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Integrable Systems Associated to a Hopf Surface

Ruxandra Moraru

Abstract. A Hopf surface is the quotient of the complex surface $\mathbb{C}^2 \setminus \{0\}$ by an infinite cyclic group of dilations of $\mathbb{C}^2$. In this paper, we study the moduli spaces $\mathcal{M}^n$ of stable $SL(2,\mathbb{C})$-bundles on a Hopf surface $\mathcal{H}$, from the point of view of symplectic geometry. An important point is that the surface $\mathcal{H}$ is an elliptic fibration, which implies that a vector bundle on $\mathcal{H}$ can be considered as a family of vector bundles over an elliptic curve. We define a map $G\colon \mathcal{M}^n \to \mathbb{P}^{2n+1}$ that associates to every bundle on $\mathcal{H}$ a divisor, called the graph of the bundle, which encodes the isomorphism class of the bundle over each elliptic curve. We then prove that the map G is an algebraically completely integrable Hamiltonian system, with respect to a given Poisson structure on $\mathcal{M}^n$. We also give an explicit description of the fibres of the integrable system. This example is interesting for several reasons; in particular, since the Hopf surface is not Kähler, it is an elliptic fibration that does not admit a section.

1 Introduction

For the past fifteen years, the moduli spaces of stable vector bundles over algebraic curves and surfaces have been studied with great success from the point of view of symplectic geometry. This was notably done by Hitchin [Hi], Bottacin [Bo2], and Markman [Mar] over a Riemann surface. In this paper, we adopt this viewpoint in the case of bundles over a Hopf surface. These moduli spaces turn out to be algebraically completely integrable Hamiltonian systems, giving a description analogous to the ones mentioned above. The techniques developed here should also apply to moduli spaces over any non-Kähler elliptic fibration.

A Hopf surface is the quotient of the complex surface $\mathbb{C}^2 \setminus \{0\}$ by an infinite cyclic group, generated by a dilation of $\mathbb{C}^2$. Let $\mathcal{M}^n$ be the moduli space of stable holomorphic $SL(2,\mathbb{C})$-bundles on a Hopf surface $\mathcal{H}$, with fixed second Chern class $c_2 = n$. An important point is that the Hopf surface admits a holomorphic fibration $\pi\colon \mathcal{H} \to \mathbb{P}^1$ whose fibre is a smooth elliptic curve T. By restricting a vector bundle E on $\mathcal{H}$ to the fibres of π, one can consider E as a family of vector bundles over T. Given Atiyah's classification of vector bundles over elliptic curves [At], one can then associate to any $SL(2,\mathbb{C})$-bundle E on $\mathcal{H}$ a divisor in $\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{P}^1 \times \operatorname{Pic}^0(T)/\pm$ that encodes the holomorphic type of E over each fibre T. This divisor is called the *graph* of the bundle E.

There exists a map

$$G\colon \mathcal{M}^n \longrightarrow |\mathcal{O}(n,1)| = \mathbb{P}^{2n+1}$$

that associates to each bundle E its graph in $\mathbb{P}^1 \times \mathbb{P}^1$. It was shown by Braam and Hurtubise [BH] that the map $G\colon \mathcal{M}^1 \to \operatorname{Im}(G)$ is a principal T-bundle, for $n = 1$, and

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that it is surjective for $n \geq 2$. In the sequel, we prove that the component functions $H_1, \dots, H_{2n+1}$ of G are functionally independent Poisson commuting functions, *i.e.*, $\{H_i, H_j\} = 0$ for every i, j, where $\{\cdot,\cdot\}$ is essentially the Poisson bracket defined by Mukai [Mu] on moduli spaces of sheaves over algebraic surfaces. In addition, we show that the generic fibre of G is an abelian variety, implying that the commuting vector fields corresponding to the functions $H_1, \dots, H_{2n+1}$ are linear on these fibres: the map G is an *algebraically* completely integrable Hamiltonian system.

Over a smooth elliptic curve, an $\mathrm{SL}(2,\mathbb{C})$-bundle is said to be *regular* if its group of automorphisms is of the smallest possible dimension. For a stable vector bundle E over a Hopf surface, there are at most a finite number of fibres over which the restriction of E is not regular. In Section 3, we study vector bundles that are regular over every fibre of π. We associate to such a bundle E a *spectral curve* S and a line bundle on S, proving subsequently that E is completely determined by these two objects. We also describe the generic fibre of the map G, which corresponds to graphs given by holomorphic maps $F\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree n that satisfy the following condition:

$$(*) \qquad \text{If } F(x) \text{ corresponds to a half-period, then } dF_x \neq 0.$$

This condition on the graph of a bundle E ensures that the restriction of E to each fibre is regular. For such a graph g, we show that there is a one-to-one correspondence between the fibre $G^{-1}(g)$ and the Jacobian of the spectral curve of E.

In Sections 4 and 5, we study vector bundles that are irregular on at least one fibre $T_x = \pi^{-1}(x)$ of the fibration π. By taking elementary modifications, we can associate to such a bundle E a rank 2 vector bundle $\bar{E}$ that is regular on each fibre: we examine how this process can be reversed. Consequently, if g is the graph of E, then the fibre $G^{-1}(g)$ of the graph map at g is the product of the Jacobian of the spectral curve of $\bar{E}$ with Picard groups of the fibres T_x and possibly some automorphism groups $\mathrm{Aut}_{\mathrm{SL}(2,\mathbb{C})}(\bar{E}|_{T_x})$. We end Section 5 with an explicit description of the fibres for $\mathcal{M}_2$. Let us note that similar techniques have been used by Friedman, Morgan and Witten [FMW] in their study of vector bundles over algebraic elliptic fibrations.

In the last section, we use Bottacin's construction [Bo1] to define a Poisson structure on the moduli spaces $\mathcal{M}^n$ and describe how such a Poisson structure degenerates, by computing the dimensions of the symplectic leaves of $\mathcal{M}^n$. In particular, we show that the rank of such a Poisson structure is generically $4n-2$, but that it "drops" at points of $\mathcal{M}^n$ given by bundles that are irregular over certain fibres. To counter this problem, we only consider bundles where the rank is $4n-2$: if Δ is the set of graphs corresponding to bundles that are irregular over at least one fibre of π, then we prove that $G\colon \mathcal{M}^n \to \mathbb{P}^{2n+1}$ is a Lagrangian fibration on the complement of Δ. We conclude by showing that the Hamiltonians $H_1, \dots, H_{2n+1}$ Poisson commute.

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2 Holomorphic Bundles on Hopf Surfaces

Let $\lambda \in \mathbb{C}$, $|\lambda| > 1$, be a fixed complex number. An action of $\mathbb{Z}$ on the complex surface $\mathbb{C}^{2*} = \mathbb{C}^2 \setminus \{(0,0)\}$ can be given by

$$\begin{aligned} \mathbb{C}^{2*} \times \mathbb{Z} &\longrightarrow \mathbb{C}^{2*} \\ \big((z_1, z_2), n\big) &\longmapsto (\lambda^n z_1, \lambda^n z_2). \end{aligned} \tag{2.1}$$

The *Hopf surface* $\mathcal{H} = \mathcal{H}_\lambda$ is then defined as the quotient $\mathbb{C}^{2*}/\mathbb{Z}$; it is a complex surface, diffeomorphic to $S^3 \times S^1$. Therefore, $b_1(\mathcal{H}) = 1$, implying that $\mathcal{H}$ does not admit a Kähler metric. It has Dolbeault cohomology

$$H^{p,q}(\mathcal{H}) = \begin{cases} \mathbb{C} & \text{if } (p,q) = (0,0), (0,1), (2,1) \text{ or } (2,2), \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

There is a holomorphic fibration

$$\pi \colon \mathcal{H} \longrightarrow \mathbb{P}^1, \tag{2.3}$$

given by the map $\mathcal{H} \to \mathbb{C}^{2*}/\mathbb{C}^*$, whose fibre is the elliptic curve

$$T \cong \mathbb{C}^*/\lambda^n \cong \mathbb{C}/\big(2\pi i \mathbb{Z} + \ln(\lambda)\mathbb{Z}\big).$$

This fibration admits a topological splitting of the complex tangent bundle of $\mathcal{H}$, giving $\mathrm{Td}(\mathcal{H}) = 1$.

Finally, we endow $\mathcal{H}$ with a hermitian metric whose associated $(1,1)$-form on the cover $\mathbb{C}^{2*}$ is

$$\omega = \frac{i}{2}\big(\mathrm{Vol}(S^3) \times \ln|\lambda|\big)^{-\frac{1}{2}} \frac{dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2}{|z_1|^2 + |z_2|^2}.$$

The constant is chosen to make the total volume 1. Furthermore, $\bar{\partial}\partial\omega = 0$: this metric is Gauduchon (see Section 2.3).

2.1 Line Bundles

The set of all holomorphic line bundles on $\mathcal{H}$ is given by the Picard group $\mathrm{Pic}(\mathcal{H})$. Referring to (2.2), one obtains

$$\mathrm{Pic}(\mathcal{H}) \cong H^1(\mathcal{H}, \mathcal{O})/H^1(\mathcal{H}, \mathbb{Z}) \cong \mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*,$$

from the exponential sheaf sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$. All line bundles on $\mathcal{H}$ have trivial Chern class, because $H^2(\mathcal{H}, \mathbb{Z}) = 0$, and $\mathrm{Pic}(\mathcal{H}) \cong \mathrm{Pic}^0(\mathcal{H})$. One can therefore realise line bundles on $\mathcal{H}$ by constant automorphy factors $\alpha \in \mathbb{C}^*$; we illustrate this by constructing the universal (Poincaré) line bundle V over $\mathcal{H} \times \mathbb{C}^*$.

One starts with a trivial line bundle $\bar{\mathbb{C}}$ on $\mathbb{C}^{2*} \times \mathbb{C}^*$ and applies to it the following $\mathbb{Z}$-action

$$\mathbb{C}^{2*} \times \mathbb{C}^* \times \mathbb{Z} \longrightarrow \mathbb{C}^{2*} \times \mathbb{C}^*$$
$$\big((z_1, z_2), \alpha, n\big) \longmapsto \big((\lambda^n z_1, \lambda^n z_2), \alpha\big),$$

which is induced from (2.1). Since this action is trivial on $\mathbb{C}^*$, the Poincaré line bundle V is obtained by identifying $s \in \bar{\mathbb{C}}_{(z_1,z_2,\alpha)}$ with $\alpha s \in \bar{\mathbb{C}}_{(\lambda z_1,\lambda z_2,\alpha)}$. In particular, the automorphy factor $\alpha = \lambda^m$ gives the line bundle $\pi^*\big(\mathcal{O}(m)\big)$ on $\mathcal{H}$, where m is an integer and $\mathcal{O}(m)$ denotes the m-fold tensor product of the hyperplane bundle on $\mathbb{P}^1$.

For any vector bundle on $\mathcal{H}$, restriction to a fibre of π is a natural operation. Since line bundles are given by constant automorphy factors, their restriction to any fibre produces the same element of $\operatorname{Pic}^0(T)$ and we have the exact sequence

$$0 \to \operatorname{Pic}(\mathbb{P}^1) \to \operatorname{Pic}^0(\mathcal{H}) \to \operatorname{Pic}^0(T) \to 0,$$
$$0 \to \mathbb{Z} \to \mathbb{C}^* \to T \to 0.$$

Note that the restriction of $\pi^*\big(\mathcal{O}(m)\big)$ to any fibre of π is trivial, for any integer m. Moreover, the canonical bundle of $\mathcal{H}$ is given by $K(\mathcal{H}) \cong \pi^*\big(\mathcal{O}(-2)\big)$.

Remark Although $\pi^*\big(\mathcal{O}(m)\big)$ is trivial over the fibres of π, one cannot define an action of $\mathbb{Z}$ on $\mathbb{C}^*$ that leaves the Poincaré line bundle V invariant. If we let $\mathbb{Z}$ act on $\mathbb{C}^*$, then multiplication by λ is defined on the fibres of $\bar{\mathbb{C}}$ as

$$\begin{aligned} &\mathbb{C}^{2*} \times \mathbb{C}^* \times \mathbb{C} \xrightarrow{\lambda} \mathbb{C}^{2*} \times \mathbb{C}^* \times \mathbb{C} \\ &(z_1, z_2, \alpha, t) \longmapsto (\lambda z_1, \lambda z_2, \lambda\alpha, \alpha t). \end{aligned} \tag{2.4}$$

On the Hopf surface, the pairs (z_1, z_2) and $(\lambda z_1, \lambda z_2)$ define the same point x. However, (2.4) indicates that λ sends $V_{(x,\alpha)}$ to $V_{(x,\lambda\alpha)} \otimes \pi^*\big(\mathcal{O}(-1)\big)$. Hence, the Poincaré line bundle is not invariant under such an action.

2.2 Rank 2 Bundles

We now consider the case of $SL(2,\mathbb{C})$-bundles over $\mathcal{H}$: rank 2 bundles E with $\Lambda^2 E \cong \mathcal{O}$. We fix $c_2(E) = n$. The description of these bundles is analogous to that of semi-stable rank 2 bundles over the protective plane $\mathbb{P}^2$. We begin by giving a quick overview of the latter case and then see how it translates to the case of bundles over the Hopf surface $\mathcal{H}$.

2.2.1 Jumping Lines

Let E be a rank 2 vector bundle over $\mathbb{P}^2$ with $c_1(E) = 0$ and $c_2(E) = n$. One can study the bundle E by considering its restriction to each line $l \cong \mathbb{P}^1$ in $\mathbb{P}^2$; by Grothendieck's fundamental theorem, any such restriction is of the form $E|_l \cong \mathcal{O}(-h) \oplus \mathcal{O}(h)$, $h \geq 0$.

A theorem of Grauert's and Mülich's states that E is semi-stable if and only if E is trivial on the generic line in $\mathbb{P}^2$. A line l is then said to be a *jumping line* of E if and only if $E|_l \cong \mathcal{O}(-h) \oplus \mathcal{O}(h)$, with $h > 0$ (or equivalently, if $h^1\big(l, E \otimes \mathcal{O}(-1)|_l\big) > 0$). Barth [Ba] has shown that the set of all jumping lines is a curve S of degree $n = c_2(E)$ in the dual $\mathbb{P}^{2*}$ and that, along with a small amount of extra data, the bundle E is completely determined by this curve.

Around a jumping line l, the bundle E can be locally considered as a rank 2 bundle over a family of lines parametrised by an open disc D in $\mathbb{C}$, *i.e.*, over $\mathbb{P}^1 \times D$ (see [H]). After possibly shrinking D, one can assume that l is the only jumping line in this family. Let $\rho\colon \mathbb{P}^1 \times D \to D$ denote the natural projection. The direct image sheaf $R^1\rho_*\big(E \otimes \mathcal{O}(-1)\big)$ is then a torsion sheaf on D supported at l. Given a resolution $0 \to \mathcal{O}^{\oplus k} \xrightarrow{h} \mathcal{O}^{\oplus k} \to R^1\rho_*\big(E \otimes \mathcal{O}(-1)\big) \to 0$, Barth defines the *multiplicity* of the jumping line l as the order of vanishing of $\det(h)$ at the point in D corresponding to l. In particular, if $E|_l = \mathcal{O}(-h) \oplus \mathcal{O}(h)$, $h > 0$, then the multiplicity of l is greater than or equal to h (see [Ba]).

The multiplicity of jumping lines has been further studied in [BHMM], by examining their formal neighbourhoods, and in [Ma], by considering elementary modifications of the bundle. In this article, we use the second approach to analyse rank 2 vector bundles over a Hopf surface.

2.2.2 The Graph of a Rank 2 Bundle over $\mathcal{H}$

Let E be a rank 2 vector bundle over $\mathcal{H}$ with $c_1(E) = 0$ and $c_2(E) = n$. To study such a bundle, one of our main tools will be restriction of the bundle to the fibres $\pi^{-1}(x) \cong T$ of the fibration $\pi\colon \mathcal{H} \to \mathbb{P}^1$, *i.e.*, E will be considered as a family, parametrised by $\mathbb{P}^1$, of bundles over the elliptic curve T. Holomorphic vector bundles over elliptic curves have been classified by Atiyah [At]; for rank 2 bundles $\mathcal{E}$ with $\Lambda^2\mathcal{E} \cong \mathcal{O}$, we have:

Proposition 2.5 **(Atiyah)** *An* $\mathrm{SL}(2,\mathbb{C})$*-bundle over an elliptic curve T is one of three possible types:*

(i) $L_0 \oplus L_0^*$, $L_0 \in \mathrm{Pic}^0(T)$.
(ii) *Non-trivial extensions* $0 \to L_0 \to \mathcal{E} \to L_0 \to 0$, $L_0^2 \cong \mathcal{O}$.
(iii) $L \oplus L^*$, $L \in \mathrm{Pic}^{-k}(T)$, $k > 0$.

Note Bundles over an elliptic curve are called *regular* if their group of automorphisms is of the smallest possible dimension. In the $\mathrm{SL}(2,\mathbb{C})$ case, only bundles of type (i) with $L_0^2 \neq \mathcal{O}$ or of type (ii) are regular.

Given an $\mathrm{SL}(2,\mathbb{C})$-bundle over $\mathcal{H}$, its restriction to a generic fibre of π is of type (i) or (ii). This is stated precisely in [BH] as:

Proposition 2.6 **(Braam-Hurtubise)** *Let E be an* $\mathrm{SL}(2,\mathbb{C})$*-bundle over $\mathcal{H}$; then, $E|_{\pi^{-1}(x)}$ is of type (iii) on at most an isolated set of points $x \in \mathbb{P}^1$.*

These isolated points correspond to the jumping lines described in the previous section; therefore, we say that the bundle *jumps* over the corresponding isolated fibres $\pi^{-1}(x)$.

Let V be a line bundle on $\mathcal{H}$ such that $h^0\big(\pi^{-1}(x), V^*E\big) = 0$, for generic x. By Proposition 2.5, the direct image sheaf $R^1\pi_*(V^*E)$ is a skyscraper sheaf supported on isolated points x, where $E|_{\pi^{-1}(x)}$ has $V|_{\pi^{-1}(x)}$ as a subline bundle, or $E|_{\pi^{-1}(x)}$ is of type (iii). The *multiplicity* of a point x is again defined, given a local resolution

$$0 \longrightarrow \mathcal{O}^{\oplus m} \xrightarrow{f(z)} \mathcal{O}^{\oplus m} \longrightarrow R^1\pi_*(V^*E) \longrightarrow 0,$$

as the order of vanishing of $\det\big(f(x)\big)$ at $x = 0$; it is equal to the complex dimension of $R^1\pi_*(V^*E)$ in a neighbourhood of x. If a is the positive generator of $H^2(\mathbb{P}^1, \mathbb{Z})$, then $\operatorname{ch}\big(R^1\pi_*(V^*E)\big) = na$ and the sheaf $R^1\pi_*(V^*E)$ is supported on n points, counting multiplicity.

To obtain a complete description of the restriction of E to the fibres of π, this construction must be repeated for every line bundle on $\mathcal{H}$; this is done by taking the direct image $R^1\pi_*$ for all line bundles simultaneously. Let π also denote the projection $\pi := \pi \times \mathrm{id} \colon \mathcal{H} \times \mathbb{C}^* \longrightarrow \mathbb{P}^1 \times \mathbb{C}^*$, where id is the identity, and let $s \colon \mathcal{H} \times \mathbb{C}^* \to \mathcal{H}$ be the projection onto the first factor. We define $L := R^1\pi_*(s^*E \otimes V)$. This sheaf is supported on a divisor $\tilde{D}$ in $\mathbb{P}^1 \times \mathbb{C}^*$, which is defined with multiplicity.

Remarks (i) Consider the $\mathbb{Z}$-action on $\mathbb{P}^1 \times \mathbb{C}^*$ induced from the one on $\mathcal{H} \times \mathbb{C}^*$. Multiplication by λ sends the fibre $L_{(z,\alpha)}$ to $L_{(z,\lambda\alpha)} \otimes \mathcal{O}(-1)$: it does not change the support of L and $\tilde{D}$ is invariant under the $\mathbb{Z}$-action. (Nevertheless, L is not invariant under this action.)

(ii) Since $\Lambda^2 E \cong \mathcal{O}$, $E \cong E^*$ and the support of $R^1\pi_*(s^*E \otimes V^*)$ is also $\tilde{D}$, *i.e.*, it is invariant under the involution on $\mathbb{C}^*$ defined by $z \mapsto 1/z$.

The quotient of $\mathbb{C}^*$ by the $\mathbb{Z}$-action is $\operatorname{Pic}^0(T) \cong T^*$; furthermore, the involution induces a two-sheeted map $\operatorname{Pic}^0(T) \to \mathbb{P}^1$ with branch points the half periods of T. Hence, $\tilde{D}$ descends to a divisor D on $\mathbb{P}^1 \times \mathbb{P}^1 = \pi(\mathcal{H}) \times \operatorname{Pic}^0(T)/\pm$, contained in the linear system $|\mathcal{O}(n,1)|$, which is pictured in Figure 1.

The above description is summarised in:

Proposition 2.7 **(Braam-Hurtubise)** *To each* $\mathrm{SL}(2,\mathbb{C})$*-bundle* E *on* $\mathcal{H}$ *with* $c_2(E) = n$*, there is associated a divisor* D *in the linear system* $|\mathcal{O}(n,1)|$ *over* $\mathbb{P}^1 \times \mathbb{P}^1$*, given by* $D := \big(\sum_{i=1}^k (\{x_i\} \times \mathbb{P}^1)\big) + \mathrm{Gr}(F)$*, where* $\mathrm{Gr}(F)$ *is the graph of a rational map* $F \colon \mathbb{P}^1 \to \mathbb{P}^1$ *of degree* $(n-k)$*. This divisor is called the* graph *of* E*.*

Remark Generically, the divisor D is the graph of a rational map of degree n.

2.2.3 The Spectral Curve of a Rank 2 Vector Bundle

Let S be the double cover in $\mathbb{P}^1 \times T^* = (\mathbb{P}^1 \times \mathbb{C}^*)/\mathbb{Z}$ of the graph D of E, *i.e.*, the quotient of the support $\tilde{D}$ of $L = R^1\pi_*(s^*E \otimes V)$ by the $\mathbb{Z}$-action. Then, S is a divisor

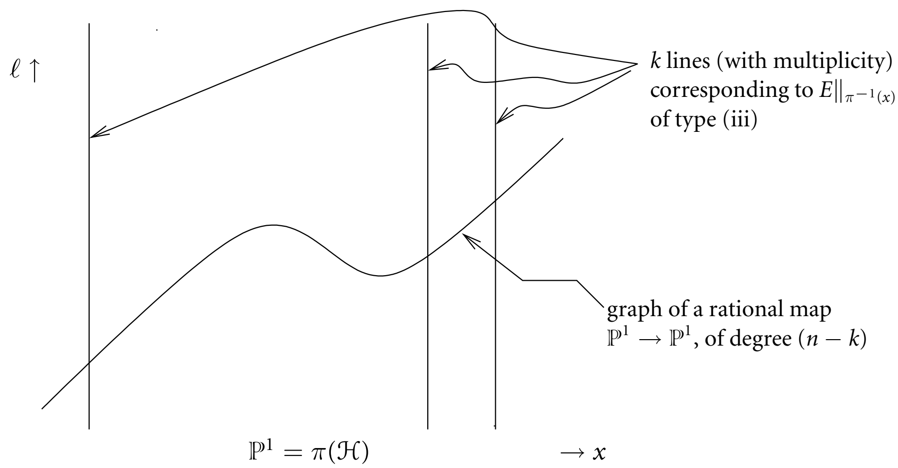

Figure 1

and it can be described as the support of a torsion sheaf $\mathcal{L}$ on $\mathbb{P}^1 \times T^*$, which we now construct.

We have seen that L is not invariant under the $\mathbb{Z}$-action, preventing it from descending to $\mathbb{P}^1 \times T^*$, because multiplication by λ sends the fibre $L_{(z,\alpha)}$ to $L_{(z,\lambda\alpha)} \otimes \mathcal{O}(-1)$. We get around this problem by constructing a sheaf $\mathcal{N}$ on $\mathbb{P}^1 \times \mathbb{C}^*$ such that $\mathcal{N}_{(z,\alpha)} \xrightarrow{\lambda} \mathcal{N}_{(z,\lambda\alpha)} \otimes \mathcal{O}(1)$; the fibres of $L \otimes \mathcal{N}$ will then be preserved by the action of $\mathbb{Z}$ and the quotient will be well-defined.

Suppose that the graph of E does not have a vertical component at the origin p_0 of $\mathbb{P}^1$; we then assume that the bundle $\mathcal{O}(1)$ is given by the divisor p_0. (If the graph has a vertical component at p_0, we choose another point in $\mathbb{P}^1$.) Let $W = (p_0 \times \mathbb{C}^*) \cap \tilde{D}$ be the set of points on $\tilde{D}$ lying above p_0. If (a, b) is a representation of the pair of points on D above p_0, then W is the set of all translates of this pair by λ: $W = \bigcup_{i\in\mathbb{Z}}(\lambda^i a, \lambda^i b)$. In addition, $(a + b)$ is a divisor on $\tilde{D}$ and we denote $T^i(a + b) := \lambda^i a + \lambda^i b$ the translate of $(a + b)$ by λ^i; we define a divisor on $\tilde{D}$ as the locally finite sum

$$B := \sum_{i\in\mathbb{Z}} iT^{-i}(a + b).$$

Let $\mathcal{N}$ be the line bundle on $\tilde{D}$ associated to the invertible sheaf $\mathcal{O}(B)$; we also denote by $\mathcal{N}$ the line bundle thought of as a sheaf on $\mathbb{P}^1 \times \mathbb{C}^*$.

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We fix a section γ of $\mathcal{O}(1)$. This section has a zero at p_0 and we use it to define the following $\mathbb{Z}$-action on the sheaf $L \otimes \mathcal{N}$ over $\mathbb{P}^1 \times \mathbb{C}^*$:

$$\begin{tikzcd}
L \otimes \mathcal{N}|_{(z,\alpha)} \arrow[rr, "(z,\alpha,s\otimes t)\mapsto(z,\lambda\alpha,\alpha s\otimes\gamma t)", "\lambda"'] \arrow[d] & & L \otimes \mathcal{N}|_{(z,\lambda\alpha)} \arrow[d] \\
\mathbb{P}^1 \times \mathbb{C}^* \arrow[rr, "\lambda", "(z,\alpha)\mapsto(z,\lambda\alpha)"'] & & \mathbb{P}^1 \times \mathbb{C}^*.
\end{tikzcd}$$

As stated above, $L \otimes \mathcal{N}$ is invariant under this action, descending to the quotient $\mathbb{P}^1 \times T^* = (\mathbb{P}^1 \times \mathbb{C}^*)/\!\sim$. The support of the quotient sheaf $\mathcal{L} := (L \otimes \mathcal{N})/\!\sim$ is S. Note that if we take the pull back of $\mathcal{L}$ to $\mathbb{P}^1 \times \mathbb{C}^*$ and tensor it by $\mathcal{N}^*$, then we recover L. This proves:

Proposition 2.8 *To each* $\mathrm{SL}(2,\mathbb{C})$*-bundle E on $\mathcal{H}$ with $c_2(E) = n$, there is associated a divisor S in $\mathbb{P}^1 \times T^*$, called the* spectral curve *of E.* ■

2.3 Degree and Stability

The degree of a vector bundle can be defined on any compact complex manifold X of dimension d. A theorem of Gauduchon's [G] states that any hermitian metric on X is conformally equivalent to a metric, called a *Gauduchon metric*, whose associated (1,1) form ω satisfies $\partial\bar{\partial}\omega^{d-1} = 0$. Suppose that X is endowed with such a metric and let L be a holomorphic line bundle over X. The *degree of L with respect to ω* is defined [Bh], up to a constant factor, by

$$\deg(L) := \int_X F \wedge \omega^{d-1},$$

where F is the curvature of a hermitian connection on L, compatible with $\bar{\partial}_L$. Any two such forms F differ by a $\partial\bar{\partial}$-exact form. Since $\partial\bar{\partial}\omega^{d-1} = 0$, the degree is independent of the choice of connection and is therefore well defined. This notion of degree is an extension of the Kähler case. If X is Kähler, we get the usual topological degree defined on Kähler manifolds; but in general, this degree is not a topological invariant, for it can take values in a continuum (see below).

Having defined the degree of holomorphic line bundles, we define the *degree* of a torsion-free coherent sheaf $\mathcal{E}$ on X by

$$\deg(\mathcal{E}) := \deg\big(\det(\mathcal{E})\big),$$

where $\det(\mathcal{E})$ is the determinant line bundle of $\mathcal{E}$, and the *slope of* $\mathcal{E}$ by

$$\mu(\mathcal{E}) := \deg(\mathcal{E})/\operatorname{rk}(\mathcal{E}).$$

The notion of stability then exists for any compact complex manifold:

A torsion-free coherent sheaf $\mathcal{E}$ on X is *stable* if and only if for every coherent subsheaf $\mathcal{S} \subset \mathcal{E}$ with $0 < \operatorname{rk}(\mathcal{S}) < \operatorname{rk}(\mathcal{E})$, we have $\mu(\mathcal{S}) < \mu(\mathcal{E})$.

Remark With this definition of stability, many of the properties from the Kähler case hold. In particular, all line bundles are stable; for rank 2 vector bundles on a surface, it is sufficient to verify stability with respect to line bundles; if the vector bundle E is stable, then $H^0\big(\mathcal{H}, \operatorname{End}(E)\big) = \mathbb{C}$ and $H^2\big(\mathcal{H}, \operatorname{End}(E)\big) = 0$.

Recall that we endowed $\mathcal{H}$ with a Gauduchon metric; we use it to compute the degrees of certain bundles on $\mathcal{H}$ and then determine which rank 2 bundles are stable.

(1) Line Bundles Suppose that $L \in \operatorname{Pic}^0(\mathcal{H})$ corresponds to the automorphy factor α. We define a metric on the trivial bundle over $\mathbb{C}^{2*}$ by

$$\|s(z)\|^2 = |s(z)|^2 \cdot |z|^{-2\ln|\alpha|/\ln|\lambda|},$$

where $|s(z)|^2$ is the standard metric. $\|\ \ \|$ descends to L over $\mathcal{H}$, and has curvature

$$F = (-\ln|\alpha|/\ln|\lambda|) \cdot (\partial\bar{\partial}\ln|z|^2).$$

Integrating F against ω, one finds that, up to a positive constant factor,

$$\deg L = \ln|\alpha|/\ln|\lambda|.$$

For example, $\deg \pi^*\big(\mathcal{O}(m)\big) = m$.

(2) $\mathrm{SL}(2,\mathbb{C})$***-Bundles*** As $\Lambda^2(E) \cong \mathcal{O}$, $\deg(E) = 0$. The stability of E is established by whether or not any line bundle of non-negative degree admits a nonzero map into E, as stated in [BH].

Proposition 2.9 **(Braam-Hurtubise)** *Suppose that the graph of E includes the graph of a non-constant map $\mathbb{P}^1 \to \mathbb{P}^1$. Then E is stable.*

Unstable bundles therefore have graphs of the form

$$\Big(\sum_{i=1}^{k}(\{x_i\} \times \mathbb{P}^1)\Big) + (\mathbb{P}^1 \times \{l\}).$$

More specifically, they are described in:

Proposition 2.10 **(Braam-Hurtubise)** *Suppose that the graph of E is given by $\big(\sum_{i=1}^{k}(\{x_i\} \times \mathbb{P}^1)\big) + (\mathbb{P}^1 \times \{l\})$. There exist line bundles K, K' on $\mathcal{H}$ such that the set of line bundles that map non-trivially to E is*

$$\big\{K \otimes \pi^*\big(\mathcal{O}(m)\big), K' \otimes \pi^*\big(\mathcal{O}(m)\big), m \leq 0\big\}.$$

If l is a half period, then $K = K'$. Furthermore, E is stable if and only if both $\deg(K)$ *and* $\deg(K')$ *are negative. The line bundles K, K' are called the* destabilising bundles of *E.*

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2.4 Instanton Moduli

2.4.1 Buchdahl's Theorem and Moduli

Let $\mathcal{M}^n$ be the moduli space of stable rank 2 $\mathrm{SL}(2,\mathbb{C})$-bundles E on $\mathcal{H}$ with $c_2(E) = n$, $n \geq 1$. Let E be in $\mathcal{M}^n$. Since $H^2(\mathcal{H},\mathbb{C}) = 0$, E is indecomposable and any connection on E is irreducible. Furthermore, as $\mu(E) = 0$, Hermitian-Einstein connections on $\mathcal{H}$ are precisely the anti-self dual ones, *i.e.*, the instantons. A theorem of Buchdahl's [Bh] then gives the following correspondence:

E is stable if and only if E admits a unique anti-self dual connection.

The moduli spaces $\mathcal{M}^n$ have been characterised in [BH] as follows:

Proposition 2.11 **(Braam-Hurtubise)** *The moduli space $\mathcal{M}^n$ of stable rank 2 $\mathrm{SL}(2,\mathbb{C})$-bundles E on $\mathcal{H}$ with $c_2(E) = n$ is a smooth, non-empty complex $4n$-dimensional manifold, diffeomorphic to the moduli space of* $\mathrm{SU}(2)$*-instantons on E.*

2.4.2 The Image of the Graph Map

Let

$$G\colon \mathcal{M}^n \longrightarrow |\mathcal{O}(n,1)| = \mathbb{P}^{2n+1}$$

be the map that associates to each bundle E its graph in $\mathbb{P}^1 \times \mathbb{P}^1$.

We have seen in Section 2.1 that $\mathrm{Pic}^0(T)$ can be described by constant automorphy factors. Setting $T = \mathbb{C}^*/\{\lambda^n\}$, the half periods of $\mathrm{Pic}^0(T)$ correspond to $1, -1, \sqrt{\lambda}, -\sqrt{\lambda}$. Let C_1 be the circle in $\mathrm{Pic}^0(T)$ given by the factors of norm 1: it projects to an interval I in $\mathrm{Pic}^0(T)/\pm = \mathbb{P}^1$, joining the two half periods +1 and -1. As proven in [BH], we have:

Theorem 2.12 **(Braam-Hurtubise)** *The* graph map $G\colon \mathcal{M}^n \to \mathbb{P}^{2n+1}$ *satisfies the following:*

(i) *For $n > 1$, G is surjective.*
(ii) *For $n = 1$, the image of G is $\mathbb{P}^3 \setminus \mathbb{P}^1 \times I$, where $\mathbb{P}^1 \times I$ denotes the set of graphs*

$$\left\{ (\{z\} \times \mathbb{P}^1) + (\mathbb{P}^1 \times \{l\}), z \in \mathbb{P}^1, l \in I \right\}.$$

A natural tool for studying bundles is the graph: one can ask how many stable bundles correspond to a given graph? The graph $G(E)$ of a rank 2 bundle with $c_2(E) = n$ is of two possible types:

I. $G(E)$ is the graph of a map $F\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree n.
II. $G(E)$ contains at least one vertical component.

The bundles of type II are said to have *jumps*, which correspond to the vertical components of the graph.

The moduli space $\mathcal{M}^1$ of instantons of charge 1 on $\mathcal{H}$ is well understood: it is the total space of a principal T-bundle given by the graph map $G\colon \mathcal{M}^1 \to \mathbb{P}^3 \setminus \mathbb{P}^1 \times I$.

Braam and Hurtubise [BH] first proved this result by classifying bundles over open subsets of $\mathcal{H}$ and then finding the different ways of glueing them together. Teleman [T] later studied $\mathcal{M}^1$ using a new approach: he associated a primary Kodaira surface to each bundle of type I and used elementary modifications to construct bundles of type II.

In the next two sections, we consider the fibre of the graph map $G\colon \mathcal{M}^n \longrightarrow \mathbb{P}^{2n+1}$, for $n \geq 2$. For bundles of type I, we approach the problem from the point of view of spectral curves: given a graph g and associated spectral curve S, a one-to-one correspondence is established between the elements of the fibre $G^{-1}(g)$ and the Jacobian of S. Bundles of type II are classified using elementary modifications on formal neighbourhoods of the jumps. Let us note that our constructions also hold for $n = 1$, in which case S is an elliptic curve.

3 Rank 2 Bundles with No Jumps

In this section, we assume that the bundle E has no jumps: its graph g is given by a holomorphic map $F\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree n. Since the map F has degree n, there are $4n$ points in $\mathbb{P}^1$ that get mapped to half periods. The spectral curve S associated to E is therefore a double cover of $\mathbb{P}^1$ with $4n$ branch points; it is given by the equation $y^2 = F(x)$, where (x, y) are local coordinates in $\mathbb{P}^1 \times T^*$. By the Riemann-Hurwitz formula, S is a compact Riemann surface of genus $2n - 1$. Hence, if $n = 1$, the curve S is elliptic, and if $n \geq 2$, it is hyperelliptic.

Referring to its equation, the spectral curve S is non-singular if and only if the map F satisfies the condition:

$$(*) \qquad \text{If } F(x) \text{ corresponds to a half period, then } dF_x \neq 0.$$

If $n = 1$, F is an automorphism and $dF_x \neq 0$ for any x. For $n \geq 2$, the differential of F vanishes at some points in $\mathbb{P}^1$; if F maps one of the zeroes of dF to a half period, then S is singular.

Remark The condition $(*)$ on the graph g implies the following condition on E:

$$\text{For any } x \text{ in } \mathbb{P}^1, \text{ we have } E|_{\pi^{-1}(x)} \not\cong L_0 \oplus L_0, L_0^2 = \mathcal{O}.$$

This is equivalent to saying that E is regular on all fibres of π. Indeed, suppose on the contrary that there is such a point x; then $R^1\pi_*(L_0E)$ is a torsion sheaf supported at x, with fibre $\mathbb{C}^m$, $m \geq 2$. If $l = \{L_0, L_0\}$, this implies that $\mathbb{P}^1 \times \{l\}$ is tangent to the graph of F, and so $dF_x = 0$.

3.1 The Fibre of the Graph Map

In the remainder of this section, we only consider non-singular spectral curves: we assume that the graph g of E is the graph of a holomorphic map $F\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree n that satisfies condition $(*)$. The restriction of the torsion sheaf $\mathcal{L}$ to S is then a line bundle. Furthermore, as the first Chern class of $\mathcal{L}$ is given by S, the Chern class of $\mathcal{L}|_S$ is completely determined by the graph. Therefore, one can associate to each element of the fibre $G^{-1}(g)$ a line bundle $\mathcal{L}|_S$ on S and these line bundles all have the same degree. Conversely, we now prove that the bundle E can be constructed using the sheaf $L = R^1\pi_{1*}(s^*E \otimes V)$; this gives us a one-to-one correspondence between the fibre $G^{-1}(g)$ of the graph map at g and line bundles on S of a fixed degree.

Remarks The construction of E is in fact more general: it holds for any rank 2 bundle that is regular over all fibres of π. Therefore, the results of this section generalise to vector bundles E with non-trivial determinant, *i.e.*, $\det E \ncong \mathcal{O}$.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two copies of the Hopf surface, with projections onto $\mathbb{P}^1$ denoted by π_1 and π_2, respectively. Let $\mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2$ be the fibred product induced by $\pi_i\colon \mathcal{H}_i \longrightarrow \mathbb{P}^1$, for $i = 1, 2$, and let $p_i\colon \mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2 \longrightarrow \mathcal{H}_i$, $i = 1, 2$, be the natural projections associated to this product. For $i = 1, 2$, we also denote by p_i the projections $p_i := p_i \times \mathrm{id}\colon \mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2 \times \mathbb{C}^* \longrightarrow \mathcal{H}_i \times \mathbb{C}^*$ and by π_i the projections $\pi_i := \pi_i \times \mathrm{id}\colon \mathcal{H}_i \times \mathbb{C}^* \longrightarrow \mathbb{P}^1 \times \mathbb{C}^*$. We have the following commutative diagram

$$\begin{tikzcd}
& \mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2 \times \mathbb{C}^* \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathcal{H}_2 \times \mathbb{C}^* \arrow[d, "\pi_2"] \\
s^*E \arrow[r] & \mathcal{H}_1 \times \mathbb{C}^* \arrow[r, "\pi_1"] \arrow[d, "s"'] & \mathbb{P}^1 \times \mathbb{C}^* \arrow[d] \\
E \arrow[r] & \mathcal{H}_1 \arrow[r, "\pi_1"] & \mathbb{P}^1.
\end{tikzcd}$$

Moreover, if $\tilde{s}$ is the canonical identification of $\mathcal{H}_2 \times \mathbb{C}^*$ with $\mathcal{H}_1 \times \mathbb{C}^*$

$$\begin{aligned}
\tilde{s}\colon \mathcal{H}_2 \times \mathbb{C}^* &\longrightarrow \mathcal{H}_1 \times \mathbb{C}^* \\
(x_2, \alpha) &\longmapsto (x_2, \alpha),
\end{aligned}$$

we also have the commutative diagram

$$\begin{tikzcd}
\mathcal{H}_1 \times \mathbb{C}^* \arrow[d, "s"'] & \mathcal{H}_2 \times \mathbb{C}^* \arrow[l, "\tilde{s}"'] \arrow[dl, "\sigma"] \\
\mathcal{H}_1. &
\end{tikzcd}$$

Let E be a regular holomorphic $\mathrm{SL}(2,\mathbb{Z})$-bundle on $\mathcal{H}$ with $c_2(E) = n$. We consider the torsion sheaf $L = R^1\pi_{1*}(s^*E \otimes V)$. Referring to Section 2.2.3, its support $\tilde{D} \subset \mathbb{P}^1 \times \mathbb{C}^*$ induces the spectral curve S associated to E. We then recover E from L in four steps.

Step 1 Pullback L to $\mathcal{H}_2 \times \mathbb{C}^*$ and then push down to $\mathcal{H}_1 \times \mathbb{C}^*$: we obtain the torsion sheaf $\tilde{s}_*(\pi_2^* L)$ whose support is now $\pi_1^{-1}(\tilde{D}) = \mathcal{H}_1 \times_{\mathbb{P}^1} \tilde{D}$.

Step 2 Tensor $\tilde{s}_*(\pi_2^* L)$ by V^* to counter the influence of V in L.

Step 3 Finally, push down to $\mathcal{H}_1$. The fibre of the map $s\colon \mathcal{H}_1 \times \mathbb{C}^* \to \mathcal{H}_1$ is not proper. However, since $\tilde{s}_*(\pi_2^* L) \otimes V^* = \tilde{s}_*\big(\pi_2^* R^1 \pi_{1*}(s^* E \otimes V)\big) \otimes V^*$, the presence of both V and V^* implies that the $\mathbb{Z}$-action, induced from $\mathcal{H}_1 \times \mathbb{C}^*$, is fibre preserving. Taking the quotient with respect to this action, we obtain a sheaf on the compact manifold $(\mathcal{H}_1 \times \mathbb{C}^*)/\mathbb{Z} = \mathcal{H}_1 \times \mathrm{Pic}^0(T)$ that we denote $L_\sim := \big(\tilde{s}_*(\pi_2^* L) \otimes V^*\big)/\!\sim$. The support of $L_\sim$ is $\mathcal{H}_1 \times_{\mathbb{P}^1} S$.

Step 4 Push down to $\mathcal{H}_1$ to obtain $E \cong s_*(L_\sim)$.

The above construction is summarised in:

Proposition 3.1 *Let E be a regular holomorphic $\mathrm{SL}(2,\mathbb{C})$-bundle on $\mathcal{H}_1$ with $c_2(E) = n$ and let S be the spectral curve associated to E. Let $L = R^1\pi_{1*}(s^* E \otimes V)$ and consider the sheaf $L_\sim := \big(\tilde{s}_*(\pi_2^* L) \otimes V^*\big)/\!\sim$, where the quotient is taken with respect to the $\mathbb{Z}$-action induced from $\mathcal{H}_1 \times \mathbb{C}$. Then, L is a torsion sheaf that restricts to a line bundle on its support. Furthermore, $E \cong s_*(L_\sim)$.*

Proof As the proof is very technical, we will state some of the results we need without proof. Their proofs will be given in the next section.

Let $\mathrm{diag}(\mathcal{H})$ be the diagonal in $\mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2$, *i.e.*, if (t_1, t_2, z) are local coordinates at a point in the fibred product, then $\mathrm{diag}(\mathcal{H}) = \{t_1 = t_2\}$. If we set $\Delta = \mathrm{diag}(\mathcal{H}) \times \mathbb{C}^* \subset \mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2 \times \mathbb{C}^*$, then Δ is an effective divisor in $\mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2 \times \mathbb{C}^*$ and there is a short exact sequence on $\mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2 \times \mathbb{C}^*$

$$0 \to p_1^*(s^* E \otimes V) \to p_1^*(s^* E \otimes V)(\Delta) \to p_1^*(s^* E \otimes V)(\Delta)|_\Delta \to 0.$$

Pushing down to $\mathcal{H}_2 \times \mathbb{C}^*$ induces the long exact sequence

$$\begin{aligned}(3.2)\quad 0 &\longrightarrow p_{2*}\big(p_1^*(s^* E \otimes V)\big) \longrightarrow p_{2*}\big(p_1^*(s^* E \otimes V)(\Delta)\big) \\ &\longrightarrow p_{2*}\big(p_1^*(s^* E \otimes V)(\Delta)|_\Delta\big) \longrightarrow R^1 p_{2*}\big(p_1^*(s^* E \otimes V)\big) \\ &\longrightarrow R^1 p_{2*}\big(p_1^*(s^* E \otimes V)(\Delta)\big) \longrightarrow \cdots .\end{aligned}$$

Lemma 3.3 *Given the above hypotheses,*

(i) $p_{2*}\big(p_1^*(s^* E \otimes V)\big) = 0$,
(ii) $R^1 p_{2*}\big(p_1^*(s^* E \otimes V)\big) = \pi_2^* L$,
(iii) $p_{2*}\big(p_1^*(s^* E \otimes V)(\Delta)\big)$ *is a locally free sheaf of rank 2,*
(iv) $R^1 p_{2*}\big(p_1^*(s^* E \otimes V)(\Delta)\big) = 0$.

The exact sequence (3.2) therefore reduces to

$$0 \longrightarrow p_{2*}\big(p_1^*(s^*E \otimes V)(\Delta)\big) \longrightarrow p_{2*}\big(p_1^*(s^*E \otimes V)(\Delta)|_\Delta\big) \longrightarrow \pi_2^*L \longrightarrow 0.$$

Since $\tilde{s}$ is a finite morphism, pushing down to $\mathcal{H}_1 \times \mathbb{C}^*$ produces another short exact sequence

$$0 \longrightarrow (\tilde{s}p_2)_*\big(p_1^*(s^*E\otimes V)(\Delta)\big) \longrightarrow (\tilde{s}p_2)_*\big(p_1^*(s^*E\otimes V)(\Delta)|_\Delta\big) \longrightarrow \tilde{s}_*(\pi_2^*L) \longrightarrow 0,$$

which further simplifies to

$$0 \longrightarrow (\tilde{s}p_2)_*\big(p_1^*(s^*E \otimes V)(\Delta)\big) \longrightarrow s^*E \otimes V \longrightarrow \tilde{s}_*(\pi_2^*L) \longrightarrow 0, \tag{3.4}$$

given the following:

Lemma 3.5 *On $\mathcal{H}_1 \times \mathbb{C}^*$, $(\tilde{s}p_2)_*\big(p_1^*(s^*E \otimes V)(\Delta)|_\Delta\big) \cong s^*E \otimes V$.*

Tensoring the exact sequence (3.4) by V^*, we obtain

$$0 \longrightarrow (\tilde{s}p_2)_*\big(p_1^*(s^*E \otimes V)(\Delta)\big) \otimes V^* \longrightarrow s^*E \longrightarrow \tilde{s}_*(\pi_2^*L) \otimes V^* \longrightarrow 0.$$

The final step is the pushdown to $\mathcal{H}_1$, but as stated above, the fibre of s is not proper. Nevertheless, the $\mathbb{Z}$-action is now well defined on every sheaf of the sequence and, taking the quotient by $\mathbb{Z}$, we have on $(\mathcal{H}_1 \times \mathbb{C}^*)/\mathbb{Z} = \mathcal{H}_1 \times \mathrm{Pic}^0(T)$ the short exact sequence

(3.6)

$$0 \to \Big((\tilde{s}p_2)_*\big(p_1^*(s^*E \otimes V)(\Delta)\big) \otimes V^*\Big)/\!\sim\, \to s^*E/\!\sim\, \to \big(\tilde{s}_*(\pi_2^*L) \otimes V^*\big)/\!\sim\, \to 0.$$

The support of $L_\sim := \big(\tilde{s}_*(\pi_2^*L) \otimes V^*\big)/\!\sim$ is the spectral curve S associated to E. Let us also denote s the canonical projection from $\mathcal{H}_1 \times \mathrm{Pic}^0(T)$ onto $\mathcal{H}_1$. We then examine the restriction of this sequence to a fibre $s^{-1}(x)$ of s. On this fibre, $s^*E/\!\sim$ restricts to the trivial bundle; the torsion sheaf $L_\sim$ is supported on the two points $p, q \in \mathrm{Pic}^0(T)$ corresponding to E over $\pi_1^{-1}(x)$, with stalk $\mathbb{C}$ at both points. By Lemma 3.3, the sheaf $\Big((\tilde{s}p_2)_*\big(p_1^*(s^*E \otimes V)(\Delta)\big) \otimes V^*\Big)/\!\sim$ is locally free of rank 2: its restriction the fibre is a rank 2 bundle of degree -2 that, by construction, must be isomorphic to $\mathcal{O}(-p) \oplus \mathcal{O}(-q)$. The sequence (3.6) then restricts to

$$0 \longrightarrow \mathcal{O}(-p) \oplus \mathcal{O}(-q) \longrightarrow \mathcal{O} \oplus \mathcal{O} \longrightarrow \mathbb{C}_{p,q} \longrightarrow 0, \tag{3.7}$$

where $\mathbb{C}_{p,q}$ is a torsion sheaf supported on $\{p, q\}$ with fibre $\mathbb{C}$. By Riemann-Roch, the holomorphic sections of the sheaf $\Big((\tilde{s}p_2)_*\big(p_1^*(s^*E \otimes V)(\Delta)\big) \otimes V^*\Big)/\!\sim$ are zero on each fibre $s^{-1}(x)$ and the pushdown of the sheaf to $\mathcal{H}_1$ vanishes. Thus, the isomorphism on global holomorphic sections of $\mathcal{O} \oplus \mathcal{O}$ and $\mathbb{C}_{p,q}$, given on the fibres of s by the exact sequence (3.7), induces an isomorphism on $\mathcal{H}_1$:

$$0 \longrightarrow s_*(s^*E/\!\sim) \cong s_*(L_\sim) \longrightarrow 0. \tag{3.8}$$

Since s is only the projection onto the first factor of a direct product and E is invariant under the $\mathbb{Z}$-action, we have $s_*(s^*E/\!\sim) \cong E$ and (3.8) gives us the required isomorphism of $s_*(L_\sim)$ with E. ∎

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3.2 Proof of Lemmas 3.3 and 3.5

We need the following:

Lemma 3.9 *If $M = \mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2 \times \mathbb{C}^*$, then*

(i) $p_{1*}(\mathcal{O}_M) \cong R^1 p_{1*}(\mathcal{O}_M) \cong \mathcal{O}_{\mathcal{H}_1 \times \mathbb{C}^*}$;
(ii) *for any $(x_1, \alpha) \in \mathcal{H}_1 \times \mathbb{C}^*$, if $T_{x_1} = p_1^{-1}(x_1, \alpha)$ and $p = \Delta \cap T_{x_1}$, then $\mathcal{O}_M(\Delta)|_{T_{x_1}} \cong \mathcal{O}_{T_{x_1}}(p)$;*
(iii) $p_{1*}\big(\mathcal{O}_M(\Delta)\big) \cong \mathcal{O}_{\mathcal{H}_1 \times \mathbb{C}^*}$;
(iv) $R^1 p_{1*}\big(\mathcal{O}_M(\Delta)\big) = 0$;
(v) $p_{1*}\big(\mathcal{O}_M(\Delta)|_\Delta\big) \cong \mathcal{O}_{\mathcal{H}_1 \times \mathbb{C}^*}$.

Proof Consider the commutative diagram

$$\begin{tikzcd} M = \mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2 \times \mathbb{C}^* \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathcal{H}_2 \times \mathbb{C}^* \arrow[d, "\pi_2"] \\ \mathcal{H}_1 \times \mathbb{C}^* \arrow[r, "\pi_1"'] & \mathbb{P}^1 \times \mathbb{C}^*. \end{tikzcd}$$

As $\mathcal{O}_M$ is isomorphic to the pullback of the structural sheaf on $\mathcal{H}_2 \times \mathbb{C}^*$, we have $R^i p_{1*}(\mathcal{O}_M) \cong R^i p_{1*}\big(p_2^*(\mathcal{O}_{\mathcal{H}_2 \times \mathbb{C}^*})\big) \cong \pi_1^*\big(R^i \pi_{2*}(\mathcal{O}_{\mathcal{H}_2 \times \mathbb{C}^*})\big)$, for $i \geq 0$, where the last equivalence is due to the flatness of the morphism π_1. Hence, it remains to show that $R^i \pi_{2*}(\mathcal{O}_{\mathcal{H}_2 \times \mathbb{C}^*}) \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{C}^*}$, or equivalently that $R^i \pi_{2*}(\mathcal{O}_{\mathcal{H}_2}) \cong \mathcal{O}_{\mathbb{P}^1}$, for $i = 0, 1$. Grauert's theorem tells us they are both invertible sheaves. The image of the global section 1 of $\mathcal{O}_{\mathbb{P}^1}$ via the structural map $\mathcal{O}_{\mathbb{P}^1} \to \pi_{2*}(\mathcal{O}_{\mathcal{H}_2})$ generates the stalk of the direct image sheaf at every point, giving $\pi_{2*}(\mathcal{O}_{\mathcal{H}_2}) \cong \mathcal{O}_{\mathbb{P}^1}$. By Grothendieck-Riemann-Roch, $R^1 \pi_{2*}(\mathcal{O}_{\mathcal{H}_2})$ is therefore an invertible sheaf on $\mathbb{P}^1$ of degree zero, proving (i).

Let (t_1, t_2, z, z') be coordinates centred at p. The divisor Δ is then given by $\{t_1 = t_2\}$ and $T_{x_1} = \{t_1 = x_1, z = \pi_1(x_1), z' = \alpha\}$. Since

$$\begin{aligned} (\Delta \cdot T_{x_1})_p &= l\Big(\mathbb{C}[t_1, t_2, z, z'] \Big/ \big(t_1 - t_2, t_1 - x_1, z - \pi_1(x_1), z' - \alpha\big)\Big) \\ &= \dim_{\mathbb{C}}(\mathbb{C}) = 1, \end{aligned}$$

$\Delta \cdot T_{x_1} = p$ and $\mathcal{O}_M(\Delta)|_{T_{x_1}} = \mathcal{O}_{T_{x_1}}(p)$ is an invertible sheaf of degree one. The inclusion $\mathcal{O}_M \hookrightarrow \mathcal{O}_M(\Delta)$ therefore induces an isomorphism on holomorphic sections and $p_{1*}\big(\mathcal{O}_M(\Delta)\big) = \mathcal{O}_{\mathcal{H}_1 \times \mathbb{C}^*}$. Furthermore, all stalks of $R^1 p_{1*}\big(\mathcal{O}_M(\Delta)\big)$ are zero and the higher direct image sheaf must vanish, proving (ii) to (iv).

There is a short exact sequence $0 \to \mathcal{O}_M \to \mathcal{O}_M(\Delta) \to \mathcal{O}_M(\Delta)|_\Delta \to 0$. Pushing down to $\mathcal{H}_1 \times \mathbb{C}^*$ and referring to (i) and (iv), we obtain

$$(3.10) \qquad 0 \to \mathcal{O}_{\mathcal{H}_1 \times \mathbb{C}^*} \to p_{1*}\big(\mathcal{O}_M(\Delta)\big) \to p_{1*}\big(\mathcal{O}_M(\Delta)|_\Delta\big) \to \mathcal{O}_{\mathcal{H}_1 \times \mathbb{C}^*} \to 0.$$

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As we have seen, the inclusion $\mathcal{O}_M \hookrightarrow \mathcal{O}_M(\Delta)$ induces an isomorphism $\mathcal{O}_{\mathcal{H}_1\times\mathbb{C}^*} \cong p_{1*}\big(\mathcal{O}_M(\Delta)\big)$, thus splitting (3.10) and giving (v). ∎

Proof of Lemma 3.3 Since π_2 is a flat morphism, the above commutative diagram implies

$$R^i p_{2*}\big(p_1^*(s^*E\otimes V)\big) \cong \pi_2^*\big(R^i\pi_{1*}(s^*E\otimes V)\big), \tag{3.11}$$

for all $i \geq 0$. The restriction of $s^*E\otimes V$ to the generic fibre of π_1 is a sum of non-trivial line bundles of degree zero. Therefore, the stalk of $\pi_{1*}(s^*E\otimes V)$ is zero at the generic point of $\mathbb{P}^1\times\mathbb{C}^*$ and $\pi_{1*}(s^*E\otimes V) = 0$; inserting this into (3.11), we get (i). As $L = R^1\pi_{1*}(s^*E\otimes V)$, (ii) also follows from (3.11).

By Lemma 3.9, the restriction of $\mathcal{O}_M(\Delta)$ to any fibre of p_2 is a line bundle of degree one, meaning that $p_1^*(s^*E\otimes V)(\Delta)$ restricts to a sum of line bundles of degree one. By Grauert's theorem, $p_{2*}\big(p_1^*(s^*E\otimes V)(\Delta)\big)$ is a locally free sheaf of rank 2 and $R^1p_{2*}\big(p_1^*(s^*E\otimes V)(\Delta)\big) = 0$, proving the lemma. ∎

Proof of Lemma 3.5 Since $\tilde{s}p_2 = p_1$ on Δ, we have

$$(\tilde{s}p_2)_*\big(p_1^*(s^*E\otimes V)(\Delta)|_\Delta\big) \cong (s^*E\otimes V)\otimes p_{1*}\big(\mathcal{O}_M(\Delta)|_\Delta\big).$$

Referring to Lemma 3.9, $p_{1*}\big(\mathcal{O}_M(\Delta)|_\Delta\big) \cong \mathcal{O}_{\mathcal{H}_1\times\mathbb{C}^*}$ and we are done. ∎

4 The local Geometry of a Jump

In this section, we consider $\mathrm{SL}(2,\mathbb{C})$-bundles with $c_2 = n$ that have a single jump of multiplicity $m \leq n$. We begin by fixing some notation. Let E be such a bundle and assume that the jump occurs over the fibre $T = \pi^{-1}(z_0)$. The graph of E in $\mathbb{P}^1\times\mathbb{P}^1$ then has two components:

$$G(E) = m(\{z_0\}\times\mathbb{P}^1) + \mathrm{Gr}(F),$$

where $\mathrm{Gr}(F)$ is the graph of a rational map $F\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree $n-m$. In addition, the restriction of E to the fibre T is of the form $L\oplus L^*$, for some $L \in \mathrm{Pic}^{-h}(T)$, $h > 0$. The integer h is called the *height* of the jump at T; the height and multiplicity satisfy the inequality $h \leq m$.

4.1 Elementary Modifications

4.1.1 Definitions and Properties

A key ingredient in our description of a jump is the notion of elementary modification of a vector bundle. Let us recall its definition. Let X be a surface and D an effective divisor on X. We denote by $i\colon D \to X$ the canonical inclusion. Let V be a rank 2 bundle on X and N a line bundle on D. Suppose that we are given a surjection

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$p\colon V \to i_*N$. The *elementary modification* of V determined by p is defined to be the sheaf W that is the kernel of the surjection $p\colon V \to i_*N$; there is an exact sequence

$$0 \to W \to V \to i_*N \to 0. \tag{4.1}$$

We give some basic properties of elementary modifications (*cf.* [F]).

1. An elementary modification is a locally free sheaf of rank 2, isomorphic to V outside D. Its Chern classes are given by

$$c_1(W) = c_1(V) - [D],$$
$$c_2(W) = c_2(V) \cdot [D] + i_*c_1(N).$$

2. This construction can be reversed to define the "inverse" elementary modification of W. If we restrict the surjection $p\colon V \to i_*N$ to the divisor D, we obtain the exact sequence

$$0 \to N' \to V|_D \to N \to 0,$$

where the line bundle N' is defined as the kernel of the map $V|_D \to N$. This in turn induces a surjection $W \to i_*N'$ and an exact sequence on X

$$0 \to U \to W \to i_*N' \to 0.$$

One easily verifies that $U = V \otimes \mathcal{O}_X(-D)$. Hence, U is considered as the "inverse" of W.

4.1.2 Reducing the Multiplicity of a Jump

Let us associate to E a vector bundle $\bar{E}$ that is isomorphic to E outside T, but has a jump at T of lower multiplicity − we do this by taking an elementary modification of E that reduces c_2. Referring to Section 4.1.1, we need to use a negative line bundle N on T for which there exists a surjection $E|_T \to N$. However, since $E|_T = L \oplus L^*$ and $\deg L < 0$, the *only* possible choice for N is L and we are left with choosing a surjection $E|_T \to L$. In this case, because two such surjections differ by a multiple of the identity, they define the same elementary modification. Let $p\colon E|_T \to L$ be the canonical projection corresponding to the splitting of E on T. It therefore determines an elementary modification of E that is *unique*, up to $\mathrm{SL}(2,\mathbb{C})$-automorphism; we denote this canonical elementary modification $\bar{E}$. In the terminology of Friedman [F], such elementary modifications are *allowable*.

Let i be the inclusion of the fibre T in $\mathcal{H}$. The surjection p induces a morphism $E \to i_*L$ of coherent sheaves over $\mathcal{H}$, also denoted p. The invariants of the induced allowable elementary modification $\bar{E} := \ker(p\colon E \to i_*L)$ are then $\det \bar{E} = \mathcal{O}(-T)$, $c_2(\bar{E}) = n - h$, and the graph

$$G(\bar{E}) = (m-h)(\{z_0\} \times \mathbb{P}^1) + \mathrm{Gr}(F).$$

The restriction of $\bar{E}$ to T is now of the form $\bar{L} \oplus \bar{L}^*$, $\bar{L} \in \mathrm{Pic}^{-r}(T)$ and $0 \le r \le h$.

Lemma 4.2 *We can associate to E a finite sequence $\{\bar{E}_1, \bar{E}_2, \dots, \bar{E}_l\}$ of allowable elementary modifications such that $\bar{E}_l$ is the only element of the sequence that does not have a jump at T.*

Proof We construct this sequence by setting $\bar{E}_1 = \bar{E}$. Then $\bar{E}_2$ is the allowable elementary modification of $\bar{E}_1$, $\bar{E}_3$ that of $\bar{E}_2$, and so on. Since each elementary modification has the effect of reducing the multiplicity of the jump, this sequence must terminate. We let l be the first integer such that $\bar{E}_l$ does not have a jump at T. ■

Definition 4.3 Consider the sequence of allowable elementary modifications constructed in Lemma 4.2. The integer l is called the *length* of the jump at T. The *jumping sequence* of T is defined as the set of integers $\{h_0, h_1, \dots, h_{l-1}\}$, where $h_0 = h$ is the height of E and h_i is the height of $\bar{E}_i$, for $0 < i \leq l-1$.

Corollary 4.4 *Let E be an $\mathrm{SL}(2, \mathbb{C})$-bundle with a single jump of multiplicity m at T and jumping sequence $\{h_0, h_1, \dots, h_{l-1}\}$. Then,*

(i) *For all $0 \leq k \leq l-1$, we have $h_{k+1} \leq h_k$.*
(ii) *Its allowable elementary modification $\bar{E}$ has a jump of length $l-1$ with jumping sequence $\{h_1, \dots, h_{l-1}\}$.*
(iii) *The multiplicity of the jump is the sum of the jumping sequence, that is, $m = \sum_{i=0}^{l-1} h_i$.* ■

Remark The jumping sequence can also be defined by considering successive formal neighbourhoods of the fibre $T = \{0\} \times T$ in a local neighbourhood $D \times T$, where D is an open disc in $\mathbb{C}$ (*cf.* [BHMM]). Let $\mathcal{O}$ be the sheaf of holomorphic functions on $D \times T$ and I be the ideal sheaf of holomorphic functions that vanish on T. The k-th *formal neighbourhood* $T^{(k)}$ of T is defined as the closed subscheme $(T, \mathcal{O}^{(k)})$ of $(D \times T, \mathcal{O})$, where $\mathcal{O}^{(k)} = \mathcal{O}/I^{k+1}$ (*cf.* [Ha]). For $k = 0$, $T^{(0)}$ is identified with the subvariety T. The elements of the jumping sequence are then defined by

$$h_k = \max\{j : \text{there exists a section of } E \otimes L_{-j} \text{ on } T^{(k)} \\ \text{that is nonzero on } T^{(0)}, \text{ for some } L_{-j} \in \mathrm{Pic}^{-j}(T)\}.$$

4.1.3 Increasing the Multiplicity of a Jump

We now reverse the process described in Section 4.1.2, to obtain an "inverse" elementary modification that increases the multiplicity of the jump at T. Note that while allowable elementary modifications are canonical, their inverses are not. To increase c_2, we use a positive line bundle N on T. A surjection $E|_T \to N$ then exits only if the degree of N is $\geq h$, where $h = -\deg L$ and $E|_T = L \oplus L^*$. There are two cases:

– If $\deg N = h$, then L^* is the *only* choice for N.
– If $\deg N > h$, then N can be any line bundle in $\mathrm{Pic}^{\deg N}(T)$.

In both instances, the surjections $E|_T \to N$ correspond to the global sections of $\mathrm{Aut}(E|_T)$. Therefore, since $h^0\big(T, \mathrm{Aut}(E|_T)\big) = 2h + 1 > 1$, the elementary modifications determined by a given positive line bundle are *not unique*.

Let r be an integer $\geq h$ and fix a line bundle $\tilde{L} \in \mathrm{Pic}^{-r}(T)$. Referring to Section 4.1.2, a surjection $\tilde{p}\colon E|_T \to \tilde{L}^*$ determines an elementary modification $\tilde{E} = \ker(E \to i_*\tilde{L}^*)$ with invariants $\det \tilde{E} = \mathcal{O}(-T)$, $c_2(\tilde{E}) = n + r$, and graph

$$G(\tilde{E}) = (m + r)(\{z_0\} \times \mathbb{P}^1) + \mathrm{Gr}(F).$$

In this case, the restriction of $\tilde{E}$ to T is isomorphic to $\tilde{L} \oplus \tilde{L}^*$.

Note The allowable elementary modification of $\tilde{E}$ is simply E. Furthermore, the jump of $\tilde{E}$ has length $l + 1$ and jumping sequence $\{r, h_0, h_1, \ldots, h_{l-1}\}$.

4.2 The Stability of a Jump

We have seen that, when the map component of the graph of a bundle is not constant, the bundle is stable. Let us then suppose that $F\colon \mathbb{P}^1 \to \mathbb{P}^1$ is a constant map and set $l_E = F(\mathbb{P}^1)$: the graph of E is

$$n(\{z_0\} \times \mathbb{P}^1) + \mathbb{P}^1 \times \{l_E\}.$$

Lemma 4.5 *Assume that E is stable and has a jump of length μ at T. Let $\bar{E}$ be the allowable elementary modification of E and K be one of the two destabilising bundles of E.*

(i) *Both E and $\bar{E}$ have the same maximal destabilising bundles.*
(ii) *If $\mu = 1$, then $\bar{E}$ is an extension of $K^* \otimes \mathcal{O}(-T)$ by K, which splits if l_E is not a half period. In particular, $\bar{E}$ is not stable.*
(iii) *If $\mu > 1$, then E is stable if and only if $\bar{E}$ is stable.*
(iv) *If l_E is not a half period, then the maximal destabilising bundles of E are K and $K^* \otimes \mathcal{O}(-\mu T)$. If l_E is a half period, then $K = K^* \otimes \mathcal{O}\big((a - \mu)T\big)$, where $0 \leq a < \mu$ is the number of fibres over which E is of type (i).*

Proof Let K be a line bundle on $\mathcal{H}$. We tensor by K^* the exact sequence $0 \to \bar{E} \to E \to i_*(L) \to 0$ and then push down to $\mathbb{P}^1$. Since $i_*(L)$ is a torsion sheaf, the resulting exact sequence gives $\pi_*(K^* \otimes E) \cong \pi_*(K^* \otimes \bar{E})$, proving (i).

Suppose that l_E corresponds to the pair of line bundles $\{L_0, L_0^*\}$. Let us assume that $\mu = 1$, so that the graph of $\bar{E}$ is $\mathbb{P}^1 \times \{l_E\}$. The restriction of $\bar{E}$ to the fibres of π is then $L_0 \oplus L_0^*$, if $L_0^2 \neq \mathcal{O}$, or generically a non-trivial extension of L_0 by L_0, if $L_0^2 = \mathcal{O}$. Let K be one of the two destabilising bundles of E and $\bar{E}$. There exists an injective bundle map $K \to \bar{E}$. As $\det \bar{E} = \mathcal{O}(-T)$, the quotient $\bar{E}/K$ must be isomorphic to $K^* \otimes \mathcal{O}(-T)$ and $\bar{E}$ is an extension of $K^* \otimes \mathcal{O}(-T)$ by K.

If $L_0^2 \neq \mathcal{O}$, then $R^i\pi_*\big(K^2 \otimes \mathcal{O}(T)\big) = 0$, for $i = 0, 1$. By the Leray spectral sequence, $H^1\big(\mathcal{H}, K^2 \otimes \mathcal{O}(T)\big) = H^1\Big(\mathbb{P}^1, \pi_*\big(K^2 \otimes \mathcal{O}(T)\big)\Big) = 0$ and the extension splits. Given the exact sequence $0 \to \bar{E} \to E \to i_*(L) \to 0$, both summands K and $K^* \otimes \mathcal{O}(-T)$ of $\bar{E}$ map injectively into E, proving that they are the destabilising line bundles of E.

If $L_0^2 = \mathcal{O}$, then $K^2 \otimes \mathcal{O}(T)$ is trivial along the fibres: it is of the form $\pi^*\big(\mathcal{O}(a)\big)$, for some integer a. Moreover, since the restriction of E to the generic fibre of π is a non-trivial extension, the sheaf $R^1\pi_*\big(K^2 \otimes \mathcal{O}(T)\big)$ has a non-trivial section. By the projection formula, $R^1\pi_*\big(K^2 \otimes \mathcal{O}(T)\big) = \mathcal{O}(a)$, thus implying that a is a non-negative integer. The destabilising bundle K must have negative degree, which forces $a = 0$ and $K^2 \otimes \mathcal{O}(T) = \mathcal{O}$, proving (ii) and (iv) (for $\mu = 1$).

If $\mu > 1$, then $\bar{E}$ is not an extension of line bundles and (iii) follows from (i). In addition, if one performs μ successive allowable elementary modifications, one obtains a sheaf $\bar{E}_\mu$ with $\det(\bar{E}_\mu) = \mathcal{O}(-\mu T)$ and graph $\mathbb{P}^1 \times \{l_E\}$. By using arguments similar to the ones used in the case $\mu = 1$, we conclude. ■

Finally, stability imposes the following conditions on the degree of the maximal destabilising bundles of a vector bundle. Suppose that E has at least one jump, *i.e.*, it has a graph of the form

$$\sum_{i=1}^{n}(\{z_i\} \times \mathbb{P}^1) + \mathbb{P}^1 \times \{l\}$$

with $n \geq 1$. By performing consecutive allowable elementary modifications, to "remove" the jumps of E, we obtain:

Corollary 4.6 *Suppose that E has k jumps of lengths $\mu_1, \dots, \mu_k$, respectively, and set $\mu = \mu_1 + \cdots + \mu_k$. Let K be one of the destabilising bundles of E.*

(i) *If l_E is not a half period, then E is stable if and only if $\deg K \in (-\mu, 0)$.*
(ii) *If l_E is a half period, then E is stable if and only if $0 \leq a < \mu$ (see Lemma 4.5).* ■

5 Rank 2 Bundles with Jumps

5.1 General Case: Bundles with $c_2 = n$

Let E be an $\mathrm{SL}(2,\mathbb{C})$-bundle with $c_2(E) = n$. Suppose that E has k jumps and that one of them occurs over the fibre $T = \pi^{-1}(z_0)$. The construction of Section 4.1.2 then generalises to define allowable elementary modifications $\bar{E}$ of any bundle E. (In fact, it also works for bundles with non-trivial determinant.) One can then remove the jump over T, by performing successive allowable elementary modifications, to obtain a bundle W that has $\det W = \mathcal{O}(-\mu T)$ (where μ is the number of elementary modifications performed) and whose graph does not have a vertical component over T.

For graphs with vertical components, the fibre of the graph map G can be described by examining how jumps can be added to vector bundles, which boils down to classifying elementary modifications with positive line bundles. Let W be a rank 2 bundle (with possibly a non-trivial determinant). Adding a jump to W over the fibre T implies two choices:

1. Picking a line bundle N of degree > 0 over T that allows at least one surjection $W|_T \to N$.
2. Choosing a surjection $W|_T \to N$.

Let us determine which epimorphisms $W|_T \to N$ give rise to distinct elementary modifications of W. Any two such epimorphisms differ by an automorphism of $W|_T$ and induce the same elementary modification, if one is a multiple of the other. We therefore restrict ourselves to $\mathrm{SL}(2,\mathbb{C})$-automorphisms of $W|_T$ and quotient out the ones that extend to global automorphisms of W. There are two cases:

Case 1 If W is stable, then its global automorphisms are constant multiples of the identity: the distinct elementary modifications of W defined using N are in one-to-one correspondence with $\mathrm{Aut}_{\mathrm{SL}(2,\mathbb{C})}(W|_T)$.

Case 2 If W is an extension of line bundles, then the elementary modification of W defined using N is *unique*, because all $\mathrm{SL}(2,\mathbb{C})$-automorphisms of $W|_T$ extend to W. This follows from:

Lemma 5.1 *Let W be an extension of line bundles in* $\mathrm{Pic}(\mathcal{H})$ *such that W does not restrict to $L_0 \oplus L_0$, $L_0^2 = \mathcal{O}$, over T. The global automorphisms of W are in one-to-one correspondence with* $\mathrm{Aut}(W|_T)$.

Proof The proof uses standard arguments (see for example the proof of Proposition 3.5 in [T]). ■

The above discussion can be summarised by a fibration we now define. Fix a graph g that contains a vertical component of multiplicity m over z_0 and a jumping sequence $\{h_0, h_1, \dots, h_{l-1}\}$. Let $\mathcal{M}^{j,n}$ be the moduli space of stable rank 2 bundles such that $\det E = \mathcal{O}(-jT)$ and $c_2(E) = n$; note that $\mathcal{M}^{0,n} = \mathcal{M}^n$, for all n. We then set

$$\mathcal{E}^j\mathcal{J}^{n,l}_{g,\{h_0,h_1,\dots,h_{l-1}\}} = \left\{ E \in \mathcal{M}^{j,n} \;\middle|\; \begin{array}{l} G(E) = g \text{ and } E \text{ has a jump at } z_0 \\ \text{that has length } l \text{ and jumping} \\ \text{sequence } \{h_0, h_1, \dots, h_{l-1}\} \end{array} \right\}.$$

In particular, $\mathcal{E}^j\mathcal{J}^{n,0}_{g,\varnothing}$ is the set of bundles in $\mathcal{M}^{j,n}$ that have graph g and no jump at z_0. For any $E \in \mathcal{M}^{j,n}$, let $\bar{E}$ be the allowable elementary modification of E; denote by g and $\bar{g}$ the graphs of E and $\bar{E}$, respectively, and their jumping sequences by $\{h_0, h_1, \dots, h_{l-1}\}$ and $\{h_1, \dots, h_{l-1}\}$, respectively. Associating to a bundle E its allowable elementary modification $\bar{E}$ therefore defines a natural map, described in:

Proposition 5.2 *Let $g \in \mathbb{P}^{2n+1}$ be the graph of a bundle that has a jump at z_0 and suppose that this jump has jumping sequence $\{h_0, \dots, h_{l-1}\}$. The fibre of the natural projection*

$$\begin{aligned} \mathcal{E}^j\mathcal{J}^{n,l}_{g,\{h_0,h_1,\dots,h_{l-1}\}} &\longrightarrow \mathcal{E}^{j+1}\mathcal{J}^{n-h_0,l-1}_{\bar{g},\{h_1,\dots,h_{l-1}\}} \\ E &\longmapsto \bar{E} \end{aligned}$$

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at W is given by

(i) $\mathrm{Aut}_{\mathrm{SL}(2,\mathbb{C})}(W|_T)$, *if $n > h_0$ and $h_0 = h_1$,*
(ii) $\mathrm{Pic}^{-h_0}(T) \times \mathrm{Aut}_{\mathrm{SL}(2,\mathbb{C})}(W|_T)$, *if $n > h_0$ and $h_0 > h_1$,*
(iii) $\mathrm{Pic}^{-n}(T)$, *if $n = h_0$.* ■

5.2 Rank 2 Bundles with $c_2 = 2$

We end this section by giving an explicit description of the fibre of the graph map $G\colon \mathcal{M}^2 \to |\mathcal{O}(2,1)| = \mathbb{P}^5$, for graphs that have more than one component; there are two types:

(α) $g = (\{z_0\} \times \mathbb{P}^1) + \mathrm{Gr}(F)$, where $\mathrm{Gr}(F)$ is the graph of a rational map $F\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree 1;
(β) $g = (\{z_0\} \times \mathbb{P}^1 + \{z_1\} \times \mathbb{P}^1) + (\mathbb{P}^1 \times \{l\})$, where we have to distinguish the cases
  (i) $z_0 \neq z_1$, and
  (ii) $z_0 = z_1$.

Proposition 5.3 *The fibre of $G\colon \mathcal{M}^2 \to \mathbb{P}^5$ at a graph g of type α is*

$$\{(W,\phi) \mid W \in T^* \text{ and } \phi \in \mathrm{Aut}_{\mathrm{SL}(2,\mathbb{C})}(W|_T)\} \times \mathrm{Pic}^{-1}(T),$$

where $T = \pi^{-1}(z_0)$. ■

Proposition 5.4 *The fibre of $G\colon \mathcal{M}^2 \to \mathbb{P}^5$ at a graph g of type β, is given in cases (i) and (ii) by*

(i) $\{(L,\phi) \mid L \in \mathrm{Pic}^{-1}(T) \text{ and } \phi \in \mathrm{Aut}_{\mathrm{SL}(2,\mathbb{C})}(L \oplus L^*)\} \times \mathrm{Pic}^{-1}(T) \times \mathbb{Z}_2$,
(ii) $\big(\{(L,\phi) \mid L \in \mathrm{Pic}^{-1}(T) \text{ and } \phi \in \mathrm{Aut}_{\mathrm{SL}(2,\mathbb{C})}(L \oplus L^*)\} \times \mathbb{Z}_2\big) \cup \mathrm{Pic}^{-2}(T)$. ■

6 Algebraically Completely Integrable Systems

6.1 Poisson Structures

6.1.1 Symplectic Geometry

We start by recalling some definitions and results of symplectic geometry, that can be found, for example, in [We].

Let X be a smooth algebraic variety over the complex field $\mathbb{C}$. A (holomorphic) *Poisson structure* on X is a Lie algebra structure $\{\cdot,\cdot\}$ on $\mathcal{O}_X$ satisfying the Leibniz identity $\{f, gh\} = \{f,g\}h + g\{f,h\}$. This structure is equivalently given by an antisymmetric contravariant 2-tensor $\theta \in H^0(X, \bigwedge^2 TX)$, where we set $\{f,g\} = \langle \theta, df \wedge dg\rangle$. If the bracket defined by θ satisfies the Jacobi identity

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0, \tag{6.1}$$

for any $f, g, h \in \Gamma(U, \mathcal{O}_X)$, then θ defines a Poisson structure on X. For any function $f \in \Gamma(U, \mathcal{O}_X)$, the map $g \mapsto \{f,g\}$ is a derivation of $\Gamma(U,\mathcal{O}_X)$ that corresponds to a holomorphic vector field H_f on U, called the *Hamiltonian vector field* associated to f.

A 2-tensor $\theta \in H^0(X, \bigwedge^2 TX)$ is equivalent to a homomorphism of vector bundles $B\colon T^*X \longrightarrow TX$ that satisfies $\langle \theta, \alpha \wedge \beta \rangle = \langle B(\alpha), \beta \rangle$ (or $\langle \alpha, B(\beta) \rangle$, up to a sign), for 1-forms α, β. If B has maximal rank everywhere, the Poisson structure is said to be *symplectic*. For any even r, let

$$X_r := \{p \in X \mid \mathrm{rk}(B) = r\}.$$

A basic result [We] then asserts that X_r is a subvariety canonically foliated into *symplectic leaves*, *i.e.*, r-dimensional subvarieties that inherit a symplectic structure.

Remark Let ω_X be the canonical bundle of X. If X is a surface, a Poisson structure on X corresponds to a global section of the anticanonical bundle ω_X^{-1}: an element of $H^0(X, \omega_X^{-1}) = H^0(X, \wedge^2 TX)$ (*cf.* [Bo1]). Since the anticanonical bundle of the Hopf surface is $\pi^*\big(\mathcal{O}(2)\big)$, Poisson structures exist on $\mathcal{H}$ (although these cannot be symplectic, because $H^2(\mathcal{H}, \mathbb{R}) = 0$).

6.1.2 Poisson Structures on $\mathcal{M}^n$

Let us fix a Poisson structure $s \in H^0(\mathcal{H}, \omega_{\mathcal{H}}^{-1})$ on the Hopf surface and denote its divisor D. If the locally free sheaf $\mathcal{O}(2)$ on $\mathbb{P}^1$ is given by the divisor $x_1 + x_2$, then $D = T_1 + T_2$, where $T_i = \pi^{-1}(x_i)$ for $i = 1, 2$.

The Poisson structure on $\mathcal{M}^n$ is constructed using the one on $\mathcal{H}$. By deformation theory, the tangent and cotangent planes to the moduli space $\mathcal{M}^n$ at a bundle E are given by

$$T_E\mathcal{M}^n \cong H^1\big(\mathcal{H}, \mathrm{sl}(E)\big)$$

and

$$T_E^*\mathcal{M}^n \cong H^1\big(\mathcal{H}, \mathrm{sl}(E) \otimes \omega_{\mathcal{H}}\big).$$

We define a Poisson structure $\theta = \theta_s \in H^0(\mathcal{M}^n, \bigotimes^2 T\mathcal{M}^n)$ on $\mathcal{M}^n$ as follows: for any bundle $E \in \mathcal{M}^n$, $\theta(E)\colon T_E^*\mathcal{M}^n \times T_E^*\mathcal{M}^n \longrightarrow \mathbb{C}$ is the composition

$$\begin{aligned}\theta(E)\colon H^1\big(\mathcal{H}, \mathrm{sl}(E) \otimes \omega_{\mathcal{H}}\big) &\times H^1\big(\mathcal{H}, \mathrm{sl}(E) \otimes \omega_{\mathcal{H}}\big)\\ &\xrightarrow{\circ} H^2\big(\mathcal{H}, \mathrm{End}(E) \otimes \omega_{\mathcal{H}}^2\big) \xrightarrow{s} H^2\big(\mathcal{H}, \mathrm{End}(E) \otimes \omega_{\mathcal{H}}\big) \xrightarrow{\mathrm{Tr}} \mathbb{C},\end{aligned}$$

where the first map is the cup-product of two cohomology classes, the second is multiplication by s, and the third is the trace map. Note that the stability of E implies that $H^0\big(\mathcal{H}, \mathrm{End}(E)\big) = \mathbb{C}$. By Serre duality, the trace map Tr is therefore an isomorphism.

Remark Bottacin [Bo1] proves that such a θ satisfies the Jacobi identity, in the case of moduli spaces of sheaves over Poisson surfaces (smooth algebraic surfaces that admit a non-zero Poisson structure). The algebraic hypothesis is, however, only used in the construction of the moduli spaces and his arguments concerning the Jacobi identity extend to any compact surface. Hence, although $\mathcal{H}$ is not algebraic, θ defines a Poisson structure on $\mathcal{M}^n$.

6.1.3 The Rank of θ

As stated above, the Poisson structure θ corresponds to a homomorphism of vector bundles $B\colon T^*\mathcal{M}^n \longrightarrow T\mathcal{M}^n$ that satisfies $\theta(\alpha\otimes\beta) = \langle B(\alpha),\beta\rangle$, on 1-forms α,β. From the definition of θ, this homomorphism is the map induced on cohomology by multiplication by the section s: at a point E of $\mathcal{M}^n$, we have

$$B(E)\colon H^1\big(\mathcal{H},\mathrm{sl}(E)\otimes\omega_{\mathcal{H}}\big) \xrightarrow{\ s\ } H^1\big(\mathcal{H},\mathrm{sl}(E)\big).$$

Let us fix a stable vector bundle E on $\mathcal{H}$. There exists a short exact sequence $0\to E\otimes\omega_{\mathcal{H}} \xrightarrow{s} E \xrightarrow{r} E|_D \to 0$, where s denotes multiplication by the section s and r restriction to D, which induces the short exact sequence

$$0\longrightarrow \mathrm{sl}(E)\otimes\omega_{\mathcal{H}} \xrightarrow{\ s\ } \mathrm{sl}(E)\longrightarrow \mathrm{sl}(E|_D)\longrightarrow 0. \tag{6.2}$$

The stability of E implies that $H^0\big(\mathcal{H},\mathrm{End}(E)\big) = \mathbb{C}$ and $H^2\big(\mathcal{H},\mathrm{sl}(E)\big) = 0$. Hence, $H^0\big(\mathcal{H},\mathrm{sl}(E)\big) = 0$ and by Serre duality, $H^2\big(\mathcal{H},\mathrm{sl}(E)\otimes\omega_{\mathcal{H}}\big) = 0$. The long exact cohomology sequence given by (6.2) is then

$$0\to H^0\big(\mathcal{H},\mathrm{sl}(E|_D)\big)\longrightarrow H^1\big(\mathcal{H},\mathrm{sl}(E)\otimes\omega_{\mathcal{H}}\big) \xrightarrow{B(E)} H^1\big(\mathcal{H},\mathrm{sl}(E)\big)\to\cdots.$$

We have thus proven:

Proposition 6.3 *The kernel of the Hamiltonian morphism $B(E)$ is given by $H^0\big(D,\mathrm{sl}(E|_D)\big)$. Since $\dim\mathcal{M}^n = 4n$, we have*

$$\mathrm{rk}\,B(E) = 4n - \dim H^0\big(D,\mathrm{sl}(E|_D)\big). \qquad \blacksquare$$

The rank of the Poisson structure θ is therefore determined by restriction to the fibres T_1 and T_2. Recall that $\mathrm{SL}(2,\mathbb{C})$-bundles over an elliptic curve T are of three possible types:

(i) $L_0\oplus L_0^*$, $L_0\in\mathrm{Pic}^0(T)$;
(ii) a non-trivial extension of L_0 by L_0, $L_0^2\cong\mathcal{O}$;
(iii) $L\oplus L^*$, $L\in\mathrm{Pic}^{-k}(T)$, $k>0$.

By Riemann-Roch, we obtain the following dimensions $h^0 = h^0\big(T_i,\mathrm{sl}(E|_{T_i})\big)$ of cohomology groups:

$$h^0 = \begin{cases} 1, & \text{if } E|_{T_i} \text{ is of type (i), with } L_0^2\not\cong\mathcal{O}, \text{ or of type (ii);}\\ 3, & \text{if } E|_{T_i} \text{ is of type (i), with } L_0^2\cong\mathcal{O};\\ 1+2k, & \text{if } E|_{T_i} \text{ is of type (iii).}\end{cases}$$

The dimension of the comohomology groups is generically 1, but "jumps" on the fibres over which the bundle is not regular. The rank of the Poisson structure is therefore generically $4n-2$ and "drops" at the points of $\mathcal{M}^n$ corresponding to bundles that are not regular over the fibres T_1 and T_2. This is summarised in:

Corollary 6.4 *Let Δ be the set of graphs in $\mathbb{P}^{2n+1}$ that contain vertical components, or that correspond to vector bundles that restrict to $L_0 \oplus L_0$, $L_0^2 \cong \mathcal{O}$, on the fibre T_1 or T_2. Let $E \in \mathcal{M}^n$.*

1. *If $G(E) \in \Delta$, then* $\operatorname{rk} B(E) \leq 4n - 4$.
2. *If $G(E) \in (\mathbb{P}^{2n+1} - \Delta)$, then* $\operatorname{rk} B(E) = 4n - 2$. ■

6.2 Integrable Systems

Let (X, θ) be a Poisson variety and B an algebraic variety. A proper morphism $h\colon X \to B$ is said to be a *Lagrangian fibration* if its generic fibre Y is a Lagrangian submanifold of X, *i.e.*, Y is contained in the closure $\bar{Z}$ of a symplectic leaf $Z \subset X$ and the intersection $Y \cap Z$ is a Lagrangian subvariety of Z. This is equivalent to requiring the components $h_1, h_2, \ldots, h_N$ of h to be functionally independent Poisson-commuting functions – they form a *completely integrable Hamiltonian system* and their corresponding Hamiltonian vector-fields are holomorphic commuting vector-fields on the generic fibre of h. If in addition, the generic fibres of a Lagrangian fibration $h\colon X \to B$ are isomorphic to abelian varieties, a Poisson structure on X is said to be an *algebraically completely integrable Hamiltonian system structure* on $h\colon X \to B$. In this case, the Hamiltonian vector-fields are linear on the fibres of h.

We have previously seen that the generic fibre of the graph map

$$G\colon \mathcal{M}^n \longrightarrow |\mathcal{O}(n, 1)| = \mathbb{P}^{2n+1}$$

is the Jacobian of a spectral curve. We consider the set ∇ of graphs in $\mathbb{P}^{2n+1}$ corresponding to vector bundles that are irregular over at least one fibre of π. If we choose a graph g in the complement of ∇, then the spectral curve S determined by g is an elliptic curve, if $n = 1$, or a hyperelliptic curve of genus $2n - 1$, if $n \geq 2$. Furthermore, the fibre $G^{-1}(g)$ of the graph map at g is isomorphic to the Jacobian of S (*cf.* Section 3). The proof of Proposition 2 in [Be] can easily be adapted to show that the fibration $G\colon \mathcal{M}^n \longrightarrow \mathbb{P}^{2n+1}$ is Lagrangian over the complement of ∇ (for details, see [Mo]).

We end this article by giving a description of the Hamiltonians of the integrable system, which correspond to the components $H_1, \ldots, H_{2n+1}$ of the graph map $G\colon \mathcal{M}^n \to \mathbb{P}^{2n+1}$. We express these components in terms of spectral curves, for it leads to a nicer description. Let E be a bundle in $\mathcal{M}^n$ and assume that its graph is given by a map $F\colon \mathbb{P}^1 \mapsto \mathbb{P}^1$ of degree n. We fix coordinates (x, y) in $\mathbb{P}^1 \times T^*$. The spectral curve S_E associated to E is then given by $y^2 = Q_E(x)$, where $Q_E(x)$ is a polynomial in x of degree $4n$, and the Hamiltonians are the coefficients of $Q_E(x)$, which are generated by the evaluation of $Q_E(x)$ at $x \in \mathbb{P}^1$.

Proposition 6.5 *The functions H_i are in involution: $\{H_i, H_j\} = 0$ for all i, j.*

Given the above notation, we define the function

$$\begin{aligned} f_x\colon \mathcal{M}^n &\to \mathbb{C} \\ E &\mapsto Q_E(x), \end{aligned}$$

for any $x \in \mathbb{P}^1$. The Hamiltonians are then generated by the functions f_x. To show that the Hamiltonians are in involution, it is therefore enough to prove it for the functions f_x and this follows directly from:

Lemma 6.6 *The differential df_x is supported on the fibre $T_x = \pi^{-1}(x)$, for all $x \in \mathbb{P}^1$.*

Proof Let s_x be a section of $\mathcal{O}(2) \to \mathbb{P}^1$ whose associated divisor is $(s_x) = 2p_x$, where p_x is the point representing x. If we also denote by s_x the pullback of s_x to $\mathcal{H}$, then $(s_x) = 2T_x$. Referring to Section 6.1.3, there is a long exact sequence on cohomology

$$\begin{aligned} 0 \longrightarrow H^0\big(T_x, \mathrm{sl}(E|_{T_x})\big) &\xrightarrow{\rho_x^*} H^1\big(\mathcal{H}, \mathrm{sl}(E) \otimes \omega_{\mathcal{H}}\big) \\ &\xrightarrow{B_{s_x}} H^1\big(\mathcal{H}, \mathrm{sl}(E)\big) \xrightarrow{dr_x} H^1\big(T_x, \mathrm{sl}(E|_{T_x})\big) \longrightarrow 0, \end{aligned}$$

where ρ_x^* is the coboundary map, B_{s_x} is multiplication by s_x, and r_x is restriction to T_x. Note that $dr_x \circ B_{s_x} = 0$. Let M_x be the moduli space of $\mathrm{SL}(2, \mathbb{C})$-bundles on T_x. The map f_x then factors through the restriction r_x: there exists a map $\psi_x \colon M_x \to \mathbb{C}$ such that the diagram

$$\begin{tikzcd} \mathcal{M}^n \arrow[rr, "r_x"] \arrow[dr, "f_x"'] & & M_x \arrow[dl, "\psi_x"] \\ & \mathbb{C} & \end{tikzcd}$$

is commutative. Thus, $df_x = d\psi_x \circ dr_x$ and for all $\alpha \in H^1\big(\mathcal{H}, \mathrm{sl}(E) \otimes \omega_{\mathcal{H}}\big)$,

$$\langle \alpha, B_{s_x}(df_x) \rangle = \langle df_x, B_{s_x}(\alpha) \rangle = \langle d\psi_x, dr_x \circ B_{s_x}(\alpha) \rangle = 0.$$

Hence, $df_x \in \mathrm{Im}(\rho_x^*)$ and $\mathrm{supp}(df_x) = T_x$. ■

Proof of Proposition 6.5 If $x \neq x'$, then $T_x \cap T_{x'} = \varnothing$ and the supports of $df_x, df_{x'}$ do not intersect (*cf.* Lemma 6.6). Thus, $\{f_x, f_{x'}\} = \langle df_x, B_s(df_{x'}) \rangle = 0$. ■

Given the above discussion, we have the following:

Theorem 6.7 *The graph map $G \colon \mathcal{M}^n \to \mathbb{P}^{2n+1}$ is an algebraically completely integrable Hamiltonian system on the moduli spaces $\mathcal{M}^n$.* ■

Remark Recall that we assumed the divisor of s to be $(s) = T_{x_1} + T_{x_2}$, for two points $x_1, x_2 \in \mathbb{P}^1$. Let $\rho \colon \mathcal{M}^n \to \mathbb{C}$ be any function. Since s is zero on T_{x_i} and df_{x_i} has support T_{x_i}, we have

$$\{f_{x_i}, \rho\} = \langle df_{x_i}, B_s(d\rho) \rangle = \langle df_{x_i}, s \cdot d\rho \rangle = 0.$$

The functions f_{x_1}, f_{x_2} are therefore Casimirs.

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Department of Mathematics and Statistics
McGill University
805 Sherbrooke Street West
Montreal, Quebec H3A 2K6
email: moraru@math.mcgill.ca