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Integrable hydrodynamic chains

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Benney hydrodynamic chain

Let us consider the remarkable Benney hydrodynamic chain

$$A_t^k = A_x^{k+1} + kA^{k-1}A_x^0, \qquad k = 0, 1, 2, \ldots$$

Let us introduce N field variables $a^k(x,t)$ such that

$$A^k = \sum_{i=1}^{N} f_{i(k)}(a^i),$$

where the functions $f_{i(k)}(a^i)$ are not yet determined.
A substitution of the above ansatz in the Benney hydrodynamic chain leads to

Benney hydrodynamic chain

$$\sum_{i=1}^{N} f'_{i(k)}(a^i)\left(a^i_t - \frac{f'_{i(k+1)}(a^i)}{f'_{i(k)}(a^i)}a^i_x - k\frac{f_{i(k-1)}(a^i)}{f'_{i(k)}(a^i)}A^0_x\right) = 0.$$

Suppose each expression (in brackets) vanishes simultaneously. It means, that

$$\varphi_i(a^i) = \frac{f'_{i(k+1)}(a^i)}{f'_{i(k)}(a^i)}, \qquad \psi_i(a^i) = k\frac{f_{i(k-1)}(a^i)}{f'_{i(k)}(a^i)}.$$

Then Benney hydrodynamic chain reduces to the form

$$\partial_t \varphi_i(a^i) = \partial_x\left(\frac{\varphi_i^2(a^i)}{2} + \sum_{m=1}^{N}\epsilon_m \varphi_m(a^m)\right),$$

where

$$\sum_{m=1}^{N}\epsilon_m = 0, \qquad \psi_i(a^i) = \frac{1}{\varphi'_i(a^i)}, \qquad A^k = \frac{1}{k+1}\sum_{n=1}^{N}\epsilon_n[\varphi_n(a^n)]^{k+1}.$$

Benney hydrodynamic chain

Under the scaling $\varphi_i(a^i) \to a^i$ the above hydrodynamic type system can be rewritten as the so-called waterbag reduction

$$a^i_t = \partial_x \left(\frac{(a^i)^2}{2} + A^0 \right),$$

where

$$A^0 = \sum_{m=1}^{N} \epsilon_m a^m.$$

Let us replace a^i by p. A corresponding equation

$$p_t = \partial_x \left(\frac{p^2}{2} + A^0 \right)$$

is nothing but a generating function of conservation laws for the Benney hydrodynamic chain.

Benney hydrodynamic chain

Indeed, let us substitute a formal series

$$p = \lambda - \frac{H_0}{\lambda} - \frac{H_1}{\lambda^2} - \frac{H_2}{\lambda^3} - ...,$$

where a parameter λ goes to infinity. Then the Benney hydrodynamic chain can be written in the conservative form

$$\partial_t H_k = \partial_x \left(H_{k+1} - \frac{1}{2} \sum_{m=0}^{k-1} H_m H_{k-1-m} \right),$$

where $H_0 = A^0, H_1 = A^1, H_2 = A^2 + (A^0)^2, H_3 = A^3 + 3A^0A^1, ...$ All conservation law densities H_k can be found by the above substitution into the inverse series

$$\lambda = p + \frac{A^0}{p} + \frac{A^1}{p^2} + \frac{A^2}{p^3} + ...$$

The Gibbons–Tsarev system

Let us suppose that moments A^n depend on N Riemann invariants r^k.Then the Benney hydrodynamic chain reduces to a family of hydrodynamic type systems

$$r_t^i = p^i(\mathbf{r})r_x^i, \qquad i = 1, 2, ..., N,$$

parameterized by N arbitrary functions of a single variable.
Each hydrodynamic type system possesses the same generating function of conservation laws

$$p_t = \partial_x \left(\frac{p^2}{2} + A^0 \right).$$

It means that a generating function of conservation law densities p satisfies the Löwner equation

$$\partial_i p = \frac{\partial_i A^0}{p^i - p},$$

where A^0 is a function of all Riemann invariants r^k.

The Gibbons–Tsarev system

The compatibility condition $\partial_i(\partial_k p) = \partial_k(\partial_i p)$ leads to the Gibbons–Tsarev system

$$\partial_i p^k = \frac{\partial_i A^0}{p^i - p^k}, \qquad \partial_{ik} A^0 = 2\frac{\partial_i A^0 \partial_k A^0}{(p^i - p^k)^2}, \qquad i \neq k,$$

whose solutions are parameterized by N arbitrary functions of a single variable

Vlasov type kinetic equations

Let us consider the so-called Vlasov kinetic equation

$$\lambda_t = \{\lambda, \mathbf{\Psi}\},$$

where $\lambda(x, t, p)$, $u(x, t)$ are unknown functions and $\mathbf{\Psi}(u, p)$ is some given function. Here the Poisson bracket is canonical

$$\{\lambda, \mathbf{\Psi}\} \equiv \frac{\partial \lambda}{\partial x}\frac{\partial \mathbf{\Psi}}{\partial p} - \frac{\partial \lambda}{\partial p}\frac{\partial \mathbf{\Psi}}{\partial x}.$$

Under the semi-hodograph transform $\lambda(x, t, p) \leftrightarrow p(x, t, \lambda)$, the above equation reduces to the conservative form

$$p_t = \partial_x \mathbf{\Psi}(u, p),$$

which is nothing but a generating function of conservation laws with respect to parameter λ.

Vlasov type kinetic equations

Suppose the above equations possesses hydrodynamic reductions

$$r_t^i = \mu^i(\mathbf{r}) r_x^i.$$

Then such hydrodynamic type systems preserve the above generating function. It means, that a generalization of the Löwner equation is given by

$$\partial_i p = \frac{\mathbf{\Psi}_u}{\mu^i - \mathbf{\Psi}_p} \partial_i u.$$

The compatibility conditions $\partial_i(\partial_k p) = \partial_k(\partial_i p)$ leads to a generalization of the Gibbons–Tsarev system.

Three distinct generating functions

Here we consider three distinct generating functions of conservation laws

$$p_t = \partial_x [u + U(p)], \qquad q_{\bar{t}} = \partial_{\bar{x}} [vV(q)], \qquad s_{\tilde{t}} = \partial_{\tilde{x}} [ws + W(s)],$$

where functions $U(p), V(q), W(s)$ satisfy to three ODE's of the second order

$$U'' = \alpha U'^2 + \beta U' + \gamma, \qquad VV'' = \alpha V'^2 + \beta V' + \gamma, \qquad sW'' = \alpha W'^2 + \beta W' + \gamma.$$

Three distinct generating functions

Corresponding semi-Hamiltonian hydrodynamic reductions

$$r_t^i = U'(p^i) r_x^i, \qquad r_{\tilde{t}}^i = v V'(q^i) r_{\tilde{x}}^i, \qquad r_{\tilde{t}}^i = [w + W'(s^i)] r_{\tilde{x}}^i, \qquad i = 1, 2, ..., N$$

are connected with generalized Löwner equations

$$\partial_i p = \frac{\partial_i u}{U'(p^i) - U'(p)}, \qquad \partial_i q = \frac{V(q) \partial_i \ln v}{V'(q^i) - V'(q)}, \qquad \partial_i s = \frac{s \partial_i w}{W'(s^i) - W'(s)},$$

where $\partial_i \equiv \partial / \partial r^i$. These generalized Löwner equations are equivalent to each other under the transformations

$$dp = \frac{dq}{V(q)} = d \ln s, \quad u = \ln v = w, \quad U'(p) = V'(q) = W'(s).$$

Three distinct generating functions

These hydrodynamic reductions written in the so-called *symmetric* form

$$a_t^i = \partial_x[u(\mathbf{a}) + U(a^i)], \qquad b_{\tilde{t}}^i = \partial_{\tilde{x}}[V(b^i)v(\mathbf{b})], \qquad c_{\tilde{t}}^i = \partial_{\tilde{x}}[c^i w(\mathbf{c}) + W(c^i)],$$

are connected with corresponding generalized Löwner-like equations

$$\partial_i p = \frac{\partial_i u}{U'(a^i) - U'(p)} \left(1 + \sum \frac{\partial_m u}{U'(a^m) - U'(p)}\right)^{-1},$$

$$\partial_i q = \frac{V(q)\partial_i \ln v}{V'(b^i) - V'(q)} \left(1 + \sum \frac{V(b^m)\partial_m \ln v}{V'(b^m) - V'(q)}\right)^{-1},$$

$$\partial_i s = s\frac{\partial_i w}{W'(c^i) - W'(s)} \left(1 + \sum \frac{c^m \partial_m w}{W'(c^m) - W'(s)}\right)^{-1},$$

due to the extra transformations

$$da^i = \frac{db^i}{V(b^i)} = d\ln c^i,$$

where $u(\mathbf{a}), v(\mathbf{b})$ and $w(\mathbf{c})$ are some functions, and $\partial_i \equiv \partial/\partial a^i, \partial_i \equiv \partial/\partial b^i, \partial_i \equiv \partial/\partial c^i$, respectively.

Explicit hydrodynamic reductions

Hydrodynamic chains associated with generating function of conservation laws can be found by the approach presented at the beginning.
1. Replacing p by N field variables a^i in the corresponding generating functions of conservation laws

$$p_t = \partial_x[u + U(p)],$$

one can obtain the symmetric hydrodynamic type system

$$a^i_t = \partial_x[u(\mathbf{a}) + U(a^i)],$$

where $u(\mathbf{a})$ is a some function, which is not determined yet.
2. The corresponding Löwner type equation

$$\partial_i p = \frac{\partial_i u}{U'(a^i) - U'(p)} \left[1 + \sum \frac{\partial_m u}{U'(a^m) - U'(p)}\right]^{-1}$$

can be integrated just in some special cases.

Explicit hydrodynamic reductions

3. Let us consider the function $u(\Delta)$, where $\Delta = \Sigma f_m(a^m)$. These functions $f_m(a^m)$ are not yet determined.
4. In such a case, the solution

$$\lambda = e^{\delta p+(\alpha+\epsilon)U(p)} \sum_{k=0}^{\infty} \frac{A^k}{U'^{k+1}(p)}$$

of the Löwner type equation can be found explicitly, where the moments A^k are given by their derivatives

$$dA^k = \sum_{i=1}^{N} \epsilon_i e^{(\beta-\delta)a^i+(\alpha-\epsilon)U(a^i)} U'^k(a^i) da^i,$$

and

$$u = \frac{1}{\epsilon} \ln \Delta \equiv \frac{1}{\epsilon} \ln A^0.$$

Explicit hydrodynamic reductions

5. A corresponding integrable hydrodynamic chain is given by

$$A_t^k = A_x^{k+1} + \frac{1}{\epsilon A^0}[(\alpha(k+1)-\epsilon)A^{k+1} + (\beta(k+1)-\delta)A^k + \gamma k A^{k-1}]A_x^0, \qquad k = 0, 1, ..$$

Remark:If the parameter $\epsilon = 0$, then

$$u = \Delta = B^0.$$

In such a case, the above equation of the Riemann surface reduces to the form

$$\lambda = \int e^{\beta p + \alpha U(p)} dp + e^{\beta p + \alpha U(p)} \sum_{k=0}^{\infty} \frac{B^k}{U'^{k+1}(p)}.$$

The corresponding hydrodynamic chain

$$B_t^k = B_x^{k+1} + (\alpha(k+1)B^{k+1} + (\beta(k+1)-\delta)B^k + \gamma k B^{k-1})B_x^0, \quad k = 0, 1, 2, ...$$

is determined by the moment decomposition

$$dB^k = \sum_{i=1}^{N} \epsilon_i e^{(\beta-\delta)a^i + \alpha U(a^i)} U'^k(a^i) da^i,$$

where $B^0 = \Delta$.

Hamiltonian formulation

This hydrodynamic chain can be written in different forms. Below, we have derive another moment decomposition assuming that hydrodynamic reductions

$$a_t^i = \partial_x[u(\mathbf{a}) + U(a^i)],$$

possess the local Hamiltonian structure

$$a_t^i = \frac{1}{\epsilon_i}\partial_x \frac{\partial h}{\partial a^i}.$$

In such a case,

$$\frac{\partial h}{\partial a^i} = \epsilon_i[u(\mathbf{a}) + U(a^i)], \qquad \frac{\partial^2 h}{\partial a^i \partial a^k} = \epsilon_i \frac{\partial u(\mathbf{a})}{\partial a^k} = \epsilon_k \frac{\partial u(\mathbf{a})}{\partial a^i}, \qquad i \neq k.$$

It means, that $u(\mathbf{a})$ is a some function of $\Sigma\epsilon_n a^n$. A substitution of this anzac in the above Löwner type equation leads to $u(\mathbf{a}) = \Sigma\epsilon_n a^n$ for $\alpha = 0$ and $u(\mathbf{a}) = \ln(\Sigma\epsilon_n a^n)/\alpha$ in a general case.

Hamiltonian formulation

The equation of the Riemann surface ($\xi = \Sigma\epsilon_n$)

$$\lambda = -\xi \int \frac{e^{\beta p + 2\alpha U(p)}}{U'(p)} dp + e^{\beta p + 2\alpha U(p)} \sum_{k=0}^{\infty} \frac{C^k}{U'^{k+1}(p)}$$

and corresponding integrable hydrodynamic chain

$$C_t^0 = \partial_x \left(C^1 + \frac{\xi}{\alpha} \ln C^0 \right), \qquad C_t^k = C_x^{k+1} + \frac{k}{\alpha C^0}(\alpha C^{k+1} + \beta C^k + \gamma C^{k-1}) C_x^0,$$

are connected with the above hydrodynamic type systems via moments C^k determined by their derivatives

$$dC^k = \sum_{i=1}^{N} \epsilon_i U'^k(a^i) da^i.$$

This hydrodynamic chain preserves a local Hamiltonian structure. A corresponding Poisson bracket is given by

$$\{C^0, C^0\} = \xi \delta'(x - x'), \qquad \{C^k, C^n\} = [\Gamma^{kn}\partial_x + \partial_x \Gamma^{nk}]\delta(x - x'), \quad k + n > 0,$$

where $\Gamma^{kn} = k(\alpha C^{k+n+1} + \beta C^{k+n} + \gamma C^{k+n-1})$.

Generalized Benney hydrodynamic chain

The generating function of conservation laws

$$p_t = \partial_x[u + U(p)]$$

associated with ODE

$$U'' = \alpha U'^2 + \beta U' + \gamma$$

can be split on four distinguished sub-cases.
1. $\alpha \neq 0$, and quadratic equation $\alpha z^2 + \beta z + \gamma$ has two distinct roots (without loss of generality we can fix $\alpha = -1, \beta = 1, \gamma = 0$). In such a case, the first generating function is reducible to

$$p_t = \partial_x[\ln(e^p + 1) - u].$$

Generalized Benney hydrodynamic chain

2. $\alpha \neq 0$, and quadratic equation $\alpha z^2 + \beta z + \gamma$ has two coincided roots (without loss of generality we can fix $\alpha = 1$). Then the first generating function is reducible to

$$p_t = \partial_x(\ln p + u).$$

3. $\alpha = 0$, but $\beta \neq 0$. Without loss of generality we can fix $\beta = 1$. Then the first generating function is reducible to

$$p_t = \partial_x(e^p + u).$$

4. $\alpha = 0$ and $\beta = 0$. Then we obtain a generating function for the remarkable Benney hydrodynamic chain

$$p_t = \partial_x\left(\frac{p^2}{2} + u\right).$$