_n Q_n - Q_nR_n(Q_{n+1} + Q_{n-1}), \tag{4}$$

$$-i\frac{d}{dt}R_n = \Delta_n R_n - Q_nR_n(R_{n+1} + R_{n-1}), \tag{5}$$

where $\Delta_n F_n \equiv F_{n+1} - 2F_n + F_{n-1}$. Equation (1) is then obtained from system (4) and (5) using the symmetry reduction

$$R_n = \mp Q^*_{-n}. \tag{6}$$

The discrete symmetry constraint (6) is new and, as we shall see, leads to interesting physical behavior as well as rich mathematical structure.

III. INFINITE NUMBER OF CONSERVED QUANTITIES AND HAMILTONIAN STRUCTURE

The infinite number of conserved quantities of (1) can be derived either directly or using scattering theory. The first few conserved quantities are listed below

$$\mathcal{C}_1 = \mp \sum_{n=-\infty}^{+\infty} Q_n Q_{1-n}^*, \tag{7}$$

$$\mathcal{C}_2 = \mp \sum_{n=-\infty}^{+\infty} \left[Q_n Q_{2-n}^* \pm \frac{1}{2}(Q_n Q_{1-n}^*)^2 \right], \tag{8}$$

$$\mathcal{C}_3 = \prod_{n=-\infty}^{+\infty} (1 \pm Q_n Q_{-n}^*). \tag{9}$$

Importantly, Eq. (1) is a Hamiltonian dynamical system with Q_n and Q_{-n}^* playing the role of coordinates and conjugate momenta respectively. The corresponding Hamiltonian and the non canonical brackets are given by

$$H = \pm \sum_{n=-\infty}^{+\infty} Q_{-n}^*(Q_{n+1} + Q_{n-1}) - 2\sum_{n=-\infty}^{+\infty} \log(1 \pm Q_n Q_{-n}^*), \tag{10}$$

$$\{Q_m, Q_{-n}^*\} = \mp i(1 \pm Q_n Q_{-n}^*)\delta_{n,m}, \tag{11}$$

$$\{Q_n, Q_m\} = \{Q_{-n}^*, Q_{-m}^*\} = 0. \tag{12}$$

With this in mind one can rewrite Eq. (1) in the form

$$i\frac{dQ_n}{dt} = \{Q_n, H\}. \tag{13}$$

IV. DIRECT SCATTERING PROBLEM

Since the potentials Q_n and R_n rapidly vanish as $n \to \pm\infty$ solutions of the scattering problem (2) can be defined in terms of the following boundary conditions

$$\lim_{n\to-\infty} \phi_n(z) = z^n \hat{e}_1, \quad \lim_{n\to-\infty} \overline{\phi}_n(z) = z^{-n}\hat{e}_2,$$
$$\lim_{n\to+\infty} \psi_n(z) = z^{-n}\hat{e}_2, \quad \lim_{n\to+\infty} \overline{\psi}_n(z) = z^n \hat{e}_1, \tag{14}$$

where $\hat{e}_1 = (1,0)^T$ and $\hat{e}_2 = (0,1)^T$. In terms of the functions $M_n(z) = z^{-n}\phi_n(z)$, $\overline{M}_n(z) = z^n\overline{\phi}_n(z)$, and $N_n(z) = z^n\psi_n(z)$, $\overline{N}_n(z) = z^{-n}\overline{\psi}_n(z)$ satisfying constant boundary conditions induced from (14) one can then obtain a recursive relations for the above functions and show that $M_n(z), N_n(z)$ are analytic outside the unit circle (in the complex z plane; $|z| > 1$) whereas $\overline{M}_n(z), \overline{N}_n(z)$ are analytic inside the unit circle ($|z| < 1$). The solutions $\phi_n(z)$ and $\overline{\phi}_n(z)$ of the scattering problem (2) with the boundary conditions (14) are, in general, linearly independent. This follows from the fact that the Wronskian, $W(v_n, w_n) = v_n^{(1)}w_n^{(2)} - v_n^{(2)}w_n^{(1)}$ of two solutions v_n and w_n to (2) is not equal to zero for all n. Similar arguments hold for $\psi_n(z)$ and $\overline{\psi}_n(z)$. Therefore because the scattering problem (2) is a second-order linear difference equation, the pairs $\{\phi_n, \overline{\phi}_n\}$ and $\{\psi_n, \overline{\psi}_n\}$ are linearly dependent and one can express one set of eigenfunctions in terms of the other:

$$\Phi_n(z) = S_L(z)\Psi_n(z), \tag{15}$$

where $\Phi_n(z) \equiv [\phi_n(z), \overline{\phi}_n(z)]$, $\Psi_n(z) \equiv [\overline{\psi}_n(z), \psi_n(z)]$, and $S_L(z)$ is the so-called left scattering matrix

$$S_L(z) = \begin{pmatrix} a(z) & b(z) \\ \overline{b}(z) & \overline{a}(z) \end{pmatrix}. \tag{16}$$

The scattering data are thus expressed as $a(z) = c_n W(\phi_n(z), \psi_n(z)), \overline{a}(z) = c_n W(\overline{\psi}_n(z), \overline{\phi}_n(z)), b(z) = c_n W(\overline{\psi}_n(z), \phi_n(z))$, $\overline{b}(z) = c_n W(\overline{\phi}_n(z), \psi_n(z))$ where $c_n \equiv c_n(Q_k, R_k) = \prod_{k=n}^{+\infty}(1 - Q_k R_k)$. Moreover, it can be shown that $a(z), \overline{a}(z)$ are respectively analytic functions outside/inside the unit circle in the z complex plane. In general, $b(z), \overline{b}(z)$ are defined only on the unit circle. As mentioned above, the nonlocal discrete NLS equation (1) is a special case of the system (4) and (5) under the symmetry reduction $R_n = \mp Q_{-n}^*$. This symmetry in the potential induces an important symmetry in the eigenfunctions that in turn imposes restrictions on the scattering data. Indeed, if $[\phi_n^{(1)}(z), \phi_n^{(2)}(z)]^T$ satisfies Eq. (2) and the symmetry (6) holds, then one can show that $g_{-n}^{\mp^*}[\phi_{1-n}^{(2)^*}(z^*), \pm \phi_{1-n}^{(1)^*}(z^*)]^T$ also satisfies the scattering problem (2) with $g_n^{\mp} \equiv \prod_{k=-\infty}^{n} \frac{1}{(1\pm Q_k Q_{-k}^*)}$. A similar symmetry result holds for $\overline{\phi}_n(z)$. Because the solutions of the scattering problem (2) are uniquely determined by their respective boundary conditions (14) we obtain the important symmetry relations valid for $R_n = \mp Q_{-n}^*$ which in turn leads to

$$N_{n-1}(z) = g_{-n}^{\mp^*} \Lambda M_{1-n}^*(z^*),$$
$$\overline{N}_{n-1}(z) = g_{-n}^{\mp^*} \Lambda \overline{M}_{1-n}^*(z^*), \tag{17}$$

where Λ is a 2×2 matrix with zeros on the main diagonal and ± 1 on the upper and one one the lower diagonal. From the Wronskian representations for the scattering data it follows $a(z) = e^{2i\theta^{\mp}} a^*(z^*)$, $\overline{a}(z) = e^{2i\theta^{\mp}} \overline{a}^*(z^*)$ and $\overline{b}(z) = \mp e^{2i\theta^{\mp}} b^*(z^*)$ where $\theta^{\mp} \equiv \arg(c_{-\infty}^{\mp})$. These relations imply that if z_j is a zero (eigenvalue) of $a(z)$ with $|z_j| > 1$ then z_j^* is a zero of $a(z)$. Similarly, if $\overline{z}_j$ is a zero of $\overline{a}(z)$ with $|\overline{z}_j| < 1$ so is $\overline{z}_j^*$. For simplicity we shall assume below that the eigenvalues $z_\ell, \overline{z}_\ell$ are all real.

V. INVERSE SCATTERING PROBLEM: LEFT-RIGHT RH APPROACH

The inverse problem consists of constructing the potential functions R_n and Q_n from the scattering data (reflection coefficients) $\rho(z,t) = e^{2i\omega(z)t} b(z,0)/a(z,0)$ and $\overline{\rho}(z,t) = e^{-2i\omega(z)t}\overline{b}(z,0)/\overline{a}(z,0)$ where $\omega(z) = (z - z^{-1})^2/2$ defined on $|z| = 1$ as well as the eigenvalues $z_j, \overline{z}_j$ and norming constants (in n) $C_j(t), \overline{C}_j(t)$. Hereafter, for ease of notation, we suppress the time dependence. It can be shown that the large z asymptotic of $N_n(z)$ depends on c_n. However, the potentials Q_n and R_n are unknown in the inverse problem. To overcome this difficulty, we introduce the new functions $N_n'(z) = J_n N_n(z), \overline{N}_n'(z) = J_n \overline{N}_n(z), M_n'(z) = J_n M_n(z)$ and $\overline{M}_n'(z) = J_n \overline{M}_n(z)$ where $J_n \equiv \mathrm{diag}(1, c_n)$. The symmetry relation (17) in turn induces a symmetry between these new modified functions and is given by

$$N_n'(z) = \begin{pmatrix} 0 & \pm g_{\infty}^{\mp^*} \\ \pm(Q_n Q_{-n}^* \pm 1) & 0 \end{pmatrix} M_{-n}'^{*}(z^*), \tag{18}$$

with similar symmetry expression relating $\overline{N}'_n(z)$ to $\overline{M}'_n(z)$. Using the RH approach, from Eq. (15) one can find equations governing the eigenfunctions $\overline{N}'_n(z), N'_n(z)$

$$\overline{N}'_n(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{J} C_j z_j^{-2n} \left[\frac{N'_n(z_j)}{z - z_j} + \frac{N'_n(-z_j)}{z + z_j} \right] - \lim_{\substack{\zeta \to z \\ |\zeta|<1}} \frac{1}{2\pi i} \oint_{|w|=1} \frac{w^{-2n}\rho(w) N'_n(w)}{w - \zeta} dw, \quad (19)$$

$$N'_n(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=1}^{\overline{J}} \overline{C}_j \overline{z}_j^{2n} \left[\frac{\overline{N}'_n(\overline{z}_j)}{z - \overline{z}_j} + \frac{\overline{N}'_n(-\overline{z}_j)}{z + \overline{z}_j} \right] + \lim_{\substack{\zeta \to z \\ |\zeta|>1}} \frac{1}{2\pi i} \oint_{|w|=1} \frac{w^{2n}\overline{\rho}(w) \overline{N}'_n(w)}{w - \zeta} dw. \quad (20)$$

The time evolution of the norming constants is given by $C_j(t) = C_j(0)e^{2i\omega(z_j)t}$, $\overline{C}_j(t) = \overline{C}_j(0)e^{-2i\omega(\overline{z}_j)t}$. To close the system we evaluate Eq. (19) at the eigenvalues $\pm\overline{z}_j$ and (20) at $\pm z_j$ and obtain a linear algebraic integral system of equations that solve the inverse problem for the eigenfunctions N'_n and $\overline{N}'_n$. To account for the symmetry condition (6), we view system (15) as a left scattering problem and supplement it with the right scattering problem

$$\Psi_n(z) = S_R(z)\Phi_n(z), \quad (21)$$

where

$$S_R(z) = \begin{pmatrix} \overline{\alpha}(z) & \overline{\beta}(z) \\ \beta(z) & \alpha(z) \end{pmatrix}. \quad (22)$$

In the same way as for the left RH above, we can formulate the corresponding RH problem on the right and find the following linear integral equations which govern the functions $M'_n(z), \overline{M}'_n(z)$:

$$\overline{M}'_n(z) = \begin{pmatrix} 0 \\ c_{-\infty} \end{pmatrix} + \sum_{j=1}^{J} B_j z_j^{2n} \left[\frac{M'_n(z_j)}{z - z_j} + \frac{M'_n(-z_j)}{z + z_j} \right] - \lim_{\substack{\zeta \to z \\ |\zeta|<1}} \frac{1}{2\pi i} \oint_{|w|=1} \frac{w^{2n} R(w) M'_n(w)}{w - \zeta} dw, \quad (23)$$

$$M'_n(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{\overline{J}} \overline{B}_j \overline{z}_j^{-2n} \left[\frac{\overline{M}'_n(\overline{z}_j)}{z - \overline{z}_j} + \frac{\overline{M}'_n(-\overline{z}_j)}{z + \overline{z}_j} \right] + \lim_{\substack{\zeta \to z \\ |\zeta|>1}} \frac{1}{2\pi i} \oint_{|w|=1} \frac{w^{-2n} \overline{R}(w) \overline{M}'_n(w)}{w - \zeta} dw. \quad (24)$$

where $R(z) = \beta(z)/\alpha(z)$ and $\overline{R}(z) = \overline{\beta}(z)/\overline{\alpha}(z)$ are the reflection coefficients. The time evolution of the norming constants (in n) are given by $B_j(t) = B_j(0)e^{-2i\omega(z_j)t}$, $\overline{B}_j(t) = \overline{B}_j(0)e^{2i\omega(\overline{z}_j)t}$. Using the relation between the two scattering matrices, i.e., $S_R(z) = S_L^{-1}(z)$ we find $R^*(z^*) = \pm\rho(z)$ and $\overline{R}^*(z^*) = \pm\overline{\rho}(z), |z| = 1$. To close the system we evaluate Eq. (23) at the eigenvalues $\pm\overline{z}_j$ and (24) at $\pm z_j$ and obtain a linear algebraic integral system of equations that solve the inverse problem for the eigenfunctions M'_n and $\overline{M}'_n$.

VI. RECOVERY OF THE POTENTIALS

To reconstruct the potentials Q_n and R_n we use the method to obtain potentials (cf. [24,25]) and find the following expressions for the potentials Q_n and R_n

$$R_n = 2\sum_{j=1}^{J} C_j z_j^{-2(n+1)} N_n'^{(2)}(z_j) + \frac{1}{2\pi i} \oint_{|w|=1} w^{-2(n+1)} \rho(w) N_n'^{(2)}(w) dw. \quad (25)$$

$$Q_{n-1} = -2\sum_{j=1}^{\overline{J}} \overline{C}_j \overline{z}_j^{2(n-1)} \overline{N}_n'^{(1)}(\overline{z}_j) + \frac{1}{2\pi i} \oint_{|w|=1} w^{2(n-1)} \overline{\rho}(w) \overline{N}_n'^{(1)}(w) dw. \quad (26)$$

VII. SOLITON SOLUTIONS

For simplicity we consider the case where the scattering data is only comprised of real eigenvalues $z_j > 1, 0 < \overline{z}_j < 1$ and $\rho(z) = 0$, $\overline{\rho}(z) = 0$ for all z. Then the inverse scattering system (19), (20) with (23), (24) subject to the symmetry relations (17) reduces to finite-dimensional linear algebraic equations for $N'_n(z_j), \overline{N}'_n(\overline{z}_j)$ and $M'_n(z_j), \overline{M}'(\overline{z}_j)$. Recall that, when $R_n = -Q^*_{-n}$, the eigenvalues appear in pairs $\{z_j, z_j^*\}$ and $\{\overline{z}_j, \overline{z}_j^*\}$. We will consider the one-soliton solution to the nonlocal NLS equation (1) corresponding to a single (real) eigenvalue ($J = \overline{J} = 1$): $z_1 = e^{\eta_1}, \overline{z}_1 = e^{-\overline{\eta}_1}$ where $\eta_1 \neq \overline{\eta}_1$ satisfying $\eta_1 > 0, \overline{\eta}_1 > 0$. Letting $C_1(0) = z_1(z_1^2 - \overline{z}_1^2)/(2\overline{z}_1)e^{i\varphi_1}$ and $\overline{C}_1(0) = (z_1^2 - \overline{z}_1^2)/(2z_1\overline{z}_1)e^{i\overline{\varphi}_1}$ then the most general one-soliton solution to Eq. (1) is given by

$$Q_n = -\frac{(\overline{z}_1 z_1)^{-1}(z_1^2 - \overline{z}_1^2)e^{i\overline{\varphi}_1}e^{-2i\overline{\omega}_1 t}\overline{z}_1^{2n}}{1 + e^{i(\overline{\varphi}_1+\varphi_1)}e^{2i(\omega_1-\overline{\omega}_1)t}\overline{z}_1^{2n} z_1^{-2n}}, \quad (27)$$

where $\overline{\omega}_1 = \omega(\overline{z}_1)$ and $\omega_1 = \omega(z_1)$. This soliton solution exhibits unique characteristics stemming from the fact that the model (1) is a PT symmetric nonlinear system. First, the power $\sum_{n=-\infty}^{\infty} |Q_n(t)|^2$ (which is not conserved) oscillates in t with period $\pi/(\omega_1 - \overline{\omega}_1)$. Such phenomenon has been observed (in the linear regime) in a PT symmetric periodic structures [9]. Interestingly enough, the soliton forms a singularity on the integers. For example, when $n = 0$ the finite time blow up is given by

$$t_{\text{singularity}} = \frac{\pi(2j+1) - \varphi_1 - \overline{\varphi}_1}{2(\omega_1 - \overline{\omega}_1)}, \quad j \in \mathbb{Z}. \quad (28)$$

VIII. COMPARISON WITH THE ABLOWITZ-LADIK MODEL

In this section we briefly contrast the properties of Eq. (1) with that of the AL equation:

$$i\dot{u}_n = (1 \pm |u_n|^2)(u_{n+1} + u_{n-1}). \tag{29}$$

In Refs. [19,20] it was shown that (29) is an integrable Hamiltonian system. Furthermore, it was found that the symmetries of the scattering data and eigenfunctions of the associated AL scattering problem are such that the scattering data and eigenfunctions outside the unit circle in the z complex plane are related to those inside the unit circle. Moreover, the eigenfunctions are related through a local in n symmetry and is independent of the potentials. This is in sharp contrast to our case where the eigenfunctions at the outside/inside the unit circle are not related. Importantly, the symmetry condition relating the eigenfunctions say, outside the unit circle, is more complicated: highly nonlocal and does depend on the potential Q_n. This new feature complicates the analysis and leads to a novel method of solution. However, if one considers Eq. (1) with initial conditions satisfying $Q_n(0) = Q_{-n}(0)$ then one obtains extra symmetries on the scattering data and eigenfunctions that are similar to the ones we find. This leads us to the important conclusion that soliton solutions to (1) will have an AL limit so long (29) admits an even solution.

IX. DISCRETE NONLOCAL PAINLEVÉ TYPE EQUATIONS

The discrete Painlevé equations are certain class of nonlinear second-order complex differential-difference equations that usually arise as reductions of the discrete soliton equations, which are solvable by IST cf. [26–28]. They are particularly interesting due to their properties in the complex plane and their associated integrability properties. In this section we propose a discrete nonlocal analogues of Painlevé-type equation. Nonlocal Painlevé type equations can also be found. For example if we let $Q_n(t) = e^{i\lambda t} Y_n$ where λ is constant then Y_n satisfies the following discrete Painlevé-type equation

$$Y_{n+1} + Y_{n-1} = -\frac{\beta Y_n}{1 + Y_n Y_{-n}^*}, \tag{30}$$

where $\beta = -2 + \lambda$. One can also take the further reduction Y_n to be real.

X. CONCLUSION

A nonlocal integrable discrete nonlinear Schrödinger equation is found from a new and simple reduction of the well-known Ablowitz-Ladik scattering problem. It is a discretization of a recently obtained integrable nonlocal nonlinear Schrödinger equation [21]. It has a Lax pair, an infinite number of conserved quantities, and is PT symmetric. The inverse scattering transform (IST) for decaying data is developed and a discrete breathing soliton solution is found. The IST requires different scattering data symmetries than that associated with the classical integrable discrete NLS equation.

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