Integrable Systems and their Recursion Operators

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Abstract

In this paper we discuss the structure of recursion operators. We show that recursion operators of evolution equations have a nonlocal part that is determined by symmetries and cosymmetries. This enables us to compute recursion operators more systematically. Under certain conditions (which hold for all examples known to us) Nijenhuis operators are well defined, i.e., they give rise to hierarchies of infinitely many commuting symmetries of the operator. Moreover, the nonlocal part of a Nijenhuis operator contains the candidates of roots and coroots.

Key words: Recursion, Nijenhuis, hereditary operator, (co)symmetries

1 Historical introduction

We first give a sketch of how the subject of integrable systems started, before we move on to the specific subject of this paper, the analysis of the structure of recursion operators.

There are several places in the literature with eye-witness descriptions such as [vdB78,Kru78,EvH81,New85,Kon87,AC91] and [Pal97]. The following narrative is based mainly on these sources.

The discovery of the physical soliton is attributed to John Scott Russell's observation in 1834: A boat was rapidly drawn along a narrow channel by

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a pair of horses. When the boat suddenly stopped before a bridge, the bow wave detached from the boat and rolled forward with great velocity assuming the form of large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel without change the original form or diminution of the speed, as observed by Scott Russell who followed on horseback.

Although Scott Russell spent a major part of his professional life carrying out experiments to determine the properties of the great wave, it is doubtful that he appreciated their true soliton properties, i.e., the ability of these waves to interact and come out of these interactions without change of form, as if they were particles. Nevertheless, the dash on the horseback exerts a powerful appeal.

The discovery of mathematical soliton started with an investigation of the solutions of nonlinear partial differential equations, such as the work of Boussinesq and Rayleigh, independently, in the 1870's.

It was in 1895 that Korteweg and de Vries derived the equation for water waves in shallow channels, which confirmed the existence of solitary waves. The equation which now bears their names [2] is of the form

$$u_t = u_3 + uu_1 \quad (\textbf{KdV equation}), \qquad \text{where } u_i = \frac{\partial^i u}{\partial x^i}. \tag{1.1}$$

This was the first stage of discovery. The primary thrust was to establish the physical and mathematical existence and robustness of the wave. The discovery of its additional properties was to await the appearance of computers.

In 1955, Fermi, Pasta and Ulam (FPU) undertook a numerical study of the one-dimensional anharmonic lattice of equal masses coupled by nonlinear springs. The computations were carried out on the Maniac I computer. They predicted that any smooth initial state would eventually reach equilibrium due to the nonlinear coupling, according to the ergodic hypothesis. Much to their surprise, the energy recollected after some time in the degree of freedom where it was when the experiment was started. Thus the experiment failed to produce the expected result. Instead it produced a difficult challenge.

Fortunately, the curious results were not ignored altogether. In 1965, Kruskal and Zabusky approached the FPU problem from the continuum viewpoint. They amazingly rederived the KdV equation and found its stable pulse-like

[2] According to R. Pego, in a letter to the Notices of the AMS, 1998, volume 45, number 3, this equation appears in a footnote of a paper by Boussinesq, *Essai sur la théorie des eaux courantes*, presented in 1872 to the French Academy and published in 1877.

waves by numerical experimentation. A remarkable property of these solitary waves was that they preserved their shapes and speeds after two of them collide, interact and then spread apart again. They named such waves **soliton**s.

The discovery by Kruskal and Zabusky attracted the attention and stimulated the curiosity of many physicists and mathematicians throughout the world. They took up the intriguing challenge of the analytical understanding of the numerical results. The stability and particle-like behavior of the solutions could only be explained by the existence of many conservation laws; this started the search for the conservation laws for the KdV equation. A conservation law has the form $D_tU + D_xF = 0$; U is called the **conserved density** and F is called **conserved flux**. The expressions for the conservation of momentum and energy were classically known:

$$D_t u - D_x(u_2 + \frac{u^2}{2}) = 0, \; D_t(\frac{u^2}{2}) - D_x(uu_2 - \frac{u_1^2}{2} + \frac{u^3}{3}) = 0.$$

Whitham found a third conserved density, which corresponds to Boussinesq's famous moment of instability. Zabusky and Kruskal continued searching and found two more densities of order 2 and 3 (the highest derivative in the expression). Since they had made an algebraic mistake, they did not find a conserved density of order 4. This caused a delay of more than a year before they went back on the right track.

Kruskal somewhat later asked Miura to search for a conserved density of order 5. Miura found one and then quickly filled in the missing order 4. After the order 6 and 7 were found, Kruskal and Miura were fairly certain that there was an infinite number. However, Miura was challenged to find the order 8 conserved density since there were rumors that order 7 was the limit. He did this during a two-week vacation in the summer of 1966. Later, it was proved that there was indeed a conserved density of each order [MGK68]. Moreover, in [SW97b] it is proven that there are no other conservation laws besides the known conservation laws of the KdV equation , see also [SW97a] for generalizations of this result.

The existence of an infinite number of conservation laws was an important link in the chain of discovery. After the search for conserved densities of the KdV equation (1.1), Miura found that the **Modified Korteweg–de Vries equation**

$$v_t = v_3 + v^2 v_1 \quad \text{(mKdV)} \tag{1.2}$$

also had an infinite number of conserved densities. He showed that

$$u_t - (u_3 + uu_1) = (2v + \sqrt{-6}D_x)(v_t - (v_3 + v^2 v_1)),$$

under the transformation $u = v^2 + \sqrt{-6}v_1$, which now bears his name. Therefore, if $v(x,t)$ is a solution of (1.2), $u(x,t)$ is a solution of (1.1). From this observation, the famous inverse scattering method was developed and the Lax pair was found [Lax68]. Gardner was the first to notice that the KdV equation could be written in a Hamiltonian framework. Later, Zakharov and Faddeev showed how this could be interpreted as a completely integrable Hamiltonian system in the same sense as finite dimensional integrable Hamiltonian systems [ZF71] where one finds for every degree of freedom a conserved quantity, the **action**.

The conserved geometric features of solitons are intimately bound up with notions of symmetry. The symmetry groups of differential equations were first studied by Sophus Lie [Lie60]. Roughly speaking, a symmetry group of a system consists of those transformations of the variables which leave the system invariant. In the classical framework of Lie, these groups consist of only geometric transformations on the space of independent and dependent variables of the system, the so-called geometric symmetries. There are four such linear independent symmetries for the KdV equation, namely arbitrary translations in x and t, Galilean boost and scaling.

In 1918, Emmy Noether proved the remarkable theorem giving a one-to-one correspondence between symmetry groups and conservation laws for the Euler–Lagrange equations [Noe18]. The question was raised how to explain the infinitely many conserved densities for the KdV equation. One started to search for the hidden symmetries, **generalized symmetries**, which are 'groups' whose infinitesimal generators depend not only on the independent and dependent variables of the system, but also the derivatives of the dependent variables.

In fact, generalized symmetries first appeared in [Noe18]. Somehow, they were neglected for many years and have since been rediscovered several times. The great advantage of searching for symmetries is that they can be found by explicit computation. Moreover, the entire procedure is rather mechanical and, indeed, several symbolic programs have been developed for this task [HZ95,Her96].

In 1977, Olver provided a method for the construction of infinitely many symmetries of evolution equations, originally due to Andrew Lenard [GGKM74, p. 99]. This is the **recursion operator** [Olv77], which maps a symmetry to a new symmetry. For the KdV equation, a recursion operator is

$$\mathfrak{R}_{KdV} = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1 D_x^{-1}.$$

Here D_x^{-1} stands for the left inverse of D_x, so the recursion operator is only defined on $\mathrm{Im}\, D_x$.

Almost at the same time, Magri studied the connections between conservation laws and symmetries from the geometric point of view [Mag78]. He observed that the object of the theory of conservation laws, the gradients of the conserved densities (covariants), was dual to that of the theory of the symmetries. This problem required the introduction of a "metric operator", called **symplectic operator** if it maps the symmetries to the cosymmetries, or called **Hamiltonian (cosymplectic) operator** in the reverse direction. He found that some systems admitted two distinct but compatible Hamiltonian structures (Hamiltonian pairs). He called them twofold Hamiltonian system, now called bi-Hamiltonian systems. The KdV equation is a bi-Hamiltonian system. It can be written

$$u_t = D_x(u_2 + \frac{1}{2}u^2) = (D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_1)u,$$

where these two operators are a Hamiltonian pair.

Actually, the two operators had made their appearance before. Lenard used them to rederive the KdV hierarchy, infinitely many equations sharing all t-independent conservation laws. Lax also used them to produce infinitely many conservation laws for the KdV equation [Lax76]. This scheme is now called the **Lenard scheme**[Dor93].

There appeared naturally a special kind of operator, called the **Nijenhuis** or **hereditary operator**. The defining relation for this operator was originally found as a necessary condition for an almost complex structure to be complex, i.e., as an integrability condition. Its important property is to construct an abelian Lie algebra. . Precisely speaking, for any given vectorfield Q_0 leaving the Nijenhuis operator $\mathfrak{R}$ invariant, the $Q_j = \mathfrak{R}^j(Q_0),\ j = 0, 1, \cdots$, leave $\mathfrak{R}$ invariant again and commute in pairs. This property was independently given by Magri [Mag80] and Fuchssteiner [Fuc79], where it was called hereditary symmetry. In the paper [GD79], the authors also introduced Nijenhuis operators, called regular structures.

Interrelations between Hamiltonian pairs and Nijenhuis operators were discovered by Gel'fand & Dorfman [GD79] and Fuchssteiner & Fokas [FF80,FF81].

For example, the recursion operator of the KdV equation

$$\mathfrak{R}_{KdV} = (D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_1)D_x^{-1}$$

is a Nijenhuis operator and it produced the higher KdV equations,

$$u_t = \mathfrak{R}^j(u_1), j = 0, 1, \cdots.$$

This is the KdV hierarchy, which shares infinitely many commuting symmetries produced by the same recursion operator.

From this point in time on, there has been an explosion of research activity in algebraic and geometric aspects of nonlinear partial differential systems, both the applications to concrete physical systems and the development of the theory itself. See [Oev84,MSS91,Olv93,Dor93,FG96,Wan98,BSW98,SW98,BSW01,SW00b] and section 6.

2 Integrable Evolution Equations

Integrable evolution equations in one space variable, like the KdV equation, are often characterized by the possession of a recursion operator, which is an operator invariant under the flow of the equation, carrying symmetries of the equation into its (new) symmetries.

Example 1 *The operator $\mathfrak{R} = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1 D_x^{-1}$ is a Nijenhuis recursion operator for the* Korteweg–de Vries *equation $u_t = u_3 + uu_1$. It is remarkable that any $\mathfrak{R}^l u_1$ is a local function ([Olv93, p. 312]), i.e., $\mathfrak{R}^l u_1 \in ImD_x$ for $l \geq 0$.*

The fact that the image under repeated application of the recursion operator is again a local function is often not proved, with the KdV equation as an exception to this rule; some proofs can be found in [Dor93], relying on the bi-Hamiltonian character of the equations. But usually one finds in the literature a few explicit calculations, followed by the remark that this goes on. It is the goal of this paper to prove that this is indeed the case for at least a large class of examples (no exception being known to us, cf. however [Li91]). We give a general theorem to that effect, valid for systems of evolution equations in one spatial variable and apply this theorem to a number of characteristic examples.

We do not use the property that the operator is a recursion operator of any given equation, only that it is a Nijenhuis operator. Of course, the fact that these operators are recursion operators makes them interesting in the study of integrable systems. We show that Nijenhuis operators of the form that one finds in the study of evolution equations are well defined, i.e., they map into their own domain under some weak conditions and can therefore produce hierarchies of symmetries which are all local. Moreover, the nonlocal part of the operator contains the candidates of roots, starting points for a hierarchy of symmetries.

Apart from the theoretical interest, this splitting (which reminds one of the factorization of the operator in symplectic and cosymplectic operators, if they exist) is useful in the actual computation of a recursion operator for a given system. For this can be done iteratively, once the order of the operator is known or guessed, treating D_x as a symbol, which is fine as long as its power is nonnegative, but fails for D_x^{-1}. It is here that one can proceed to split off a symmetry, and move the remaining part after the D_x^{-1}. This turns out to be an effective algorithm to compute recursion operators.

3 Construction of recursion operators

We show that if the recursion operator is of a specific form, its nonlocal part, namely containing the D_x^{-1} term(s), can be written as the outer products of symmetries and cosymmetries.

Let us start by sketching the way things are set up abstractly, mainly to fix notations. We construct, following [GD79,Dor93], a Lie algebra complex. First we consider the representation space $\mathcal{A}$ as functions in t, x and $u, u_1, \cdots$ (where each function depends only on a finite number of variables, the so-called *local functions*), divided out by the image of the differentiation operator

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} u_{k+1} \frac{\partial}{\partial u_k}.$$

The equivalence class of a local function g is called a **functional** and denoted by $\int g$, the idea being that one can do *partial integration*, since

$$0 \equiv \int D_x(gh) = \int h D_x g + \int g D_x h.$$

Next the Lie algebra $\mathfrak{h}$, the algebra of **evolutionary vectorfields**, is defined as follows. Take the Lie algebra of vertical vectorfields of the form

$$\sum_{k=0}^{\infty} f_k \frac{\partial}{\partial u_k},$$

where the f_k are local functions, and divide out the image of the Lie derivative of D_x, where the Lie derivative is given by the commutator of the vectorfields.

This construction has as an immediate consequence that elements in $\mathfrak{h}$ commute with D_x and that if $\sum_{k=0}^{\infty} f_k \frac{\partial}{\partial u_k} \in \mathfrak{h}$, then $f_k = D_x^k f_0$. This implies that the elements in the Lie algebra are determined by a local function f_0. Under this identification the Lie derivative now looks like

$$L_{k_0} f_0 = D_{f_0}[k_0] - D_{k_0}[f_0],$$

where $D_{f_0}[k_0]$ is the Fréchet derivative

$$D_{f_0}[k_0] = \sum_{k=0}^{\infty} \frac{\partial f_0}{\partial u_k} D_x^k k_0.$$

To each element in $\mathfrak{h}$ one can associate an evolution equation of the form

$$u_t = f_0,$$

although, strictly speaking, this association is not so innocent as it looks, since one associates to the evolution equation the operator

$$\hat{f}_0 = \frac{\partial}{\partial t} + \sum_{k=0}^{\infty} D_x^k f_0 \frac{\partial}{\partial u_k}.$$

To distinguish the evolutionary vectorfields from the operator associated to the evolution equation, we introduce the notation

$$\bar{f}_0 = \sum_{k=0}^{\infty} D_x^k f_0 \frac{\partial}{\partial u_k} \in \mathfrak{h}.$$

This leads to: $\bar{f}_0 k_0 = D_{k_0}[f_0]$. As long as things are time-independent, one does not see the difference, but in the time dependent case one really has to treat the equation and its symmetries as living in different spaces. Since everything is defined in terms of the Lie derivative of D_x, the natural ring of coefficients $\mathcal{C}$ is formed by the functions of t alone, since they commute with D_x. In the sequel, tensor products will be with respect to $\mathcal{C}$, unless otherwise indicated. We now have a ring of coefficients, a Lie algebra and a representation space, with the Lie derivative as the representation operator. From here on we can construct a Lie algebra complex. Let us give the first steps here, since we do not need the general theory.

We denote the space $\mathcal{C}$-multilinear n-forms by $C^n(\mathfrak{h}, \mathcal{A})$. One defines, starting with a 0-form in $C^0(\mathfrak{h}, \mathcal{A}) = \mathcal{A}$, i.e. a functional $\int g$, a 1-form

$$d^0 \int g(\sum_{k=0}^{\infty} D_x^k f_0 \frac{\partial}{\partial u_k}) = \int \sum_{k=0}^{\infty} D_x^k f_0 \frac{\partial g}{\partial u_k} \equiv \int f_0 \sum_{k=0}^{\infty} (-1)^k D_x^k \frac{\partial g}{\partial u_k} = \int f_0 \mathbf{E}(g)$$

in $C^1(\mathfrak{h},\mathcal{A})$, where $\mathbf{E}$ is the so-called Euler operator, cf [Olv93]. One could write this as

$$d^0\int g = \mathbf{E}(\int g).$$

When $\alpha \in C^1(\mathfrak{h},\mathcal{A})$ is of the form $\alpha = d^0 \int g$, one calls $\int g$ the **density** of α.

Next one defines, starting from a 1-form $\alpha \in C^1(\mathfrak{h},\mathcal{A})$, a 2-form $d^1\alpha \in C^2(\mathfrak{h},\mathcal{A})$ by

$$\iota^2(H)d^1\alpha = L_H\alpha - d^0\int \alpha\cdot H = L_H\alpha - d^0\iota^1(H)\alpha,$$

where $\iota^n(H)\beta(H_1,\cdots,H_n) = \beta(H,H_1,\cdots,H_n)$, with $H, H_i \in \mathfrak{h}, i = 1,\cdots .n$.

In this Lie algebra complex we identify all the objects of interest to the study of integrable systems, like symmetries, cosymmetries, conservation laws, recursion operators etc., which are all characterized by the fact that their Lie derivative with respect to the evolution equation vanishes. For the complete theory of this Lie algebra complex we refer to [Wan98].

In this paper we make the blanket assumptions that

- The operators and vectorfields are t-independent. However, with care we can treat the time-dependent case, see example 17. For some new developments concerning time-dependent vectorfields we refer to [SW01].
- There exists an universal scaling symmetry. E.g., for the KdV equation

$$u_t = K[u] = u_3 + uu_1$$

  we have a scaling symmetry $\overline{xu_1 + 2u}$, such that $\lambda_{\hat{K}} = 3$ and $\lambda_{\mathfrak{R}[u]} = 2$. Here λ_Y is defined by $L_{\overline{xu_1+2u}}Y = \lambda_Y Y, \quad \lambda_Y \in \mathcal{C}$.
- For any operator $\mathfrak{R} : \mathfrak{h} \to \mathfrak{h}$ of the form

$$\mathfrak{R} = \sum_{i=0}^{n} \mathfrak{R}^{(i)} D_x^i + \sum_{j\in\Gamma} h^{(j)} \otimes D_x^{-1}\xi^{(j)},$$

  where $\mathfrak{R}^{(i)} \in \mathfrak{h}\otimes C^1(\mathfrak{h},\mathcal{A})$, $h^{(j)} \in \mathfrak{h}$, $\xi^{(j)} \in C^1(\mathfrak{h},\mathcal{A})$ and Γ is the set of gradings, i.e., $\lambda_{h^{(j)}} = j$. Remark that this does not hold for the $\xi^{(j)}$, but we have $j + \lambda_{\xi^{(j)}} = 1 + \lambda_{\mathfrak{R}}$. We denote $\mathfrak{R} \sim \sum_{j\in\Gamma} h^{(j)} \otimes D_x^{-1}\xi^{(j)}$. In order to be able to give some estimates later, we introduce the **gap length**

$$\gamma(\mathfrak{R}) = max_{j\in\Gamma} j - min_{j\in\Gamma} j.$$

  We denote by $\mathcal{G}_{\mathfrak{R}}^{\perp}$ the space of all $g \in \mathfrak{h}$ such that $\xi^{(j)}(g) \equiv 0$, where the $\xi^{(j)}$ are given in the recursion operator.

Lemma 2 *Let $\mathfrak{R} = \sum_{i=0}^{n} \mathfrak{R}^{(i)} D_x^i + \sum_{j\in\Gamma} h^{(j)} \otimes D_x^{-1}\xi^{(j)}$ be a recursion operator of the equation $u_t = K$, i.e. $L_{\hat{K}}\mathfrak{R} = 0$, with $|\lambda_{\hat{K}}| > \gamma(\mathfrak{R})$. Then the $h^{(j)}$ are symmetries of the equation and the $\xi^{(j)}$ are cosymmetries for any $j \in \Gamma$, that is*

$$L_{\hat{K}} h^{(j)} = 0, \qquad L_{\hat{K}} \xi^{(j)} = 0.$$

PROOF. *Notice that*

$$L_{\hat{K}}\mathfrak{R} \sim \sum_{j\in\Gamma} \left(h^{(j)} \otimes D_x^{-1} L_{\hat{K}} \xi^{(j)} + L_{\hat{K}} h^{(j)} \otimes D_x^{-1} \xi^{(j)} \right).$$

We have either $\lambda_{K\partial_u} > \gamma(\mathfrak{R})$, in which case $\lambda_{L_{\hat{K}} h^{(j)}} = \lambda_{\hat{K}} + j > \gamma(\mathfrak{R}) + j \geq max_{j\in\Gamma} j$, or $\lambda_{\hat{K}} < -\gamma(\mathfrak{R})$ and then $\lambda_{L_{\hat{K}} h^{(j)}} = \lambda_{\hat{K}} + j < -\gamma(\mathfrak{R}) + j \leq min_{j\in\Gamma} j$. Therefore $L_{\hat{K}} h^{(j)} = 0$ and $L_{\hat{K}} \xi^{(j)} = 0$.

Remark 3 *The only case known to us that a recursion operator is not in the form covered by our blanket assumption, is in one of the noncommutative mKdV equations, see [GKS99].*

Remark 4 *We mention that this result also appeared in the paper [Bil93]. The author gave the condition that $h^{(j)}$ and $\xi^{(j)}$ are independent differential functions, which seems not enough for the proof.*

Remark 5 *A similar result holds for symplectic and cosymplectic operators i.e., their D_x^{-1} part (if it exists) can be written as the product of cosymmetries and symmetries, respectively. This observation is of great help in computing the splitting of a recursion operator, see section 5.7,* The new Nijenhuis operator (3D), *for an example.*

4 Hierarchies of symmetries

It is shown that under certain conditions (which hold for all examples known to us) Nijenhuis operators are well defined, i.e., they give rise to hierarchies of infinitely many commuting symmetries of the operator. Moreover, the nonlocal part of a Nijenhuis operator contains the candidates of roots and coroots.

Definition 6 *We say that A is an* **invariant** *of X if $L_X A = 0$. If A is in $\mathcal{A}$, then $L_X A = XA$, if $A \in \mathfrak{h}$ then $L_X A = [X, A]$. For the definition of the Lie derivative L_X on other objects, see [Olv93, Wan98, Oev84].*

We make a distinction between $\mathfrak{R}$ being an invariant of $\hat{K}$ and of $Y \in \mathfrak{h}$. In the first case, we say $\mathfrak{R}$ is a **recursion operator** of $u_t = K$ but in the second case that Y is a **symmetry of** $\mathfrak{R}$. When $\mathfrak{R}$ is t-independent, the $\frac{\partial}{\partial t}$ in $\hat{K}$ does not play a role. These two act in the same way. The operator $\mathfrak{R}$ is $\mathcal{C}$-linear. If Y is a symmetry of $\mathfrak{R}$, then fY, with $f \in \mathcal{C}$, is also a symmetry of $\mathfrak{R}$. However, when Y is a symmetry of $\hat{K}$, in general fY will not be a symmetry unless $f \in \mathbb{R}$.

We finally remark that $\bar{u}_1$ is a trivial symmetry for any operator and that any t-independent operator is a recursion operator of the equation $u_t = u_1$.

Definition 7 *An operator $\mathfrak{R}$ is called a* **Nijenhuis operator** *(or* **hereditary operator***) if*

$$L_{\mathfrak{R}Y}\mathfrak{R} = \mathfrak{R}L_Y\mathfrak{R} \quad \forall Y \in \mathfrak{h}$$

Corollary 8 *If $Y \in \mathfrak{h}$ is a symmetry of $\mathfrak{R}$, then the $\mathfrak{R}^l Y, l \in \mathbb{N}$ are commuting symmetries.*

Theorem 9 *Let $\mathfrak{R} = \sum_{i=0}^n \mathfrak{R}^{(i)} D_x^i + \sum_{j\in\Gamma} h^{(j)} \otimes D_x^{-1}\xi^{(j)}$ be a Nijenhuis operator with $d^1\xi^{(r)} = 0$ for $r \in \Gamma$. If $Q_0 \in \mathfrak{h}$ is a symmetry of the Nijenhuis operator with $|\lambda_{Q_0}| > \gamma(\mathfrak{R})$ and $\lambda_{\mathfrak{R}}\lambda_{Q_0} \geq 0$, then $Q_l = \mathfrak{R}^l Q_0 \in \mathcal{G}_{\mathfrak{R}}^{\perp}$ for all $l \geq 0$. Moreover, the Q_l commute.*

PROOF. *Since $\mathfrak{R}$ is a Nijenhuis operator, for any $l \geq 0$ and any symmetry $Q_0 \in \mathfrak{h}$ i.e., $L_{Q_0}\mathfrak{R} = 0$, it follows from corollary 8 that $Q_l \in \mathfrak{h}$ satisfies $L_{Q_l}\mathfrak{R} = 0$. We have*

$$L_{Q_l}\mathfrak{R} \sim \sum_{j\in\Gamma} \left(h^{(j)} \otimes D_x^{-1} L_{Q_l}\xi^{(j)} + L_{Q_l}h^{(j)} \otimes D_x^{-1}\xi^{(j)} \right).$$

Due to the assumption that $\lambda_{\mathfrak{R}}\lambda_{Q_0} \geq 0$, $|\lambda_{Q_l}| = |l\lambda_{\mathfrak{R}} + \lambda_{Q_0}| \geq |\lambda_{Q_0}| > \gamma(\mathfrak{R})$. Therefore $L_{Q_l}h^{(j)} = 0$ and $L_{Q_l}\xi^{(j)} = 0$ by the same reason as in the proof of lemma 2. We have, using the definition of d^1,

$$d^0 \int \xi^{(j)} \cdot Q_l = d^0 \iota^1(Q_l)\xi^j = L_{Q_l}\xi^{(j)} - \iota^2(Q_l) d^1 \xi^j = 0.$$

This implies $\xi^{(j)} \cdot Q_l \in Im D_x$ and we prove $Q_l \in \mathcal{G}_{\mathfrak{R}}^{\perp}$ by induction. The commuting of Q_l follows from corollary 8.

Remark 10 *The operator $\mathfrak{R}$ can often be written as*

$$\sum_{i=0}^n \mathfrak{R}^{(i)} D_x^i + \sum_{j\in\Gamma} h^{(j)} \otimes D_x^{-1} d^0 T^{(j)},$$

where the $T^{(j)}$ are the densities of $\xi^{(j)}$, i.e., $\mathbf{E}(T^{(j)}) = \xi^{(j)}$.

Our assumptions for lemma 2 and theorem 9 are not sharp. For a given operator which is t-dependent, the proof may still go through (cf. example 17).

Recursion operators of nonevolution equations appear to have a similar form. Compare with [vBGKS97], where the same splitting of the D_x^{-1} and D_y^{-1} terms is found. It would be interesting to see whether one can obtain similar results.

Corollary 11 *If, moreover, $\mathfrak{R}$ is a recursion operator of an equation and Q_0 is a symmetry of the equation, all Q_l for $l \geq 0$ consist of a hierarchy of commuting symmetries of the equation.*

Definition 12 *Let $\mathfrak{R}$ be a Nijenhuis operator. If the $\mathfrak{R}^l Q_0 \neq 0$ exist for all $l \geq 0$ and are commuting symmetries of $\mathfrak{R}$, we call $Q_0 \in \mathfrak{h}$ but $\notin Im\mathfrak{R}$ a* **root of $\mathfrak{R}$**.

Definition 13 *Let $\mathfrak{R}$ be a recursion operator of a given equation. If the $\mathfrak{R}^l Q_0 \neq 0$ exist for all $l \geq 0$ and are commuting symmetries of the equation, we call $Q_0 \in \mathfrak{h}$ but $\notin Im\mathfrak{R}$ a* **root of symmetries** *for the equation.*

This way, a hierarchy of commuting symmetries (of $\mathfrak{R}$ or the equation itself, respectively) can be characterized by the pair $(\mathfrak{R}, Q_0)$.

Definition 14 *We define $\mathfrak{R}^\star : C^1(\mathfrak{h}, \mathcal{A}) \to C^1(\mathfrak{h}, \mathcal{A})$, the conjugate of $\mathfrak{R}$, by*

$$(\mathfrak{R}^\star \alpha)Y = \alpha(\mathfrak{R}Y), \ \ Y \in \mathfrak{h}.$$

Similarly, we define $Q^0 \in C^1(\mathfrak{h}, \mathcal{A})$ to be the **coroot of a Nijenhuis operator $\mathfrak{R}$**, from which we produce a hierarchy consisting of elements satisfying $d^1 \mathfrak{R}^{\star l} Q^0 = 0$ for all $l \geq 0$ and **coroot of covariants** for the equation, when the operator $\mathfrak{R}$ is its recursion operator, from which we produce a hierarchy of covariants (furthermore, conserved densities) for the equation.

The question we try to answer now is: are the conditions in theorem 9 really necessary or does everything follow from the fact that one has a Nijenhuis operator? Our answer is not complete, but definitely a step in the right direction.

The reader should also notice that the proofs of theorems 9 and 15 are advertisements for the cohomological methods which are the basis for our setup. Written out in terms of Fréchet derivatives the proofs take many pages.

Theorem 15 *Define*

$$\Delta = \{H \in dom\ \mathfrak{R} \mid |\lambda_{\mathfrak{R}H}| > \gamma(\mathfrak{R})\},$$

and let $\mathfrak{R} = \sum_{i=0}^{n} \mathfrak{R}^{(i)} D_x^i + \sum_{j\in\Gamma} h^{(j)} \otimes D_x^{-1}\xi^{(j)}$ *be a Nijenhuis operator. Assume that* $\mathfrak{R}h^{(j)}$ *exist and that* $d^1\xi^{(j)} = 0$ *for* $j \in \Gamma$*. Then* $L_{h^{(j)}}\mathfrak{R}\mid_{\Delta}= 0$ *and* $\iota^2(H)d^1\mathfrak{R}^{\star}\xi^{(j)} = 0,\ H \in \Delta$*.*

PROOF. We know that $\mathfrak{R}$ is a Nijenhuis operator, that is $L_{\mathfrak{R}H}\mathfrak{R} = \mathfrak{R}L_H\mathfrak{R}$ for $H \in dom\ \mathfrak{R}$. We have

$$\begin{aligned}
\mathfrak{R}L_H\mathfrak{R} \sim & \sum_{r\in\Gamma} \mathfrak{R}L_H h^{(r)} \otimes D_x^{-1}\xi^{(r)} + \mathfrak{R}h^{(r)} \otimes D_x^{-1}L_H\xi^{(r)} \\
& + \sum_{j\in\Gamma} h^{(j)} \otimes D_x^{-1}(L_H\mathfrak{R})^{\star}\xi^{(j)}, \\
L_{\mathfrak{R}H}\mathfrak{R} \sim & \sum_{j\in\Gamma} L_{\mathfrak{R}H}h^{(j)} \otimes D_x^{-1}\xi^{(j)} + \sum_{j\in\Gamma} h^{(j)} \otimes D_x^{-1}L_{\mathfrak{R}H}\xi^{(j)}.
\end{aligned}$$

Since $H \in dom\ \mathfrak{R}$, $d_0^0 \int \xi^{(j)} \cdot H = 0$. We know that

$$d^0 \int \xi^{(j)} \cdot H = d^0\iota^1(H)\xi^{(j)} = L_H\xi^{(j)} - \iota^2(H)d^1\xi^{(j)}.$$

Therefore, $L_H\xi^{(j)} = 0$.

Due to the same analysis as in the proof of lemma 2, for any $H \in \Delta$, we draw the following conclusions:

$$\mathfrak{R}L_H h^{(j)} = L_{\mathfrak{R}H}h^{(j)}, \tag{4.1}$$
$$(L_H\mathfrak{R})^{\star}\xi^{(j)} = L_{\mathfrak{R}H}\xi^{(j)}. \tag{4.2}$$

Formula (4.1) implies that $(L_{h^{(j)}}\mathfrak{R})H = L_{h^{(j)}}(\mathfrak{R}H) - \mathfrak{R}L_{h^{(j)}}H = 0$. Hence $L_{h^{(j)}}\mathfrak{R}\mid_{\Delta}= 0$.

Notice that $(L_H\mathfrak{R})^{\star} = L_H\mathfrak{R}^{\star}$. Therefore, formula (4.2) implies that

$$L_H(\mathfrak{R}^{\star}\xi^{(j)}) - \mathfrak{R}^{\star}L_H\xi^{(j)} = L_{\mathfrak{R}H}\xi^{(j)},$$

i.e., $L_H(\mathfrak{R}^{\star}\xi^{(j)}) = L_{\mathfrak{R}H}\xi^{(j)}$ since $L_H\xi^{(j)} = 0$. This leads to

$$\begin{aligned}
&\iota^2(H)d^1\mathfrak{R}^{\star}\xi^{(j)} = L_H(\mathfrak{R}^{\star}\xi^{(j)}) - d^0 \int \mathfrak{R}^{\star}\xi^{(j)} \cdot H = \\
&= L_{\mathfrak{R}H}\xi^{(j)} - d^0 \int \xi^{(j)} \cdot \mathfrak{R}H = \iota^2(\mathfrak{R}H)d^1\xi^{(j)} = 0.
\end{aligned}$$

The statement is proved now.

Remark 16 *This theorem theoretically gives us the candidates of roots and coroots. One notices that the restriction on the space Δ is due to the technical problem in the proof. In practice, they are indeed roots and coroots, since the statement of the theorem can be checked by computer algebra on arbitrary $H \in dom\ \mathfrak{R}$, without any use of the condition that restricts one to Δ.*

Example 17 *Consider the* **Cylindrical Korteweg–de Vries equation**

$$u_t = u_3 + uu_1 - \frac{u}{2t}$$

and let a Nijenhuis recursion operator be given by

$$\mathfrak{R} = t(D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1 D_x^{-1}) + \frac{1}{3}x + \frac{1}{6}D_x^{-1} \sim (\frac{t}{3}u_1 + \frac{1}{6})D_x^{-1}.$$

There exists a scaling symmetry $-3t\partial_t + \overline{xu_1 + 2u}$ such that

$$\lambda_{\hat{u}_t} = 3,\ \lambda_{\mathfrak{R}} = -1,\ \lambda_{\overline{\frac{t}{3}u_1 + \frac{1}{6}}} = -2.$$

However, $h^{(-2)} = \frac{t}{3}u_1 + \frac{1}{6}$ is **not** *a symmetry and $\xi^{(-2)} = 1$ is* **not** *a cosymmetry of the equation. Lemma 2 fails since $L_X \xi^{(-2)} = -\frac{1}{2t}\xi^{(-2)}$ and $-\frac{1}{2t} \in \mathcal{C}$ which commutes with D_x^{-1}. We now compute $L_X \kappa_{-2} = -\frac{1}{2t}\kappa_{-2}$, to find $\frac{\partial \kappa_{-2}}{\partial t} = -\frac{1}{2t}\kappa_{-2}$. This implies $\kappa_{-2} = \frac{1}{\sqrt{t}}$.*

We rewrite the nonlocal part of $\mathfrak{R}$ as $\mathfrak{R} \sim \sqrt{t}(\frac{t}{3}u_1 + \frac{1}{6})D_x^{-1}\frac{1}{\sqrt{t}}$, and we have that $\lambda_{\sqrt{t}(\frac{u_1}{3} + \frac{1}{6t})} = -\frac{7}{2}$ and $\lambda_{\frac{1}{\sqrt{t}}} = \frac{5}{2}$. By direct computation, we know that $h^{(-\frac{7}{2})}$ is a symmetry of the equation and $\xi^{(-\frac{7}{2})}$ is a cosymmetry.

Notice that $L_{h^{(-\frac{7}{2})}}\mathfrak{R} = 0$ and the proof of theorem 9 can be applied. Therefore, it produces a hierarchy of symmetries of the operator $\mathfrak{R}$. So we conclude that $\sqrt{t}(\frac{u_1}{3} + \frac{1}{6t})$ is a root of symmetries for the equation and $\frac{1}{\sqrt{t}}$ is a coroot.

The trivial symmetry u_1 of $\mathfrak{R}$ cannot produce a hierarchy of symmetries, consistent with the fact $\lambda_{u_1}\lambda_{\mathfrak{R}} = -1$ and it does not satisfy the conditions of theorem 9.

5 Examples

A number of examples are given, exhibiting the structure of the Nijenhuis operator and proving the existence of the hierarchies.

In the following examples we do not check whether the recursion operator is in fact a Nijenhuis recursion operator except for Burgers equation. Proofs in the literature are usually in the forms 'after a long and boring calculation it follows that...'. We show they satisfy the other conditions of the theorem.

5.1 Burgers equation

Consider Burgers equation [Olv93, p.315], [Oev84, p.38]

$$u_t = f = u_2 + uu_1.$$

We explicitly check all the conditions we need in order to prove that a hierarchy of the symmetries for the equation and the operator exists.

First we check that $\mathfrak{R} = D_x + \frac{1}{2}u + \frac{1}{2}u_1 D_x^{-1}$ is a recursion operators of the equation, i.e., that $L_{\hat{f}}\mathfrak{R}$ is equal to zero. We have

$$\begin{aligned}
L_X\mathfrak{R} &= \frac{\partial \mathfrak{R}}{\partial t} + D_{\mathfrak{R}}[f] - D_f\mathfrak{R} + \mathfrak{R}D_f \\
&= \frac{1}{2}f + \frac{1}{2}D_x(f)D_x^{-1} - (D_x^2 + uD_x + u_1)(D_x + \frac{1}{2}u + \frac{1}{2}u_1D_x^{-1}) \\
&\quad +(D_x + \frac{1}{2}u + \frac{1}{2}u_1D_x^{-1})(D_x^2 + uD_x + u_1) \\
&= \frac{u_2 + uu_1}{2} + \frac{u_3 + uu_2 + u_1^2}{2}D_x^{-1} \\
&\quad -(D_x^3 + \frac{3u}{2}D_x^2 + \frac{5u_1 + u^2}{2}D_x + \frac{3u_2 + 3uu_1}{2} + \frac{u_3 + uu_2 + u_1^2}{2}D_x^{-1}) \\
&\quad +(D_x^3 + \frac{3u}{2}D_x^2 + \frac{5u_1 + u^2}{2}D_x + u_2 + uu_1) \\
&= 0.
\end{aligned}$$

Furthermore, there exists a vectorfield $\overline{xu_1 + u}$ as a scaling symmetry such that $\lambda_{\overline{u_2+uu_1}} = 2,\ \lambda_{\mathfrak{R}} = 1,\ \lambda_{\overline{u_1}} = 1$. It is easy to see that for $\mathfrak{R}$, the conditions of lemma 2 are satisfied since $\gamma(\mathfrak{R}) = 0$. So, u_1 is a symmetry of the equation.

Now we check $\mathfrak{R}$ is a Nijenhuis operator. For all $H \in dom\mathfrak{R}$, i.e., $H = D_xP$, we compute

$$\begin{aligned}
L_{\mathfrak{R}H}\mathfrak{R} = {} & D_{\mathfrak{R}}[D_x(H) + \frac{1}{2}D_x(uP)] - (D_xD_H + \frac{1}{2}D_x \cdot (uD_P + P))\mathfrak{R} \\
& +\mathfrak{R}(D_xD_H + \frac{1}{2}D_x \cdot (uD_P + P))
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}(D_x(H) + \frac{1}{2}D_x(uP)) + \frac{1}{2}(D_x^2(H) + \frac{1}{2}D_x^2(uP))D_x^{-1} \\
&\quad -(\mathfrak{R}D_H + \frac{1}{2}(PD_x + H))\mathfrak{R} + \mathfrak{R}(\mathfrak{R}D_H + \frac{1}{2}(PD_x + H)) \\
&= \mathfrak{R}^2 D_H - \mathfrak{R}D_H\mathfrak{R} + \frac{1}{2}D_x(H) + \frac{1}{4}uH + \frac{1}{4}u_1P \\
&\quad + \frac{1}{2}(D_x^2(H) + \frac{1}{2}(uD_x(H) + 2u_1H + u_2P))D_x^{-1} \\
&\quad - \frac{1}{4}u_1P - \frac{1}{4}(u_2P + u_1H)D_x^{-1} + \frac{1}{2}HD_x + \frac{1}{2}D_x(H) \\
&= \mathfrak{R}^2 D_H - \mathfrak{R}D_H\mathfrak{R} + \frac{1}{2}HD_x + D_x(H) + \frac{1}{4}uH \\
&\quad + \frac{1}{2}(D_x^2(H) + \frac{1}{2}(uD_x(H) + u_1H))D_x^{-1}, \\
\mathfrak{R}L_H\mathfrak{R} &= \mathfrak{R}(\frac{1}{2}H + \frac{1}{2}D_x(H)D_x^{-1} - D_H\mathfrak{R} + \mathfrak{R}D_H) \\
&= \mathfrak{R}^2 D_H - \mathfrak{R}D_H\mathfrak{R} + \frac{1}{2}HD_x + D_x(H) + \frac{1}{4}uH \\
&\quad + \frac{1}{2}(D_x^2(H) + \frac{1}{2}(uD_x(H) + u_1H))D_x^{-1}.
\end{aligned}
$$

Therefore, $L_{\mathfrak{R}H}\mathfrak{R} = \mathfrak{R}L_H\mathfrak{R}$, for all $H \in dom\mathfrak{R}$.

Notice that $\lambda_{\mathfrak{R}}\lambda_{u_1} > 0$ and $L_{u_1}\mathfrak{R} = 0$. From theorem 9 and its corollary, a hierarchy of symmetries of the equation are $\mathfrak{R}^l u_1$ for $l \geq 0$. This confirms our remark 16 that u_1 is the root of $\mathfrak{R}$. There is no coroot for it since $\mathfrak{R}^\star(1) = 0$. This reflects the fact there is only one conservation law for Burgers equation.

5.2 Krichever–Novikov equation

The **Krichever–Novikov equation** [Dor93, p. 121] is given by

$$u_t = u_3 - \frac{3}{2}u_1^{-1}u_2^2$$

and it has a Nijenhuis recursion operator of the form

$$\mathfrak{R} = D_x^2 - 2u_1^{-1}u_2D_x + (u_1^{-1}u_3 - u_1^{-2}u_2^2) + u_1D_x^{-1}\xi^{(1)},$$

where $\xi^{(1)} = 3u_1^{-4}u_2^3 - 4u_1^{-3}u_2u_3 + u_1^{-2}u_4 = \mathbf{E}(\frac{1}{2}u_1^{-2}u_2^2)$. First we have $\gamma(\mathfrak{R}) = 0$ and $\lambda_{\mathfrak{R}} = 2$, $\lambda_{u_1} = 1$, $\lambda_{u_t\partial_u} = 3$ with respect to a scaling symmetry $xu_1 \in \mathfrak{h}$. Therefore, u_1 is a symmetry and $\xi^{(1)}$ is a cosymmetry of the equation by lemma 2. Moreover, we compute $L_{u_1}\mathfrak{R} = 0$ and $\mathfrak{R}^\star\xi^{(1)} = \mathbf{E}(-\frac{1}{2}u_1^{-2}u_3^2 + \frac{3}{8}u_1^{-4}u_2^4)$. We conclude that u_1 is a root of a hierarchy and $\xi^{(1)}$ is a coroot generating a hierarchy of covariants for the equation.

5.3 *Diffusion system*

We consider the **Diffusion system** [Oev84, p. 41]

$$\begin{cases} u_t = u_2 + v^2 \\ v_t = v_2, \end{cases}$$

with Nijenhuis recursion operator given by

$$\mathfrak{R}[u,v] = \begin{pmatrix} D_x & vD_x^{-1} \\ 0 & D_x \end{pmatrix} \sim \begin{pmatrix} v \\ 0 \end{pmatrix} \otimes D_x^{-1} \begin{pmatrix} 0, & 1 \end{pmatrix}.$$

We have $\Gamma = \{1\}$, $\lambda_{\mathfrak{R}} = 1$ with respect to the scaling symmetry $\begin{pmatrix} xu_1 \\ xv_1 + v \end{pmatrix}$ and $L_{h^{(1)}}\mathfrak{R} = 0$, where obviously $d^1\xi^{(1)} = 0$. Thereby, $h^{(1)}$ fulfills the conditions of lemma 2 and theorem 9. It is a root of $\mathfrak{R}$ and also produces a hierarchy of symmetries of the equation.

Another candidate is the trivial symmetry $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$ and it satisfies the conditions of theorem 9, so it is a root of a hierarchy of symmetries, which includes the equation itself, for both the equation and $\mathfrak{R}$.

5.4 *Boussinesq system*

We consider the **Boussinesq system** [Olv93, p. 459]

$$\begin{cases} u_t = v_1 \\ v_t = \frac{1}{3}u_3 + \frac{8}{3}uu_1, \end{cases}$$

with Nijenhuis recursion operator given by

$$\mathfrak{R}(u,v) =$$
$$\begin{pmatrix} 3v + 2v_1 D_x^{-1} & D_x^2 + 2u + u_1 D_x^{-1} \\ \frac{1}{3}D_x^4 + \frac{10}{3}uD_x^2 + 5u_1 D_x + 3u_2 + \frac{16}{3}u^2 + 2v_t D_x^{-1} & 3v + v_1 D_x^{-1} \end{pmatrix}$$
$$\sim \begin{pmatrix} 2v_1 \\ \frac{2}{3}u_3 + \frac{16}{3}uu_1 \end{pmatrix} \otimes D_x^{-1} \begin{pmatrix} 1, & 0 \end{pmatrix} + \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \otimes D_x^{-1} \begin{pmatrix} 0, & 1 \end{pmatrix}.$$

There exists a scaling symmetry $\begin{pmatrix} xu_1 + 2u \\ xv_1 + 3v \end{pmatrix}$ such that $\Gamma = \{1, 2\}$ and $\lambda_{\mathfrak{R}} = 3$. Since the equation is t-independent, it is also a symmetry of $\mathfrak{R}$. Notice that $\gamma(\mathfrak{R}) = 1$ and $d^1\xi^{(1)} = d^1\xi^{(2)} = 0$. So $h^{(2)}$ obeys the estimates in theorem 9 and is therefore a root since $\lambda_{\mathfrak{R}} > 2$.

For $h^{(1)}$ we explicitly compute

$$\mathfrak{R}h^{(1)} = \begin{pmatrix} v_3 + 4u_1 v + 4uv_1 \\ \frac{1}{3}u_5 + 4uu_3 + 8u_1 u_2 + \frac{32}{3}u^2 u_1 + 4vv_1 \end{pmatrix}$$

and $\lambda_{\mathfrak{R}h^{(1)}} = 4$. So $\mathfrak{R}h^{(1)}$ satisfies the conditions of theorem 9. Therefore the $h^{(j)},\ j = 1, 2$ are roots of hierarchies of symmetries for the equation (of $\mathfrak{R}$).

5.5 Derivative Schrödinger system

Consider the **Derivative Schrödinger system** [Oev84, p. 103]

$$\begin{cases} u_t = -v_2 - (u^2 + v^2)u_1 \\ v_t = u_2 - (u^2 + v^2)v_1, \end{cases}$$

with Nijenhuis recursion operator given by

$$\mathfrak{R}(u,v) =$$
$$\begin{pmatrix} vD_x^{-1} \cdot v_1 - u_1 D_x^{-1} \cdot u - \frac{u^2+v^2}{2} & -D_x - u_1 D_x^{-1} \cdot v - vD_x^{-1} \cdot u_1 \\ D_x - v_1 D_x^{-1} \cdot u - uD_x^{-1} \cdot v_1 & uD_x^{-1} \cdot u_1 - v_1 D_x^{-1} \cdot v - \frac{u^2+v^2}{2} \end{pmatrix}$$
$$\sim \begin{pmatrix} vD_x^{-1} \cdot v_1 - u_1 D_x^{-1} \cdot u & -u_1 D_x^{-1} \cdot v - vD_x^{-1} \cdot u_1 \\ -v_1 D_x^{-1} \cdot u - uD_x^{-1} \cdot v_1 & uD_x^{-1} \cdot u_1 - v_1 D_x^{-1} \cdot v \end{pmatrix}$$

$$= \begin{pmatrix} v \\ -u \end{pmatrix} \otimes D_x^{-1} \begin{pmatrix} v_1, & -u_1 \end{pmatrix} + \begin{pmatrix} -u_1 \\ -v_1 \end{pmatrix} \otimes D_x^{-1} \begin{pmatrix} u, & v \end{pmatrix}.$$

We find that $\lambda_{\mathfrak{R}} = 1$ and $\Gamma = \{0,1\}$ with respect to the scaling symmetry $\begin{pmatrix} xu_1 + \frac{u}{2} \\ xv_1 + \frac{v}{2} \end{pmatrix}$, implying $\gamma(\mathfrak{R}) = 1$. And we have that $d^1\xi^{(0)} = d^1\xi^{(1)} = 0$. However, the condition for $h^{(0)}$ and $h^{(1)}$ in theorem 9 is not satisfied. First we notice that $h^{(1)} = -\mathfrak{R}h^{(0)}$. So all we have to check explicitly are the conditions for $h^{(1)}$. First all, $L_{h^{(1)}}\mathfrak{R} = 0$ trivially. Secondly, $\mathfrak{R}h^{(1)} = \begin{pmatrix} -u_t \\ -v_t \end{pmatrix} \neq 0$ and $\lambda_{\mathfrak{R}h^{(1)}} = 2$. We now see that $\mathfrak{R}h^{(1)}$ satisfies the conditions. So $h^{(0)} = \begin{pmatrix} v \\ -u \end{pmatrix}$ is a root of a hierarchy.

5.6 Sine–Gordon equation in the laboratory coordinates

Consider the **Sine–Gordon equation** [CLL87] in the form of

$$\begin{cases} u_t = v \\ v_t = u_2 - \alpha \sin(u), \alpha \in \mathbb{R}. \end{cases}$$

with Nijenhuis recursion operator given by

$$\begin{aligned} \mathfrak{R}(u,v) &= \begin{pmatrix} \mathfrak{R}_{11} & \mathfrak{R}_{12} \\ \mathfrak{R}_{21} & \mathfrak{R}_{22} \end{pmatrix} \\ &\sim \begin{pmatrix} u_1 + v \\ u_2 + v_1 - \alpha \sin(u) \end{pmatrix} \otimes D_x^{-1} \begin{pmatrix} -(u_2 + v_1 - \alpha \sin(u)), & u_1 + v \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \mathfrak{R}_{11} =& 4D_x^2 - 2\alpha \cos(u) + (u_1 + v)^2 - (u_1 + v)D_x^{-1}(u_2 + v_1 - \alpha \sin(u)), \\ \mathfrak{R}_{12} =& 4D_x + (u_1 + v)D_x^{-1}(u_1 + v), \\ \mathfrak{R}_{21} =& 4D_x^3 + (u_1 + v)^2 D_x - 4\alpha \cos(u)D_x + 2u_1\alpha \sin(u) + (u_2 + v_1)(u_1 + v) \\ & - (u_2 + v_1 - \alpha \sin(u))D_x^{-1}(u_2 + v_1 - \alpha \sin(u)), \end{aligned}$$

$$\mathfrak{R}_{22} = 4D_x^2 + (u_1+v)^2 - 2\alpha\cos(u) + (u_2+v_1-\alpha\sin(u))D_x^{-1}(u_1+v).$$

This is one of the 'new' operators which appeared in [FOW87], p. 53, when $\alpha = 1$. The system is not homogeneous in 2-dimensional space, namely u and v. Let us consider the extended system

$$\begin{cases} u_t = v \\ v_t = u_2 - \alpha\sin(u) \\ \alpha_t = 0. \end{cases}$$

Then

$$\tilde{\mathfrak{R}}(u,v,\alpha) = \begin{pmatrix} \mathfrak{R}_{11} & \mathfrak{R}_{12} & 0 \\ \mathfrak{R}_{21} & \mathfrak{R}_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim h^{\tilde{(1)}} \otimes D_x^{-1}\xi^{\tilde{(1)}}$$

and $\lambda_{\mathfrak{R}} = 2$ with respect to the scaling symmetry $\begin{pmatrix} xu_1 \\ xv_1 + v \\ x\alpha_x + 2\alpha \end{pmatrix}$. We see that $\gamma(\mathfrak{R}) = 0$, $d^1\xi^{\tilde{(1)}} = 0$ and $L_{h^{\tilde{(1)}}}\tilde{\mathfrak{R}} = 0$. So $h^{\tilde{(1)}}$ is a root of a hierarchy of symmetries of the Nijenhuis operator $\tilde{\mathfrak{R}}$. This leads to the same result if we take α as a constant, i.e., $h = \begin{pmatrix} u_1 + v \\ u_2 + v_1 - \alpha\sin(u) \end{pmatrix}$ is a root of $\mathfrak{R}$.

In fact $\mathfrak{R}(h) = 2\mathfrak{R}(Q^0)$, where $Q^0 = \begin{pmatrix} u_1, & v_1 \end{pmatrix}$ which is considered as a start point in [FOW87] (so this is not in contradiction, since the same hierarchy will be generated by h and Q^0).

5.7 The new Nijenhuis operator (3D)

Consider the following Nijenhuis operator [FOW87, p. 54]

$$\mathfrak{R}(u,\phi,\psi)=\begin{pmatrix} 4u^2 & 0 & 1 \\ D_x & -2u^2 & 0 \\ 4u\psi & D_x-4u\phi & -2u^2 \end{pmatrix}$$

$$+4\begin{pmatrix} \phi \\ \psi \\ u_1-6u^2\phi \end{pmatrix}\otimes D_x^{-1}\begin{pmatrix} \psi+6u^3, & -\phi, & u \end{pmatrix}.$$

We see that $\lambda_{\mathfrak{R}}=\frac{2}{3}$ and $\mathfrak{R}\sim 4h^{(\frac{1}{3})}\otimes D_x^{-1}\xi^{(\frac{1}{3})}$ under the scaling symmetry

$$\begin{pmatrix} xu_1+\frac{u}{3} \\ x\phi_1+\frac{2\phi}{3} \\ x\psi_1+\psi \end{pmatrix}.$$

Furthermore, we have $L_{h^{(\frac{1}{3})}}\mathfrak{R}=0$ and $d^1\xi^{(\frac{1}{3})}=0$. So the conditions of theorem 9 are satisfied for $h^{(\frac{1}{3})}$. Therefore from $h^{(\frac{1}{3})}$ a hierarchy of symmetries of $\mathfrak{R}$ is generated.

In fact $\mathfrak{R}(h^{(\frac{1}{3})})=\begin{pmatrix} 4u_1, & 4\phi_1, & 4\psi_1 \end{pmatrix}$, which is considered as a starting point in [FOW87]. $h^{(\frac{1}{3})}$ must be a root, since the only (using a scaling argument and the fact that it should be in the domain of $\mathfrak{R}$) vectorfield that could generate $h^{(\frac{1}{3})}$ is $\begin{pmatrix} 0, & u, & \phi \end{pmatrix}$, and this gives not $h^{(\frac{1}{3})}$, but $h^{(\frac{1}{3})}-\begin{pmatrix} 0, & \psi+2u^3, & 0 \end{pmatrix}$ and their difference is not in $\operatorname{Ker}\mathfrak{R}$.

Assume $\mathfrak{H}_2=\mathfrak{R}\mathfrak{H}_1$ where $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are both cosymplectic operators. We know that $\mathfrak{H}_2$ must have the following term as its nonlocal part

$$\begin{pmatrix} \phi \\ \psi \\ u_1-6u^2\phi \end{pmatrix}\otimes D_x^{-1}\begin{pmatrix} \phi \\ \psi \\ u_1-6u^2\phi \end{pmatrix}.$$

This means $\Big(\psi + 6u^3,\ -\phi,\ u\Big)\,\mathfrak{H}_1 \equiv \Big(\phi,\ \psi,\ u_1 - 6u^2\phi\Big)$. Therefore

$$\mathfrak{H}_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 6u^2 \\ 0 & -6u^2 & -D_x \end{pmatrix}.$$

One can easily check that $\mathfrak{H}_1$ and $\mathfrak{H}_2$ form an cosymplectic pair (cf. [Olv93] pp. 444–454)

5.8 Landau–Lifshitz system

Consider the **Landau–Lifshitz system** [vBK91]

$$\begin{cases} u_t = -\sin(u)v_2 - 2\cos(u)u_1v_1 + (J_1 - J_2)\sin(u)\cos(v)\sin(v) \\ v_t = \frac{u_2}{\sin(u)} - \cos(u)v_1^2 + \cos(u)(J_1\cos^2(v) + J_2\sin^2(v) - J_3), \end{cases}$$

with Nijenhuis recursion operator given by

$$\begin{aligned} &\mathfrak{R}(u, v) = \\ &= \begin{pmatrix} \mathfrak{R}_{11} & \mathfrak{R}_{12} \\ \mathfrak{R}_{21} & \mathfrak{R}_{22} \end{pmatrix} \\ &\sim \begin{pmatrix} u_t \\ v_t \end{pmatrix} \otimes D_x^{-1}\Big(\sin(u)v_1,\ -\sin(u)u_1\Big) - \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \otimes D_x^{-1}\Big(S_1,\ S_2,\Big), \end{aligned}$$

where

$$\begin{aligned} \mathfrak{R}_{11} = &-D_x^2 - 2\sin^2(u)v_1^2 - u_1^2 + v_1^2 - (J_1 - J_2)\sin^2(u)\sin^2(v) \\ &+(J_1 - J_3)\sin^2(u) + J_3 - J_2 + u_tD_x^{-1}\cdot(\sin(u)v_1) - u_1D_x^{-1}\cdot S_1, \\ \mathfrak{R}_{12} = &\,2\cos(u)\sin(u)v_1D_x + \cos(u)\sin(u)v_2 - 3\sin^2(u)u_1v_1 + 2u_1v_1 \\ &+u_tD_x^{-1}\cdot(-\sin(u)u_1) - u_1D_x^{-1}\cdot S_2, \\ \mathfrak{R}_{21} = &-2\cos(u)v_1D_x - \cos(u)v_2 + u_1v_1 \\ &+v_tD_x^{-1}\cdot(\sin(u)v_1) - v_1D_x^{-1}\cdot S_1, \\ \mathfrak{R}_{22} = &-D_x^2 - 2\cos(u)u_1D_x - \cos(u)u_2 - (J_1 - J_2)\sin(u)\sin^2(v) \\ &-2\sin^2(u)v_1^2 + v_1^2 + (J_1 - J_3)\sin^2(u) + J_3 - J_2 \end{aligned}$$

$$
\begin{aligned}
&+v_t D_x^{-1} \cdot (-\sin(u)u_1) - v_1 D_x^{-1} \cdot S_2, \\
S_1 = &\,(J_1 - J_2)\cos(u)\sin(u)\sin^2(v) - (J_1 - J_3)\cos(u)\sin(u) \\
&+ \cos(u)\sin(u)v_1^2 - u_2, \\
S_2 = &\,(J_1 - J_2)\cos(v)\sin^2(u)\sin(v) - 2\cos(u)\sin(u)u_1v_1 - \sin^2(u)v_2.
\end{aligned}
$$

As we did in section 5.6, for the extended system, we have that $\lambda_{J_i} = 2$ for $i = 1, 2, 3$, $\lambda_{\mathfrak{R}} = 2$ and $\Gamma = \{1, 2\}$ with

$$
h^{(1)} = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \; h^{(2)} = \begin{pmatrix} u_t \\ v_t \end{pmatrix}.
$$

So $\gamma(\mathfrak{R}) = 1$. Notice that $d^1\xi^{(1)} = d^1\xi^{(2)} = 0$. It follows that $h^{(2)}$ is a root of a hierarchy. For $h^{(1)}$, we have

$$
\begin{aligned}
&\xi^{(1)} \cdot h^{(1)} = \begin{pmatrix} S_1, \; S_2 \end{pmatrix} \cdot \begin{pmatrix} u_1 \; v_1 \end{pmatrix} \\
&= (J_1 - J_2)\cos(u)\sin(u)\sin^2(v)u_1 + (J_1 - J_2)\cos(v)\sin^2(u)\sin(v)v_1 \\
&\quad -(J_1 - J_3)\cos(u)\sin(u)u_1 - u_1u_2 - \cos(u)\sin(u)u_1v_1^2 - \sin^2(u)v_1v_2 \\
&= \frac{1}{2}D_x((J_1 - J_2)\sin^2(u)\sin^2(v) - (J_1 - J_3)sin^2(u) - u_1^2 - \sin^2(u)v_1^2)
\end{aligned}
$$

and

$$
\xi^{(2)} \cdot h^{(1)} = \begin{pmatrix} \sin(u)v_1 & -\sin(u)u_1 \end{pmatrix} \cdot \begin{pmatrix} u_1, \; v_1 \end{pmatrix} = 0.
$$

So $h^{(1)} \cdot \xi^{(j)} \in ImD_x$ for $j = 1, 2$. This implies that $\mathfrak{R}h^{(1)} \neq 0$ exists. Notice that $\lambda_{\mathfrak{R}h^{(1)}} = 3$. By theorem 9, we conclude that $h^{(1)}$ is a root of a hierarchy.

6 Plans for the future

Nowadays the field of integrable systems is so widely extended that it would be difficult to list all developments, let alone sketch their future direction. So here we restrict ourself to our own views.

It is surprising, at least to us, how many questions, both on a fundamental and philosophical level and on a practical computational level are still unanswered, in spite of more than three decades of rather intensive activity, which attracted a sizeable part of the mathematical and physical research community.

For one thing, the connections between the different kind of integrability, while quite convincing by their overwhelming evidence, are still far from being understood. And with any new pattern that is found, there is none that gives a complete and convincing explanation.

On a more mundane level, there are two kinds of computations that never seem to have been completely incorporated in the theoretic development, namely the time-dependent and the nonlocal systems and symmetries. For instance, in [SW01], we found that the time dependent recursion operator for Burgers equation gives the wrong result! The source of many of the problems seems to be that the relation

$$D_x^{-1} D_x = 1$$

is not valid on $\operatorname{Ker} D_x$. In [OSW01] we address these problems, extending the theory in such a way that we get rid of $\operatorname{Ker} D_x$. We hope to be able to analyze by Lie algebraic methods the nonlocal KP equation, and more specifically its time-dependent symmetries using representation theory of $\mathfrak{sl}(2, \mathbb{R})$, see [SW00a].

It is hoped that classifications of integrable systems, according to the different definitions, will clarify their connections. At least it will provide us with definitive material to start thinking about the larger issues. A good source on this is [Zak91]. More recently we were able to classify a large class of scalar equations [SW98,SW00b] and a not so large class of systems of evolution equations [BSW98,BSW01], using the symbolic method and number theory. We are presently working on the classification of homogeneous polynomial systems.

Another area of current interest is noncommutative equations. The similarity and difference between the commutative and noncommutative case is rather unexpected and unpredictable. A list of scalar equations with one symmetry was found in [OS98] and these equations were proved to be integrable in [OW00]. The technique of the proof in this case is completely similar to that in the commutative case.

It is often assumed, implicitly or explicitly, that the occurrence of one generalized (non-contact) symmetry of an evolution equation implies the existence of infinitely many. As a rule, this seems to be the case. However, in [vdKS99] an example was found of a two-dimensional system with exactly two such symmetries. This means that the conjecture of Fokas, which states that if the number of such symmetries equals the dimension of the system, the system must by integrable, has to be refined in order to describe the situation correctly. Clearly, theorems to this effect would be of great practical importance, since they reduce the integrability question to a search of a finite number of symmetries. Of course, bounds on the order of possible symmetries are also important in this respect.

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