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Integrable probability: From representation theory to Macdonald processes

Alexei Borodin

Department of Mathematics, Massachusetts Institute of Technology
77 Massachusetts ave., Cambridge, MA 02139, USA
and Institute for Information Transmission Problems
Bolshoy Karetny per. 19, Moscow, 127994, Russia
e-mail: borodin@math.mit.edu

and

Leonid Petrov

Department of Mathematics, Northeastern University
360 Huntington ave., Boston, MA 02115, USA
and Institute for Information Transmission Problems
Bolshoy Karetny per. 19, Moscow, 127994, Russia
e-mail: lenia.petrov@gmail.com

Abstract: These are lecture notes for a mini-course given at the Cornell Probability Summer School in July 2013. Topics include lozenge tilings of polygons and their representation theoretic interpretation, the (q,t)-deformation of those leading to the Macdonald processes, nearest neighbor dynamics on Macdonald processes, their limit to semi-discrete Brownian polymers, and large time asymptotic analysis of polymer's partition function.

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1. Introduction

One way to describe the content of these lecture notes is to say that they give a proof of the following statement (up to certain technical details that can be looked up in suitable articles).

Theorem 1.1. *Let $B_1, B_2, \dots$ be independent standard Brownian motions. Define*

$$Z_N^t = \int\limits_{0<s_1<\dots<s_{N-1}<t} e^{B_1(s_1)+\left(B_2(s_2)-B_2(s_1)\right)+\dots+\left(B_N(\tau)-B_N(s_{N-1})\right)} ds_1\dots ds_{N-1}.$$

Then, for any $\varkappa>0$,

$$\lim_{N\to\infty} \mathbf{P}\left\{\frac{\log Z_N^{\varkappa N}-c_1(\varkappa)N}{c_2(\varkappa)N^{1/3}}\leq u\right\}=F_2(u)$$

with certain explicit $\varkappa$-dependent constants $c_1, c_2 > 0$, where $F_2(\cdot)$ is the distribution function of the GUE Tracy-Widom distribution.

The quantity Z_N^t was introduced by O'Connell-Yor [68], and it can be viewed as the partition function of a semi-discrete Brownian polymer (also sometimes referred to as the "O'Connell-Yor polymer"). The limit relation above shows that this polymer model belongs to the celebrated Kardar-Parisi-Zhang (KPZ) universality class, see Corwin [34] for details on the KPZ class and §1.6 of [13] for more explanations, consequences, and references concerning the polymer interpretation.

The exact value of $c_1(\varkappa)$ was conjectured by O'Connell-Yor [68] and proven by Moriarty-O'Connell [63], and the above limit theorem was proven by Borodin-Corwin [13] for a restricted range of $\varkappa$ and by Borodin-Corwin-Ferrari [16] for all $\varkappa > 0$. A nice physics-oriented explanation of $c_2(\varkappa)$ was given by Spohn [78].

Although the most direct proof of this theorem would likely be quite a bit shorter than these notes, brevity was not our goal. Despite the probabilistic appearance of the statement, any of the known approaches to the proof would involve a substantial algebraic component, and the appearance of algebra at first seems at least slightly surprising. The goal of these lecture notes is to suggest the most logically straightforward path (in authors' opinion) that leads to the desired result, minimizing as much as possible the number of *ad hoc* steps one takes along the way. (For an interested reader we remark that a shorter proof of Theorem 1.1 can be obtained via combining Corollary 4.2 of [66], Theorem 2 of [20], and asymptotic analysis of [13].)

As we travel along our path (that naturally starts on the algebraic side — in representation theory of unitary groups), we encounter other probabilistic models that are amenable to similar tools of analysis. The approach that we develop has a number of other applications as well. It was so far used for (we refer the reader to the indicated references for further explanations)

- asymptotics of the KPZ equation with a certain class of initial conditions [16];
- asymptotics of Log-Gamma fully discrete random directed polymers [20];
- asymptotics of q-TASEP and ASEP [21, 42];
- analysis of new integrable (1+1)d interacting particle systems — discrete time q-TASEPs of [15], q-PushASEP [27, 37], and (q, μ, ν)-TASEP [35];
- establishing a law of large numbers for infinite random matrices over a finite field [29] (conjectured by Vershik and Kerov, see [45]);
- Gaussian Free Field asymptotics of the general beta Jacobi corners process [25];
- developing spectral theory for the q-Boson particle system [19];
- asymptotics of probabilistic models originating from representation theory of the infinite-dimensional unitary group $U(\infty)$ [10, 11, 28, 12].

The emerging domain of studying such models that enjoy the benefits of a rich algebraic structure behind, is sometimes called *Integrable Probability*, and we refer the reader to the introduction of Borodin-Gorin [24] for a brief discussion

of the domain and of its name (the integrable nature of the semi-discrete polymer of Theorem 1.1 was first established by O'Connell [66]). To a certain extent, the present text may be considered as a continuation of [24], but it can be also read independently.

In contrast with [24], in our exposition below we do not shy away from the representation theoretic background and intuition that were essential in developing the subject. We also focus on proving a single theorem, rather than describing the variety of other related problems listed above, in order to discuss in depth the analytic difficulties arising in converting an algebraic formalism into analytic statements. These difficulties are related to the phenomenon of *intermittency* and popular yet highly non-rigorous and sometimes dangerous *replica trick* favoured by physicists, and one of our goals is to show how raising the amount of "algebraization" of the problem can be used to overcome them.

The notes are organized as follows.

In Section 2, we explain how lozenge tilings of a class of polygons on the triangular lattice can be interpreted via representation theory of the unitary groups, and how this leads to contour integral formulas for averages of various observables.

In Section 3, we show, in a specific example, how the steepest descent analysis of the obtained contour integrals yields meaningful probabilistic information about lozenge tilings.

Section 4 describes an approach to constructing local Markov dynamics on lozenge tilings, and how (1+1)-dimensional interacting particle systems (like usual and long range Totally Asymmetric Simple Exclusion Processes (TASEPs)) arise as marginals of such dynamics. The approach we describe is relatively recent; it was developed in Borodin-Petrov [27] (an extension of the method will appear in [29]).

Section 5 deals with a two-parameter (Macdonald, (q,t)-) generalization of the previous material.

In Section 6 we show how simple-minded asymptotics of the contour integrals in the q-deformation of lozenge tilings leads to semi-discrete Brownian polymers. The contour integrals describe the q-moments of the q-TASEP, an integrable deformation of the usual TASEP.

Section 7 explains difficulties which arise if one straightforwardly tries to describe the distribution of the polymer partition function using its moments. The latter come out naturally as limits of the q-TASEP's q-moments.

In final Section 8 we demonstrate how those difficulties can be overcome through considering the Laplace transform of the polymer partition function and its q-analog for the q-TASEP particle locations.

2. Lozenge tilings and representation theory

We begin with a discussion of a well-known probabilistic model of randomly tiling a hexagon drawn on the triangular lattice, and explain its relation to representation theory of unitary groups. This relation produces rather natural tools for analysis of uniformly random lozenge tilings of the hexagon.

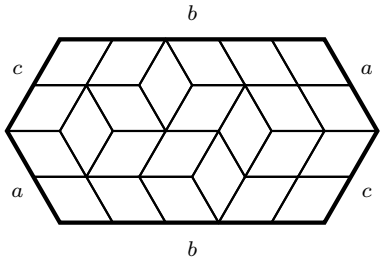


FIG 1. *An example of a lozenge tiling of the hexagon with sides* a, b, c, a, b, c*, where* $a = 2$*,* $b = 5$*, and* $c = 2$*.*

2.1. Lozenge tilings of a hexagon

Consider the problem of tiling a hexagon with sides of length a, b, c, a, b, c drawn on the triangular lattice by *lozenges* that are defined as pairs of triangles glued together (see Fig. 1a). Here a, b, and c are any positive integers, and we assume that the side of an elementary triangle has length 1. There are three different types of lozenges:  ,  , and  . Such tilings (that are in a bijective correspondence with *boxed plane partitions*) can be interpreted in a variety of ways (see Fig. 2):

(1) As *dimers* (or *perfect matchings*) on the dual hexagonal lattice.
(2) As sets of nonintersecting Bernoulli paths following lozenges of two types ( and ) with prescribed beginnings and ends.
(3) As stepped surfaces made of $1 \times 1 \times 1$ cubes.
(4) As interlacing configurations of lattice points — centers of lozenges of one of the types, say,  , as on Fig. 2. Such configurations must have a prescribed number of points in each *horizontal* section that may depend on the section.

Our first goal is to match this combinatorial object with a basic representation theoretic one.

2.2. Representations of unitary groups

Denote by $\mathcal{U}(N)$ the (compact Lie) group of all the unitary matrices[1] of size N. A (finite-dimensional) *representation* of $\mathcal{U}(N)$ is a continuous map

$$T\colon \mathcal{U}(N) \to GL(m, \mathbb{C})$$

(for some $m = 1, 2, \ldots$) which respects the group structure: $T(UV) = T(U)T(V)$, $U, V \in \mathcal{U}(N)$. A representation is called *irreducible* if it has no *invariant* subspaces $E \subset \mathbb{C}^m$ ($E \neq 0$ or $\mathbb{C}^m$), i.e., such that $T(\mathcal{U}(N))E \subset E$.

[1] $U^* = U^{-1}$, where U^* is the conjugate transpose.

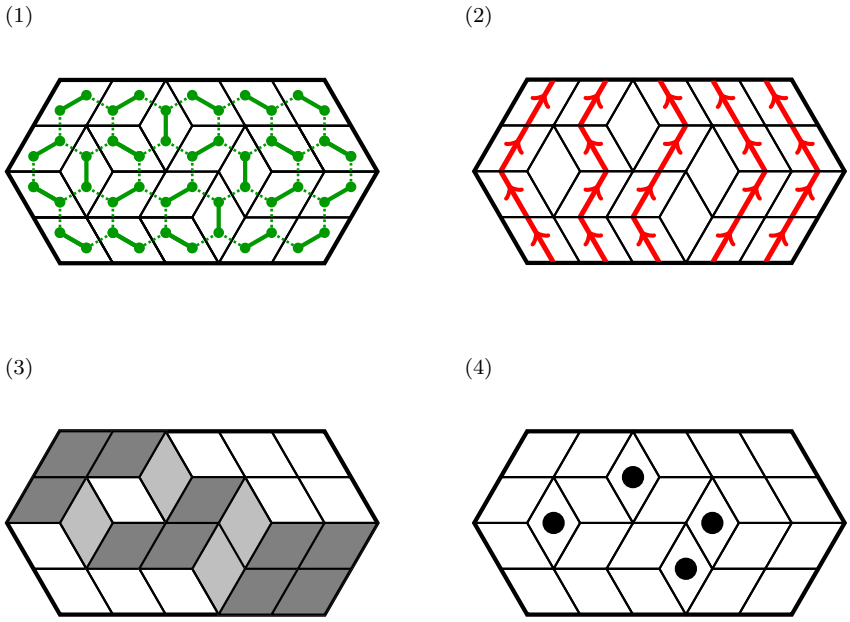


FIG 2. *Various interpretations of a lozenge tiling.*

The classification of irreducible representations of $\mathcal{U}(N)$ (equivalently, of $GL(N,\mathbb{C})$ by analytic continuation — "unitary trick" of H. Weyl) is one of high points of the classical representation theory. It is due to Hermann Weyl in mid-1920's. In order to understand how it works, let us restrict T to the abelian subgroup of diagonal unitary matrices

$$\mathcal{H}_N := \left\{\operatorname{diag}(e^{\mathbf{i}\varphi_1},\dots,e^{\mathbf{i}\varphi_N})\colon \varphi_1,\dots,\varphi_N\in\mathbb{R}\right\}.$$

Any commuting family of (diagonalizable[2]) matrices can be simultaneously diagonalized. In particular, this is true for $T(\mathcal{H}_N)$. Hence, for $1\le j\le m$,

$$\mathbb{C}^m=\bigoplus_{j=1}^m \mathbb{C}v_j,\qquad T\left(\operatorname{diag}(e^{\mathbf{i}\varphi_1},\dots,e^{\mathbf{i}\varphi_N})\right)v_j=t_j(e^{\mathbf{i}\varphi_1},\dots,e^{\mathbf{i}\varphi_N})\cdot v_j,$$

where each t_j is a continuous homomorphism $\mathcal{H}_N\to\mathbb{C}$. Any such homomorphism has the form

$$t(z_1,\dots,z_N)=z_1^{k_1}\dots z_N^{k_N},\qquad k_1,\dots,k_N\in\mathbb{Z}.$$

Each N-tuple $(k_1,\dots,k_N)\in\mathbb{Z}^N$ for $t=t_j$, $1\le j\le m$, is called a *weight* of the representation T. There is a total of m weights (which is the dimension of the representation).

[2]Any finite-dimensional representation of a finite or compact group, in particular, $U(N)$, is unitary in a suitable basis (e.g., see [84]), hence all our matrices are diagonalizable.

Theorem 2.1 (H. Weyl, see, e.g., [82, 84])**.** *Irreducible representations of* $\mathcal{U}(N)$ *are in one-to-one correspondence with ordered* N*-tuples* $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N) \in \mathbb{Z}^N$*.*

The correspondence is established by requiring that λ *is the unique highest (in lexicographic order) weight of the corresponding representation. Then the generating function of all weights of this representation* T_λ *can be written as*

$$\begin{aligned}&\operatorname{Trace}\Big(T_\lambda\big(\operatorname{diag}(z_1,\dots,z_N)\big)\Big)\\&\qquad=\sum_{(k_1,\dots,k_N)\ \text{weight of } T_\lambda} z_1^{k_1}\dots z_N^{k_N}=\frac{\det\left[z_i^{N+\lambda_j-j}\right]_{i,j=1}^N}{\det\left[z_i^{N-j}\right]_{i,j=1}^N}.\end{aligned}\tag{1}$$

Note that the denominator in (1) is the Vandermonde determinant which evaluates to

$$\det\left[z_i^{N-j}\right]_{i,j=1}^N=\prod_{1\leq i<j\leq N}(z_i-z_j).$$

The numerator in (1) is necessarily divisible by the denominator because of its skew-symmetry with respect to $z_i \leftrightarrow z_j$, and thus the ratio is a finite linear combination of the monomials of the form $z_1^{k_1}\dots z_N^{k_N}$, $k_1,\dots,k_N \in \mathbb{Z}$ (i.e., an element of $\mathbb{C}[z_1^{\pm1},\dots,z_N^{\pm1}]^{S(N)}$).

The polynomials $\operatorname{Trace}(T_\lambda)$ are called *Schur polynomials*, after Issai Schur, who used them in the representation theory of the symmetric group in his thesis around 1900. However, one of the earliest appearances of them dates back to Cauchy [32] and Jacobi [53], over 100 years before Weyl's work. The Schur polynomials are denoted by $s_\lambda = \operatorname{Trace}(T_\lambda)$. Schur polynomials are, generally speaking, symmetric homogeneous Laurent polynomials in N variables.

While the ratio of determinants formula (1) is beautiful and concise (it is a special case of *Weyl's character formula* which works for any compact semi-simple Lie group), is does not describe the set of weights explicitly. To do that, we need an elementary

Lemma 2.2. *For any* $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N) \in \mathbb{Z}^N$*,*

$$s_\lambda(z_1,\dots,z_N)=\sum_{\mu\prec\lambda} s_\mu(z_1,\dots,z_{N-1})z_N^{|\lambda|-|\mu|},\tag{2}$$

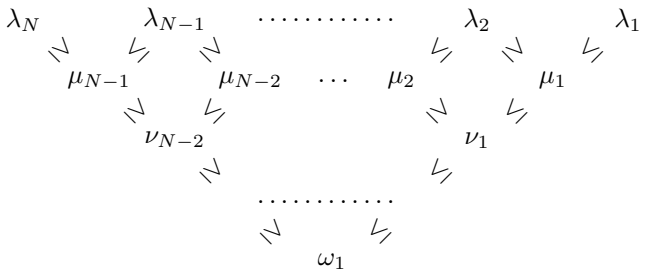
where the sum is taken over $\mu=(\mu_1,\dots,\mu_{N-1}) \in \mathbb{Z}^{N-1}$*, the notation* $\mu \prec \lambda$ *means the* interlacing

$$\lambda_N \leq \mu_{N-1} \leq \lambda_{N-1} \leq \cdots \leq \lambda_2 \leq \mu_1 \leq \lambda_1,$$

and $|\lambda| = \sum_{j=1}^N \lambda_j$*,* $|\mu| = \sum_{j=1}^{N-1} \mu_j$*.*

Proof. Clear the denominators in (2) and compare coefficients by each of the monomials $z_1^{k_1}\dots z_N^{k_N}$. □

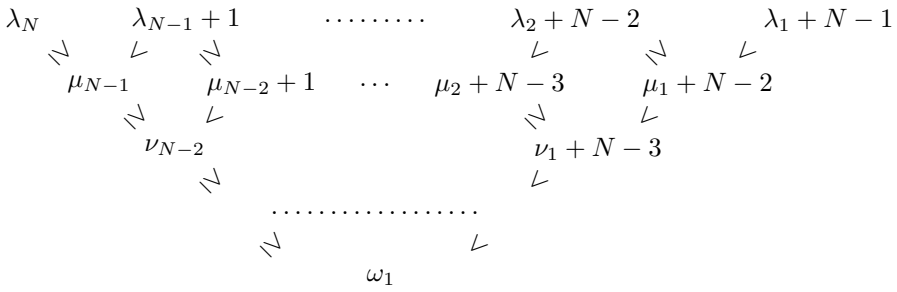
Applying this lemma N times, we see that the weights are in one-to-one correspondence with interlacing triangular arrays of integers



(3)

Such arrays are called *Gelfand–Tsetlin schemes/patterns*, and they will play a prominent role in what follows.

Observe that if we shift all leftmost entries of a Gelfand–Tsetlin scheme of depth (or height) N by 0, the second to left ones by 1, etc., then in the end we obtain a similar array where some of the inequalities become strict:



(4)

Proposition 2.3. *Lozenge tilings of the hexagon with sides a, b, c, a, b, c (Fig. 1) are in one-to-one correspondence with weights of the irreducible representation of the unitary group $\mathcal{U}(a+b)$ with highest weight*

$$\lambda = \big(\underbrace{b, b, \dots, b}_{a \text{ times}}, \underbrace{0, 0, \dots, 0}_{c \text{ times}}\big). \tag{5}$$

Proof. By picture, see Fig. 3. If we coordinatize by taking the centers of the vertical lozenges in the coordinate system on the picture, then we read off the shifted Gelfand–Tsetlin schemes (4). □

The total number of weights of T_λ (or, equivalently, the dimension of the representation) was denoted by $m = m(\lambda)$ above, and it is given by

Proposition 2.4. *For any $\lambda = (\lambda_1 \ge \dots \ge \lambda_N) \in \mathbb{Z}^N$,*

$$\dim T_\lambda = s_\lambda\big(\underbrace{1, 1, \dots, 1}_{N}\big) = \prod_{1 \le i < j \le N} \frac{(\lambda_i - i) - (\lambda_j - j)}{j - i}. \tag{6}$$

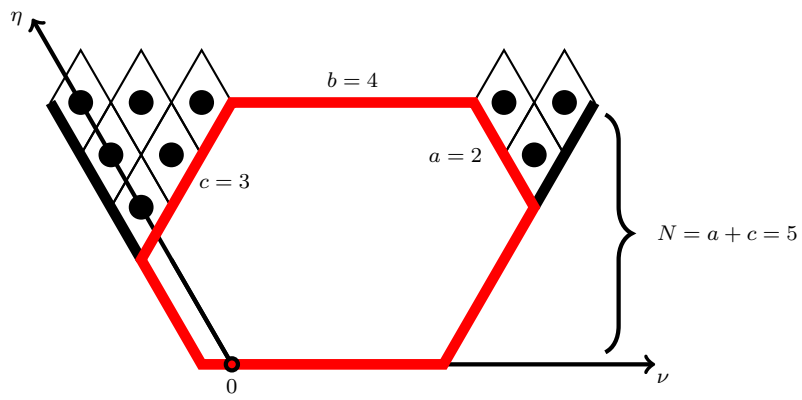


FIG 3. *On the correspondence between lozenge tilings of a hexagon and weights.*

This is a special case of *Weyl's dimension formula* (which again works for any compact semi-simple Lie group).

Proof. Follows from (1) either directly (by the L'Hôpital's rule), or through the substitution $(z_1, \dots, z_N) = (1, q, q^2, \dots, q^{N-1})$ and the limit $q \to 1$. □

2.3. Distribution of lozenges on a horizontal slice

Consider the uniform probability measure $\mathsf{Prob}_{a,b,c}$ on the space of all lozenge tilings of the hexagon with sides a, b, c, a, b, c (see Fig. 1). The normalizing factor in the measure $\mathsf{Prob}_{a,b,c}$ (the so-called *partition function*) is given in (6) with λ as in (5).

Remark 2.5. For tilings of the hexagon with sides of length a, b, c, a, b, c the partition function was first computed in a nicer product form

$$s_{(b,b,\dots,b,0,0,\dots,0)} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

by MacMahon [61].

Let us focus on what happens when we consider one horizontal slice of our uniformly random lozenge tiling. As above, we coordinatize it by locations of centers of the vertical lozenges. It is easiest to assume that the slice is close enough to one of the two horizontal boundaries, say, the lowest one (see Fig. 4).

Proposition 2.6. *For any h, $0 \le h \le \min(a, c)$, the distribution of lozenges on the horizontal slice at height h has the form*

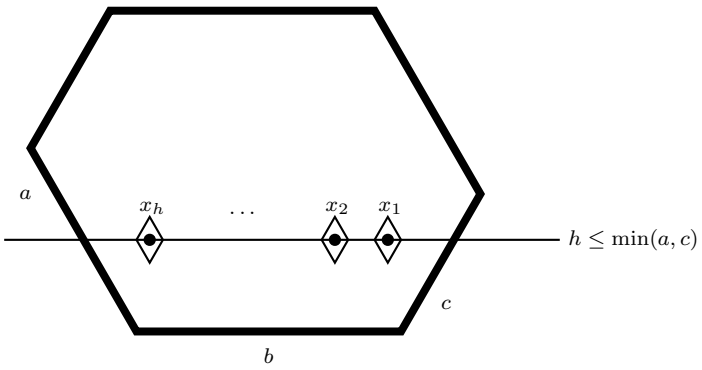


FIG 4. *Horizontal slice of a lozenge tiling.*

$$\mathsf{Prob}_{a,b,c,h}\{x_1,\dots,x_h\} = \mathrm{const}(a,b,c,h)\cdot \prod_{1\le i<j\le h}(x_i-x_j)^2 \prod_{i=1}^{h} w_{a,b,c,h}(x_i), \tag{7}$$

where

$$w_{a,b,c,h}(x) = \frac{(b+c-1-x)!\,(a-h+x)!}{x!\,(b+h-1-x)!}. \tag{8}$$

Remark 2.7. Probability measures of the form (7) with arbitrary positive weight function $w(\cdot)$ are known as *orthogonal polynomial ensembles* as they are closely related to the orthogonal polynomials with weight w. The measure (7) itself is often referred to as the Hahn orthogonal polynomial ensemble, as this particular weight w (8) corresponds to the classical Hahn orthogonal polynomials. See, e.g., [58] and references therein for details.

Proof. We can cut the enumeration problem into two that look like those on Fig. 5, and then multiply the results. Each of the two problems (compute the number of tilings of the corresponding region with fixed top row) is solved by the dimension formula (6). □

The computation of Proposition 2.6 already allows to see asymptotic transitions for a fixed h (for example, $h = 1$). We can rewrite (8) as

$$w_{a,b,c,h}(x) = \frac{(b+c-1)!\,(a-h)!}{(b+h-1)!}\times \tag{9}$$
$$\times\left[\frac{1}{x!}\cdot\frac{(a-h+1)\dots(a-h+x)\cdot(b+h-1)\dots(b+h-x)}{(c+b-1)\dots(c+b-x)}\right].$$

One can consider the following limit regimes:

(1) If $a, b, c \to \infty$ so that $ab/c \to t$, the first term just contributes to a constant, while the second one converges to $t^x/x!$.

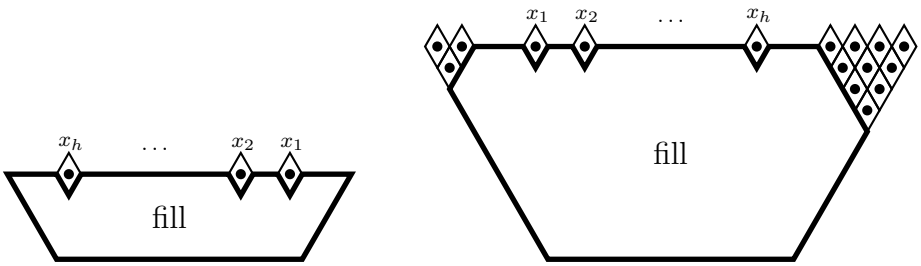


FIG 5. *Computing the distribution of lozenges on a horizontal slice (cf. Fig. 4) amounts to two enumeration problems.*

(2) In a similar way, if we keep a finite and send $b, c \to \infty$ in such a way that $b/(b+c) \to \xi$, $0 < \xi < 1$, then we see that the relevant part of (9) converges to

$$\frac{(a-h+1)(a-h+2)\dots(a-h+x)}{x!}\xi^x.$$

(3) A slightly more complicated limit transition would be to take $a, b, c \to \infty$ so that the triple ratio $a : b : c$ has a finite limit. Then Stirling's formula shows that after proper shifting and scaling of $(x_1, \dots, x_h)$, which would not affect the factor $\prod_{1\le i<j\le h}(x_i - x_j)^2$, the nontrivial part of $w_{a,b,c,h}(x)$ converges to a Gaussian weight $e^{-x^2/2}$, $x \in \mathbb{R}$.

There is also a representation-theoretic way to view these results. Restricting to a fixed horizontal slice means that we only care about the restriction of our representation of $\mathcal{U}(a+c)$ (recall Proposition 2.3) to the subgroup $\mathcal{U}(h)$ of matrices which are nontrivial (i.e., different from Id) only in the top-left $h \times h$ corner. In terms of weights, we only care about powers of $z_1, \dots, z_h$ and substitute $z_{h+1} = \dots = z_N = 1$. This is equivalent to saying that the probability (7) of $(x_1, \dots, x_h) = (\mu_1 + h - 1, \mu_2 + h - 2, \dots, \mu_h)$ is the normalized coefficient of $s_\mu(z_1, \dots, z_h)$ in the identity

$$\frac{s_\lambda(z_1, \dots, z_h, 1, \dots, 1)}{s_\lambda(1, \dots, 1)} = \sum_{\mu_1 \ge \dots \ge \mu_h} \mathsf{Prob}_{a,b,c,h}\{\mu\} \cdot \frac{s_\mu(z_1, \dots, z_h)}{s_\mu(1, \dots, 1)}, \tag{10}$$

where λ is as in (5), and we are dividing by the normalizing constants to have the "Prob" coefficients add up to 1. This corresponds to looking at relative dimensions of *isotypical* subspaces (i.e., those that transform according to fixed irreducible representation) rather than the actual ones.

The first two of the above three limit transitions turn (10) into

$$\frac{ab}{c} \to t: \qquad \prod_{i=1}^{h} e^{t(z_i - 1)} \qquad = \sum_{\mu_1 \ge \dots \ge \mu_h} \mathsf{Prob}_{t,h}\{\mu\} \cdot \frac{s_\mu(z_1, \dots, z_h)}{s_\mu(1, \dots, 1)}; \tag{11}$$

$$\frac{b}{b+c}\to\xi:\qquad \prod_{i=1}^{h}\frac{(1-\xi)^a}{(1-\xi z_i)^a}\quad=\sum_{\mu_1\geq\cdots\geq\mu_h}\mathsf{Prob}_{a,\xi,h}\{\mu\}\cdot\frac{s_\mu(z_1,\dots,z_h)}{s_\mu(1,\dots,1)}. \tag{12}$$

In fact, these two limits $ab/c\to t$ and $b/(b+c)\to\xi$ correspond to certain infinite-dimensional representations of the infinite-dimensional unitary group $\mathcal{U}(\infty)=\varinjlim\mathcal{U}(N)$.

The third (Gaussian) limit is the eigenvalue projection of the matrix Fourier transform identity

$$\int\limits_{\mathrm{Herm}(N)} e^{\mathbf{i}\,\mathrm{Trace}(AB)}\mathrm{M}(dB)=e^{-\,\mathrm{Trace}(A^2)/2},$$

where $\mathrm{Herm}(N)$ is the space of $N\times N$ Hermitian ($H^*=H$) matrices, $A\in\mathrm{Herm}(N)$, and $\mathrm{M}(dB)$ is the probability measure on $\mathrm{Herm}(N)$ with the density $e^{-\,\mathrm{Trace}(B^2)/2}dB$ also known as the *Gaussian Unitary Ensemble* (or *GUE*). This limit is a special case of the so-called quasi-classical limit in representation theory that degenerates "large" representations to probability measures on (coadjoint orbits of) the associated Lie algebra, e.g., see [76, 51, 50]. For a broad survey of quantization ideas in representation theory see e.g. [57] and references therein.

2.4. Scalar operators and observables

We are interested in more complex limit transitions than those in §2.3, and for accessing them the following representation theoretic thinking is useful. Our probability weights (7) arise as relative dimensions of the isotypical subspaces in the representation space for $\mathcal{U}(N)$. Moreover, these subspaces are blocks of identical irreducibles with respect to the action of the smaller group $\mathcal{U}(h)$.

2.4.1. Locally scalar operators

The problem of decomposing a representation on irreducible components is often referred to as the *problem of* (*noncommutative*) *harmonic analysis*. It can be viewed as a noncommutative Fourier transform — an analogue of the classical Fourier transform when $\mathbb{R}$ acts by shifts on $L^2(\mathbb{R})$. The "best" way to solve such a problem would be to find operators in the representation space which project to a given isotypical component. For the classical Fourier transform, these operators have the form

$$f\mapsto\int_{-\infty}^{+\infty}e^{-\mathbf{i}xp}f(x)dx.$$

For the action of the symmetric group, such operators are known under the name *Young symmetrizers*, they date back to the earliest days of representation theory. However, even if one can construct such operators, they are quite

complicated. The "next best" thing is to find operators which are scalar in each irreducible representation (the projection operators take value 1 in one irreducible representation, and 0 in all other irreducible representations). By a simple Schur's lemma, such operators are exactly those that commute with the action of the group.

2.4.2. Dilation operators

Observe that $\mathcal{U}(h)$ has a nontrivial center — scalar matrices of the form $e^{\mathbf{i}\varphi}\cdot\mathbf{1}$, $\varphi\in\mathbb{R}$. Their action on elements $\operatorname{diag}(z_1,\dots,z_h)\in\mathcal{H}_h$ amounts to multiplying each z_j by $e^{\mathbf{i}\varphi}$, and their action on a vector of weight $(k_1,\dots,k_h)\in\mathbb{Z}^h$ is the multiplication by $e^{\mathbf{i}\varphi|k|}=e^{\mathbf{i}\varphi(k_1+\dots+k_h)}$ (see §2.2). Hence, using the homogeneity of the Schur polynomials we see that on an irreducible representation of $\mathcal{U}(h)$ with highest weight $\mu=(\mu_1\ge\dots\ge\mu_h)$ such an operator acts as the scalar operator $e^{\mathbf{i}\varphi|\mu|}\cdot\mathbf{1}$.

Let us now apply such an operator, viewed simply as the dilation operator $(\mathcal{D}_\varphi f)(z_1,\dots,z_h)=f(e^{\mathbf{i}\varphi}z_1,\dots,e^{\mathbf{i}\varphi}z_h)$, to the decomposition identity (11) defining $\mathsf{Prob}_{t,h}\{\mu\}$:

$$\prod_{j=1}^{h}e^{t(e^{\mathbf{i}\varphi}z_j-1)}=\sum_{\mu_1\ge\dots\ge\mu_h}e^{\mathbf{i}\varphi|\mu|}\,\mathsf{Prob}_{t,h}\{\mu\}\cdot\frac{s_\mu(z_1,\dots,z_h)}{s_\mu(1,\dots,1)}$$

(clearly, $\mathcal{D}_\varphi\, s_\mu=e^{\mathbf{i}\varphi|\mu|}s_\mu$). Setting $z_1=\dots=z_h=1$ above, we get

$$e^{ht(e^{\mathbf{i}\varphi}-1)}=\sum_{\mu_1\ge\dots\ge\mu_h}e^{\mathbf{i}\varphi|\mu|}\,\mathsf{Prob}_{t,h}\{\mu\}.$$

This immediately tells us ("for free"), that $|\mu|$ has the Poisson distribution with parameter ht because the left-hand side is the characteristic function of that distribution.

2.4.3. Quadratic Casimir–Laplace operator

Going further, the first nontrivial example of an operator which commutes with the action of $\mathcal{U}(h)$ is the so-called *quadratic Casimir–Laplace operator* $\mathcal{C}_2$. Its action on functions on $\mathcal{H}_h$ is given by

$$(\mathcal{C}_2 f)(z_1,\dots,z_h)$$
$$=\prod_{1\le i<j\le h}(z_i-z_j)^{-1}\sum_{r=1}^{h}\left(z_r\frac{\partial}{\partial z_r}\right)^2\prod_{1\le i<j\le h}(z_i-z_j)\,f(z_1,\dots,z_h).$$

Such operators exist for all semi-simple Lie groups and are one of the basic representation-theoretic objects. Also,

$$\mathcal{C}_2-\sum_{j=1}^{h-1}j^2 \tag{13}$$

is the (projection to eigenvalues of the) generator of the Brownian motion on $\mathcal{U}(h)$. In other words, (13) is the generator of the circular Dyson Brownian motion [40, 39]. See also §4.1 below for a related Markov dynamics.

It is immediate to see (using the ratio of determinants formula (1) and the fact that $(z\frac{\partial}{\partial z})z^k = kz^k$) that the action of the quadratic Casimir–Laplace operators on the Schur polynomials is diagonal, and

$$\mathcal{C}_2 s_\mu = \sum_{i=1}^{h} (\mu_i + h - i)^2 s_\mu.$$

We could now proceed with the application of $\mathcal{C}_2$ to (11). However, let us first note that the dilation operators $\mathcal{D}_\varphi$ can be written in a form rather similar to $\mathcal{C}_2$:

$$(\mathcal{D}_\varphi f)(z_1,\dots,z_h) = \prod_{1\le i<j\le h} (z_i - z_j)^{-1}\, e^{\mathbf{i}\varphi\left(\sum_{r=1}^{h} z_r \frac{\partial}{\partial z_r} - \frac{h(h-1)}{2}\right)} \prod_{1\le i<j\le h} (z_i - z_j)\, f(z_1,\dots,z_h).$$

Indeed, the desired eigenrelation $\mathcal{D}_\varphi\, s_\mu = e^{\mathbf{i}\varphi|\mu|} s_\mu$ again follows from (1) and the fact that $e^{\mathbf{i}\varphi z\frac{\partial}{\partial z}} z^k = e^{\mathbf{i}\varphi k} z^k$.

2.4.4. A q-deformation

Let us now note that we have a general recipe on our hands of constructing operators which have Schur functions as their eigenfunctions. Namely, for any operator of the form

$$(\mathcal{D}\, f)(z_1,\dots,z_h) = \prod_{1\le i<j\le h} (z_i - z_j)^{-1} \Big(\sum_{r=1}^{h} \mathcal{D}^{(z_r)}\Big) \prod_{1\le i<j\le h} (z_i - z_j)\, f(z_1,\dots,z_h) \tag{14}$$

with

$$\mathcal{D}^{(z)}\, z^k = d_k z^k,$$

we have

$$\mathcal{D}\, s_\mu = \Big(\sum_{i=1}^{h} d_{\mu_i+h-i}\Big) s_\mu. \tag{15}$$

For example, we can take

$$\mathcal{D}^{(z)} = \mathcal{T}_{q,z}, \qquad (\mathcal{T}_{q,z}\, f)(z) = f(qz), \qquad d_k = q^k,$$

where $q \in \mathbb{C}$ is a parameter.

Then using (14) we obtain a q-difference operator that can be rewritten in the form

$$\mathcal{D}^{(1)} = \sum_{i=1}^{h} \prod_{j \neq i} \frac{qz_i - z_j}{z_i - z_j} \, \mathcal{T}_{q,z_i}, \tag{16}$$

and (15) gives

$$\mathcal{D}\, s_\mu = \Big(\sum_{i=1}^{h} q^{\mu_i + h - i} \Big) s_\mu. \tag{17}$$

2.5. Contour integrals and the density function

We can now apply $\mathcal{D}^{(1)}$ (16) to the identity (11) defining the measure $\mathsf{Prob}_{t,h}$. This gives

$$\begin{aligned} e^{t \sum_{r=1}^{h} (z_r - 1)} \sum_{i=1}^{h} \prod_{j \neq i} \frac{qz_i - z_j}{z_i - z_j} \, e^{t(qz_i - z_i)} \\ = \sum_{\mu_1 \geq \cdots \geq \mu_h} \Big(\sum_{r=1}^{h} q^{\mu_r + h - r} \Big) \mathsf{Prob}_{t,h}\{\mu\} \cdot \frac{s_\mu(z_1, \ldots, z_h)}{s_\mu(1, \ldots, 1)}. \end{aligned} \tag{18}$$

As before, we would like to substitute $z_1 = \cdots = z_h = 1$ in the above identity. However, observe that the left-hand side is not well-suited for that. A standard trick helps — the left-hand side can be rewritten as a simple contour integral:

Lemma 2.8. *Let $f \colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Then*

$$\sum_{r=1}^{h} \prod_{j \neq r} \frac{qz_r - z_j}{z_r - z_j} \cdot \frac{f(qz_r)}{f(z_r)} = \frac{1}{2\pi \mathbf{i}} \oint_{\{w\}} \prod_{j=1}^{h} \frac{qw - z_j}{w - z_j} \frac{1}{qw - w} \frac{f(qw)}{f(w)} dw,$$

where the integration contour goes around $z_1, \ldots, z_h$ is the positive direction.

Using the above lemma and setting $z_1 = \cdots = z_h = 1$, we read from (18):

$$\begin{aligned} \frac{1}{2\pi \mathbf{i}} \oint_{|w-1|=\varepsilon} \left(\frac{qw - 1}{w - 1} \right)^h \frac{1}{(q-1)w} e^{t(q-1)w} dw \\ = \sum_{\mu_1 \geq \cdots \geq \mu_h} \Big(\sum_{r=1}^{h} q^{\mu_r + h - r} \Big) \mathsf{Prob}_{t,h}\{\mu\}. \end{aligned} \tag{19}$$

The quantity in the right-hand side of (19) does not seem very probabilistic, but we can now use the arbitrariness of the parameter q. For any $n \in \mathbb{Z}$, we can compare the coefficients of q^n in both sides of (19). This amounts to integrating the left-hand side again (with dq/q^{n+1}), and thus yields:

Theorem 2.9. *For any $t \geq 0$ and $h = 1, 2, \ldots$,*

$$\mathsf{Prob}_{t,h}\left\{n \in \{\mu_i + h - i\}_{i=1}^{h}\right\} \tag{20}$$
$$= \frac{1}{(2\pi\mathbf{i})^2} \oint_{|q|=\varepsilon} \frac{dq}{q^{n+1}} \oint_{|w-1|=\varepsilon} \left(\frac{qw-1}{w-1}\right)^h \frac{e^{t(q-1)w}}{(q-1)w} dw.$$

The left-hand side of (20) is a very meaningful probabilistic quantity — it is the probability of seeing a vertical lozenge at any given location on the horizontal slice (cf. Fig. 4). This is the so-called *density function* of the measure $\mathsf{Prob}_{t,h}$. Furthermore, we see that the right-hand side of (20) is well-suited for asymptotics. We perform the asymptotic analysis in the next section.

Remark 2.10. Theorem 2.9 is a special case of a more general formula that represents correlation functions of the so-called *Schur measures* as multiple contour integrals. See [24] and references therein for details.

3. Asymptotics of tiling density via double contour integrals

Here we perform an asymptotic analysis of the density function (20) of the measure $\mathsf{Prob}_{t,h}$ on the hth horizontal slice in the regime

$$t = \tau L, \qquad n = \nu L, \qquad h = \eta L, \qquad L \to \infty, \tag{21}$$

where n is the point of observation in (20), which also must be scaled to yield nontrivial asymptotics. The limit regime (21) is quite nontrivial and is not achievable via elementary tools (in contrast with the limit transitions in §2.3). The reader may want to peek at Figures 10 and 11 below to see what type of description we are aiming at.

Changing variables $q \mapsto v = wq$, $dq = dv/w$ in the left-hand side of (20) gives

$$\mathsf{Prob}_{t,h}\left\{n \in \{\mu_i + h - i\}_{i=1}^{h}\right\}$$
$$= \frac{1}{(2\pi\mathbf{i})^2} \oint_{|v|=\varepsilon} \oint_{|w-1|=\varepsilon} \frac{dv \cdot w^{-1}}{v^{n+1} w^{-n-1}} \left(\frac{v-1}{w-1}\right)^h \frac{e^{t(v-w)} dw}{v-w}$$
$$= \frac{1}{(2\pi\mathbf{i})^2} \oint_{\Gamma_0} \frac{dv}{v} \oint_{\Gamma_1} dw \frac{e^{tv}(v-1)^h v^{-n}}{e^{tw}(w-1)^h w^{-n}} \frac{1}{v-w}.$$

Here by Γ_0 and Γ_1 we have denoted small positively oriented contours around 0 and 1, respectively. Further analysis uses the original idea of Okounkov [69] and largely follows [22]. We observe that the integrand above has the form

$$\frac{e^{L(F(v)-F(w))}}{v(v-w)}, \qquad F(z) := \tau z + \eta \ln(z-1) - \nu \ln z.$$

If we manage to deform the contours in such a way that $\Re(F(v) - F(w)) < 0$ on them except for possibly finite number of points (where $\Re$ denotes the real part),

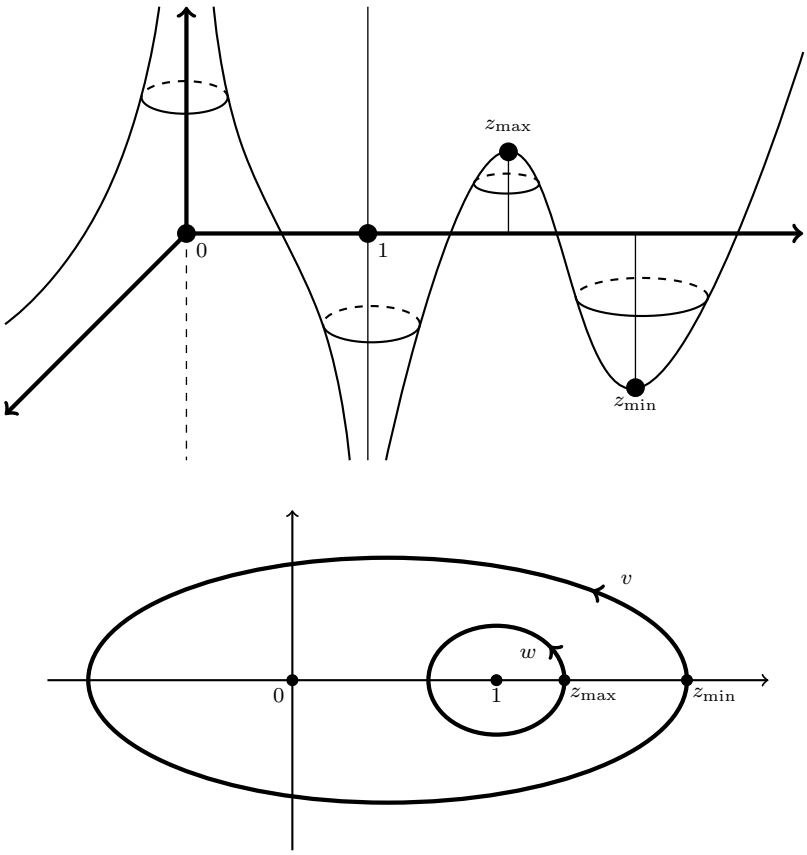


FIG 6. *Case 1. Plot of* $\Re(F(z))$ *(top), and the deformed contours of integration (bottom). In this case the two critical points are real and* > 1.

then our integral would asymptotically vanish as $L \to \infty$. The deformation depends on the location of the critical points of $F(z)$, i.e., of the roots of the equation

$$F'(z) = \frac{\tau z(z-1) + \eta z - \nu(z-1)}{z(z-1)} = 0. \tag{22}$$

The discriminant of the numerator has the form

$$\text{discr} = \big(\nu - (\sqrt{\tau} - \sqrt{\eta})^2\big)\big(\nu - (\sqrt{\tau} + \sqrt{\eta})^2\big). \tag{23}$$

We will now consider all possible cases one by one.

Case 1. $\sqrt{\nu} > \sqrt{\tau} + \sqrt{\eta}$**.** In this case both roots of (22) are real and greater than 1. The plot of $\Re(F(z))$ looks as on Fig. 6 (top). Moving the v contour to the level line $\Re(F(v)) = \Re(F(z_{\min}))$ and the w contour to the level line $\Re(F(w)) =$

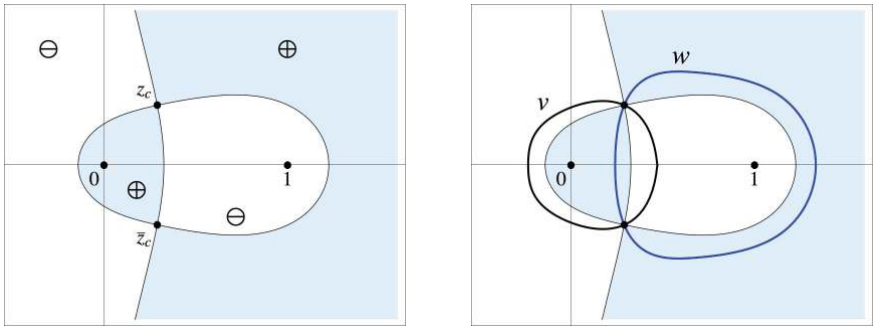


FIG 7. *Case 2. Contour plot of* $\Re(F(z)-F(z_c))$ *(left), and the deformed contours (right).*

$\Re(F(z_{\max}))$, we achieve that $\Re(F(v)-F(w)) = F(z_{\min})-F(z_{\max}) < 0$, which implies the desired vanishing. However, in the process of deformation, the v contour, which was originally a small circle around the origin, has swallowed the w contour, see Fig. 6 (bottom). Because of $(v-w)^{-1}$ in the integrand, we have to compensate the result of moving the contours by subtracting the residue

$$-\frac{1}{2\pi\mathbf{i}}\oint_{\Gamma_1} dw \operatorname*{Res}_{v=w}\frac{1}{v}\frac{e^{L(F(v)-F(w))}}{v-w} = -\frac{1}{2\pi\mathbf{i}}\oint_{\Gamma_1}\frac{dw}{w} = 0.$$

Thus, we see that for $\sqrt{\nu} > \sqrt{\tau}+\sqrt{\eta}$, the density of vertical lozenges asymptotically vanishes.

Case 2. $|\sqrt{\tau}-\sqrt{\eta}| < \sqrt{\nu} < \sqrt{\tau}+\sqrt{\eta}$**.** In this case, two critical points — solutions of (22) — are complex conjugate. Consider the contour plot of $\Re(F(z)-F(z_c))$, where we have shifted $F(z)$ by the value of F at the upper critical point, $F'(z_c) = 0$, $\Im(z_c) > 0$ ($\Im$ denotes the imaginary part). This contour plot looks like Fig. 7 (left). Deforming the w contour into the region where $\Re(F(w))$ is greater than $\Re(F(z_c))$, and the v contour into the region where $\Re(F(v))$ is less than $\Re(F(z_c))$, we again achieve that $\Re(F(v)-F(w)) < 0$ on the deformed contours. However, in the process of deformation, we pick up the residue

$$-\frac{1}{2\pi\mathbf{i}}\int_{\bar{z}_c}^{z_c} dw \operatorname*{Res}_{v=w}\frac{1}{v}\frac{e^{L(F(v)-F(w))}}{v-w} = \frac{1}{2\pi\mathbf{i}}\int_{\bar{z}_c}^{z_c}\frac{dw}{w} = \frac{\arg(z_c)}{\pi},$$

which is the limiting density function for vertical lozenges in this regime.

Case 3. $0 < \sqrt{\nu} < |\sqrt{\tau}-\sqrt{\eta}|$**.** This final case contains two subcases depending on whether $\sqrt{\tau} > \sqrt{\eta}$ or $\sqrt{\tau} < \sqrt{\eta}$.

In the first one, the plot of $\Re(F(z))$ looks as on Fig. 8 (upper). Deforming the integration contours to level lines (similarly to what was done before in Case 1) requires no residue-picking. Thus, the limiting density is zero for the subcase $\sqrt{\tau} > \sqrt{\eta}$.

In the second subcase, the picture is slightly different, see Fig. 8 (lower). The familiar deformation of the contours to the level lines now requires that the w

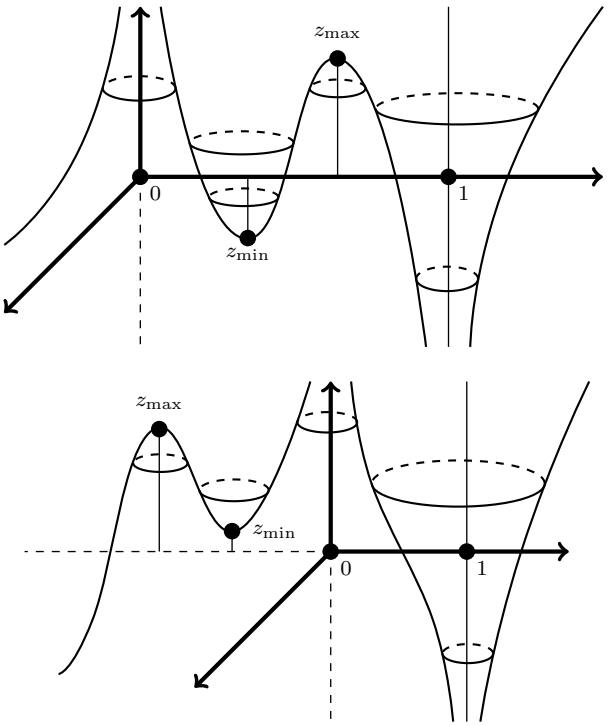


FIG 8. *Case 3. Plots of $\Re(F(z))$ for two subcases, $\tau > \eta$ (upper) and $\tau < \eta$ (lower).*

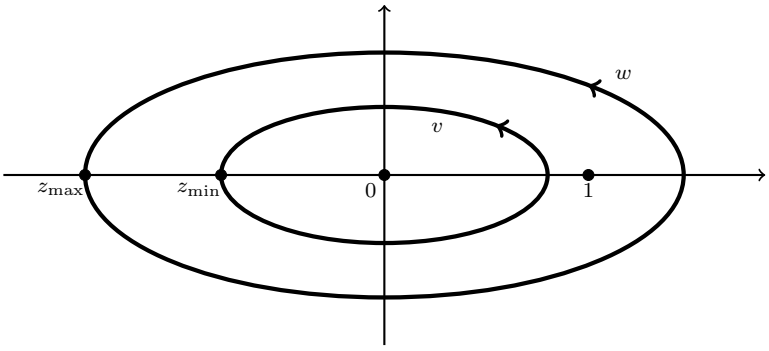


FIG 9. *Case 3. Deformed contours when $\tau < \eta$.*

contour swallows the v contour (see Fig. 9). This results in the extra residue

$$-\frac{1}{2\pi\mathbf{i}}\oint_{\Gamma_0}\frac{dv}{v}\operatorname*{Res}_{w=v}\frac{e^{L(F(v)-F(w))}}{v-w}=\frac{1}{2\pi\mathbf{i}}\oint_{\Gamma_0}\frac{dv}{v}=1.$$

Thus, the limiting density is 1 in this case.

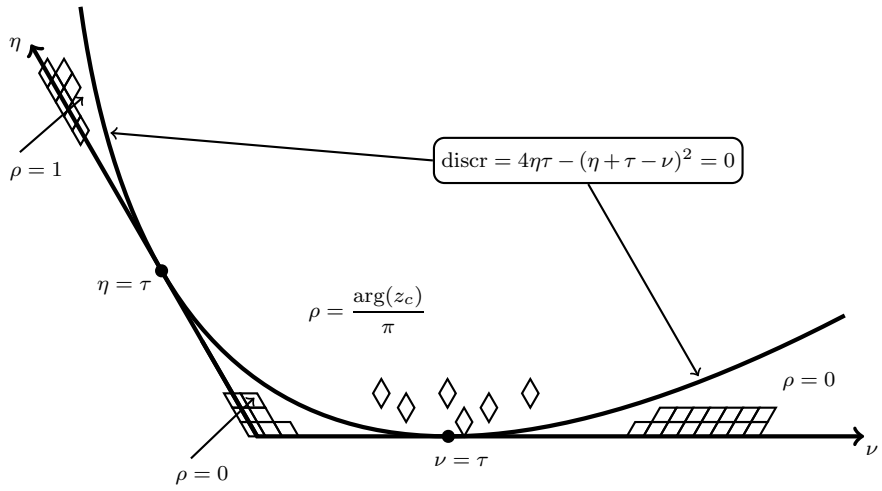


FIG 10. *Limiting density of the vertical lozenges in the* (ν, η)*-plane.*

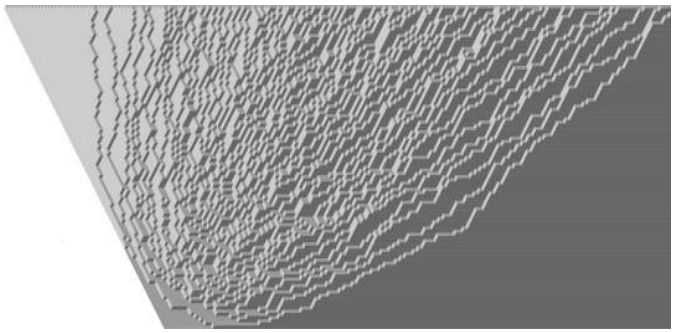


FIG 11. *Simulation of the limiting distribution of lozenges. See also [41].*

Summarizing, we see that the asymptotic density of the vertical lozenges is nontrivial for each given τ, inside the parabola $\mathrm{discr} = 0$ (23) in the (ν, η)-plane. Outside of this parabola, the density of the vertical lozenges either vanishes or tends to 1, signaling the *frozen parts* (*facets*) of the limit shape, see Figures 10 and 11.

In a similar way, using products of operators $\mathcal{D}^{(1)}$ with different values of q, one can extract integral representations for higher *correlation functions* of vertical lozenges (i.e., probabilities that a given set of locations is occupied by vertical lozenges). Those integral representations can be analyzed exactly in the same fashion as above, this was done in [22]. Indeed, if one knows (here h is fixed, but one can also handle different h's)

$$\sum_{\mu_1 \geq \cdots \geq \mu_h} \Bigl(\sum_{i=1}^{h} q_1^{\mu_i + h - i} \Bigr) \cdots \Bigl(\sum_{i=1}^{h} q_s^{\mu_i + h - i} \Bigr) \mathsf{Prob}_{t,h}\{\mu\}$$

for any $q_1, \dots, q_s \in \mathbb{C}$, one can extract the order s correlation function by looking at coefficients of monomials $q_1^{n_1} q_2^{n_2} \dots q_s^{n_s}$. The result reproduces known formulas for the correlation functions of the so-called *Schur processes*, e.g., see [24] and references therein.

It should also be possible to carry out a similar program for the case of the growing hexagon with sides a, b, c, a, b, c when the triple ratio $a : b : c$ remains constant. This would require analyzing the asymptotics of ratios of the form

$$\frac{s_\lambda(q_1, \dots, q_s, 1, \dots, 1)}{s_\lambda(1, 1, \dots, 1)}$$

with growing λ as in (5), which can probably be done via recently developed techniques of [46].

In a different way, integral representations for the correlation functions in the hexagon were recently obtained and asymptotically analyzed in [70, 71, 72].

4. Markov dynamics

Our next goal is to add an extra dimension to our probabilistic models by introducing suitable Markov evolutions on them. This is not obvious and requires preliminary work.

4.1. Dyson Brownian motion and its discrete counterparts

A hint at the existence of a nontrivial Markov dynamics comes from the relation to random matrices mentioned before (in particular, see the third limit regime in §2.3). Indeed, a GUE matrix of size $N \times N$ has density with respect to the Lebesgue measure on the linear space $\mathrm{Herm}(N)$ of Hermitian $N \times N$ matrices given by

$$\mathrm{M}(dX) = e^{-\operatorname{Trace}(X^2)/2} dX = \prod_{i=1}^{N} e^{-x_{ii}^2/2} \prod_{1 \le i < j \le N} e^{-(\Re x_{ij})^2} e^{-(\Im x_{ij})^2} dX,$$

where $X = [x_{ij}]_{i,j=1}^N$. Thus, the N^2 quantities

$$(x_{ii}, \sqrt{2} \cdot \Re x_{ij}, \sqrt{2} \cdot \Im x_{ij}) \tag{24}$$

are independent identically distributed standard normal random variables. Following Dyson [39], one can replace these variables by standard Brownian motions. A nontrivial computation shows that the corresponding Markov process on Hermitian matrices projects to a *Markov* process on the spectra of matrices. The generator of the process on the spectra is given by (here $\operatorname{Spec}(X) = (x_1, \dots, x_N)$):

$$\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{N} \Big(\sum_{j \ne i} \frac{1}{x_i - x_j} \Big) \frac{\partial}{\partial x_i} \tag{25}$$

$$= \prod_{1\le i<j\le N} (x_i - x_j)^{-1} \circ \frac{1}{2}\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \circ \prod_{1\le i<j\le N} (x_i - x_j).$$

Here on the right, $\circ$ means composition of operators: First, we multiply by the Vandermonde determinant $\prod_{i<j}(x_i - x_j)$, then apply the Laplacian, and after that divide by the Vandermonde determinant, similarly to (14) above. The projection of the (random) matrix $X(t) \in \mathrm{Herm}(N)$ (evolving according to standard Brownian motions of its elements (24)) to the spectrum then has the distribution density (see for example [4])

$$\text{const} \cdot \prod_{1\le <j\le N} (x_i - x_j)^2 \prod_{i=1}^{N} e^{-x_i^2/2t}.$$

The dynamics with generator (25) (called the *Dyson Brownian motion*) can be easily mimicked for all the ensembles of the form $\text{const}\cdot\prod_{i<j}(x_i-x_j)^2\prod_i w(x_i)$ considered in §2.3. Let us focus on the Poisson ($ab/c \to t$) case, when $w(x) = t^x/x!$, $x \in \mathbb{Z}_{\ge 0}$. Consider a Markov jump process with generator

$$L^{(N)}_{\text{Poisson}} = \prod_{1\le i<j\le N} (x_i - x_j)^{-1} \circ \sum_{i=1}^{N} \nabla_i \circ \prod_{1\le i<j\le N} (x_i - x_j), \tag{26}$$

where $(\nabla f)(x) = f(x+1) - f(x)$ is the generator of the standard Poisson process, and ∇_i acts as ∇ on the ith coordinate. One easily checks that the measures with $w(x) = t^x/x!$ are generated by the above Markov process started from the initial condition $(x_1, \dots, x_N) = (N-1, N-2, \dots, 1, 0)$.

The process with generator (26) can be obtained by conditioning independent Poisson processes not to intersect until time $+\infty$, and also to grow at the same rate:

$$\lim_{t\to\infty} \frac{x_1}{t} = \cdots = \lim_{t\to\infty} \frac{x_N}{t}.$$

(Different growth rates of different x_i's will result in conjugating $\sum_{i=1}^N \nabla_i$ by a different function, cf. [59, 26].) This is similar to the stationary version of the Dyson Brownian motion being obtained from independent one-dimensional standard Brownian motions by conditioning on the event that they never intersect, and, moreover, stay within the distance $o(\sqrt{\text{time}})$ from the origin as time goes to plus or minus infinity.

4.2. Gibbs property and stochastic links

There is also another "perpendicular" Markovian structure on the measures from §2.3. Observe that the uniform measure on lozenge tilings has the following property: If we pick a domain inside the hexagon, then fixing the boundary lozenge configuration induces the uniform measure on tilings of the interior.

This seemingly trivial observation becomes useful when the hexagon becomes infinitely large in some way (as in §2.3). Then the global uniform measure makes no sense, but this property survives. We will refer to it as to the *Gibbs property*.

In particular, fixing h vertical lozenges on the horizontal slice of height h (as on Fig. 4) induces the uniform measure on the set of all configurations of lozenges between this slice and the lower border (height zero). Thus, given locations $x^{(h)} = (x_1^{(h)}, \dots, x_h^{(h)})$ of the vertical lozenges on the h-th slice, the distribution of $h-1$ vertical lozenges at height $h-1$ is given by the ratio (assuming that $x^{(h-1)}$ interlaces $x^{(h)}$)

$$\begin{aligned}&\mathsf{Prob}\left\{x^{(h-1)} \mid x^{(h)}\right\}\\ &\quad= \frac{\#\text{ of GT schemes of depth } h-1 \text{ with top row } x^{(h-1)}}{\#\text{ of GT schemes of depth } h \text{ with top row } x^{(h)}}\\ &\quad= \frac{\prod\limits_{1\le i<j\le h-1} \dfrac{x_i^{(h-1)}-x_j^{h-1}}{j-i}}{\prod\limits_{1\le i<j\le h} \dfrac{x_i^{(h)}-x_j^{h}}{j-i}} = (h-1)!\cdot \frac{\prod\limits_{1\le i<j\le h-1}(x_i^{(h-1)}-x_j^{h-1})}{\prod\limits_{1\le i<j\le h}(x_i^{(h)}-x_j^{h})}.\end{aligned} \tag{27}$$

(we have used Proposition 2.4). We will denote the above probabilities by $\Lambda^h_{h-1}(x^{(h)}; x^{(h-1)})$.

Note that the horizontal slices of measures that we obtain in §2.3 by taking limits $ab/c \to t$ and $b/(b+c) \to \xi$ of the hexagon are also related by these *stochastic links* Λ^h_{h-1}. In the GUE limit, the formula remains the same, except that the $x_i^{(h-1)}, x_i^{(h)}$ are now reals, not integers. In this case the above formula (27) gives the density of a Markov kernel with respect to the Lebesgue measure.

4.3. Example of a two-dimensional dynamics

The two Markov processes discussed above (the Dyson Brownian motion and its discrete analogue $L^{(N)}_{\text{Poisson}}$) are quite canonical, but they have one deficiency — they are one-dimensional (in the sense that the state space consists of particle configurations in $\mathbb{Z}^1$ or $\mathbb{R}^1$). We would like to construct a two-dimensional process which has interlacing two-dimensional arrays (3) as its state space, and that "stitches together" the above one-dimensional processes in a natural way. We begin by considering one such process which is constructed as follows.

Consider random words built from the alphabet $\{1, 2, \dots, N\}$ as follows: Each letter j is appended at the end of the word according to a standard (= rate 1) Poisson process' jumps, and different letters appear independently. We can encode this as on Fig. 12: We draw a star $(*)$ in row j at the time moment when a new letter j is added. The stars in each row form a Poisson process, and different rows are independent.

From these data we construct a Gelfand–Tsetlin scheme (3) of depth N written as

$$\boldsymbol{\lambda} = (\lambda^{(1)} \prec \lambda^{(2)} \prec \dots \prec \lambda^{(N)}), \qquad \lambda^{(h)} = (\lambda_1^{(h)} \ge \dots \ge \lambda_h^{(h)}),$$

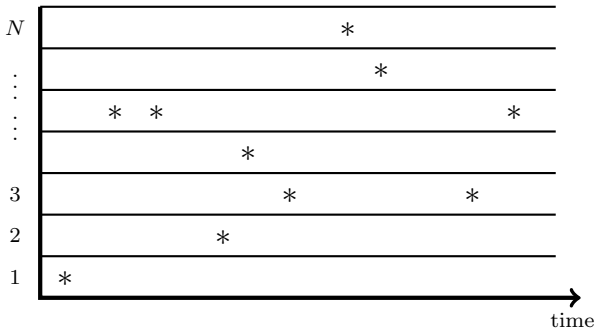


FIG 12. *Encoding random words.*

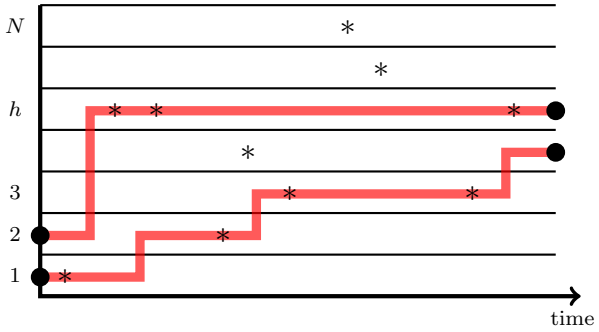


FIG 13. *Nonintersecting paths used to determine* $\lambda_1^{(h)}+\lambda_2^{(h)}+\cdots+\lambda_j^{(h)}$, *see* (28). *On the picture,* $h=5$, $j=2$, *and* $\lambda_1^{(5)}+\lambda_2^{(5)}=7$.

as follows (see Fig. 13):

$$\lambda_1^{(h)}+\lambda_2^{(h)}+\cdots+\lambda_j^{(h)}=\begin{pmatrix}\text{the maximal number of }(*)\text{ one can collect on}\\ \text{Fig. 12 along } j \text{ nonintersecting up-right paths}\\ \text{that connect points } (1,2,\dots,j) \text{ on the left bor-}\\ \text{der (time} = 0), \text{ and } (h-j+1,h-j+2,\dots,h)\\ \text{on the right border (time} = t>0)\end{pmatrix}. \tag{28}$$

In particular, we see that $\lambda_1^{(h)}$, $h=1,\dots,N$, is the length of the longest increasing subsequence of letters in the subword made of letters $\{1,2,\dots,h\}$. Moreover, $\lambda_1^{(h)}+\cdots+\lambda_h^{(h)}$ is the total number of letters $1,2,\dots,h$ in our word.

Proposition 4.1. *After time* t, *the distribution of the Gelfand–Tsetlin scheme* λ *defined by* (28) *is the same as the* $ab/c\to t$ *limit of the uniform measure on tilings of hexagon. That is, to obtain the measure on Gelfand–Tsetlin schemes,*

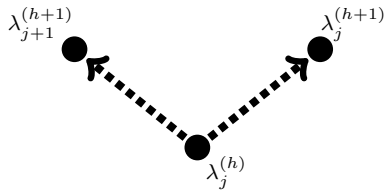


FIG 14. *Possible directions of move propagation, see Proposition 4.2.*

one takes the following distribution of the top row $\lambda^{(N)}$*:*

$$\text{const} \cdot \prod_{1\le i<j\le N} (\lambda_i^{(N)} - \lambda_j^{(N)})^2 \prod_{i=1}^{N} \frac{t^{\lambda_j^{(N)}}}{\lambda_j^{(N)}!},$$

and projects it down by the stochastic links, i.e., multiplies it by

$$\Lambda_{N-1}^{N}(\lambda^{(N)}, \lambda^{(N-1)})\Lambda_{N-2}^{N-1}(\lambda^{(N-1)}, \lambda^{(N-2)}) \dots \Lambda_1^2(\lambda^{(2)}, \lambda^{(1)}).$$

Proof. This is essentially Greene's theorem for the Robinson–Schensted correspondence coupled with explicit formulas for the number of standard and semistandard Young tableaux. See, e.g., [49, 73, 79]. □

As we are interested in time evolution, the following statement is relevant:

Proposition 4.2. *The Markov process on random words (i.e., the process of adding new letters according to standard Poisson processes) projects to a Markov process on Gelfand–Tsetlin schemes defined above. It can be described by the following rules:*

- *Each "particle"* $\lambda_1^{(h)}$ *has an independent Poissonian clock of rate 1. When the clock rings, the particle jumps by 1, i.e.,* $\lambda_1^{(h)} \mapsto \lambda_1^{(h)} + 1$*.*
- *When any particle* $\lambda_j^{(h)}$ *moves by 1, it triggers either the move* $\lambda_j^{(h+1)} \mapsto \lambda_j^{(h+1)} + 1$*, or* $\lambda_{j+1}^{(h+1)} \mapsto \lambda_{j+1}^{(h+1)} + 1$ *(exactly one of them), see Fig. 14. The second one is chosen generically, while the first one is chosen only if* $\lambda_j^{(h+1)} = \lambda_j^{(h)}$*, i.e., if the move* $\lambda_j^{(h)} \mapsto \lambda_j^{(h)} + 1$ *violated the interlacing constraint.*

Proof. See [27] (in particular, §7) and references therein. □

The Markov dynamics on Gelfand–Tsetlin schemes from Proposition 4.2 turns out to have the following properties:

(I) For each $h \ge 1$ and any initial condition, the evolution of $\{\lambda^{(1)}, \dots, \lambda^{(h)}\}$ is Markovian (i.e., lower rows do not care about the upper ones).

(II) For each $h \ge 1$, the evolution preserves the Gibbs property (§4.2) on $\{\lambda^{(1)}, \dots, \lambda^{(h)}\}$. That is, if one starts with an initial condition of the form

$$\mathsf{Prob}\{\lambda^{(1)}, \dots, \lambda^{(h)}\} = m_h(\lambda^{(h)})\Lambda_{h-1}^{h}(\lambda^{(h)}, \lambda^{(h-1)}) \dots \Lambda_1^2(\lambda^{(2)}, \lambda^{(1)}),$$

then after running the dynamics for any time $t > 0$, the distribution of $\{\lambda^{(1)}, \dots, \lambda^{(h)}\}$ will be of the same form with a different probability measure $\tilde{m}_h$.

(III) For each $h \geq 1$, the map $m_h \mapsto \tilde{m}_h$ is the time t evolution of the Markov process with the generator $L^{(h)}_{\mathrm{Poisson}}$ (26).

While (I) is obvious, (II) and (III) are not; they follow e.g. from Theorem 4.5 below.

There is one more property which can be easily observed from the random words description of the dynamics. Namely, the projection of the process of Proposition 4.2 to the rightmost particles $\lambda^{(1)}_1, \dots, \lambda^{(N)}_1$ is *Markov*. It is more convenient to describe it in shifted strictly ordered coordinates $y_1 = \lambda^{(1)}_1 < y_2 = \lambda^{(2)}_1 + 1 < \cdots < y_N = \lambda^{(N)} + N - 1$ (cf. (4)). Then each y_j jumps to the right by 1 independently with rate 1, and pushes y_{j+1} over by 1 if y_{j+1} occupies the target location of y_j (i.e., if we had $y_{j+1} = y_j + 1$ before the jump). We call this process the *PushTASEP*, i.e., the *Pushing Totally Asymmetric Simple Exclusion Process* (it was introduced in [77] under the name *long-range TASEP*, see also [23]).

Remark 4.3. Definition (28) is powered by what is known as Robinson–Schensted algorithm in Combinatorics. Questions related to application of various insertion algorithms (including the general Robinson–Schensted algorithm) to random input were considered in, e.g., [6, 54], [55, §5], and [43], and can be traced back to the work of Vershik and Kerov [81] in mid-1980's. The dynamical perspective has been substantially developed by O'Connell [64, 65, 66] and Biane–Bougerol–O'Connell [9] (see also Chhaibi [33]).

4.4. General construction of two-dimensional dynamics

The existence of the Markov dynamics (of Proposition 4.2) satisfying (I)–(III) is remarkable, yet its above construction is fairly complicated. We would like to access it in a different way.

Let us search for all continuous-time Markov jump processes on Gelfand–Tsetlin schemes which satisfy conditions (I)–(III) of §4.3. They must have the following structure: Each particle $\lambda^{(h)}_j$ jumps to the right by 1 with a certain rate (potentially dependent on $\lambda^{(1)}, \dots, \lambda^{(h)}$), and its jump triggers further moves on the higher levels $\lambda^{(h+1)}, \dots, \lambda^{(N)}$. Indeed, because of (III) and the fact that $L^{(h)}_{\mathrm{Poisson}}$ moves one particle at a time, no two particles on the same level can jump simultaneously. Moreover, because of (I), moves can propagate only upwards.

In order to reach a reasonable classification, we need to restrict the class further by requiring *nearest neighbor interactions*: A move of $\lambda^{(h)}_j$ can only trigger (potentially with certain probabilities) moves of the immediate top right neighbor $\lambda^{(h+1)}_j$ and the the immediate top left neighbor $\lambda^{(h+1)}_{j+1}$ (see Fig. 14), which can trigger moves on level $h+2$, and so on. Actually, it is better to extend

the notion of the top right nearest neighbor from $\lambda_j^{(h+1)}$ to the first particle in the sequence $\lambda_j^{(h+1)}, \lambda_{j-1}^{(h+1)}, \dots, \lambda_1^{(h+1)}$ whose jump does not violate interlacing. We will additionally assume (extending the nearest neighbor hypothesis) that the individual jump rates of particles at level h may only depend on $\lambda^{(h-1)}$ and $\lambda^{(h)}$, and that the same is true for left and right probabilities of move propagation from level $h-1$ to level h.

Let us now parametrize our possibilities. Fix $h \geq 2$, and denote by $w_j = w_j(\lambda^{(h-1)}, \lambda^{(h)})$ the jump rate of $\lambda_j^{(h)}$, $1 \leq j \leq h$. Also, denote by l_j the *conditional* probabilities that, given that the jth part of $\lambda^{(h-1)}$ has just increased by 1, this move propagates to the top left neighbor $\lambda_{j+1}^{(h)}$ of $\lambda_j^{(h-1)}$:

$$l_j(\lambda^{(h-1)}, \lambda^{(h)}) = \mathsf{Prob}\left(\lambda^{(h)} \mapsto \lambda^{(h)} + \mathsf{e}_{j+1} \,|\, \lambda^{(h-1)} \mapsto \lambda^{(h-1)} + \mathsf{e}_j\right),$$

where e_j is the vector having zeros at each position except the jth where it has 1. Similarly, let

$$r_j(\lambda^{(h-1)}, \lambda^{(h)}) = \mathsf{Prob}\left(\lambda^{(h)} \mapsto \lambda^{(h)} + \mathsf{e}_{\xi(j)} \,|\, \lambda^{(h-1)} \mapsto \lambda^{(h-1)} + \mathsf{e}_j\right),$$

where $\xi(j)$ is the lower index of the nearest top right neighbor of $\lambda_j^{(h-1)}$ that is free to jump (typically, $\xi(j) = j$).

Proposition 4.4. *Assume that $\lambda^{(h-1)}$ and $\lambda^{(h)}$ interlace with strict inequalities instead of weak ones. Then properties* (II), (III) *imply*

$$r_m + l_{m-1} + w_m = 1, \qquad 1 \leq m \leq h,$$

where we set $r_h = l_0 = 0$.

Proof. Consider the Gibbs measure on the first h levels that projects to the delta measure at $\lambda^{(h)}$ on level h. The rate of any jump $\lambda^{(h)} \mapsto \lambda^{(h)} + \mathsf{e}_m$, $1 \leq m \leq h$, can be computed in two different ways, using $L^{(h)}_{\text{Poisson}}$ or using the two-dimensional dynamics on the array. The projection of the array dynamics to levels $(h-1, h)$ looks as follows: On level $h-1$ we have the process driven by $L^{(h-1)}_{\text{Poisson}}$ whose jumps may propagate to level h with probabilities (l_j, r_j). Moreover, particles on level h can jump independently according to the jump rates w_j. Comparing the two ways to describe the rate of $\lambda^{(h)} \mapsto \lambda^{(h)} + \mathsf{e}_m$ yields the desired relations. □

The system of equations of Proposition 4.4 needs to be modified if the inequalities between parts of $\lambda^{(h-1)}$ and $\lambda^{(h)}$ are not strict. Indeed, if $\lambda_j^{(h)}$ is "blocked" by $\lambda_{j-1}^{(h-1)}$, i.e., $\lambda_j^{(h)} = \lambda_{j-1}^{(h-1)}$, then w_j must be zero, and also l_{j-1} and r_{j-1} make no sense as $\lambda^{(h-1)}$ could not have just come from the jump $\lambda^{(h-1)} - \mathsf{e}_{j-1} \mapsto \lambda^{(h-1)}$. The modification looks as follows.

Theorem 4.5. *For any $\lambda^{(h-1)} \prec \lambda^{(h)}$, let*

$$\left\{j_1 + 1 < j_2 + 1 < \cdots < j_\kappa + 1\right\}$$

be the set of indices such that each particle $\lambda^{(h)}_{j_m+1}$ *is free to move, i.e.,* $\lambda^{(h)}_{j_m+1} < \lambda^{(h-1)}_{j_m}$. *Then*

$$r_{j_{m+1}} + l_{j_m} + w_{j_m+1} = 1, \qquad 1 \le m \le \kappa, \tag{29}$$

with the agreement that $r_{j_\kappa+1} = l_{j_0} = 0$.

Solving these equations for all pairs $\lambda^{(h-1)} \prec \lambda^{(h)}$, $h = 2, 3, \ldots$, *under the conditions* $r_j(\lambda^{(h-1)}, \lambda^{(h)}) \ge 0$, $l_j(\lambda^{(h-1)}, \lambda^{(h)}) \ge 0$, $w_j(\lambda^{(h-1)}, \lambda^{(h)}) \ge 0$, *and* $r_j(\lambda^{(h-1)}, \lambda^{(h)}) + l_j(\lambda^{(h-1)}, \lambda^{(h)}) \le 1$, *is equivalent to constructing a nearest neighbor Markov dynamics as defined above satisfying conditions* (I)–(III), *with an additional "forced move" rule: If* $\lambda^{(h-1)}_j = \lambda^{(h)}_j$ *and* $\lambda^{(h-1)}_j$ *moves (by 1), then* $\lambda^{(h)}_j$ *also moves.*[3]

Proof. Similar to Proposition 4.4, cf. [27, §6]. □

It is easy to describe linear spaces of solutions to the above linear systems. Any combination of them, for every pair $\lambda^{(h-1)} \prec \lambda^{(h)}$, gives us a Markov process with desired properties. One can choose such combinations to design different processes.

4.5. Further examples of two-dimensional dynamics

We give three examples below, see [27] for more.

Example 1. All $l_j \equiv 1$, all $r_j \equiv 0$, and

$$w_j = \begin{cases} 1, & j = 1; \\ 0, & \text{otherwise.} \end{cases}$$

This is the dynamics described above in §4.3 via nonintersecting paths.

Example 2. All $r_j \equiv 1$, all $l_j \equiv 0$, and

$$w_j = \begin{cases} 1, & j = h; \\ 0, & \text{otherwise.} \end{cases}$$

This dynamics can be viewed as coming from the column insertion algorithm (as opposed to the row insertion algorithm corresponding to the dynamics of §4.3). Observe that the restriction of this dynamics to the left-most particles $\{\lambda^{(h)}_h\}_{h=1}^N$ is Markovian. Via the shift $y_h = \lambda^{(h)}_h - h$ this restriction matches the well-known *Totally Asymmetric Simple Exclusion Process* (*TASEP*). This dynamics was first introduced in [64].

Example 3. All $r_j \equiv 0$, all $l_j \equiv 0$, and all $w_j \equiv 1$. This dynamics has minimal pushing and maximal "noise" (coming from individual jumps). It can be viewed as a two-dimensional growth model; in terms of the stepped surfaces interpretation, independently with rate one this dynamics adds all possible "sticks"

[3] The forced move corresponds to the only possibility of having $\lambda^{(h-1)}$ and $\lambda^{(h)}$ not interlacing after a move on level $h-1$.

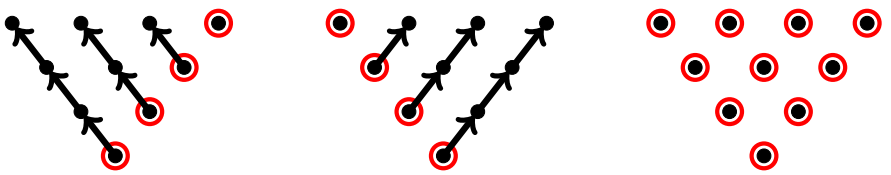


FIG 15. *Examples 1 (left), 2 (center), and 3 (right) of two-dimensional dynamics on interlacing arrays. Circles correspond to $w_j = 1$, and right and left arrows to r_j or $l_j = 1$.*

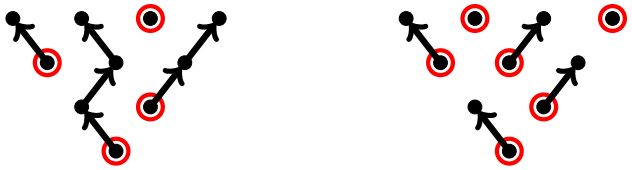


FIG 16. *Other examples of a two-dimensional dynamics.*

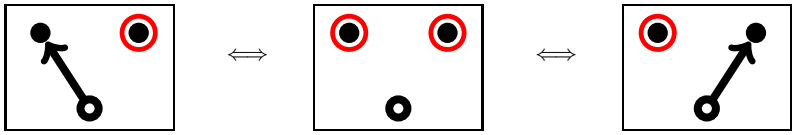
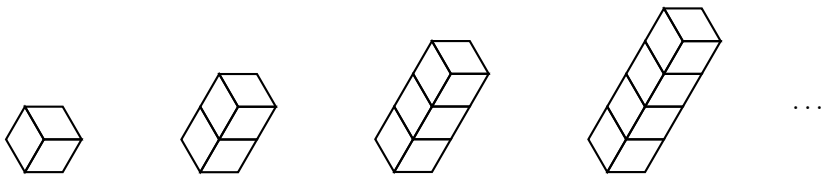


FIG 17. *Local "flips" are defined for any three particles $\lambda_j^{(h+1)}$, $\lambda_{j+1}^{(h+1)}$, and $\lambda_j^{(h)}$ in the array such that the are no outside arrows pointing to any of the upper particles. The type of the lower particle $\lambda_j^{(h)}$ can be arbitrary (i.e., it can jump independently or be pushed/pulled). The "flip" operation allows to replace one of the three local pictures by any of the two remaining ones.*

(directed columns) of the form



directed as shown (no overhangs allowed).

Projection of this dynamics to the leftmost particles $\{\lambda_h^{(h)}\}_{h=1}^N$ gives TASEP, and projection to the rightmost particles $\{\lambda_1^{(h)}\}_{h=1}^N$ gives PushTASEP. See [22] for more details on this dynamics.

Pictorially, the three examples can be represented as on Fig. 15. Pictures like Fig. 16 provide other interesting examples of two-dimensional dynamics, cf. [27, §7]. Note that due to (29), pictures corresponding to various "fundamental" dynamics for which two of the three quantities $r_{j_{m+1}}$, l_{j_m}, and w_{j_m+1} in each of the equations (29) are zero (and the remaining quantity is 1) can be transformed one into the other by a sequence of local "flips" as on Fig. 17.

4.6. Conclusion

We have seen how to construct random growth models in $(1+1)$ dimension (TASEP, PushTASEP) and $(2+1)$ dimension, and in §3 we have seen how these models can be analyzed at large times. We will now move on to a q-deformation of this picture, which will eventually lead us to directed polymers in random media.

5. The (q,t)-generalization

5.1. Historical remarks: Deformations of Schur polynomials

The developments of the previous sections were heavily based on properties of the Schur polynomials

$$s_\lambda(x_1,\dots,x_N)=\frac{\det\left[x_i^{N+\lambda_j-j}\right]_{i,j=1}^N}{\det\left[x_i^{N-j}\right]_{i,j=1}^N}$$
$$=\sum_{\lambda^{(1)}\prec\lambda^{(2)}\prec\ldots\prec\lambda^{(N)}} x_1^{|\lambda^{(1)}|}x_2^{|\lambda^{(2)}|-|\lambda^{(1)}|}\dots x_N^{|\lambda^{(N)}|-|\lambda^{(N-1)}|}.$$

We would like to add parameters to the theory.

It is easy to deform (= add parameters to) our model viewed as a probabilistic object. However, most such deformations would lack solvability properties of the original model based on Schur polynomials. The reason is that the Schur polynomials are algebraic objects, and algebraic structures (in contrast with probabilistic ones) are usually very rigid. Thus, to find meaningful (solvable) deformations of the model requires nontrivial algebraic work.

Historically, first two different one-parameter deformations of the Schur polynomials were suggested: around 1960 by algebraists Philip Hall and D.E. Littlewood,[4] and around 1970 by a statistician Henry Jack.

The *Hall–Littlewood polynomials* naturally arose in finite group theory and were later shown to be indispensable in representation theory of $GL(n)$ over finite and p-adic fields.

The *Jack polynomials* extrapolated the so-called zonal spherical functions arising in harmonic analysis on Riemannian symmetric spaces from three distinguished parameter values that correspond to spaces over $\mathbb{R},\mathbb{C}$, and $\mathbb{H}$. They are also known as eigenfunctions of the trigonometric Calogero–Sutherland integrable system.

In mid-1980's, in a remarkable development Ian Macdonald united the two deformations into a two-parameter deformation known as *Macdonald polynomials*. The two parameters are traditionally denoted as q and t. We will soon set t to 0, so it will not interfere with the time variable in our Markov processes.

[4] This is not the most famous mathematician with this last name, that would be J.E. Littlewood.

The Hall–Littlewood polynomials arise when $q = 0$, and the Jack polynomials correspond to the limit regime $t = q^\theta \to 1$, where $\theta > 0$. Schur polynomials correspond to $q = t$. Other significant values are: Schur's Q-functions (for $q = 0$, $t = -1$); monomial symmetric functions ($q = 0, t = 1$); and (the most important for us) q-Whittaker functions arising for $t = 0$.

5.2. Definition of Macdonald polynomials

The shortest way[5] to define Macdonald polynomials is to say that these are elements of $\mathbb{Q}(q,t)[x_1,\dots,x_N]^{S(N)}$ (this is the algebra of symmetric polynomials in variables $x_1,\dots,x_N$ whose coefficients are rational functions in q and t), that diagonalize the following first order q-difference operator:

$$\begin{aligned}\mathcal{D}^{(1)} &= \sum_{i=1}^{N}\left[\prod_{1\le i<j\le N}\frac{1}{x_i-x_j}\,\mathcal{T}_{t,x_i}\prod_{1\le i<j\le N}(x_i-x_j)\right]\mathcal{T}_{q,x_i}\\ &= \sum_{i=1}^{N}\prod_{j\ne i}\frac{tx_i-x_j}{x_i-x_j}\,\mathcal{T}_{q,x_i},\end{aligned}\tag{30}$$

where, as before, $(\mathcal{T}_{q,x}\,f)(x) = f(qx)$. It is immediately recognized as a deformation of the $q = t$ operator (16) from §2.4.4.

The operator $\mathcal{D}^{(1)}$ from (30) is called the *first Macdonald difference operator*. There are also higher order ones,

$$\mathcal{D}^{(k)} = \sum_{\substack{I\subset\{1,2,\dots,N\}\\ |I|=k}}\prod_{\substack{i\in I\\ j\notin I}}\frac{tx_i-x_j}{x_i-x_j}\prod_{i\in I}\mathcal{T}_{q,x_i}\,.\tag{31}$$

The operators $\mathcal{D}^{(k)}$ are diagonalized by the same polynomial basis [60, Ch. VI].

As Schur polynomials, the Macdonald polynomials in N variables are parametrized by $\lambda = (\lambda_1 \ge \dots \ge \lambda_N)$. We denote the (*monic*, i.e., with coefficient 1 of the lexicographically largest monomial, which is $x_1^{\lambda_1}x_2^{\lambda_2}\dots x_N^{\lambda_N}$) Macdonald polynomials by P_λ. They satisfy

$$\mathcal{D}^{(k)}\,P_\lambda = e_k(q^{\lambda_1}t^{N-1}, q^{\lambda_2}t^{N-2},\dots,q^{\lambda_N})P_\lambda,\tag{32}$$

where

$$e_k(y_1,\dots,y_N) = \sum_{1\le i_1<i_2<\dots<i_k\le N} y_{i_1}y_{i_2}\dots y_{i_k}$$

are the elementary symmetric polynomials. The eigenrelation (32) for $k = 1$ and $q = t$ was used in §2 to compute the density function of the vertical lozenges.

[5] The exposition below is very brief; a much more detailed one can be found in [60, Ch. VI].

5.3. q-Whittaker facts

Developing the (beautiful) theory of Macdonald polynomials requires significant efforts, and we will not pursue this here. An excellent resource is the Macdonald's book [60]. Instead, we will focus on the q-Whittaker ($t = 0$) case, where, for a story parallel to the Schur case (§§2–4), we need the following facts.

Proposition 5.1 (q-analogue of Lemma 2.2)**.** *For any $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N) \in \mathbb{Z}^N$, we have*

$$P_\lambda(x_1,\dots,x_N) = \sum_{\mu\colon \mu\prec\lambda} P_\mu(x_1,\dots,x_{N-1}) \frac{\prod_{i=1}^{N-1}(\lambda_i-\lambda_{i+1})!_q}{\prod_{i=1}^{N-1}(\lambda_i-\mu_i)!_q(\mu_i-\lambda_{i+1})!_q} x_N^{|\lambda|-|\mu|},$$

where $k!_q = \frac{(1-q)(1-q^2)\dots(1-q^k)}{(1-q)^k}$ is the q-analogue of the factorial.

We thus see that the interlacing structure (Gelfand–Tsetlin schemes (3)) remains intact, but the Gibbs property is q-deformed. We will now say that a probability measure on Gelfand–Tsetlin schemes $\boldsymbol{\lambda} = (\lambda^{(1)} \prec \dots \prec \lambda^{(N)})$ is *Gibbs* if for any $2 \leq h \leq N$,

$$\begin{aligned}&\mathsf{Prob}\left\{\lambda^{(1)},\dots,\lambda^{(h-1)} \mid \lambda^{(h)}\right\}\\ &\qquad = \Lambda^h_{h-1}(\lambda^{(h)},\lambda^{(h-1)})\Lambda^{h-1}_{h-2}(\lambda^{(h-1)},\lambda^{(h-2)})\dots\Lambda^2_1(\lambda^{(2)},\lambda^{(1)}),\end{aligned}$$

with the q-deformed stochastic links

$$\Lambda^h_{h-1}(\lambda,\mu) = \frac{P_\mu(1,\dots,1)}{P_\lambda(1,\dots,1)} \cdot \frac{\prod_{i=1}^{N-1}(\lambda_i-\lambda_{i+1})!_q}{\prod_{i=1}^{N-1}(\lambda_i-\mu_i)!_q(\mu_i-\lambda_{i+1})!_q}.$$

Recall that in the Schur case we had simply $\frac{s_\mu(1,\dots,1)}{s_\lambda(1,\dots,1)}$ which was then explicitly evaluated in (27).

Proposition 5.2 (q-analogue of the limit $ab/c \to t$ and $L^{(h)}_{\mathrm{Poisson}}$ of §4.1)**.** *If we define the coefficients $\mathsf{Prob}_{t,h}\{\mu\}$ by expanding*[6]

$$\prod_{i=1}^{h} e^{t(x_i-1)} = \sum_{\mu_1\geq\dots\geq\mu_h} \mathsf{Prob}_{t,h}\{\mu\} \frac{P_\mu(x_1,\dots,x_h)}{P_\mu(1,\dots,1)}, \tag{33}$$

then $\mathsf{Prob}_{t,h}\{\mu\} \geq 0$ for any μ. Moreover, these probability measures on $\{\mu_1 \geq \dots \geq \mu_h \geq 0\}$ are time t distributions of a jump Markov process with jump rates

$$L^{(h;q)}_{\mathrm{Poisson}}(\lambda \mapsto \lambda+\mathsf{e}_j) = \frac{P_{\lambda+\mathsf{e}_j}(1,\dots,1)}{P_\lambda(1,\dots,1)} \cdot (1-q^{\lambda_{j-1}-\lambda_j}), \qquad 1 \leq j \leq h, \tag{34}$$

where for $j = 1$ the last factor is omitted.

A proof (in the more general (q,t)-setting) can be found in [13, §2.3]. In the Schur case, (34) reduces to the ratio of Vandermonde determinants, cf. (26) above.

[6] Such a decomposition exists as $\{P_\mu\}_{\mu_1\geq\dots\geq\mu_h\geq 0}$ form a basis in the linear space of symmetric polynomials in $x_1,\dots,x_h$. Thus, (33) is a variant of the Taylor expansion.

5.4. q-deformed Markov dynamics

Propositions 5.1 and 5.2 give us sufficient data to run the same search (as in §4) for nearest neighbor Markov processes which preserve the Gibbs measures, and which on each level are described by $L^{(h;q)}_{\mathrm{Poisson}}$ (34). This immediately leads to the following q-analogue of Theorem 4.5:

Theorem 5.3 ([27]). *For any $\lambda^{(h-1)} \prec \lambda^{(h)}$, define*

$$T_i(\lambda^{(h-1)},\lambda^{(h)}) := \frac{\left(1-q^{\lambda_i^{(h-1)}-\lambda_{i+1}^{(h)}}\right)\left(1-q^{\lambda_{i-1}^{(h-1)}-\lambda_i^{(h-1)}+1}\right)}{\left(1-q^{\lambda_i^{(h)}-\lambda_i^{(h-1)}+1}\right)},$$

$$S_j(\lambda^{(h-1)},\lambda^{(h)}) := \frac{\left(1-q^{\lambda_{j-1}^{(h-1)}-\lambda_j^{(h)}}\right)\left(1-q^{\lambda_j^{(h)}-\lambda_{j+1}^{(h)}+1}\right)}{\left(1-q^{\lambda_j^{(h)}-\lambda_j^{(h-1)}+1}\right)},$$

with $1 \le i \le k-1$, $1 \le j \le k$. If there is a $\lambda^{(\cdot)}$ whose indices make no sense, then the corresponding factor is omitted:

$$T_1(\lambda^{(h-1)},\lambda^{(h)}) = \frac{1-q^{\lambda_1^{(h-1)}-\lambda_2^{(h)}}}{1-q^{\lambda_1^{(h)}-\lambda_1^{(h-1)}+1}}, \quad S_1(\lambda^{(h-1)},\lambda^{(h)}) = \frac{1-q^{\lambda_1^{(h)}-\lambda_2^{(h)}+1}}{1-q^{\lambda_1^{(h)}-\lambda_1^{(h-1)}+1}},$$

$$S_h(\lambda^{(h-1)},\lambda^{(h)}) = 1-q^{\lambda_{h-1}^{(h-1)}-\lambda_h^{(h)}}.$$

Let $\{j_1+1 < j_2+1 < \cdots < j_\kappa+1\}$ be all the indices such that particle $\lambda^{(h)}_{j_m+1}$ is free to move, i.e., $\lambda^{(h)}_{j_m+1} < \lambda^{(h-1)}_m$. Then

$$T_{j_{m+1}} r_{j_{m+1}} + T_{j_m} l_{j_m} + w_{j_m+1} = S_{j_m+1}, \qquad 1 \le m \le \kappa,$$

with agreement $r_{j_{\kappa+1}} = l_{j_0} = 0$, and also $T_h = 0$. Solving these equations for all pairs $\lambda^{(h-1)} \prec \lambda^{(h)}$ is equivalent to constructing nearest neighbor Markov dynamics satisfying the q-versions of conditions (I)–(III) *of §4.3.*

5.5. Examples of q-deformed two-dimensional dynamics

Using Theorem 5.3, we can now explore the same three examples as in §4.5:

Example 1. We enforce the almost sure move propagation (i.e., $r_j + l_j \equiv 1$), and also

$$w_j = \begin{cases} 1, & j = 1; \\ 0, & \text{otherwise.} \end{cases}$$

This gives a unique solution

$$r_j = \frac{S_1 + \cdots + S_j - T_1 - \cdots - T_{j-1}}{T_j}, \qquad l_j = 1 - r_j,$$

for all j such that $\lambda^{(h)}_{j+1}$ is free. In fact, this expression telescopes to give

$$r_j = \begin{cases} q^{\lambda^{(h)}_1 - \lambda^{(h-1)}_1}, & j = 1; \\ q^{\lambda^{(h)}_j - \lambda^{(h-1)}_j} \dfrac{1 - q^{\lambda^{(h-1)}_{j-1} - \lambda^{(h)}_j}}{1 - q^{\lambda^{(h-1)}_{j-1} - \lambda^{(h-1)}_j}}, & 2 \le j \le h. \end{cases}$$

We observe that for $0 \le q < 1$, all the probabilities r_j and l_j are nonnegative, and the projection to the rightmost particles $\{\lambda^{(h)}_1\}_{h=1}^N$ is *Markovian*. In the shifted variables $y_h = \lambda^{(h)}_1 + h$, it can be described as follows: Each particle jumps to the right by 1 independently with Poisson clock of rate 1. If the jth particle moved, it triggers the move of $(j+1)$st one with probability q^{gap}, where gap is the number of empty spots in front of the jth particle before the move (which in its turn may trigger the move of the $(j+2)$nd particle, etc.). Note that the probability q^{gap} is 1 if $\mathsf{gap} = 0$. We call this particle system the *q-PushTASEP*, it was first introduced in [27]. Its generalization (called *q-PushASEP*) with particles moving in both directions can be found in [37].

Example 2. Now we again enforce $l_j + r_j \equiv 1$, and

$$w_j = \begin{cases} 1, & j = h; \\ 0, & \text{otherwise.} \end{cases}$$

This gives

$$r_j = 1 + \frac{q^{\lambda^{(h-1)}_j - \lambda^{(h)}_{j+1}}}{T_j}, \qquad l_j = 1 - r_j,$$

for all j such that $\lambda^{(h)}_{j+1}$ is free. Obviously, this gives negative probabilities, and we do not pursue this example further.

Example 3. Here we enforce $l_j = r_j \equiv 0$. This clearly gives $w_j = S_j$, and for $0 \le q < 1$ this is a well-defined Markov process without long-range interactions. It was first constructed in [13], and it is closely related to the q-Boson stochastic particle system of [75], see also [21, 19]. While the projection of this process to the rightmost particles $\{\lambda^{(h)}_1\}_{h=1}^N$ does not appear to be Markovian (because S_1 depends on $\lambda^{(h)}_2$), the projection to the leftmost particles $\{\lambda^{(h)}_h\}_{h=1}^N$ *is* Markovian. In the shifted coordinates $y_h = \lambda^{(h)}_h - h$ it can be described as follows: Each particle jumps to the right by 1 independently of the others with rate $1 - q^{\mathsf{gap}}$, where gap is (as before) the number of empty spaces in front of y_h before the jump. Note that this rate vanishes when $\mathsf{gap} = 0$, which correspond to a TASEP-like blocking of the move. We call this interacting particle system the *q-TASEP*.

Obviously, as $q \to 0$, the q-PushTASEP turns into the usual PushTASEP, and q-TASEP becomes the usual TASEP.

5.6. Conclusion

We have thus obtained q-deformations of the random growth models from the Schur case. Our next task will be to investigate their asymptotic behavior at large times.

6. Asymptotics of q-deformed growth models

Our main tool in studying asymptotics will be the Macdonald difference operators (§5.2).

6.1. A contour integral formula for expectations of observables

By $\mathsf{Prob}_{t,h}$ we mean the measure defined in Proposition 5.2.

Proposition 6.1. *For any $1 \le h \le N$,*

$$\sum_{\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_N \ge 0} q^{\lambda_N + \dots + \lambda_{N-k+1}} \mathsf{Prob}_{t,N}\{\lambda\} = \frac{(-1)^{\frac{k(k-1)}{2}}}{(2\pi \mathbf{i})^k k!} \oint \dots \oint \prod_{1 \le A < B \le k} (z_A - z_B)^2 \prod_{j=1}^{k} \frac{e^{(q-1)tz_j}}{(1-z_j)^N} \frac{dz_j}{z_j^k}, \tag{35}$$

where all the integrals are taken over small positively oriented closed contours around 1.

Proof. In the proof we need to use the second Macdonald parameter $t \neq 0$. For this, let us in this proof denote the time variable by τ to avoid the confusion.

We apply the kth order Macdonald operator $\mathcal{D}^{(k)}$ (31) to the series expansion (33) defining our measures, which now looks as

$$e^{\tau \sum_{i=1}^{N} (x_i - 1)} = \sum_{\lambda_1 \ge \dots \ge \lambda_N \ge 0} \mathsf{Prob}_{\tau,h}\{\lambda\} \frac{P_\lambda(x_1, \dots, x_h)}{P_\lambda(1, \dots, 1)}. \tag{36}$$

We then replace the sum in the left-hand side by the residue expansion of the integral (see [13, §2.2.3] for more detail)

$$\frac{\mathcal{D}^{(k)} e^{\tau \sum_{i=1}^{N} (x_i - 1)}}{e^{\tau \sum_{i=1}^{N} (x_i - 1)}} = \frac{1}{(2\pi \mathbf{i})^k k!} \oint \dots \oint \frac{\prod_{1 \le A < B \le k} (tz_A - tz_B)(z_B - z_A)}{\prod_{A,B=1}^{k} (tz_A - z_B)} \times \times \prod_{j=1}^{k} \left(\prod_{m=1}^{N} \frac{tz_j - x_m}{z_j - x_m} e^{(q-1)\tau z_j} dz_j \right),$$

where the contours encircle $\{x_1, \dots, x_N\}$ and no other poles (i.e., the residues are taken at $z_j = x_{m_j}$, $j = 1, \dots, k$, $1 \le m_j \le N$). Note that in fact

$$\frac{\prod_{1 \le A < B \le k} (tz_A - tz_B)(z_B - z_A)}{\prod_{A,B=1}^{k} (tz_A - z_B)} = \det \left[\frac{1}{tz_A - z_B} \right]_{A,B=1}^{k}$$

via the Cauchy determinant formula.

In the right-hand side of (36) we use the eigenrelation (see §5.2):

$$\begin{aligned}\mathcal{D}^{(k)}\,P_\lambda &= e_k(q^{\lambda_1}t^{N-1},q^{\lambda_2}t^{N-2},\dots,q^{\lambda_N})P_\lambda\\ &= \left(q^{\lambda_N+\lambda_{N-1}+\cdots+\lambda_{N-k+1}}t^{\frac{k(k-1)}{2}}+\text{higher powers of } t\right)P_\lambda.\end{aligned}$$

We then divide both sides of (36) by $t^{\frac{k(k-1)}{2}}$, take the limit as $t\to 0$, and also set $x_j=1$. □

There are two simple limit transitions that one can observe in the right-hand side of the contour integral formula (35). We consider them in §6.2 and §6.3 below.

6.2. Gaussian limit

The first limit regime is $q=e^{-\varepsilon}\to 1$, $t=\tau\varepsilon^{-1}\to\infty$, z_j's do not change. Looking at the left-hand side of (35), which is $\mathbb{E}(q^{\lambda_N+\cdots+\lambda_{N-k+1}})$, it is natural to expect that each λ_j grows as ε^{-1} so that the quantities q^{λ_j} have finite limits (which may still be random variables). Looking at higher powers of Macdonald operators indeed reveals that this is a Gaussian limit: $\varepsilon\lambda_j$ has a law of large numbers with Gaussian fluctuations of size $\varepsilon^{-1/2}$, and the Markov dynamics we constructed converge to Gaussian processes. We do not pursue this limit regime here, its detailed exposition will appear in [17]. Another, structurally similar appearance of Gaussian processes can be found in [25].

6.3. Polymer limit

The second limit is a bit more complicated. We again take $q=e^{-\varepsilon}\to 1$, but now $t=\tau\varepsilon^{-2}\to\infty$ (i.e., we wait for a longer time than in the Gaussian limit of §6.2). Then to see a nontrivial limit, we have to take the z_j's of distance $O(\varepsilon)$ from 1: $z_j=1+\varepsilon w_j$. Then we have

$$\begin{gathered}e^{(q-1)tz_j}=e^{(-\varepsilon+\varepsilon^2/2-\cdots)\tau\varepsilon^{-2}(1+\varepsilon w_j)}=e^{-\tau\varepsilon^{-1}}e^{\tau/2-\tau w_j},\\ (1-z_j)^N=(-\varepsilon w_j)^N=(-1)^N\varepsilon^N w_j^N,\\ \prod_{1\le A<B\le k}(z_A-z_B)^2=\varepsilon^{k(k-1)}\prod_{1\le A<B\le k}(w_A-w_B)^2,\\ \prod_{j=1}^N dz_j=\varepsilon^k\prod_{j=1}^N dw_j,\end{gathered}$$

and we see that the right-hand side of (35) becomes $e^{-\tau k\varepsilon^{-1}}\varepsilon^{k(k-1)-kN+k}$ times an asymptotically finite expression. To figure out the limiting behavior of $\lambda_N+\cdots+\lambda_{N-k+1}$, we now have to take $\log_q$ of this expression, or take the natural logarithm and multiply by $-\varepsilon^{-1}$. This gives

$$\lambda_N+\cdots+\lambda_{N-k+1}\sim\tau k\varepsilon^{-2}+(-kN+k^2)\frac{\ln\varepsilon^{-1}}{\varepsilon}+\frac{R_{N,k}}{\varepsilon},$$

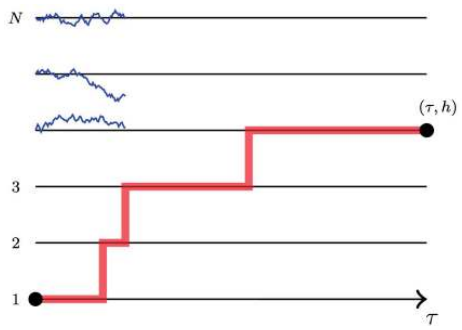


FIG 18. *Semi-discrete Brownian polymer.*

where the remainder $R_{N,k}$ is supposed to be a finite random variable. Equivalently,

$$\lambda_j = \tau\varepsilon^{-2} - (N+1-2j)\frac{\ln\varepsilon}{\varepsilon} + \frac{T_{N,j}}{\varepsilon}, \qquad 1 \le j \le N, \tag{37}$$

with some limiting random variables $T_{N,j}$. (Note that at this moment this is simply a *guess*!)

We can now test what is happening with our dynamics under this conjectural scaling. For example, consider the q-PushTASEP. The asymptotics of the pushing probability is

$$q^{\lambda_1^{(h)}-\lambda_1^{(h-1)}} \sim e^{-\varepsilon\left(-\frac{\ln\varepsilon}{\varepsilon}+\frac{T_{h,1}-T_{h-1,1}}{\varepsilon}\right)} = \varepsilon e^{T_{h-1,1}-T_{h,1}}.$$

The increment of $\lambda_1^{(h)}$ over time $d\tau = \varepsilon^{-2}dt$ must then be (1) the increment coming from its own jumps, which is $\varepsilon^{-2}dt + \varepsilon^{-1}dB_h$, where B_h is a Brownian motion, and (2) the increment coming from pushing, which is $\varepsilon e^{T_{h-1,1}-T_{h,1}}$ times the increment of $\lambda_1^{(h-1)}$. Collecting terms of order ε^{-1}, we conclude that

$$dT_{h,1} = dB_h + e^{T_{h-1,1}-T_{h,1}}d\tau, \qquad 1 \le h \le N,$$

where $B_1, B_2, \dots, B_N$ are independent standard Brownian motions (for $h = 1$ the last term is omitted). This system of stochastic differential equations (SDEs, for short) is solved by

$$T_{h,1} = \log \int\limits_{0<s_1<\cdots<s_{h-1}<\tau} e^{B_1(s_1)+\cdots+\left(B_h(\tau)-B_h(s_{h-1})\right)} ds_1 \dots ds_{h-1}. \tag{38}$$

The integral in the right-hand side can be viewed as the logarithm of the partition function (i.e., the *free energy*) of a semi-discrete Brownian polymer, see Fig. 18, and also [13, Chapter 5] for a general discussion of directed polymers in random media. More precisely, to any Poisson-type up-right path

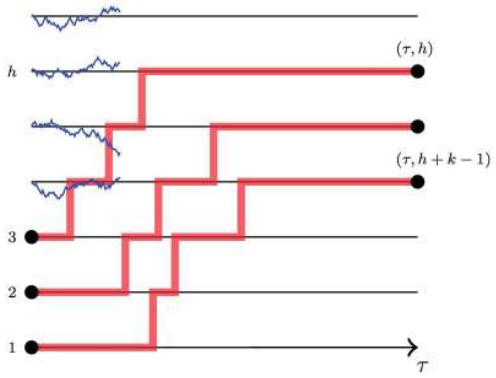


FIG 19. *Nonintersecting Poisson paths* $\phi_1, \dots, \phi_k$.

$\phi_{(0,1)\to(\tau,h)}$ that travels from 1 to h during time τ with jumps at moments $0 < s_1 < \cdots < s_{h-1} < \tau$, assign the energy

$$E(\phi_{(0,1)\to(\tau,h)}) = B_1(s_1) + \big(B_2(s_2) - B_2(s_1)\big) + \cdots + \big(B_h(\tau) - B_h(s_{h-1})\big).$$

Then $T_{h,1}$ is the logarithm of the integral of the Boltzmann factor

$$\exp\big(E(\phi_{(0,1)\to(t,h)})\big)$$

over the Lebesgue measure on all such paths (the inverse temperature can be absorbed into the rescaling of τ with the help of the Brownian scaling).

Similar empirical scaling arguments show that the Markov process on Gelfand–Tsetlin schemes of depth N that lead to the q-PushTASEP (Example 1 in §5.5) converges to a solution of the following system of SDEs:

$$dT_{h,k} = \mathbf{1}_{k=1} dB_h + \mathbf{1}_{k\neq 1} dT_{h-1,k-1} + \big(e^{T_{h-1,k}-T_{h,k}} - e^{T_{h-1,k-1}-T_{h,k-1}}\big)\, d\tau. \tag{39}$$

Theorem 6.2. *Under the above scaling* (37)*, the measure* $\mathsf{Prob}_{t,h}\{\lambda\}$ *weakly converges to a probability measure on arrays*[7] $\{T_{h,k}\} \in \mathbb{R}^{\frac{N(N+1)}{2}}$ *that can be written in the form*

$$T_{h,1} + \cdots + T_{h,k} = \log \int \dots \int e^{E(\phi_1)+\cdots+E(\phi_k)} d\phi_1 \dots d\phi_k, \tag{40}$$

the integral taken over the Lebesgue measure on the polytope of k*-tuples of nonintersecting Poisson-type paths* $\phi_1, \dots, \phi_k$ *joining* $\{(0,1),(0,2),\dots,(0,k)\}$ *with* $\{(\tau, h-k+1),(\tau,h-k+2),\dots,(\tau,h)\}$*, see Fig. 19. The limit measure is invariant under the flip* $\{T_{h,k} \leftrightarrow -T_{h,h-k+1}\}$.

[7] Note that there is no interlacing in this limit!

Proof. The Markov dynamics related to the semi-discrete directed Brownian polymer was introduced in [66]. The weak convergence was proven in [13, §4]. □

It is also known that the right-hand side of (40) satisfies the system of SDEs (39). Thus, the convergence in the above theorem should extend to a trajectory-wise statement, but, to our best knowledge, this has not been worked out in full detail yet.

Observe the similarity of (40) and the row Robinson–Schensted construction of §4.3 (in particular, see Figures 13 and 19). In fact, (40) arises via a *geometric* lifting of the Robinson–Schensted(–Knuth) correspondence, see [66, 36, 67].

It is worth noting that other (Brownian-type) scaling limits of growth models discussed here are considered in [47, 48].

7. Moments of q-Whittaker processes

From now on we focus on the asymptotic behavior of the free energy $T_{N,1} \stackrel{d}{=} -T_{N,N}$ (equality in distribution) of the semi-discrete Brownian polymer, see §6.3 (recall that this is the logarithm of the polymer's partition function $e^{-T_{N,N}}$). The free energy can be viewed as a limit of either $\lambda_1^{(N)}$ (thus, q-PushTASEP), or of $\lambda_N^{(N)}$ (corresponding to the q-TASEP), and either one can be used for the analysis. (Note that to obtain the polymer's partition function, N must remain fixed.) We will employ the q-TASEP, as this is a bit more straightforward, and there are more details on the q-TASEP in the literature.

7.1. Moments of the q-TASEP

We start by employing products of first order Macdonald operators rather than a single one to obtain moments (of all orders) of the q-TASEP particle locations.

Proposition 7.1. *Consider the random Gelfand–Tsetlin schemes of depth $N \geq 1$ distributed according to*

$$\mathsf{Prob}_{t,N}\{\lambda^{(N)}\}\Lambda^{N}_{N-1}(\lambda^{(N)},\lambda^{(N-1)})\Lambda^{N-1}_{N-2}(\lambda^{(N-1)},\lambda^{(N-2)})\dots\Lambda^{2}_{1}(\lambda^{(2)},\lambda^{(1)})$$

(this is the Gibbs measure with the top row distributed according to $\mathsf{Prob}_{t,N}$, see §5.3). Then for any $N \geq N_1 \geq N_2 \geq \dots \geq N_k \geq 1$,

$$\mathbb{E}\left(q^{\lambda^{(N_1)}_{N_1}+\dots+\lambda^{(N_k)}_{N_k}}\right) = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi\mathbf{i})^k}\oint\dots\oint\prod_{1\leq A<B\leq k}\frac{z_A-z_B}{z_A-qz_B}\prod_{j=1}^{k}\frac{e^{(q-1)tz_j}}{(1-z_j)^{N_j}}\frac{dz_j}{z_j}, \tag{41}$$

where the integral is taken over positively oriented, nested *contours around 1: The z_k contour encircles 1 and no other poles, the z_{k-1} contour encircles 1 and the contour $\{qz_k\}$, and so on; the z_1 contour encircles $\{1, qz_k, qz_{k-1}, \dots, qz_2\}$, see Fig. 20.*

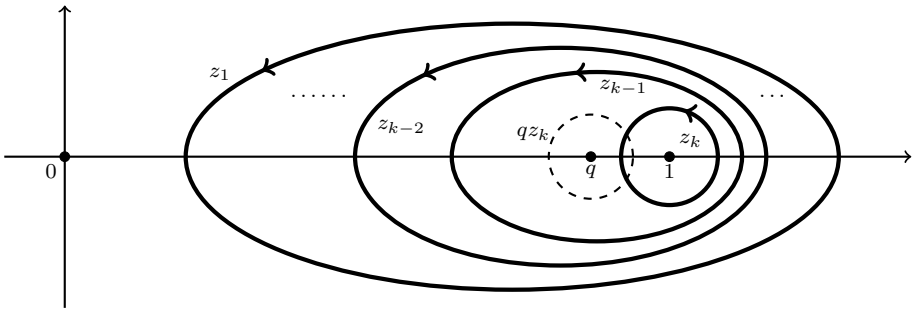


FIG 20. Nested *contours of integration in* (41).

Proof. We consider $k = 2$, for larger k the argument is similar. We start with the defining identity (33):

$$e^{t\sum_{i=1}^{N_1}(x_i-1)} = \sum_{\lambda^{(N_1)}} \mathsf{Prob}_{t,N_1}\{\lambda^{(N_1)}\}\widetilde{P}_{\lambda^{(N_1)}}(x_1,\dots,x_{N_1}), \tag{42}$$

where $\tilde{P}_\lambda$ means the normalization of P_λ by itself evaluated at all $x_j \equiv 1$. Apply the first Macdonald operator in N_1 variables to (42) (note that the second Macdonald parameter t is zero, and it has no relation to the time t in (42)). This operator has the form

$$\mathcal{D}^{(1)}_{N_1} = \sum_{i=1}^{N_1}\prod_{j\neq i}\frac{-x_j}{x_i-x_j}\,\mathcal{T}_{q,x_i}\,.$$

Because of the eigenrelation $D^{(1)}_{N_1}\tilde{P}_{\lambda^{(N_1)}} = q^{\lambda^{(N_1)}_{N_1}}P_{\lambda^{(N_1)}}$, in the right-hand side of (42) we observe

$$\sum_{\lambda^{(N_1)}} q^{\lambda^{(N_1)}_{N_1}}\,\mathsf{Prob}_{t,N_1}\{\lambda^{(N_1)}\}\widetilde{P}_{\lambda^{(N_1)}}(x_1,\dots,x_{N_1}).$$

On the other hand, the left-hand side of (42) is, by residue expansion, equal to the integral

$$-\frac{x_1,\dots,x_{N_1}}{2\pi\mathbf{i}}\oint\frac{e^{t\sum_{i=1}^{N_1}(x_i-1)}e^{(q-1)tz}}{(x_1-z)\dots(x_{N_1}-z)}\frac{dz}{z}, \tag{43}$$

over a positively oriented contour around the simple poles $x_1,\dots,x_{N_1}$.

The above argument works for applying $\mathcal{D}^{(1)}_{N_1}$ to any multiplicative function $F(x_1)\dots F(x_{N_1})$ with the exponentials $e^{t\sum_{i=1}^{N_1}(x_i-1)}e^{(q-1)tz}$ in (43) replaced by $F(x_1)\dots F(x_{N_1})\frac{F(qz)}{F(z)}$.

In the next step, we apply $\mathcal{D}^{(1)}_{N_2}$ to the result of application of $\mathcal{D}^{(1)}_{N_1}$ to (42) after setting $x_{N_2+1} = \dots = x_{N_1} = 1$ (the order $N_2 \le N_1$ is important). For the

right-hand side of the resulting expression, we use Proposition 5.1 that gives

$$P_{\lambda^{(N_1)}}(x_1,\dots,x_{N_2},\underbrace{1,\dots,1}_{N_1-N_2}) = \sum_{\lambda^{(N_2)}} \mathsf{Prob}_{t,N_1}\{\lambda^{(N_1)}\}\Lambda^{N_1}_{N_2}(\lambda^{(N_1)},\lambda^{(N_2)})\tilde{P}_{\lambda^{(N_2)}}(x_1,\dots,x_{N_2}).$$

Here $\Lambda^{N_1}_{N_2}$ means the matrix product $\Lambda^{N_1}_{N_1-1}\dots\Lambda^{N_2+1}_{N_2}$.

This implies, together with the eigenrelation $\mathcal{D}^{(1)}_{N_2}P_{\lambda^{(N_2)}} = q^{\lambda^{(N_2)}_{N_2}}P_{\lambda^{(N_2)}}$, that the right-hand side (after setting $x_1=\dots=x_{N_2}=1$) gives $\mathbb{E}\,q^{\lambda^{(N_1)}_{N_1}+\lambda^{(N_2)}_{N_2}}$. On the other hand, in the left-hand side, the x-dependence in (43), after setting $x_{N_2+1}=\dots=x_{N_1}=1$, is in the form of $G(x_1)\dots G(x_{N_2})$ with $G(x)=\frac{x}{x-z}e^{t(x-1)}$. Hence, we can apply the same residue expansion (for computing the application of $\mathcal{D}^{(1)}_{N_2}$ to (43)) using the fact that

$$\frac{G(qw)}{G(w)} = q\frac{z-w}{z-qw}e^{t(q-1)w}.$$

Here w is the new integration variable whose contour has to encircle $x_1,\dots,x_{N_2}$, but not any other poles (in particular, not $q^{-1}z$). Renaming $(z,w)\mapsto(z_1,z_2)$, we obtain the desired formula for $k=2$. For larger k the proof is similar. □

The above iterative argument is due to V. Gorin, see [18] for a more general version. The original proof of (41) in [21] involved a discretization of the quantum delta Bose gas, see also [19] and §7.2 below.

7.2. Moments of the semi-discrete Brownian polymer

Since we already know the scaling which takes us from $\lambda^{(N)}_N$ to the polymer partition function (§6.3), we can immediately do the limit in the integral of Proposition 7.1. This is very similar to the limit that we took in §6.3. That is, let us use

$$t=\tau\varepsilon^{-2},\qquad q=e^{-\varepsilon},\qquad z_j=1-\varepsilon w_j.$$

This leads to the following formula for the moments of the polymer partition function: For $N_1\ge N_2\ge\dots\ge N_k\ge 0$ (as in Proposition 7.1), we have

$$\mathbb{E}\left(e^{-T_{N_1,N_1}-\dots-T_{N_k,N_k}}\right) = \frac{e^{\tau k/2}}{(2\pi\mathbf{i})^k}\oint\dots\oint\prod_{1\le A<B\le k}\frac{w_A-w_B}{w_A-w_B-1}\prod_{j=1}^{k}\frac{e^{\tau w_j}}{w_j^{N_j}}dw_j, \tag{44}$$

where the integrals are now over nested contours around 0: the w_k contour contains only 0, the w_{k-1} contour contains $\{w_k+1\}$ and 0, and so on; the w_1 contour contains $\{0,w_k+1,w_{k-1}+1,\dots,w_2+1\}$. This limit transition from (41),

however, is *not* a proof of the formula (44). Indeed, Theorem 6.2 only claims weak convergence, and we have exponential moments under the expectation (that is, expectations of *unbounded* functions).

Sketch of the proof of (44). Observe that if we define (for *any*, not necessarily ordered $N_1,\dots,N_k\geq 0$)

$$F(\tau;N_1,\dots,N_k):=\frac{1}{(2\pi\mathbf{i})^k}\oint\dots\oint\prod_{1\leq A<B\leq k}\frac{w_A-w_B}{w_A-w_B-1}\prod_{j=1}^{k}\frac{e^{\tau w_j}}{w_j^{N_j}}dw_j$$

(with contours as in (44)), then

$$\frac{d}{d\tau}F(\tau;N_1,N_2\dots,N_k)=F(\tau;N_1-1,N_2,\dots,N_k)\\+F(\tau;N_1,N_2-1,\dots,N_k)+\dots+F(\tau;N_1,N_2,\dots,N_k-1).$$

Further, for $N_k=0$, F vanishes (because there are no poles inside the smallest contour), and for $N_i=N_{i+1}$,

$$F(\tau;\dots,N_i-1,N_{i+1},\dots)-F(\tau;\dots,N_i,N_{i+1}-1,\dots)\\-F(\tau;\dots,N_i,N_{i+1},\dots)=0,$$

because when we write out the integral for this linear combination, the integrand will be skew-symmetric in w_i and w_{i+1}, and the two corresponding contours can be taken to be the same (the obstacle to both these properties in (44) is the factor $(w_i-w_{i+1}-1)$, and the linear combination above exactly cancels this factor out). These properties together with initial condition $F(0;N_1,\dots,N_k)=\mathbf{1}_{N_1=\dots=N_k=1}$ uniquely determine $F(\tau;N_1,N_2\dots,N_k)$ for $N_1\geq\dots\geq N_k\geq 0$. Thus, it suffices to check that the moments of the polymer partition function (see (40)) satisfy the same properties. This check is fairly straightforward. □

A discussion of how the evolution equation and the boundary conditions at $N_i=N_{i+1}$ used above relate to a discrete quantum Bose gas can be found in [13, 21, 19].

7.3. Continuous Brownian polymer

There is a further limit that takes the semi-discrete polymer to a fully continuous one. In that case, one defines

$$Z(t,x)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}\,\mathbb{E}_{\substack{\text{Brownian bridge}\\ b:\,(0,t)\to\mathbb{R},\ b(0)=0,\ b(t)=x}}:\exp:\left\{\int_0^t\xi(s,b(s))ds\right\},$$

where ξ is the space-time white noise, and : exp : means normally ordered exponential, e.g., see [1] for an explanation. Equivalently (via the Feynman-Kac formula), Z solves the *stochastic heat equation with multiplicative noise*

$$\frac{\partial}{\partial t}Z(t,x)=\frac{1}{2}\frac{\partial^2}{\partial x^2}Z(t,x)+\xi(t,x)Z(t,x),\qquad Z(0,x)=\delta(x).$$

Then, defining

$$u(t; x_1, \dots, x_k) = \mathbb{E}_{\text{white noise}} \left(Z(t, x_1) \dots Z(t, x_k) \right),$$

we have

$$\left(\frac{\partial}{\partial t} - \frac{1}{2} \sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2} \right) u = 0 \tag{45}$$

away from diagonal subset, and

$$\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}} - 1 \right) u = 0 \qquad \text{if } x_i = x_{i+1}. \tag{46}$$

The observations (45)–(46) are not hard and were recorded at least as far back as the end of 1980's by Kardar [56] and Molchanov [62]. We believe that a rigorous proof can be extracted from the results of [8]. In particular, case $N = 2$ was treated in [2, I.3.2]. To the best of our knowledge, the general case has not been worked out in full detail yet.

A concise way to write (45)–(46) is

$$\frac{\partial}{\partial t} u = Hu, \qquad H = \frac{1}{2} \left(\sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2} + \sum_{i \neq j} \delta(x_i - x_j) \right).$$

H is called the *delta-Bose gas* (or *Lieb–Liniger*) *Hamiltonian*. The nested contour integral then takes the form

$$u(t; x_1, \dots, x_k) = \int_{\alpha_1 - \mathbf{i}\infty}^{\alpha_1 + \mathbf{i}\infty} dz_1 \dots \int_{\alpha_k - \mathbf{i}\infty}^{\alpha_k + \mathbf{i}\infty} dz_k \prod_{1 \le A < B \le k} \frac{z_A - z_B}{z_A - z_B - 1} \prod_{j=1}^{k} e^{\frac{t}{2} z_j^2 + x_j z_j}, \tag{47}$$

where $\alpha_1 > \alpha_2 + 1 > \dots > \alpha_k + (k-1)$, and $x_1 \le \dots \le x_k$ (the above formula is not true without the order assumption on the x_j's). For more detail, e.g., see [13, §6.2] and references therein.

7.4. Intermittency

By setting $N_1 = \dots = N_k = N$ in (44), we see that the nested contour integrals provide us with all moments of the polymer partition function $e^{-T_{N,N}}$. One might expect that this is sufficient to find its distribution or, equivalently, the distribution of the free energy $T_{N,N}$. It turns out that in this particular situation this is not true. The distribution of the polymer partition function displays *intermittency*, which we now discuss.

This term appeared in studying the velocity and temperature fields in a turbulent medium [7], and describes structures that appear in random media having

the form of peaks that arise at random places and at random time moments. The phenomenon is widely discussed in physics literature, with magnetic hydrodynamics (like on the surface of the Sun) and cosmology (theory of creation of galaxies) being two well-known examples, e.g., see [83, 62].

The main property that allows one to detect an intermittent distribution is anomalous behavior (as compared to the Gaussian case, for example) of ratios of successive moments.

Toy example. Consider a sequence of independent identically distributed random variables $\xi_1, \xi_2, \ldots$, each taking value 0 or 2 with probability 1/2. Set $\xi = \xi_1 \ldots \xi_N$. Then, clearly,

$$\mathbb{E}\,\xi = 1, \qquad \mathbb{E}(\xi^2) = 2^N, \qquad \ldots, \qquad \mathbb{E}(\xi^p) = 2^{(p-1)N}.$$

The growth speed can be measured by the quantities

$$\gamma_p = \frac{\log \mathbb{E}(\xi^p)}{N} = (p-1)\ln 2.$$

On the other hand, computing a similar quantity for the sum of the ξ_j's (which is asymptotically Gaussian) gives[8]

$$\gamma_p = \lim_{N\to\infty} \frac{\log \mathbb{E}\left((\xi_1 + \cdots + \xi_N)^p\right)}{\log N} = p.$$

The key difference of these two cases is that $\frac{\gamma_p}{p} < \frac{\gamma_{p+1}}{p+1}$ in the first case, while $\frac{\gamma_p}{p} = \frac{\gamma_{p+1}}{p+1}$ in the second one.

In general, imagine that one has a time-dependent nonnegative random variable $Z(t)$ which grows in t roughly exponentially (or $\log Z$ grows roughly linearly). There are many ways to measure such growth; we mostly follow [31] in the exposition below. Define:

- *Almost sure Lyapunov exponent*

$$\tilde{\gamma}_1 := \lim_{t\to\infty} \frac{\ln Z(t)}{t},$$

  if the a.s. limit exists (the "a.s." requirement can be weakened to convergence in probability).
- *Moment* (or *annealed*) *Lyapunov exponents*

$$\gamma_p := \lim_{t\to\infty} \frac{\ln \mathbb{E}\left(Z(t)\right)^p}{t}$$

  (assuming that limits exist).

[8] We divide by $\log N$ because the random variable in question grows roughly linearly in N, and ξ above has exponential growth.

Hölder's inequality implies that

$$\frac{\gamma_p}{p} \le \frac{\gamma_{p+1}}{p+1}$$

(because $(\mathbb{E}\, Z^p)^{\frac{1}{p}} \le (\mathbb{E}\, Z^{p+1})^{\frac{1}{p+1}}$). The strict inequalities will be referred to as *intermittency*. Note that we also obviously have $\tilde{\gamma}_1 \le \gamma_1$. In a typical situation, when the distribution of $Z(t)$ does not deviate much from its mean,

$$\tilde{\gamma}_1 = \gamma_1 = \frac{\gamma_2}{2} = \frac{\gamma_3}{3} = \cdots = \frac{\gamma_p}{p} = \cdots .$$

Lemma 7.2. *If there exists $k \ge 1$ such that*

$$\frac{\gamma_k}{k} < \frac{\gamma_{k+1}}{k+1},$$

then for all $p \ge k$,

$$\frac{\gamma_p}{p} < \frac{\gamma_{p+1}}{p+1}.$$

Proof. Hölder's inequality with $\frac{1}{2} + \frac{1}{2} = 1$ gives

$$(\mathbb{E}\, Z^k)^2 \le \mathbb{E}\, Z^{k+h} \cdot \mathbb{E}\, Z^{k-h}, \qquad h = 1, 2, \ldots, k,$$

which implies that $\gamma_k \le \frac{\gamma_{k+h}+\gamma_{k-h}}{2}$. Replacing k by $k+1$ and taking $h = 1$, we have

$$\gamma_{k+1} \le \frac{\gamma_{k+2} + \gamma_k}{2} < \frac{1}{2}\left(\gamma_{k+2} + \frac{k}{k+1}\gamma_{k+1}\right)$$

(we used the hypothesis of the lemma). Rearranging terms gives the needed inequality for $p = k+1$. Repeating inductively, we obtain the desired claim. □

Let us now show how the definition of intermittency relates to peaks. If we pick α such that $\frac{\gamma_p}{p} < \alpha < \frac{\gamma_{p+1}}{p+1}$, then for large enough t (below we omit t in the notation for $Z(t)$):

- $\mathsf{Prob}\{Z > e^{\alpha t}\} > 0$, because otherwise we would have $(\mathbb{E}\, Z^{p+1})^{\frac{1}{p+1}} \le \alpha$.
- An overwhelming contribution to $\mathbb{E}\, Z^{p+1}$ comes from the region where $Z > e^{\alpha t}$. Indeed,

  $$\mathbb{E}\, Z^{p+1} = \mathbb{E}\left(Z^{p+1}\mathbf{1}_{Z \le e^{\alpha t}}\right) + \mathbb{E}\left(Z^{p+1}\mathbf{1}_{Z > e^{\alpha t}}\right).$$

  The first term is $\le e^{\alpha(p+1)t} \ll e^{\gamma_{p+1}t}$, and we know that the left-hand side behaves exactly as $e^{\gamma_{p+1}t}$.
- $\mathsf{Prob}\{Z > e^{\alpha t}\} \le e^{-(\alpha - \gamma_p/p)pt}$ because $\mathbb{E}\, Z^p \ge e^{\alpha p t}\, \mathsf{Prob}\{Z > e^{\alpha t}\}$.

Hence, we observe a hierarchy of higher and higher peaks concentrated on smaller and smaller sets (that are actually exponentially small in probability),

and higher peaks contribute overwhelmingly to high enough moments. In the situation of random fields when ergodicity allows to replace computing expectations by space averaging, at each fixed large time one can then observe a hierarchy of islands with exponentially (in time) high values that dominate moment computations.

Intermittency is a characteristic feature of products of a large number of independent random variables (cf. the toy example above). Indeed, by the central limit theorem, let us check that random variables of the form $e^{\xi_1+\cdots+\xi_t} \sim e^{\mathcal{N}(\mu t,\sigma^2 t)}$ (for example, with independent identically distributed ξ_j's) are intermittent. We have

$$\mathbb{E}\left(\left[e^{\mathcal{N}(\mu t,\sigma^2 t)}\right]^p\right) = e^{t\left(p\mu+\frac{p^2}{2}\sigma^2\right)},$$

which implies that

$$\tilde{\gamma}_1 = \mu < \gamma_1 = \mu + \frac{1}{2}\sigma^2 < \frac{\gamma_2}{2} = \mu + \sigma^2 < \frac{\gamma_3}{3} = \mu + \frac{3}{2}\sigma^2 < \cdots .$$

7.5. Moment problem and intermittency

Since under intermittency the moments are dominated by increasingly atypical behavior (i.e., observed with small probability), it is hard to expect that the moments would determine the distribution. For example, for the exponential of the standard Gaussian $\mathcal{N}(0,1)$ they do not: Any distribution with density

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}}\frac{1}{x}\exp\left(-\frac{(\ln x)^2}{2}\right)\cdot(1+\varepsilon h(x)), & x>0;\\ 0, & x\le 0,\end{cases}$$

with $h(x) := \sin(2\pi\ln(x))$, $-1 \le \varepsilon \le 1$, gives the same moments. When $\varepsilon = 0$, this is the density of the log-normal random variable $e^{\mathcal{N}(0,1)}$. See [80] for more detail.

We will now check if the polymer partition function is intermittent.

Theorem 7.3. *The moment Lyapunov exponents for the Brownian polymer are given by*

- *In the semi-discrete case (§6.3 and §7.2), $\gamma_p = H_p(z_{0,p})$, where (for $N = t$)*

$$H_p(z) = \frac{p^2}{2} + pz - \log(z(z+1)\dots(z+p-1)),$$

  and $z_{0,p}$ is the unique solution to $H_p'(z) = 0$ on $(0,+\infty)$.
- *In the fully continuous case (§7.3), for paths between $(0,0)$ and $(t,0)$,*

$$\gamma_p = \frac{p^3-p}{24}.$$

Proof. Steepest descent for contour integral representations of moments given before (§7.2 and §7.3). See [14] for details. □

In the fully continuous setting, the above result is due to Kardar [56] (non-rigorously), and Bertini–Cancrini [8].

7.6. Replica trick

Given our previous discussion on peak domination in moments, it seems hopeless that this limit behavior of moments would carry any information about the behavior of the main bulk of the distribution. This is, however, a suitable moment to demonstrate the (in)famous *replica trick* widely used in physics.

Note that, at least formally,

$$\log Z(t) = \lim_{p\to 0} \frac{(Z(t))^p - 1}{p}.$$

Averaging both sides, dividing by t and taking $t \to \infty$ suggests

$$\tilde{\gamma}_1 = \lim_{p\to 0} \lim_{t\to\infty} \frac{1}{t} \frac{e^{t\gamma_p} - 1}{p} = \lim_{p\to 0} \frac{\gamma_p}{p},$$

where we tacitly use a (non-unique!) analytic continuation of γ_p off nonnegative integers. For the semi-discrete polymer, $H_p(z) = \frac{p^2}{2} + pz - \log \frac{\Gamma(z+p)}{\Gamma(z)}$, and

$$\lim_{p\to 0} \frac{H_p(z)}{p} = z - \psi(z), \qquad \psi(z) = (\log \Gamma(z))'.$$

Taking the value of $z-\psi(z)$ at the only critical point of this function on $(0, +\infty)$, gives

$$\tilde{\gamma}_1 = \inf_{z\in(0,+\infty)} (z - \psi(z)).$$

This actually *is* the correct answer! It was conjectured in [68] and proven in [63].

Similarly, in the fully continuous case,

$$\tilde{\gamma}_1 = \lim_{p\to 0} \frac{1}{p} \frac{p^3 - p}{24} = -\frac{1}{24},$$

which is also correct [3, 74].

We thus see that this very nonrigorous procedure, quite remarkably, lead us to the correct almost sure behavior! In the next section we show how to access these results rigorously, and our approach will also explain in a way why the replica trick worked in this particular situation.

8. Laplace transforms

8.1. Setup

As we have seen in §7, the intermittency phenomenon prevents us from recovering the distribution of the polymer partition function from its moments. However, this is not so in the q-setting. Namely, the q-moments $\mathbb{E}\, q^{k\lambda_N^{(N)}}$, $k = 1, 2, \dots,$

uniquely determine the distribution of $\lambda_N^{(N)}$ (because $\lambda_N^{(N)} \geq 0$ and $q \in (0,1)$, so these are moments of a bounded random variable).

Our plan is thus to convert the q-moment formulas that we have (Proposition 7.1) into a formula for the expectation of a one-parameter family of observables that remain bounded (unlike the moments $\mathbb{E}\, q^{k\lambda_N^{(N)}}$) in the $q \to 1$ which leads to polymers. Since this will involve q-moments with $k \to \infty$, it is inconvenient to use nested contours in integral representations as their positions depend on k. There are two ways to "un-nest" the contours: (1) to deform all of them to identical large concentric circles $|z| = R > 1$; or (2) to deform all of them to identical small concentric circles $|z-1| = r < \varepsilon$. The first way is easier to realize, but it is harder to turn the result into a meaningful asymptotic information. Thus, we proceed with the second one. The following lemma is nontrivial and very useful:

Lemma 8.1. *Let f be a meromorphic function and $\mathbb{A}$ be its singular set which must not include 0. Assume that $q^m\mathbb{A}$ is disjoint from $\mathbb{A}$ for all integers $m \geq 1$. Then*

$$\mu_k := \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi\mathbf{i})^k} \oint \dots \oint \prod_{1\leq A<B\leq k} \frac{z_A - z_B}{z_A - qz_B} \frac{f(z_1)\dots f(z_k)}{z_1 \dots z_k} dz_1 \dots dz_k$$

$$= k!_q \sum_{\substack{\lambda=(\lambda_1\geq\lambda_2\geq\dots\geq\lambda_\ell>0)\\ \lambda_1+\dots+\lambda_\ell=k \\ \lambda=1^{m_1}2^{m_2}\dots}} \frac{1}{m_1!m_2!\dots} \frac{(1-q)^k}{(2\pi\mathbf{i})^\ell} \oint \dots \oint \det\left[\frac{1}{w_i q^{\lambda_i} - w_j}\right]_{i,j=1}^{\ell} \times$$

$$\times \prod_{j=1}^{\ell} f(w_j) f(qw_j) \dots f(q^{\lambda_j - 1} w_j) dw_j, \tag{48}$$

where each z_p contour contains $\{qz_j\}_{j>p}$ and $\mathbb{A}$, but not 0 (thus, the z contours are nested), and the w_j contours contain $\mathbb{A}$ and no other poles (so, the w contours can be taken to be all the same).

Note that for $f(z) = \frac{e^{(q-1)tz}}{(1-z)^N}$, (48) becomes the formula for $\mathbb{E}\, q^{k\lambda_N^{(N)}}$, see Proposition 7.1. In this case, $\mathbb{A} = \{1\}$.

Proof. This lemma is the result of mere bookkeeping of the residues when we shrink the contours and take into account the poles at $z_A = qz_B$. The fact that the result is rather nice[9] is nontrivial, and takes origin in harmonic analysis on Riemannian symmetric spaces and Hecke algebras, cf. [52]. A proof of the lemma can be found in [13, Prop. 3.2.1], see also [19, Lemma 3.3 and Prop. 7.4]. □

A limiting case of this lemma, as $q = e^{-\varepsilon} \to 1$, $z_j = 1 + \varepsilon\tilde{z}_j$, $w_j = 1 + \varepsilon\tilde{w}_j$, is at the heart of the moments asymptotics which were stated in §7.5.

[9] The number of residues involved is much larger than the number of terms in (48).

8.2. Generating functions

The form of the right-hand side of (48) suggests that one could take a generating function of such expressions over different k. More exactly, it easily implies that

$$\sum_{k\geq 0}\mu_k\frac{\zeta^k}{k!_q} = \sum_{\ell\geq 0}\frac{1}{\ell!}\sum_{n_1,\dots,n_\ell\geq 1}\frac{1}{(2\pi\mathbf{i})^\ell}\oint\dots\oint\det\left[\frac{1}{q^{n_i}w_i-w_j}\right]_{i,j=1}^{\ell}\times \\ \times\prod_{j=1}^{\ell}(1-q)^{n_j}\zeta^{n_j}f(w_j)\dots f(q^{n_j-1}w_j)dw_j. \tag{49}$$

We will now take $f(z)=\frac{e^{(q-1)tz}}{(1-z)^N}$, with all the integration contours above being small enough positively oriented contours around 1. Here $(n_1,\dots,n_\ell)$ are simply permuted values of $(\lambda_1\geq\dots\geq\lambda_\ell)$ in (48), and the change of the combinatorial factor from $\frac{1}{m_1!m_2!\dots}$ to $\frac{1}{\ell!}$ is due to that un-ordering.[10]

Now, using the q-exponential identity (e.g., see [5, 44])

$$\sum_{k\geq 0}\frac{a^k}{k!_q}=\prod_{m\geq 0}\frac{1}{1-(1-q)aq^m}=:\frac{1}{\big((1-q)a;q\big)_\infty},$$

we can rewrite the left-hand side of (49) as

$$\mathbb{E}\,\frac{1}{\big((1-q)q^{\lambda_N^{(N)}}\zeta;q\big)_\infty}.$$

One should expect that in a suitable scaling limit as $q\to 1$ (which we can predict by looking at the moment asymptotics), the q-moment generating function would converge to the Laplace transform of the polymer partition function. The latter *does* define the distribution uniquely, with or without intermittency. The real question now is how to take a similar limit in the right-hand side of (49). Observe that termwise limit would produce a moment generating series, and we already know that it diverges!

8.3. Case $N=1$ and the Mellin–Barnes integral representation

Let us consider the case $N=1$ in which the problem of convergence is already there. Then μ_k's are the q-moments of the simple continuous-time one-sided random walk started from 0 at $t=0$. We expect their q-generating function to converge (as $q\to 1$) to the Laplace transform of the lognormal distribution (i.e., $e^{\mathcal{N}(0,\tau)}$). Indeed, we should expect that because

$$\frac{1}{\big((1-q)q^{\lambda_1^{(1)}(t)}\zeta;q\big)_\infty}\xrightarrow{\varepsilon\to+0}e^{-ue^{-T_{1,1}(\tau)}},\quad \lambda_1^{(1)}\sim\frac{\tau}{\varepsilon^2}+\frac{T_{1,1}(\tau)}{\varepsilon},\quad \zeta=e^{\tau\varepsilon^{-1}}u,$$

where $T_{1,1}(\tau)\sim\mathcal{N}(0,\tau)$.

[10] Indeed, $\frac{\ell!}{m_1!m_2!\dots}$ is the number of different ways to obtain a given $\lambda=1^{m_1}2^{m_2}\dots$ with $|\lambda|=\ell$ from $(n_1,\dots,n_\ell)\in\mathbb{Z}_{\geq 1}^{\ell}$.

Observe that we only have first order poles at $w_j = 1$ in the right-hand side of (49) for $N = 1$. Because of vanishing of the $\det[\frac{1}{q^{n_i} w_i - w_j}]_{i,j=1}^{\ell}$ for equal values of the w_j's, we conclude that only $\ell \le 1$ give a nontrivial contribution. This contribution is

$$1+\sum_{n\ge 1}\frac{1}{2\pi\mathbf{i}}\oint_{\Gamma_1}\frac{dw}{(q^n-1)w}(1-q)^n\zeta^n\frac{e^{(q^n-1)tw}}{(1-w)(1-qw)\dots(1-q^{n-1}w)}$$
$$=1+\sum_{n\ge 1}\frac{(1-q)^n\zeta^n}{1-q^n}\frac{e^{(q^n-1)t}}{(1-q)\dots(1-q^{n-1})}=\sum_{n\ge 0}\frac{\big((1-q)\zeta\big)^n e^{(q^n-1)t}}{(1-q)\dots(1-q^{n-1})}. \tag{50}$$

We now need to take the $q \to 1$ limit in the above sum, and we cannot do that termwise as this would result in a divergent series. A standard tool of the theory of special functions used for dealing with such a limit is the *Mellin–Barnes integral representation* which dates back to the end of the 19th century. In its simplest incarnation, it says that

$$\sum_{n\ge 0} g(q^n)\zeta^n = \frac{1}{2\pi\mathbf{i}}\oint_{\Gamma_{0,1,2,\dots}}\Gamma(-s)\Gamma(1+s)(-\zeta)^s g(q^s)ds, \qquad |\arg(\zeta)|<\pi,$$

where the integral in the right-hand side goes in the negative direction around the poles $s = 0, 1, 2, \dots$.[11] Indeed,

$$-\operatorname*{Res}_{s=n}\Gamma(-s)\Gamma(1+s)(-\zeta)^s g(q^s) = g(q^n)\zeta^n,$$

where we assume z^s to be defined with the branch cut $(-\infty, 0)$. Omitting convergence and contour deformation justifications (which can be performed), we rewrite the series in (50) as

$$\sum_{n\ge 0}\frac{\big((1-q)\zeta\big)^n e^{(q^n-1)t}}{(1-q)\dots(1-q^{n-1})} \tag{51}$$
$$=\frac{1}{2\pi\mathbf{i}}\int_{\delta-\mathbf{i}\infty}^{\delta+\mathbf{i}\infty}\Gamma(-s)\Gamma(1+s)\big((q-1)\zeta\big)^s e^{(q^s-1)t}\frac{\prod_{m\ge 1}(1-q^{s+m})}{\prod_{m\ge 1}(1-q^m)}ds,$$

where $0 < \delta < 1$ and the integration is taken over a contour as on Fig. 21. We can now take the needed limit. We note that

$$\Gamma_q(x) = \prod_{m\ge 1}\frac{1-q^m}{1-q^{x+m-1}}(1-q)^{1-x}$$

is the q-analogue of the Euler Γ-function, and that (e.g., see [5])

$$\lim_{q\to 1}\Gamma_q(x) = \Gamma(x), \qquad x \notin \{0, -1, -2, \dots\}.$$

[11] Note that $\Gamma(-s)\Gamma(1+s) = -\frac{\pi}{\sin(\pi s)}$.

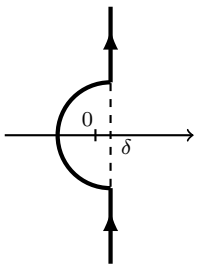


FIG 21. *Integration contour in* (51).

We take the scaling

$$q = e^{-\varepsilon} \to 1, \qquad t = \frac{\tau}{\varepsilon^2}, \qquad \zeta = e^{\tau\varepsilon^{-1}} u,$$

so that

$$\big((q-1)\zeta\big)^s = \big((e^{-\varepsilon}-1)e^{\tau\varepsilon^{-1}}(-u)\big)^s \sim \varepsilon^s e^{\tau s\varepsilon^{-1}} u^s, \qquad e^{(q^s-1)t} \sim e^{-\tau s\varepsilon^{-1}+\frac{\tau s^2}{2}},$$

and

$$\prod_{m\geq 1} \frac{1-q^{s+m}}{1-q^m} = \frac{(1-q)^{-s}}{\Gamma_q(s+1)} \sim \varepsilon^{-s} \frac{1}{\Gamma(s+1)}.$$

We see that the limit of the integral in (51) is

$$\frac{1}{2\pi\mathbf{i}} \int_{-\frac{1}{2}-\mathbf{i}\infty}^{-\frac{1}{2}+\mathbf{i}\infty} \Gamma(-s)\Gamma(1+s) \frac{u^s}{\Gamma(1+s)} e^{\tau s^2/2} ds = \frac{1}{2\pi\mathbf{i}} \int_{-\frac{1}{2}-\mathbf{i}\infty}^{-\frac{1}{2}+\mathbf{i}\infty} \Gamma(-s) u^s e^{\tau s^2/2} ds,$$

which is a correct expression for the Laplace transform of the lognormal random variable $e^{\mathcal{N}(0,\tau)}$, as we expected.

8.4. Asymptotics of the generating function for any $N \geq 1$

The same Mellin–Barnes integral representation works for any $N \geq 1$. The summations over $n_1, \dots, n_\ell \geq 1$ are replaced by integrals over $\frac{1}{2} + \mathbf{i}\mathbb{R}$ with $\Gamma(-s_j)\Gamma(1+s_j)$ inside, and using scaling of §6.3 together with Theorem 6.2 (which guarantees convergence of expectations of bounded functions), as well as the asymptotic relations above (setting $v_j = q^{w_j}$), we obtain the following generating function for the semi-discrete Brownian polymer's partition function (§6.3):

Theorem 8.2. *Fix $N \geq 1$, $0 < \delta_2 < 1$, and $0 < \delta_1 < \delta_2/2$. Then*[12]

$$\mathbb{E}\, e^{-ue^{-T_{N,N}(t)}} = 1 + \sum_{\ell\geq 1} \frac{1}{\ell!} \oint_{|v_1|=\delta_1} dv_1 \dots \oint_{|v_\ell|=\delta_1} dv_\ell \int_{\delta_2-\mathbf{i}\infty}^{\delta_2+\mathbf{i}\infty} ds_1 \dots \int_{\delta_2-\mathbf{i}\infty}^{\delta_2+\mathbf{i}\infty} ds_\ell$$

[12]Note that the time parameter t in (52) was denoted by τ in §6.3 and §8.3.

FIG 22. *The integration contours in (53) for variables a_i (left) and b_j (right).*

$$\times \prod_{j=1}^{\ell} \Gamma(-s_j)\Gamma(1+s_j) \left(\frac{\Gamma(v_j)}{\Gamma(s_j+v_j)}\right)^N \times \\ \times \frac{u^{s_j+v_j}}{u^{v_j}} \frac{e^{\frac{t}{2}(s_j+v_j)^2}}{e^{\frac{t}{2}v_j^2}} \det\left[\frac{1}{v_i+s_i-v_j}\right]_{i,j=1}^{\ell}. \tag{52}$$

The expression in the right-hand side above is actually well-suited for further asymptotic analysis. Let us first state the final result:

Theorem 8.3 ([13, 16])**.** *For any $\varkappa > 0$, define*

$$f_\varkappa = \min_{s>0}(\varkappa s - \psi(s)), \qquad s_\varkappa = \operatorname*{argmin}_{s>0}(\varkappa s - \psi(s)), \qquad g_\varkappa = -\psi''(s_\varkappa) > 0$$

(as before, $\psi(z) = (\log\Gamma(z))'$). Then for $t = \varkappa N$, we have

$$\lim_{N\to\infty} \mathsf{Prob}\left\{\frac{-T_{N,N}(t) - Nf_\varkappa}{N^{1/3}} \le r\right\} = F_{GUE}\left(\left(\frac{g_\varkappa}{2}\right)^{-\frac{1}{3}} r\right),$$

where F_{GUE} is the GUE Tracy–Widom distribution.

Recall that $-T_{N,N}(t)$ can be identified with the logarithm of the polymer partition function as in (38) (and that $-T_{N,N}(t) \stackrel{d}{=} T_{N,1}(t)$).

Note that Theorem 8.3 proves the value of the almost sure Lyapunov exponent $\tilde\gamma_1$ that we guessed (for $\varkappa = 1$, but this could have been for any $\varkappa$) using replica trick in §7.6.

The Tracy–Widom distribution in the right-hand side of (52) arises as the series

$$F_{GUE}\left(\left(\frac{g_\varkappa}{2}\right)^{-\frac{1}{3}} r\right) = 1 + \sum_{\ell\ge 1} \frac{1}{\ell!} \frac{1}{(2\pi\mathbf{i})^{2\ell}} \int\dots\int da_1\dots da_\ell \int\dots\int db_1\dots db_\ell \\ \times \prod_{j=1}^{\ell} \frac{1}{a_j - b_j} \frac{\exp\left(-\frac{g_\varkappa}{6}a_j^3 + ra_j\right)}{\exp\left(-\frac{g_\varkappa}{6}b_j^3 + rb_j\right)} \det\left[\frac{1}{b_i - a_j}\right]_{i,j=1}^{\ell}, \tag{53}$$

where the a_i and the b_j contours are as on Fig. 22. The identification of (53) with a traditional formula for F_{GUE} is explained in [13] (after formula (4.51)).

The way one reaches (53) from the right-hand side of (52) is fairly straightforward. By changing the variables $s_j \to y_j = s_j + v_j$, one rewrites the part of the integrand that depends on the large parameter N as

$$\prod_{j=1}^{\ell} \exp\Big(N\big(G(v_j) - G(y_j)\big)\Big), \qquad G(z) = \ln\Gamma(z) - z\frac{\ln u}{N} - \frac{\varkappa}{2}z^2.$$

Since

$$e^{-ue^{-T_{N,N}(t)}} = e^{-e^{-T_{N,N}(t)+\log u}},$$

we take $\log u \sim -Nf_{\varkappa} - rN^{\frac{1}{3}}$, and then we see that

$$G(z) \sim \ln\Gamma(z) + f_{\varkappa}z - \frac{\varkappa}{2}z^2.$$

The analysis then follows the scheme explained in §3, with v contours being deformed to the domain with $\Re G(v) < 0$, and y contours — to the domain with $\Re G(y) > 0$. The limiting expression arises in the situation when $G(z)$ has a double critical point $G'(z_c) = G''(z_c) = 0$, and through a local change of integration variables near the critical point; the constant $-g_{\varkappa}$ is actually $G'''(z_c)$. Details can be found in [13] and [16].

Let us conclude by observing that if we expand the right-hand side of (52) into residues at $s_j = 1, 2, \ldots$, we get back the divergent generating series for the moments of the polymer partition function that we found before. This shows that a more sophisticated replica trick than the one from §7.6 can actually be used to obtain the limiting distribution, and not only the law of large numbers (i.e., $\tilde{\gamma}_1$). Namely, one can obtain the moments by solving the equations (the delta Bose gas of §7.3) that they satisfy, write down the series for the Laplace transform through moments (despite the fact that this series diverges), make sense of this series via the Mellin–Barnes integral representation, and then proceed with the asymptotic analysis. This approach was successfully carried out in physics papers [38, 30]. However, the only plausible explanation we have at the moment as to why such an approach leads to the correct answer, is that it is a limiting case of the q-deformed situation, where all the steps are legal and indeed lead to a proof of the GUE edge fluctuations.

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