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Integrable Nonlinear Klein-Gordon Equations and Toda Lattices

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Abstract. We present a class of nonlinear Klein-Gordon systems which are soluble by means of a scattering transform. More specifically, for each $N \geqq 2$ we present a system of $(N-1)$ nonlinear Klein-Gordon equations, together with the corresponding $N \times N$ matrix scattering problem which can be used to solve it. We illustrate these with some special examples. The general system is shown to be closely related to the equations of the periodic Toda lattice. We present a Bäcklund transformation and superposition formula for the general system.

Introduction

Part of the current folk-lore in soliton theory is that the only integrable, nonlinear Klein-Gordon equations in one variable θ:

$$\theta_{xt} = F(\theta) \tag{1.1}$$

are those for which

$$\frac{d^2F}{d\theta^2} + kF = 0\,. \tag{1.2}$$

This belief stems from two distinct lines of argument; one is the study of self-Bäcklund transformations, and the other is the question of the existence of an infinite number of polynomial conserved densities (p.c.d.s).

A well known result is that Eq. (1.1) possesses a self-Bäcklund transformation if and only if condition (1.2) is satisfied [1–3].

The question of the existence of p.c.d.s is more complicated, however. The sufficiency of (1.2) for the existence of an infinite number of p.c.d.s is well known, but the necessity has never been proven. Kruskal [4] looked for a conserved density with leading order term θ_{xx}^2 (of weight 4) for Eq. (1.1), and found (1.2) to be necessary for its existence. Dodd and Bullough [5] later found several higher order p.c.d.s for the equation:

$$\theta_{xt} = e^{2\theta} - e^{-\theta} \tag{1.3}$$

but believed, at the time, that only a finite number of those existed.

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It is of interest that Eq. (1.3) arises in the context of Lie-Bäcklund contact transformations [6–8], which are, of course, intimately related to conservation laws through Noether's theorem [9].

In this paper we present integrable systems of nonlinear Klein-Gordon Eq. (1.1) which are more complicated than the scalar equations satisfying (1.2). Indeed, for each $N \geqq 2$ we present a $N \times N$ matrix scattering problem with an isospectral flow whose typical form is an $(N-1)$-component nonlinear Klein-Gordon system:

$$\begin{aligned}\theta^{(n)}_{xt} = A_{n\,n-1} &\exp(\theta^{(n)} - \theta^{(n-1)}) \\ &- A_{n+1\,n} \exp(\theta^{(n+1)} - \theta^{(n)}),\end{aligned} \tag{1.4}$$

where $A_{n\,n-1}$ are constants and $n = 1, 2, \ldots, N(\mathrm{mod}\, N)$. However, there exist restrictions of these systems, among which is the scalar Eq. (1.3).

We also show that these systems are closely related to the periodic Toda lattice equations. Using the Bäcklund transformations of the Toda lattice as our model, we write down the Bäcklund transformation for our systems of nonlinear Klein-Gordon equations.

2. The Eigenvalue Problem

Recently we [10, 11] have considered the factorisation of third order scalar differential operators into the product of three first order factors.

The general Nth order elliptic scalar differential operator

$$L = \partial^N + u_{N-2}\partial^{N-2} + \ldots + u_1\partial + u_0 \tag{2.1}$$

may be written as

$$L' = (\partial - \theta^{(N)}_x) \ldots (\partial - \theta^{(1)}_x), \tag{2.1a}$$

where the functions in the linear factors are written in potential form for the purpose of this paper $\left(\text{here and in what follows } \partial \equiv \partial_x \equiv \dfrac{\partial}{\partial x}\right)$.

The Nth order scalar eigenvalue problem

$$L\psi_1 = \zeta^N \psi_1 \tag{2.2}$$

may then be written as

$$\partial \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix} = \begin{pmatrix} \theta^{(1)}_x & \zeta & 0 & \ldots & 0 \\ 0 & \theta^{(2)}_x & \zeta & 0 & \ldots \\ & & \ddots & \ddots & \\ & & & \ddots & \zeta \\ \zeta & & & & \theta^{(N)}_x \end{pmatrix} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}, \tag{2.2a}$$

where the trace $\sum_{n=1}^{N} \theta^{(n)}_x$ vanishes because $u_{N-1} = 0$.

Given the eigenvalue problem (2.2a) we could consider time evolutions of the column vector $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_N)^T$ which are of the form

$$\partial_t \boldsymbol{\psi} = M\boldsymbol{\psi}, \tag{2.3}$$

where M is an $N \times N$ matrix whose components M_{ij} depend upon ζ, the functions $\theta^{(n)}$ and, possibly, their x-derivatives. In order that the time evolution (2.3) should lead to an isospectral flow of Klein-Gordon type, M is chosen inversely proportional to ζ:

$$M_{ij} = \frac{1}{\zeta} a_{ij}, \tag{2.4}$$

where the a_{ij} are independent of ζ. The details of the ensuing calculation would be out of place here (see [12, 11] for the 2×2 and 3×3 cases respectively), so we just present the time evolution:

$$\partial_t \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix} = \frac{1}{\zeta} \begin{pmatrix} 0 & \cdots & \cdots & a_{1N} \\ a_{21} & & & \\ & \ddots & & \\ & & a_{NN-1} & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}, \tag{2.5}$$

where a_{nn-1} and a_{1N} are the only non-zero elements.

The integrability conditions between (2.2a) and (2.5) are

$$\frac{\partial a_{n+1\,n}}{\partial x} = a_{n+1\,n}(\theta_x^{(n+1)} - \theta_x^{(n)}) \tag{2.6}$$

and

$$\theta_{xt}^{(n)} = a_{n\,n-1} - a_{n+1\,n}, \tag{2.7}$$

where $n = 1, \ldots, N$, and the indices are considered modulo N.

From Eq. (2.6) we deduce

$$a_{n+1\,n} = A_{n+1\,n} \exp(\theta^{(n+1)} - \theta^{(n)}) \tag{2.8}$$

so that Eq. (2.7) implies

$$\begin{aligned} \theta_{xt}^{(n)} = A_{n\,n-1} &\exp(\theta^{(n)} - \theta^{(n-1)}) \\ &- A_{n+1\,n} \exp(\theta^{(n+1)} - \theta^{(n)}), \end{aligned} \tag{2.9}$$

where $A_{n\,n-1}$ are constants and $n = 1, \ldots, N \pmod N$.

Equations (2.9) are a system of nonlinear Klein-Gordon equations which can be solved exactly by means of the scattering problem (2.2a) and (2.5).

The most interesting case is when all the A_n are positive; without loss of generality we may then set $A_n = 1$:

$$\theta_{xt}^{(n)} = \exp(\theta^{(n)} - \theta^{(n-1)}) - \exp(\theta^{(n+1)} - \theta^{(n)}) \tag{2.10}$$

which will be considered in more detail in the next section.

3. Some Examples

Second Order

The simplest examples of the systems introduced here are those derived from a 2×2 scattering problem

$$\partial \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \theta_x & \zeta \\ \zeta & -\theta_x \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{3.1}$$

where, as in Sect. 2, we have used the condition that the trace, $\sum \theta_x^{(n)}$, should vanish.

The time evolution of the column vector $\boldsymbol{\psi} = (\psi_1, \psi_2)^T$ is given by

$$\partial_t \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{\zeta} \begin{pmatrix} 0 & A_{12} e^{2\theta} \\ A_{21} e^{-2\theta} & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{3.2}$$

Taking $A_{12} = A_{21} = 1$, we obtain the integrability condition

$$\theta_{xt} = 2 \sinh 2\theta, \tag{3.3}$$

which is the well-known sinh-Gordon equation; it was solved independently by Ablowitz et al. [13] and Takhtadzhyan and Faddeev [14].

By allowing θ to assume imaginary values, $\theta = i\tilde{\theta}$, we obtain the sine-Gordon equation,

$$\tilde{\theta}_{xt} = 2 \sin 2\tilde{\theta}. \tag{3.4}$$

As is well known, this equation, having degenerate vacua, possesses multisoliton solutions.

By taking $A_{21} = 0$, $A_{12} = 1$, we obtain the Liouville equation,

$$\theta_{xt} = e^{2\theta}. \tag{3.5}$$

Third Order

Some more interesting examples may be derived by considering the 3×3 scattering problem:

$$\partial \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \theta_x + \phi_x & \zeta & 0 \\ 0 & -2\phi_x & \zeta \\ \zeta & 0 & -\theta_x + \phi_x \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \tag{3.6}$$

where we have put

$$\begin{aligned} \theta^{(1)} &= \theta + \phi \\ \theta^{(2)} &= -2\phi \\ \theta^{(3)} &= -\theta + \phi \end{aligned} \tag{3.7}$$

in order that the trace, $\sum \theta_x^{(n)}$, should vanish.

The time evolution of $\boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3)^T$ is given by

$$\partial_t \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \frac{1}{\zeta} \begin{pmatrix} 0 & 0 & A_{13} \exp(2\theta) \\ A_{21} \exp(-\theta - 3\phi) & 0 & 0 \\ 0 & A_{32} \exp(3\phi - \theta) & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}. \tag{3.8}$$

If we set $A_{13}=A_{21}=A_{32}=1$, then we obtain, as integrability conditions

$$\begin{aligned}\theta_{xt} &= e^{2\theta}-e^{-\theta}\cosh 3\phi\\ \phi_{xt} &= e^{-\theta}\sinh 3\phi\,.\end{aligned} \tag{3.9}$$

A particular case of the system (3.9) is

$$\begin{aligned}\phi &\equiv 0\\ \theta_{xt} &= e^{2\theta}-e^{-\theta}\,.\end{aligned} \tag{3.10}$$

This is an example of a single component nonlinear Klein-Gordon equation which is completely integrable. However, as will be seen in Sect. 4, and as is to be expected from the results summarised in Sect. 1, this equation does not possess a self-Bäcklund transformation; the Bäcklund transformation (4.8) relates solutions of (3.10) not to one another, but rather to solutions of the more complicated system (3.9).

Further, on calculating the hierarchy of polynomial conserved densities of (3.9), by expanding the logarithm of the transmission coefficient of (3.6) in an asymptotic series for large ζ, one finds; first, that every third such density is trivial, being an exact derivative; and second, that every second density (of odd parity with respect to x) is an exact derivative when ϕ is identically zero. The result of this is that the next simplest conserved quantity after the "momentum"

$$I_2=\int\left(\tfrac{1}{2}\theta_x^2+\tfrac{3}{2}\phi_x^2\right)dx\,\|_{\phi\equiv 0}=\int\tfrac{1}{2}\theta_x^2\,dx \tag{3.11}$$

is (for $\phi\equiv 0$)

$$I_6=-\int\left(\tfrac{1}{2}\theta_{3x}^2+\tfrac{5}{6}\theta_{2x}^3+\tfrac{5}{2}\theta_{2x}^2\theta_x^2+\tfrac{1}{6}\theta_x^6\right)dx\,. \tag{3.12}$$

This quantity is, on identifying θ_x with v, the Hamiltonian of the modified Kupershmidt equation

$$v_t=-\left(-v_{4x}+5v_xv_{2x}+5vv_x^2+5v^2v_{2x}-v^5\right)_x, \tag{3.13}$$

which is discussed in [10].

By letting ϕ assume imaginary values, $\phi=i\tilde{\phi}$, we may transform the system (3.9) into

$$\begin{aligned}\theta_{xt} &= e^{2\theta}-e^{-\theta}\cos 3\tilde{\phi}\\ \tilde{\phi}_{xt} &= e^{-\theta}\sin 3\tilde{\phi}\,.\end{aligned} \tag{3.14}$$

Like the sine-Gordon equation, this system, possessing degenerate vacua, has "kink" solutions; these, as well as two-kink and kink-antikink solutions have been obtained by Hirota's method [15]. The single kink, or soliton solution is

$$\begin{aligned}\theta &= \frac{1}{2}\ln\left[\frac{1+e^{3\eta}}{(1+e^{\eta})^3}\right]\\ \tilde{\phi} &= \arctan\left[\frac{\sqrt{3}\,e^{\eta}}{2-e^{\eta}}\right],\end{aligned} \tag{3.15}$$

where $\eta=kx+3t/k$.

To generate the multi-soliton solutions to (3.14), one could use the superposition formula (4.10) which we will derive below.

There remain some problems concerning the stability of the system (3.14), however; this is because the momentum integral (3.11), which, being positive definite, guarantees the stability of the system (3.9), is transformed, for the system (3.14), into the form:

$$I_2=\int(\tfrac{1}{2}\theta_x^2-\tfrac{3}{2}\tilde{\phi}_x^2)dx\,. \tag{3.16}$$

However, some preliminary numerical results of Eilbeck [16] suggest that this system is indeed stable.

Higher Orders

The behaviour of these systems becomes more complicated as N is increased; for $N \geqq 4$ we find that the linearised normal modes have different dispersion relations. For example, taking $N=4$, we find that the equations may be expressed as

$$\begin{aligned}\theta_{xt}&=e^{2\theta}\cosh 2\phi-e^{-2\theta}\cosh 2\psi\\ \phi_{xt}&=e^{2\theta}\sinh 2\phi\\ \psi_{xt}&=e^{-2\theta}\sinh 2\psi\,.\end{aligned} \tag{3.17}$$

In field theoretic terms, the field θ has mass 2, while ϕ and ψ have mass $\sqrt{2}$.

There are various restrictions of this system which make sense; either or both of ϕ and ψ may be suppressed. If one, say ϕ, is retained, we get

$$\begin{aligned}\theta_{xt}&=e^{2\theta}\cosh 2\phi-e^{-2\theta}\\ \phi_{xt}&=e^{2\theta}\sinh 2\phi\end{aligned} \tag{3.18}$$

which is equivalent to a system brought to our attention by Kupershmidt [17]. The other restrictions of the system reduce to sinh-Gordon equations.

For $N=5$, taking $\theta^{(1)}=-\theta^{(5)}=\theta$, $\theta^{(2)}=-\theta^{(4)}=\phi$, $\theta^{(3)}=0$ we get

$$\begin{aligned}\theta_{xt}&=e^{2\theta}-e^{\phi-\theta}\\ \phi_{xt}&=e^{\phi-\theta}-e^{\phi}\,.\end{aligned} \tag{3.19}$$

The full system, with four independent field variables, is more complicated. It may be seen that the "masses" of the normal modes of the linearised system are given by

$$m^2=2-\varrho-\varrho^{-1}\,, \tag{3.20}$$

where ϱ is a nontrivial Nth root of unity.

4. The Relationship with the Toda Lattice

The system (2.10) clearly bears some similarity to the well known Toda lattice equations [18, 19]. Indeed, introducing

$$\tau=t-x\,,\qquad \theta^{(n)}(x,t)=-\phi^{(n)}(\tau) \tag{4.1}$$

we obtain:

$$\phi^{(n)}_{\tau\tau}=\exp(\phi^{(n-1)}-\phi^{(n)})-\exp(\phi^{(n)}-\phi^{(n+1)}) \tag{4.2}$$

which are the Toda lattice equations, for which a scattering problem was found by Flaschka [20]. The variable τ introduced above is the time in laboratory co-ordinates. By transforming to laboratory co-ordinates, and removing a factor of $\exp(-\frac{1}{2}\phi^{(n)})$ from the wave function ψ_n, one may easily recover Flaschka's Lax pair representation of the Toda equations.

Kac-van Moerbeke Equations

A closely related system to the Toda lattice is that of Kac and van Moerbeke [21, 22]

$$u^{(n)}_{\tau}=\exp(u^{(n+1/2)})-\exp(u^{(n-1/2)})\,, \tag{4.3}$$

where, for convenience, we have taken the indices to be half-integers. Clearly

$$\begin{aligned}u^{(n)}_{\tau\tau}&=\exp(u^{(n+1/2)})(\exp(u^{(n+1)})-\exp(u^{(n)}))\\&\quad-\exp(u^{(n-1/2)})(\exp(u^{(n)})-\exp(u^{(n-1)}))\,.\end{aligned} \tag{4.4}$$

By setting

$$u^{(n)}+u^{(n+1/2)}=\phi^{(n+1)}-\phi^{(n)} \tag{4.5}$$

we obtain the Toda lattice Eqs. (4.2) from (4.4).

Thus the first order system (4.3) models two Toda lattices, interpolated one between the other. The variables with integer and half-odd integer indices respectively satisfy autonomous systems of Toda equations.

Putting (for integer n, and arbitrary $A>0$)

$$\begin{aligned}u^{(n+1/2)}&=\phi^{(n+1)}-\bar{\phi}^{(n)}-\ln A\\u^{(n)}&=\bar{\phi}^{(n)}-\phi^{(n)}+\ln A\end{aligned} \tag{4.6}$$

(where $\bar{\phi}^{(n)}=\phi^{(n+1/2)}$) we solve (4.5) and get

$$\begin{aligned}&\frac{d}{d\tau}(\phi^{(n+1)}-\bar{\phi}^{(n)})=A[\exp(\bar{\phi}^{(n+1)}-\phi^{(n+1)})-\exp(\bar{\phi}^{(n)}-\phi^{(n)})]\\&\frac{d}{d\tau}(\bar{\phi}^{(n)}-\phi^{(n)})=\frac{1}{A}[\exp(\phi^{(n+1)}-\bar{\phi}^{(n)})-\exp(\phi^{(n)}-\bar{\phi}^{(n-1)})]\end{aligned} \tag{4.7}$$

which is the usual Bäcklund transformation relating a pair of solutions ϕ and $\bar{\phi}$ of (4.2).

Bäcklund Transformation

Analogously, for the nonlinear Klein-Gordon equations discussed above we may write down

$$\begin{aligned}\frac{\partial}{\partial x}(\bar{\theta}^{(n)}-\theta^{(n+1)})&=-\kappa[\exp(\theta^{(n+1)}-\bar{\theta}^{(n+1)})-\exp(\theta^{(n)}-\bar{\theta}^{(n)})]\\ \frac{\partial}{\partial t}(\theta^{(n)}-\bar{\theta}^{(n)})&=\frac{1}{\kappa}[\exp(\bar{\theta}^{(n)}-\theta^{(n+1)})-\exp(\bar{\theta}^{(n-1)}-\theta^{(n)})]\end{aligned}\tag{4.8}$$

which are easily seen to be a Bäcklund transformation for (2.10).

These were obtained by requiring that under the identifications (4.1), Eqs. (4.7) should be recovered.

For the more general system (2.9), for which some of the A_n may be zero [see, for example, Eq. (3.10)] a slightly different form of Bäcklund transformation is needed.

The Bäcklund transformation (4.8) may be regarded as arising from a system which is related to (2.10) in the same way as the Kac-van Moerbeke equations are related to the Toda lattice. This system is

$$\begin{aligned}\frac{\partial}{\partial x}u^{(n+1/2)}&=\exp u^{(n)}-\exp u^{(n+1)}\\ \frac{\partial}{\partial t}u^{(n)}&=\exp u^{(n+1/2)}-\exp u^{(n-1/2)},\end{aligned}\tag{4.9}$$

where the variable n assumes integer values (mod N). It is likely that this system is also completely integrable.

Superposition Formula

It is possible to derive a nonlinear "superposition formula" for the systems discussed above, by requiring that two successive Bäcklund transformations should commute [23].

We let a Bäcklund transformation with parameter $\bar{\kappa}$ take a solution θ to a solution $\bar{\theta}$, and another Bäcklund transformation of parameter $\hat{\kappa}$ take θ to $\hat{\theta}$. We then require that these transformations take $\hat{\theta}$ and $\bar{\theta}$ respectively into a single new solution $\tilde{\theta}$. Some algebraic manipulation then yields the result:

$$\exp(\theta^{(n+1)}+\tilde{\theta}^{(n)})=\left[\frac{\bar{\kappa}\exp(-\bar{\theta}^{(n)})-\hat{\kappa}\exp(-\hat{\theta}^{(n)})}{\bar{\kappa}\exp(-\bar{\theta}^{(n+1)})-\hat{\kappa}\exp(-\hat{\theta}^{(n+1)})}\right]\cdot\exp(\bar{\theta}^{(n)}+\hat{\theta}^{(n)})\tag{4.10}$$

as the condition that both the space and time halves of the Bäcklund transformations commute.

5. Conclusions

In this paper we have exhibited integrable systems of nonlinear Klein-Gordon equations, of which the usual sine- and sinh-Gordon equations are very special cases. These equations were shown to be closely related to those of the periodic Toda lattice. The simple form of these systems stems from the special structure of their corresponding eigenvalue problems, which arise from the factorisation of scalar differential operators, and for which they are isospectral flows. More general scattering problems should lead to more complicated systems (see [24] for the 3×3 case).

Further generalisations of the systems discussed here are those related to the generalised Toda lattices discovered by Bogoyavlensky [25] and discussed in detail by Kostant [26]. We consider these systems in another paper [27].

Among the integrable systems presented here is the new scalar nonlinear Klein-Gordon Eq. (3.10). While there seem to be no other such equations in the present class of systems, it would be rash to assert that the more general systems mentioned above could not give rise to any other such scalar equations.

As mentioned above, Eilbeck [16] has obtained numerical results concerning the stability of some of the systems of Sect. 3. In a future paper, we intend to study these systems in more detail, both numerically and analytically.

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