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# Integrable Systems on Quad-Graphs

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## 1 Introduction

Discrete (lattice) systems constitute a well-established part of the theory of integrable systems. They came up already in the early days of the theory (see, e.g. [11, 12]), and took gradually more and more important place in it (cf. a review in [18]). Nowadays many experts in the field agree that discrete integrable systems are in many respects even more fundamental than the continuous ones. They play a prominent role in various applications of integrable systems such as discrete differential geometry (see, e.g., a review in [9]).

Traditionally, independent variables of discrete integrable systems are considered as belonging to a regular square lattice $\mathbb{Z}^2$ (or its multidimensional analogs $\mathbb{Z}^d$). Only very recently, there appeared first hints on the existence of a rich and meaningful theory of integrable systems on nonsquare lattices and, more generally, on arbitrary graphs. The relevant publications are almost exhausted by [2, 3, 5, 6, 16, 20, 21, 22].

We define integrable systems on graphs as flat connections with the values in loop groups. This is very natural definition, and experts in discrete integrable systems will not only immediately accept it, but might even consider it trivial. Nevertheless, it crystallized only very recently, and seems not to appear in the literature before [3, 5, 6]. (It should be noted that a different framework for integrable systems on graphs is being developed by Novikov with collaborators [16, 20, 21].)

We were led to considering such systems by our (with Hoffmann) investigations of circle patterns as objects of discrete complex analysis: in [5, 6] we demonstrated that certain classes of circle patterns with the combinatorics of regular hexagonal lattice

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are directly related to integrable systems. To realize this, it was necessary to formulate the notion of an integrable system on a graph which is not a square lattice (the regular hexagonal lattice is just one example of such a graph).

Another context where such systems appear naturally is the differential geometry of discretized surfaces, see [9]. It was realized in the latter reference that natural domains, where discrete analogs of parametrized surfaces are defined, are constituted by *quad-graphs*, that is, cellular decompositions of surfaces whose two-cells are all quadrilaterals (see [9, Figure 1]). However, a systematic study of integrable systems on quad-graphs was not put forward in [9].

Finally, one of the sources of the present development is the theory of integrable discretizations of one-dimensional lattice systems of the relativistic Toda type [23]. It was demonstrated by Adler [2] that these discretizations are naturally regarded as integrable systems on the regular triangular lattice, which eventually led him to the discovery of systems of the Toda type on arbitrary graphs [3].

In the present paper, we put forward the viewpoint according to which integrable systems on quad-graphs are very fundamental and explain many known phenomena in the field. In particular, we present the *derivation* of zero curvature representations with a spectral parameter for such systems from a very simple and fundamental principle, namely the consistency on a three-dimensional quad-graph. It should be noted that usually the zero curvature (or Lax) representation is considered as a rather transcendental attribute of a given integrable system, whose existence is easy to verify but very difficult to guess just by looking at the equation. Our approach shows how to derive a zero curvature representation for a given discrete system provided it possesses the property of the three-dimensional consistency. Taking into account that, as widely believed, the whole universe of two-dimensional continuous integrable systems (described by differential-difference or partial differential equations) may be obtained from the discrete ones via suitable limit procedures, we come to the conclusion that there is nothing mysterious in zero curvature representations anymore: they can be found in a nearly algorithmic way. Also, we solve a lesser mystery of the existence of Toda type systems on arbitrary graphs [3], in that we trace their origin back to much more simple and fundamental systems on quad-graphs.

The structure of the paper is as follows: in Section 2 we present the definition of integrable systems on graphs in more details. In Section 3 we consider quad-graphs and discuss their relation to more general graphs. Section 4 contains generalities on integrable systems on quad-graphs (four-point equations) and their relation to the discrete Toda type systems (equations on stars). Sections 5, 6, and 7 are devoted to a detailed account of several concrete integrable systems on quad-graphs, namely the *cross-ratio* *system* and two of its generalizations. This account includes the algorithmic derivation of the zero curvature representation, discussion of the duality and related constructions, derivation of the corresponding systems of the Toda type along with their zero curvature representations, and a geometric interpretation in terms of circle patterns. Finally, conclusions and perspectives for the further research are given in Section 8.

## 2 Discrete flat connections on graphs

We consider integrable systems on graphs as flat connections with the values in loop groups. More precisely, this notion includes the following ingredients:

• A *cellular decomposition* $\mathcal{G}$ of an oriented surface. The set of its vertices will be denoted $V(\mathcal{G})$, the set of its edges will be denoted $E(\mathcal{G})$, and the set of its faces will be denoted $F(\mathcal{G})$. For each edge, one of its possible orientations is fixed.

• A *loop group* $G[\lambda]$, whose elements are functions from $\mathbb{C}$ into some group $G$. The complex argument $\lambda$ of these functions is known in the theory of integrable systems as the spectral parameter.

• A *wave function* $\Psi : V(\mathcal{G}) \mapsto G[\lambda]$, defined on the vertices of $\mathcal{G}$.

• A collection of *transition matrices* $L : E(\mathcal{G}) \mapsto G[\lambda]$ defined on the edges of $\mathcal{G}$.

It is supposed that for any oriented edge $\mathfrak{e} = (v_1, v_2) \in E(\mathcal{G})$ the values of the wave functions in its ends are connected via

$$\Psi(v_2, \lambda) = L(\mathfrak{e}, \lambda)\Psi(v_1, \lambda). \tag{2.1}$$

Therefore, the following *discrete zero curvature condition* is supposed to be satisfied. Consider any closed contour consisting of a finite number of edges of $\mathcal{G}$

$$\mathfrak{e}_1 = (v_1, v_2), \mathfrak{e}_2 = (v_2, v_3), \dots, \mathfrak{e}_n = (v_n, v_1). \tag{2.2}$$

Then

$$L(\mathfrak{e}_n, \lambda) \cdots L(\mathfrak{e}_2, \lambda) L(\mathfrak{e}_1, \lambda) = I. \tag{2.3}$$

In particular, for any edge $\mathfrak{e} = (v_1, v_2)$, if $\mathfrak{e}^{-1} = (v_2, v_1)$, then

$$L(\mathfrak{e}^{-1}, \lambda) = \big(L(\mathfrak{e}, \lambda)\big)^{-1}. \tag{2.4}$$

Actually, in applications the matrices $L(\mathfrak{e}, \lambda)$ depend also on a point of some set $X$ (the phase space of an integrable system), so that some elements $x(\mathfrak{e}) \in X$ are attached to the edges $\mathfrak{e}$ of $\mathcal{G}$. In this case the discrete zero curvature condition (2.3) becomes equivalent to the collection of equations relating the fields $x(\mathfrak{e}_1), \ldots, x(\mathfrak{e}_p)$ attached to the edges of each closed contour. We say that this collection of equations admits a *zero curvature representation*.

For an arbitrary graph, the analytical consequences of the zero curvature representation for a given collection of equations are not clear. However, in case of regular graphs, like those generated by the square lattice $\mathbb{Z} + i\mathbb{Z} \subset \mathbb{C}$, or by the regular triangular lattice $\mathbb{Z} + e^{2\pi i/3}\mathbb{Z} \subset \mathbb{C}$, such representation may be used to determine conserved quantities for suitably defined Cauchy problems, as well as to apply powerful analytical methods for finding concrete solutions.

## 3 Quad-graphs

Our point in this paper is that, although one can consider integrable systems on very different kinds of graphs, there is one kind—quad-graphs—supporting the most fundamental integrable systems.

Definition 3.1. A cellular decomposition $\mathcal{G}$ of an oriented surface is called a *quad-graph*, if all its faces are quadrilateral.

In the present paper, we are interested mainly in the local theory of integrable systems on quad-graphs, therefore, in order to avoid the discussion of some subtle boundary effects, we shall always suppose that the surface carrying the quad-graphs has no boundary.

Before we turn to illustrating this claim by concrete examples, we would like to give a construction which produces from an arbitrary cellular decomposition a certain quad-graph. Towards this aim, we first recall the notion of the *dual graph*, or, more precisely, of the *dual cellular decomposition* $\mathcal{G}^*$. The vertices from $V(\mathcal{G}^*)$ are in one-to-one correspondence to the faces from $F(\mathcal{G})$ (actually, they can be chosen as some points inside the corresponding faces, cf. Figure 3.1). Each $\mathfrak{e} \in E(\mathcal{G})$ separates two faces from $F(\mathcal{G})$, which in turn correspond to two vertices from $V(\mathcal{G}^*)$. A path between these two vertices is then declared as an edge $\mathfrak{e}^* \in E(\mathcal{G}^*)$ dual to $\mathfrak{e}$. Finally, the faces from $F(\mathcal{G}^*)$ are in a one-to-one correspondence with the vertices from $V(\mathcal{G})$: if $v_0 \in V(\mathcal{G})$, and $v_1, \ldots, v_n \in V(\mathcal{G})$ are its neighbors connected with $v_0$ by the edges $\mathfrak{e}_1 = (v_0, v_1), \ldots, \mathfrak{e}_n = (v_0, v_n) \in E(\mathcal{G})$, then the face from $F(\mathcal{G}^*)$ corresponding to $v_0$ is defined by its boundary $\mathfrak{e}_1^* \cup \cdots \cup \mathfrak{e}_n^*$ (cf. Figure 3.2).

Now, following [17], we introduce a new complex, the *double* $\mathcal{D}$, constructed from $\mathcal{G}$, $\mathcal{G}^*$ (notice that our terminology here is different from that of [17]: the complex $\mathcal{D}$

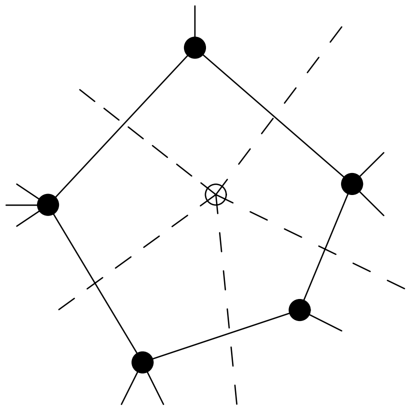

**Figure 3.1** The vertex from $V(\mathcal{G}^*)$ dual to the face from $F(\mathcal{G})$.

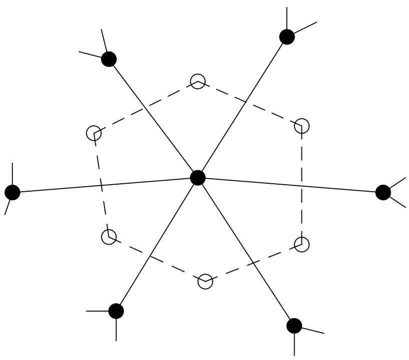

**Figure 3.2** The face from $F(\mathcal{G}^*)$ dual to the vertex from $V(\mathcal{G})$.

is called a *diamond* there, while the term *double* is used for another object). The set of vertices of the double $\mathcal{D}$ is $V(\mathcal{D}) = V(\mathcal{G}) \cup V(\mathcal{G}^*)$. Each pair of dual edges, say $\mathfrak{e} = (v_1, v_2)$ and $\mathfrak{e}^* = (f_1, f_2)$, as in Figure 3.3, defines a quadrilateral $(v_1, f_1, v_2, f_2)$, and all these quadrilaterals constitute the faces of the cell decomposition (quad-graph) $\mathcal{D}$. We stress that the edges of $\mathcal{D}$ belong neither to $E(\mathcal{G})$ nor to $E(\mathcal{G}^*)$. See Figure 3.3.

Quad-graphs $\mathcal{D}$ coming as doubles have the following property: the set $V(\mathcal{D})$ may be decomposed into two complementary halves, $V(\mathcal{D}) = V(\mathcal{G}) \cup V(\mathcal{G}^*)$ (black and white vertices), such that two ends of each edge from $E(\mathcal{D})$ are of different colours. Equivalently, any closed loop consisting of edges of $\mathcal{D}$ has an even length. We will call such quad-graphs *even*.

Conversely, any even quad-graph $\mathcal{D}$ may be considered as a double of some cellular decomposition $\mathcal{G}$. The edges from $E(\mathcal{G})$, say, are defined then as paths joining two

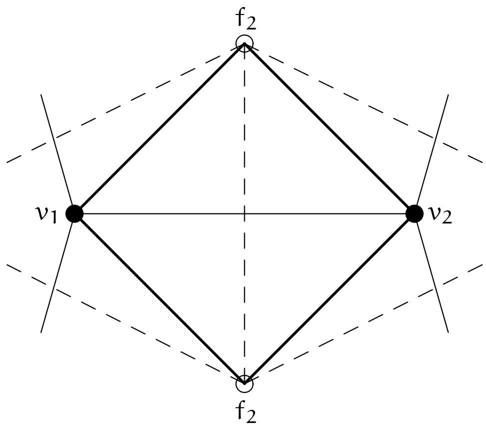


**Figure 3.3** A face of the double.

black vertices of each face from $F(\mathcal{D})$. (This decomposition of $V(\mathcal{D})$ into $V(\mathcal{G})$ and $V(\mathcal{G}^*)$ is unique, up to interchanging the roles of $\mathcal{G}$ and $\mathcal{G}^*$.)

Notice that if $\mathcal{D}$ is not even (i.e., if it admits loops consisting of an odd number of edges), then one can easily produce from $\mathcal{D}$ a new even quad-graph $\mathcal{D}'$, simply by refining each of the quadrilaterals from $F(\mathcal{D})$ into four smaller ones.

Again since we are interested mainly in the local theory, we would like to avoid global considerations. Therefore, we always assume (without mentioning it explicitly) that our quad-graphs are cellular decompositions of an open topological disc. In particular, our quad-graphs $\mathcal{D}$ are always even, so that $\mathcal{G}$ and $\mathcal{G}^*$ are well defined.

We will consider in this paper several integrable systems on quad-graphs. For all these systems the fields $z : V(\mathcal{D}) \mapsto \widehat{\mathbb{C}}$ will be attached to the *vertices* rather than to the edges of the graph, so that for an edge $\mathfrak{e} = (v_0, v_1) \in E(\mathcal{D})$ the transition matrix will be written as

$$L(\mathfrak{e}, \lambda) = L\big(z_1, z_0, \lambda\big). \tag{3.1}$$

Here and below $z_k = z(v_k)$. It is easy to understand that condition (2.4) for the transition matrices reads in the present setup as

$$L\big(z_0, z_1, \lambda\big) = \big(L\big(z_1, z_0, \lambda\big)\big)^{-1}, \tag{3.2}$$

while the discrete zero curvature condition (2.3) in the case of quad-graphs is equivalent to

$$L\big(z_2, z_1, \lambda\big)L\big(z_1, z_0, \lambda\big) = L\big(z_2, z_3, \lambda\big)L\big(z_3, z_0, \lambda\big). \tag{3.3}$$

In all our examples this matrix equation will be equivalent to a single scalar equation

$$\Phi(z_0, z_1, z_2, z_3) = 0, \tag{3.4}$$

relating four fields sitting on the four vertices of an arbitrary face from $F(\mathcal{D})$. Moreover, in all our examples it will be possible to uniquely solve (4.4) for any field $z_0, \ldots, z_3$ in terms of other three ones.

## 4 Equations on quad-graphs and discrete systems of the Toda type

Of course, the above constructions might seem well known, and there are several four-point integrable equations of the type (4.4) available in the literature (see, e.g., [1, 7, 8, 10, 11, 12, 18, 19]). However, these equations were always considered only on the simplest possible quad-graph, namely on the square lattice $V(\mathcal{D}) = \mathbb{Z}^2$. We will see, however, that the novel feature introduced in the present paper, namely considering quad-graphs of arbitrary combinatorics, has interesting and far reaching consequences. One of such consequences is the derivation of *equations of the discrete Toda type on arbitrary graphs* which were first introduced in [3].

At this point we have to specify the setup further. We will assume that the transition matrices depend additionally on a *parameter* $\alpha = \alpha(\mathfrak{e})$ assigned to each (nonoriented) edge

$$L(\mathfrak{e}, \lambda) = L(z_1, z_0, \alpha, \lambda). \tag{4.1}$$

So, actually, an integrable system will be characterized by a function $\alpha : E(\mathcal{D}) \mapsto \mathbb{C}$. For some reasons which will become clear soon, the function $\alpha$ will always satisfy an additional condition.

Definition 4.1. A *labelling* of $E(\mathcal{D})$ is a function $\alpha : E(\mathcal{D}) \mapsto \mathbb{C}$ satisfying the following condition: the values of $\alpha$ on two opposite edges of any quadrilateral from $F(\mathcal{D})$ are equal to one another.

This is illustrated in Figure 4.1. Obviously, there exist infinitely many labellings, all of them may be constructed as follows: choose some value of $\alpha$ for an arbitrary edge of $\mathcal{D}$, and assign consecutively the same value to all parallel edges along a strip of quadrilaterals, according to the definition of labelling. After that, take an arbitrary edge still without a label, choose some value of $\alpha$ for it, and extend the same value along the corresponding strip of quadrilaterals. Proceed similarly, till all edges of $\mathcal{D}$ are exhausted.

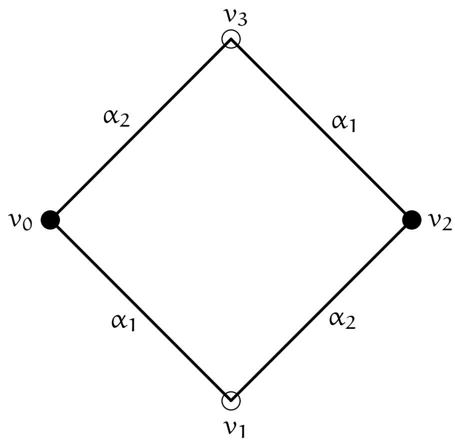


**Figure 4.1** A face of the labelled quad-graph.

Condition (3.2) for the transition matrices reads in the present setup as

$$L(z_0, z_1, \alpha, \lambda) = \big(L(z_1, z_0, \alpha, \lambda)\big)^{-1}, \tag{4.2}$$

while the discrete zero curvature condition (3.3) in the notations of Figure 4.1 reads as

$$L(z_2, z_1, \alpha_2, \lambda)L(z_1, z_0, \alpha_1, \lambda) = L(z_2, z_3, \alpha_1, \lambda)L(z_3, z_0, \alpha_2, \lambda). \tag{4.3}$$

The left-hand side of the scalar equation (3.4) also depends on the parameters

$$\Phi(z_0, z_1, z_2, z_3, \alpha_1, \alpha_2) = 0. \tag{4.4}$$

We will require two additional features of (4.4). The first one is quite natural and is satisfied by all known examples: the function $\Phi$ is symmetric with respect to the cyclic shift $(z_0, z_1, z_2, z_3) \mapsto (z_1, z_2, z_3, z_0)$ accompanied by interchanging the parameters $\alpha_1 \leftrightarrow \alpha_2$. The second feature might not seem natural at all, at least on the first sight, but also holds, for some mysterious reasons, for the majority of (if not all) the known examples. Namely, suppose that (4.4) may be equivalently rewritten in the form

$$f(z_0, z_1, \alpha_1) - f(z_0, z_3, \alpha_2) = g(z_0, z_2, \alpha_1, \alpha_2). \tag{4.5}$$

We call (4.5) the *three-leg form* of (4.4). Indeed, its three terms may be associated with three legs $(v_0, v_1)$, $(v_0, v_3)$, and $(v_0, v_2)$. Notice that while the first two legs belong to $E(\mathcal{D})$, the third one does not; instead, it belongs to $E(\mathcal{G}^*)$ (recall that our quad-graph $\mathcal{D}$ comes as double constructed from $\mathcal{G}$, $\mathcal{G}^*$). Now consider $n$ faces from $F(\mathcal{D})$ having common black

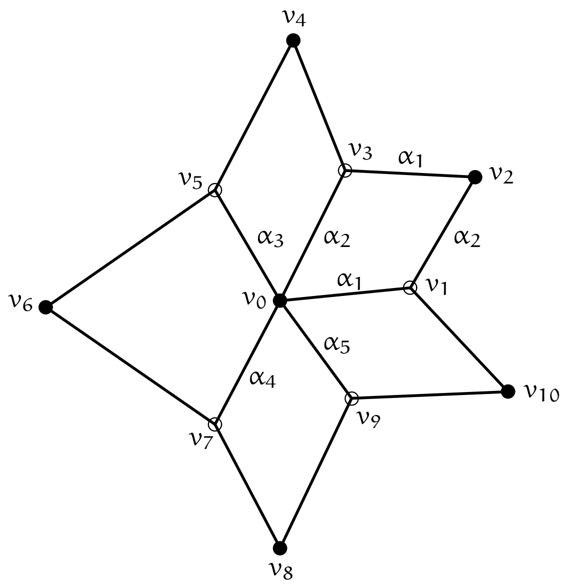


**Figure 4.2** Faces from $F(\mathcal{D})$ around the vertex $v_0$.

vertex $v_0 \in V(\mathcal{G}) \subset V(\mathcal{D})$ (cf. Figure 4.2, where $n = 5$). Summing up equations (4.5) for these $n$ faces, that is,

$$f(z_0, z_{2k-1}, \alpha_k) - f(z_0, z_{2k+1}, \alpha_{k+1}) = g(z_0, z_{2k}, \alpha_k, \alpha_{k+1}), \tag{4.6}$$

we end up with the equation which includes only the fields $z$ in the black vertices

$$\sum_{k=1}^{n} g(z_0, z_{2k}, \alpha_k, \alpha_{k+1}) = 0. \tag{4.7}$$

The latter formula may be interpreted as a system on the graph $\mathcal{G}$: there is one equation per vertex $v_0 \in V(\mathcal{G})$, and this equation relates the fields on the *star* of $v_0$ consisting of the edges from $E(\mathcal{G})$ incident to $v_0$, see Figure 4.3. The parameters $\alpha_k$ are then viewed as assigned to the corners of the faces of $\mathcal{G}$. Observe that Definition 4.1 of a labelling of $E(\mathcal{D})$ may be reformulated in terms of $\mathcal{G}$ alone as follows: the corner parameters of two faces adjacent to the edge $(v_0, v_{2k}) \in E(\mathcal{G})$ at the vertex $v_0$ are the same as at the vertex $v_{2k}$ (the ordering of the parameters at $v_0$ when looking towards $v_{2k}$ is the same as the ordering of the parameters at $v_{2k}$ when looking towards $v_0$, cf. Figure 4.3).

The collection of equations (4.7) constitutes the *discrete Toda type system on* $\mathcal{G}$. Obviously, the nature of this system is formally somewhat different from the general construction of Section 2 (the former relates fields on stars, while the latter relates fields on closed contours). However, our construction suggests the following type of a zero

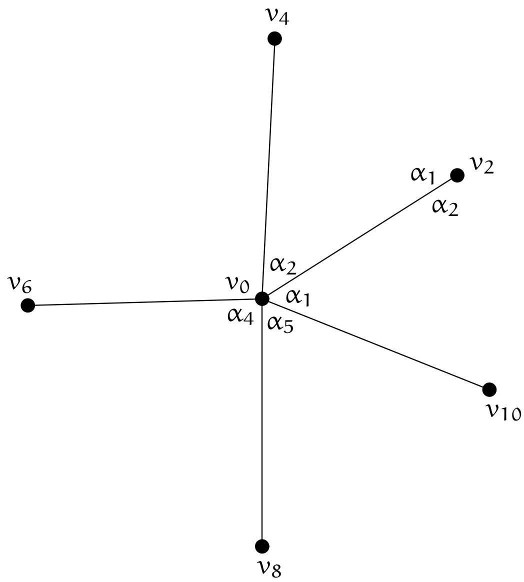


**Figure 4.3** The star (in $\mathcal{G}$) of the vertex $v_0$.

curvature representation for (4.7). The wave function $\Psi : V(\mathcal{G}^*) \mapsto G[\lambda]$ is defined on the white vertices $v_{2k-1}$ (in the notations of Figure 4.2), that is, on the vertices of the dual graph. The transition matrix along the edge $\mathfrak{e}^* = (v_{2k-1}, v_{2k+1}) \in E(\mathcal{G}^*)$ of the dual graph is equal to

$$\begin{aligned} &L(z_{2k+1}, z_{2k}, \alpha_k, \lambda) L(z_{2k} \cdot z_{2k-1}, \alpha_{k+1}, \lambda) \\ &\quad = L(z_{2k+1}, z_0, \alpha_{k+1}, \lambda) L(z_0, z_{2k-1}, \alpha_k, \lambda). \end{aligned} \tag{4.8}$$

If one prefers to consider everything in terms of $\mathcal{G}$ alone, one can think of $\Psi$ as defined on the faces of $\mathcal{G}$, and of the above matrix as corresponding to the transition *across* the edge $\mathfrak{e} = (v_0, v_{2k}) \in E(\mathcal{G})$. One would like this matrix to depend only on the fields $z_0$, $z_{2k}$. In all our examples this can and will be achieved by a certain gauge transformation $\Psi(v_k) \mapsto A(v_k)\Psi(v_k)$.

Next, notice that in the above reasoning the roles of $\mathcal{G}$ and $\mathcal{G}^*$ may be interchanged. Hence, one can derive a Toda type system on $\mathcal{G}^*$ as well. The distribution of the corner parameters on $\mathcal{G}$ and $\mathcal{G}^*$ is illustrated in Figure 4.4.

We arrive at the following statement.

**Proposition 4.2.** (a) Let system (4.4) on a quad-graph $\mathcal{D}$ admit a three-leg form. Any solution $z : V(\mathcal{D}) \mapsto \mathbb{C}$ of this system, restricted to $V(\mathcal{G})$, respectively to $V(\mathcal{G}^*)$, delivers a solution of the discrete Toda type system (4.7) on $\mathcal{G}$, respectively on $\mathcal{G}^*$, with the corner parameters given by the labelling of $E(\mathcal{D})$.

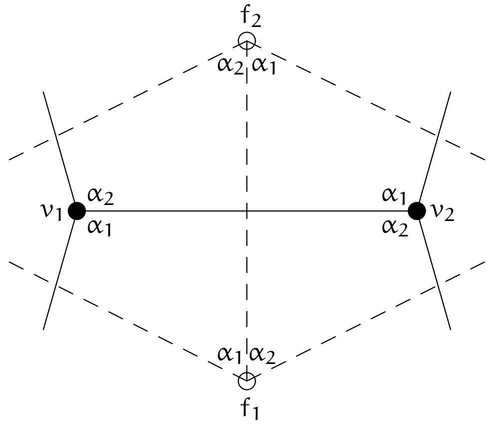


**Figure 4.4** Corner parameters of the dual graph.

(b) Conversely, given a solution $z : V(\mathcal{G}) \mapsto \mathbb{C}$ of the discrete Toda type system (4.7) on $\mathcal{G}$, there exists a one-parameter family of its extensions to a solution of system (4.4) on $\mathcal{D}$. □

Proof. The first statement is already proved. As for the second one, suppose we know the solution of the Toda type system on $V(\mathcal{G})$, that is, in all black vertices. Fixing the value of the field $z$ in one white vertex arbitrarily, we can extend it successively to all white vertices in virtue of (4.4). The consistency of this procedure is assured by (4.5). ■

Remark 4.3. Proposition 4.2 implies that from a solution of the Toda type system on $\mathcal{G}$ one gets a one-parameter family of solutions of the Toda type system on $\mathcal{G}^*$. This may be regarded as a sort of the *Bäcklund transformation*, especially when the graphs $\mathcal{G}$ and $\mathcal{G}^*$ are isomorphic, which is the case, for example, for the regular square lattice.

## 5 Cross-ratio system and additive rational Toda system

### 5.1 Definition

The cross-ratio system is one of the simplest and at the same time the most fundamental and important integrable systems on quad-graphs.

Definition 5.1. Given a labelling of $E(\mathcal{D})$, the *cross-ratio system* on $\mathcal{D}$ is the collection of equations for the function $z : V(\mathcal{D}) \mapsto \widehat{\mathbb{C}}$, one equation per face of $\mathcal{D}$, which read, in notations as in Figure 4.1 and $z_k = z(v_k)$

$$q(z_0, z_1, z_2, z_3) \stackrel{\text{def}}{=} \frac{(z_0 - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z_0)} = \frac{\alpha_1}{\alpha_2}. \tag{5.1}$$

Several simple remarks are here in order. Although (5.1) is a well-defined equation for each face of $\mathcal{D}$, the cross-ratio itself is not a function on $F(\mathcal{D})$ since

$$q(z_0, z_1, z_2, z_3) = \frac{1}{q(z_1, z_2, z_3, z_0)}. \tag{5.2}$$

On the other hand, $F(\mathcal{D})$ is in a one-to-one correspondence with $E(\mathcal{G})$, as well as with $E(\mathcal{G}^*)$. It is sometimes convenient to consider the cross-ratio as a function on $E(\mathcal{G})$ or on $E(\mathcal{G}^*)$, setting (see Figure 4.1)

$$\begin{aligned} q(\mathfrak{e}) &= q(z_0, z_1, z_2, z_3), \quad \text{where} \quad \mathfrak{e} = (v_0, v_2) \in E(\mathcal{G}), \\ q(\mathfrak{e}^*) &= q(z_1, z_2, z_3 \cdot z_0), \quad \text{where} \quad \mathfrak{e}^* = (v_1, v_3) \in E(\mathcal{G}^*). \end{aligned} \tag{5.3}$$

Sometimes a sort of inverse problem is of interest: given a function $z : V(\mathcal{D}) \mapsto \widehat{\mathbb{C}}$, can it be considered as a solution of the cross-ratio system for *some* labelling? The answer follows from a simple lemma.

**Lemma 5.2.** Let, as usual, $\mathcal{D}$ be a cellular decomposition of an open topological disc, and therefore a double for some pair of dual cellular decompositions $\mathcal{G}, \mathcal{G}^*$. Let $Q : E(\mathcal{G}) \mapsto \widehat{\mathbb{C}}$ be some function, and extend it to $E(\mathcal{G}^*)$ by the formula $Q(\mathfrak{e}^*) = 1/Q(\mathfrak{e})$. Then the necessary and sufficient condition for the existence of a labelling $\alpha : E(\mathcal{D}) \mapsto \mathbb{C}$ such that, in the notations of Figure 4.1,

$$Q(\mathfrak{e}) = Q(v_0, v_2) = \frac{\alpha_1}{\alpha_2} \Longleftrightarrow Q(\mathfrak{e}^*) = Q(v_1, v_3) = \frac{\alpha_2}{\alpha_1}, \tag{5.4}$$

is given (in the notations of Figure 4.3) by the equations

$$\begin{aligned} \prod_{\mathfrak{e} \in \text{star}(v_0)} Q(\mathfrak{e}) = 1 \quad \text{for all } v_0 \in V(\mathcal{G}), \\ \prod_{\mathfrak{e}^* \in \text{star}(v_1)} Q(\mathfrak{e}^*) = 1 \quad \text{for all } v_1 \in V(\mathcal{G}^*). \end{aligned} \tag{5.5}$$

□

Proof. The necessity is obvious. To prove sufficiency, we construct $\alpha$ by assigning an arbitrary value (say, $\alpha = 1$) to some edge from $E(\mathcal{D})$, and then extending it successively using either of the equations (5.4) and the definition of labelling. Conditions (5.5) assure the consistency of this procedure. ■

**Corollary 5.3.** Let $z : V(\mathcal{D}) \mapsto \widehat{\mathbb{C}}$ be some map. The necessary and sufficient condition for the existence of a labelling $\alpha : E(\mathcal{D}) \mapsto \mathbb{C}$ such that $z$ is a solution of the corresponding cross-ratio system is given (in the notations of Figure 4.2) by the equation

$$\prod_{k=1}^{n} q\big(z_0, z_{2k-1}, z_{2k}, z_{2k+1}\big) = 1 \tag{5.6}$$

for all vertices $v_0 \in V(\mathcal{D})$. □

## 5.2 Zero curvature representation

**Proposition 5.4.** The cross-ratio system on $\mathcal{D}$ admits a zero curvature representation in the sense of Section 2, with the transition matrices for $\mathfrak{e} = (v_0, v_1)$

$$L(\mathfrak{e}, \lambda) = L\big(z_1, z_0, \alpha, \lambda\big) = (1 - \lambda\alpha)^{-1/2} \begin{pmatrix} 1 & z_0 - z_1 \\ \dfrac{\lambda\alpha}{z_0 - z_1} & 1 \end{pmatrix}. \tag{5.7}$$

Here $\alpha = \alpha(\mathfrak{e})$, and, as usual, $z_k = z(v_k)$. □

Proof. Verifying (4.2) and (4.3) is a matter of a simple computation. Notice that the entry 12 of the matrix equation (4.3) gives an identity, the entries 11 and 22 give (5.1) as it stands, while the entry 21 gives

$$\frac{\alpha_1}{z_0 - z_1} + \frac{\alpha_2}{z_1 - z_2} = \frac{\alpha_2}{z_0 - z_3} + \frac{\alpha_1}{z_3 - z_2}, \tag{5.8}$$

which is a consequence of (5.1). ■

The zero-curvature representation of Proposition 5.4 was introduced in [18], of course only in the case of the square lattice and constant labelling. (Cf. also a gauge equivalent form in [7], where a more general situation of discrete isothermic surfaces in $\mathbb{R}^3$ is considered (in this context the fields $z$ take values in $\mathbb{H}$—the ring of quaternions).)

## 5.3 Zero curvature representation from three-dimensional consistency

Now we demonstrate how to *derive* the zero curvature representation of Proposition 5.4 starting from an additional deep property of the cross-ratio system. To this end we extend the planar quad-graph $\mathcal{D}$ into the third dimension. Formally speaking, we consider the second copy $\mathcal{D}'$ of $\mathcal{D}$ and add edges connecting each vertex $v \in V(\mathcal{D})$ with its

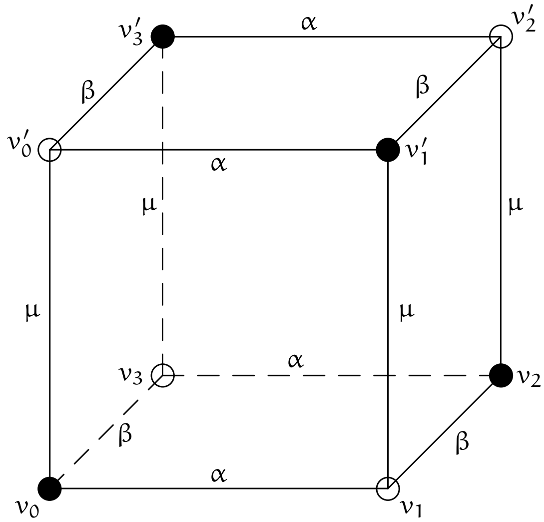


**Figure 5.1** Elementary cube of $\mathcal{D} \cup \mathcal{D}'$.

copy $v' \in V(\mathcal{D}')$. This way we obtain a "three-dimensional quad-graph" $\mathbf{D}$, whose set of vertices is

$$V(\mathbf{D}) = V(\mathcal{D}) \cup V(\mathcal{D}'), \tag{5.9}$$

whose set of edges is

$$E(\mathbf{D}) = E(\mathcal{D}) \cup E(\mathcal{D}') \cup \{(v, v') : v \in V(\mathcal{D})\}, \tag{5.10}$$

and whose set of faces is

$$F(\mathbf{D}) = F(\mathcal{D}) \cup F(\mathcal{D}') \cup \{(v_0, v_1, v_1', v_0') : v_0, v_1 \in V(\mathcal{D})\}. \tag{5.11}$$

Elementary building blocks of $\mathbf{D}$ are cubes $(v_0, v_1, v_2, v_3, v_0', v_1', v_2', v_3')$, as shown in Figure 5.1. Clearly, we still can consistently subdivide the vertices of $\mathbf{D}$ into black and white ones, assigning to each $v' \in V(\mathcal{D}')$ the colour opposite to the colour of its counterpart $v \in V(\mathcal{D})$.

Next, we define a labelling on $E(\mathbf{D})$ in the following way: each edge $(v_0', v_1') \in E(\mathcal{D}')$ carries the same label as its counterpart $(v_0, v_1) \in E(\mathcal{D})$, while all vertical edges $(v, v')$ carry one and the same label $\mu$. Now, the fundamental property of the cross-ratio system mentioned above is the *three-dimensional consistency*. This should be understood as follows. Consider an elementary cube of $\mathbf{D}$, as in Figure 5.1. Suppose that the values of the field $z$ are given at the vertex $v_0$ and at its three neighbors $v_1$, $v_3$, and $v_0'$. Then the cross-ratio equation (5.1) uniquely determines the values of $z$ at $v_2, v_1'$, and $v_3'$. After that the cross-ratio equation delivers three a priori different values for the value of the field $z$ at the vertex $v_2'$, coming from the faces $(v_1, v_2, v_2', v_1')$, $(v_2, v_3, v_3', v_2')$, and $(v_0', v_1', v_2', v_3')$, respectively. The three-dimensional consistency means exactly that *these three values for* $z(v_2')$ *actually coincide*. The verification of this claim consists of a straightforward computation. Because of the importance of this property we formulate it in a separate proposition.

**Proposition 5.5.** The cross-ratio system is consistent on the three-dimensional quad-graph $\mathbf{D}$. □

Now we can derive a zero curvature representation for the cross-ratio equation on $\mathcal{D}$, using the arbitrariness of the label $\mu$ assigned to the vertical edges $(v, v')$ of $\mathbf{D}$. For this aim, consider the cross-ratio equation on the vertical face $(v_0, v_1, v_1', v_0')$. Denote $\lambda = \mu^{-1}$, then the equation reads

$$\frac{(z_1' - z_0')(z_0 - z_1)}{(z_0' - z_0)(z_1 - z_1')} = \lambda\alpha. \tag{5.12}$$

This gives $z_1'$ as a fractional-linear (Möbius) transformation[1] of $z_0'$ with the coefficients depending on $z_0$, $z_1$, and the (arbitrary) parameter $\lambda$

$$z_1' = \widetilde{L}(z_1, z_0, \alpha, \lambda)\left[z_0'\right], \tag{5.13}$$

where

$$\widetilde{L}(z_1, z_0, \alpha, \lambda) = \begin{pmatrix} 1 + \dfrac{\lambda\alpha z_1}{z_0 - z_1} & -\dfrac{\lambda\alpha z_0 z_1}{z_0 - z_1} \\[2ex] \dfrac{\lambda\alpha}{z_0 - z_1} & 1 - \dfrac{\lambda\alpha z_0}{z_0 - z_1} \end{pmatrix} \tag{5.14}$$

$$\begin{aligned} &= I + \frac{\lambda\alpha}{z_0 - z_1}\begin{pmatrix} z_1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ -z_0 \end{pmatrix}^{T} \\ &= (1 - \lambda\alpha)I + \frac{\lambda\alpha}{z_0 - z_1}\begin{pmatrix} z_0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ -z_1 \end{pmatrix}^{T}. \end{aligned} \tag{5.15}$$

[1] We use the following matrix notation for the action of Möbius transformations:

$$\frac{az + b}{cz + d} = L[z], \quad \text{where } L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In principle, the matrices $\widetilde{L}(z_1, z_0, \alpha, \lambda)$, after multiplying by $(1-\lambda\alpha)^{-1/2}$ (which is nothing but normalizing the determinants) are the sought after transition matrices of the discrete flat connection on $\mathcal{D}$, as assured by Proposition 5.5. To recover the transition matrices from Proposition 5.4, we perform a gauge transformation defined on the level of the wave functions $\Psi : V(\mathcal{D}) \mapsto G$ as

$$\Psi(v) \longmapsto A\big(z(v)\big)\Psi(v), \tag{5.16}$$

where

$$A(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}. \tag{5.17}$$

On the level of transition matrices this results in

$$\begin{aligned} \widetilde{L}(z_1, z_0, \alpha, \lambda) \longmapsto L(z_1, z_0, \alpha, \lambda) &= A^{-1}(z_1)\widetilde{L}(z_1, z_0, \alpha, \lambda)A(z_0) \\ &= \begin{pmatrix} 1 & z_0 - z_1 \\ \dfrac{\lambda\alpha}{z_0 - z_1} & 1 \end{pmatrix}. \end{aligned} \tag{5.18}$$

Notice that the gauge transformation (5.16) and (5.17) may be interpreted in terms of the variables $z(v')$ as a shift of each $z(v')$ by $z(v)$, so that the interpretation of the matrix $L(z_1, z_0, \alpha, \lambda)$ is as follows:

$$z_1' - z_1 = L\big(z_1, z_0, \alpha, \lambda\big)\big[z_0' - z_0\big]. \tag{5.19}$$

From our point of view, the property of the three-dimensional consistency lies at the heart of the two-dimensional integrability. Not only does it allow us to derive the spectral-parameter dependent zero-curvature representation for the two-dimensional discrete system. We can consider the function $z : V(\mathcal{D}') \mapsto \widehat{\mathbb{C}}$ as a *Bäcklund transformation* of the solution $z : V(\mathcal{D}) \mapsto \widehat{\mathbb{C}}$. Moreover, if $\mathcal{D}$ is generated by a square lattice, then, as well known, by refining this lattice in one or two directions, we can obtain certain integrable differential-difference or partial differential equations, the latter being the famous Schwarzian Korteweg-de Vries equation

$$z_t = z_{xxx} - \frac{3}{2}\frac{z_{xx}^2}{z_x}, \tag{5.20}$$

see [18]. So, the three-dimensional consistency delivers all these equations and their Bäcklund transformations in one construction. For instance, if in (5.12) $z_0$ is a solution of the Schwarzian KdV, $z_1$ its Bäcklund transformation, and $z_0'$, $z_1'$ correspond to the $\epsilon$-shift of these solutions in the $x$-direction, where $\lambda = \epsilon^{-2}$, then in the limit $\epsilon \to 0$ one recovers the well-known formula [1]

$$(z_0)_x (z_1)_x = \alpha^{-1}(z_0 - z_1)^2. \tag{5.21}$$

Similarly, when considered on a single face of the double, as in Figure 4.1, the cross-ratio equation (5.1) expresses nothing but the commutativity of the Bianchi diagram (Figure 4.1) for Bäcklund transformations of the Schwarzian KdV. Indeed, multiplying the equations

$$(z_0)_x (z_1)_x = \alpha_1^{-1}(z_0 - z_1)^2, \qquad (z_3)_x (z_2)_x = \alpha_1^{-1}(z_3 - z_2)^2, \tag{5.22}$$

and dividing by two similar equations corresponding to the edges $(v_0, v_3)$ and $(v_1, v_2)$ (with the parameter $\alpha_2$), we arrive at the (squared) equation (5.1). So, one can say that (5.1) on the cube Figure 5.1 contains *everything* about the Schwarzian KdV and its Bäcklund transformations.

### 5.4 Additive rational Toda system

The further remarkable feature of the cross-ratio equation is the existence of the *three-leg form*. To derive it, notice that (5.1) is equivalent to

$$\alpha_1 \frac{z_1 - z_2}{z_0 - z_1} - \alpha_2 \frac{z_3 - z_2}{z_0 - z_3} = 0 \iff \alpha_1 \frac{z_0 - z_2}{z_0 - z_1} - \alpha_2 \frac{z_0 - z_2}{z_0 - z_3} = \alpha_1 - \alpha_2, \tag{5.23}$$

or to

$$\frac{\alpha_1}{z_0 - z_1} - \frac{\alpha_2}{z_0 - z_3} = \frac{\alpha_1 - \alpha_2}{z_0 - z_2}, \tag{5.24}$$

which obviously is of the form (4.5).

According to the construction of Section 4, the corresponding system of the Toda type reads (in the notations of Figure 4.3)

$$\sum_{k=1}^{n} \frac{\alpha_k - \alpha_{k+1}}{z_0 - z_{2k}} = 0. \tag{5.25}$$

It will be called the *additive rational Toda system* on $\mathcal{G}$. For the cross-ratio system on $\mathcal{D}$ and additive rational Toda systems on $\mathcal{G}$, $\mathcal{G}^*$ there holds Proposition 4.2.

Remark 5.6. Notice that the Toda system on $\mathcal{G}$ remains unchanged upon the *simultaneous additive shift* of all corner parameters $\alpha_k \mapsto \alpha_k + a$. However, such a shift changes the cross-ratio system essentially. Hence, Proposition 4.2 yields also a nontrivial transformation between the solutions of two different cross-ratio systems on $\mathcal{D}$ whose labellings differ by a common additive shift.

Next, we derive a zero curvature representation for the additive rational Toda system from such a representation for the cross-ratio system. Consider the additive rational Toda system on the black vertices from $V(\mathcal{G})$, that is, on $v_{2k}$ in the notations of Figure 4.2. The transition matrix across the edge $\mathfrak{e}_k = (v_0, v_{2k}) \in E(\mathcal{G})$, that is, along the edge $\mathfrak{e}_k^* = (v_{2k-1}, v_{2k+1}) \in E(\mathcal{G}^*)$, is given by

$$\widetilde{\mathcal{L}}(z_{2k}, z_0, \lambda) = \widetilde{L}(z_{2k+1}, z_0, \alpha_{k+1}, \lambda)\widetilde{L}(z_0, z_{2k-1}, \alpha_k, \lambda). \tag{5.26}$$

We use here the matrices (5.15), gauge equivalent to (5.7), because of the remarkable fact underlined by the notation for the matrices on the left-hand side of the last equation: they actually depend only on $z_0, z_{2k}$, but not on $z_{2k-1}$, $z_{2k+1}$. (This property would fail, if we would use the matrices (5.7).)

**Proposition 5.7.** The following formulas hold:

$$\widetilde{\mathcal{L}}(z_{2k}, z_0, \lambda) = I + \frac{\lambda}{z_0 - z_{2k}} \begin{pmatrix} \alpha_{k+1} z_{2k} - \alpha_k z_0 & (\alpha_k - \alpha_{k+1}) z_0 z_{2k} \\ \alpha_{k+1} - \alpha_k & \alpha_k z_{2k} - \alpha_{k+1} z_0 \end{pmatrix} \tag{5.27}$$

$$= (1 - \lambda\alpha_k) I + \lambda \frac{\alpha_{k+1} - \alpha_k}{z_0 - z_{2k}} \begin{pmatrix} z_{2k} \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -z_0 \end{pmatrix}^{\mathrm{T}} \tag{5.28}$$

$$= (1 - \lambda\alpha_{k+1}) I + \lambda \frac{\alpha_{k+1} - \alpha_k}{z_0 - z_{2k}} \begin{pmatrix} z_0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -z_{2k} \end{pmatrix}^{\mathrm{T}}. \tag{5.29}$$

□

Proof. Use the second formula in (5.15) for $\widetilde{L}(z_0, z_{2k-1}, \alpha_k, \lambda)$ and the first formula in (5.15) for $\widetilde{L}(z_{2k+1}, z_0, \alpha_{k+1}, \lambda)$. This leads, after a remarkable cancellation of terms quadratic in $\lambda$, to the following expression for the matrix (5.26):

$$\widetilde{\mathcal{L}}(z_{2k}, z_0, \lambda) = (1 - \lambda\alpha_k) I + \lambda \xi \begin{pmatrix} 1 \\ -z_0 \end{pmatrix}^{\mathrm{T}}, \tag{5.30}$$

where

$$\xi = \frac{\alpha_{k+1}}{z_0 - z_{2k+1}} \begin{pmatrix} z_{2k+1} \\ 1 \end{pmatrix} - \frac{\alpha_k}{z_0 - z_{2k-1}} \begin{pmatrix} z_{2k-1} \\ 1 \end{pmatrix}. \tag{5.31}$$

A simple calculation based on formula (5.24), written now as

$$\frac{\alpha_{k+1}}{z_0 - z_{2k+1}} - \frac{\alpha_k}{z_0 - z_{2k-1}} = \frac{\alpha_{k+1} - \alpha_k}{z_0 - z_{2k}}, \tag{5.32}$$

shows that

$$\xi = \frac{\alpha_{k+1} - \alpha_k}{z_0 - z_{2k}} \begin{pmatrix} z_{2k} \\ 1 \end{pmatrix}. \tag{5.33}$$

This proves (5.28). ■

The matrix (5.27) coincides with the transition matrix for the additive rational Toda lattice found by Adler in [3].

## 5.5 Duality

Next, we discuss a duality transformation for the cross-ratio system. This comes into play, if we rewrite the cross-ratio equation (5.8) in the form

$$\frac{\alpha_1}{z_0 - z_1} + \frac{\alpha_2}{z_1 - z_2} + \frac{\alpha_1}{z_2 - z_3} + \frac{\alpha_2}{z_3 - z_0} = 0. \tag{5.34}$$

This guarantees (locally, and therefore in the case when $\mathcal{D}$ is a cellular decomposition of a topological disc) that one can introduce the function $Z : V(\mathcal{D}) \mapsto \widehat{\mathbb{C}}$ such that

$$\begin{aligned} Z_1 - Z_0 &= \frac{\alpha_1}{z_0 - z_1}, \qquad & Z_2 - Z_1 &= \frac{\alpha_2}{z_1 - z_2}, \\ Z_3 - Z_2 &= \frac{\alpha_1}{z_2 - z_3}, \qquad & Z_0 - Z_3 &= \frac{\alpha_2}{z_3 - z_0}. \end{aligned} \tag{5.35}$$

In other words, for an arbitrary $\mathfrak{e} = (v_1, v_2) \in E(\mathcal{D})$, one sets

$$Z(v_2) - Z(v_1) = \frac{\alpha(\mathfrak{e})}{z(v_1) - z(v_2)}. \tag{5.36}$$

**Proposition 5.8.** For any solution $z : V(\mathcal{D}) \mapsto \widehat{\mathbb{C}}$ of the cross-ratio system on $\mathcal{D}$, formula (5.36) correctly defines a new (dual) solution of the same system. □

Proof. The above argument shows that formula (5.36) correctly defines some function on $V(\mathcal{D})$. It can be easily seen that this function also solves the cross-ratio system. ■

Combining Propositions 4.2 and 5.8, we see that, to any solution of the additive rational Toda system on $\mathcal{G}$ there corresponds a one-parameter family of solutions of the additive rational Toda system on $\mathcal{G}^*$, and any such pair of corresponding solutions yields (almost uniquely—up to an additive constant) dual solutions for the additive rational Toda systems on $\mathcal{G}$ and $\mathcal{G}^*$. But actually this duality relation between Toda systems is much more stiff. Indeed, adding the first and the fourth equations in (5.35) and using (5.24), we arrive at the formula

$$Z_1 - Z_3 = \frac{\alpha_2 - \alpha_1}{z_2 - z_0}. \tag{5.37}$$

Therefore, a solution of the additive rational Toda system on $\mathcal{G}$ determines the dual solution of the additive rational Toda system on $\mathcal{G}^*$ almost uniquely (up to an additive constant).

It is also instructive to regard the function $w : V(\mathcal{D}) \mapsto \widehat{\mathbb{C}}$ which coincides with $z$ on $V(\mathcal{G})$ and with $Z$ on $V(\mathcal{G}^*)$. Then formula (5.37) may be considered as a system of equations on the quad-graph, reading on the face $(v_0, v_1, v_2, v_3)$ as

$$w_1 - w_3 = \frac{\alpha_2 - \alpha_1}{w_2 - w_0}. \tag{5.38}$$

Interestingly (but not very surprisingly), (5.38) also constitute an integrable system on the quad-graph. In the case when $\mathcal{D}$ is generated by a square lattice and the labelling is constant, this system is known under the name of the *discrete KdV equation* [11, 18]. Actually, it appeared even earlier in the numerical analysis as the $\varepsilon$-*algorithm* [24]. The transition matrices for (5.38) are given by

$$L(\mathfrak{e}, \mu) = L(w_1, w_0, \alpha, \mu) = (\mu - \alpha)^{-1/2} \begin{pmatrix} w_0 & -w_0 w_1 + \alpha - \mu \\ 1 & -w_1 \end{pmatrix}. \tag{5.39}$$

This transition matrix may be derived in exactly the same way as for the cross-ratio system, based on the three-dimensional consistency. Namely, this is the matrix of the Möbius transformation $v_0' \mapsto v_1'$, see Figure 5.1

$$\begin{aligned} &(w_1 - w_0')(w_1' - w_0) + \alpha - \mu = 0 \\ &\Longrightarrow \quad w_1' = \frac{w_0 w_0' - w_0 w_1 + \alpha - \mu}{w_0' - w_1} = L(w_1, w_0, \alpha, \mu)[w_0']. \end{aligned} \tag{5.40}$$

Again, the three-dimensional consistency is the central principle for the whole integrability theory of the discrete, differential-difference and partial differential equations related to the (potential) KdV equation

$$w_t = w_{xxx} - 6w_x^2, \tag{5.41}$$

$$
\begin{tikzcd}
\text{Toda on } \mathcal{G} & \text{cross-ratio on } \mathcal{D} \arrow[l] \arrow[r] \arrow[dd, leftrightarrow, "\textit{duality}"] & \text{Toda on } \mathcal{G}^* \\
\text{discr. KdV on } \mathcal{D} \arrow[u] \arrow[d] \arrow[rr, leftrightarrow, "\textit{duality}"'] & & \text{discr. KdV on } \mathcal{D} \arrow[u] \arrow[d] \\
\text{dual Toda on } \mathcal{G}^* & \text{dual cross-ratio on } \mathcal{D} \arrow[l] \arrow[r] & \text{dual Toda on } \mathcal{G}
\end{tikzcd}
$$

**Figure 5.2** Relations between different systems.

as well as their Bäcklund transformations. In particular, (5.38) expresses nothing but the commutativity of the Bianchi's diagram (Figure 4.1) for the Bäcklund transformations of the potential KdV. Indeed, recall that the Bäcklund transformations corresponding to the edges $(v_0, v_1)$ and $(v_3, v_2)$ of the diagram in Figure 4.1 are defined by the equations

$$(w_0)_x + (w_1)_x = (w_1 - w_0)^2 + \alpha_1, \qquad (w_3)_x + (w_2)_x = (w_2 - w_3)^2 + \alpha_1. \quad (5.42)$$

Adding these two equations and subtracting two similar equations corresponding to the edges $(v_0, v_3)$ and $(v_1, v_2)$ (with the parameter $\alpha_2$), we arrive at

$$(w_2 - w_0)(w_1 - w_3) + \alpha_1 - \alpha_2 = 0, \quad (5.43)$$

which is nothing but (5.38).

Notice that (5.38) is already in the three-leg form, and restrictions of its solutions to $V(\mathcal{G})$ and to $V(\mathcal{G}^*)$ are solutions of the rational additive Toda system on the corresponding graphs. It is easy to check that the pairwise products of the matrices (5.39), like in (5.26), lead to the same transition matrices (5.27) for the additive rational Toda system as before.

The relations between various systems discussed in the present section are summarized in Figure 5.2.

The thick lines here are for the restriction maps, the thin ones for the duality maps. The duality for the cross-ratio system is described by formula (5.36). Its transcription in terms of the variables $w$ and the dual ones $W$ reads

$$W(v_2) - w(v_1) = \frac{\alpha(\mathfrak{e})}{W(v_1) - w(v_2)}. \quad (5.44)$$

It is not difficult to prove the statement similar to Proposition 5.8: for any solution $w : V(\mathcal{D}) \mapsto \widehat{\mathbb{C}}$ of the discrete KdV system on $\mathcal{D}$, formula (5.44) correctly defines a one-parameter family of (dual) solutions $W : V(\mathcal{D}) \mapsto \widehat{\mathbb{C}}$ of the same system.

### 5.6 Cross-ratio system and circle patterns

We close the discussion of the cross-ratio system on an arbitrary quad-graph $\mathcal{D}$ by giving its geometric interpretation. For this aim, we consider circle patterns with arbitrary combinatorics.

Definition 5.9. Let $\mathcal{G}$ be an arbitrary cellular decomposition. We say that a map $z : V(\mathcal{G}) \mapsto \widehat{\mathbb{C}}$ defines a *circle pattern with the combinatorics of* $\mathcal{G}$, if the following condition is satisfied. Let $f \in F(\mathcal{G})$ be an arbitrary face of $\mathcal{G}$, and let $v_1, v_2, \ldots, v_n$ be its consecutive vertices. Then the points $z(v_1), z(v_2), \ldots, z(v_n) \in \widehat{\mathbb{C}}$ lie on a circle, and their circular order is just the listed one. We denote this circle by $C(f)$, thus putting it into a correspondence with the face $f$, or, equivalently, with the respective vertex of the dual decomposition $\mathcal{G}^*$.

As a consequence of this condition, if two faces $f_1, f_2 \in F(\mathcal{G})$ have a common edge $(v_1, v_2)$, then the circles $C(f_1)$ and $C(f_2)$ intersect at the points $z(v_1)$ and $z(v_2)$. In other words, the edges from $E(\mathcal{G})$ correspond to pairs of neighboring (intersecting) circles of the pattern. Similarly, if several faces $f_1, f_2, \ldots, f_m \in F(\mathcal{G})$ meet at one point $v_0 \in V(\mathcal{G})$, then the corresponding circles $C(f_1), C(f_2), \ldots, C(f_m)$ also have a common intersection point $z(v_0)$. An example of a circle pattern is given in Figure 5.3.

Given a circle pattern with the combinatorics of $\mathcal{G}$, we can extend the function $z$ to the vertices of the dual graph, simply setting

$$z(f) = \text{ Euclidean center of the circle } C(f), \quad f \in F(\mathcal{G}) \simeq V(\mathcal{G}^*). \tag{5.45}$$

A slightly more general extension is achieved by setting

$$z(f) = \text{ conformal center of the circle } C(f), \quad f \in F(\mathcal{G}) \simeq V(\mathcal{G}^*). \tag{5.46}$$

Recall that the conformal center of a circle $C$ is a reflection of some fixed point $P_\infty \in \widehat{\mathbb{C}}$ in $C$. In particular, if $P_\infty = \infty$, then the conformal center coincides with the Euclidean one.

Thus, after either extension, the map $z$ is defined on all of $V(\mathcal{D}) = V(\mathcal{G}) \cup V(\mathcal{G}^*)$, where $\mathcal{D}$ is the double of $\mathcal{G}$. Consider a face of the double. Clearly, its vertices $v_0, v_1, v_2, v_3$ correspond to the intersection points and the (conformal) centers of two neighboring

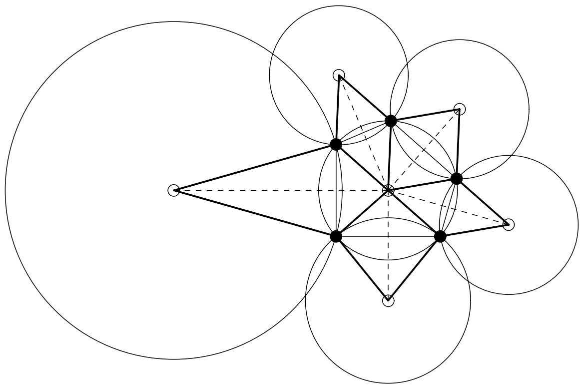

**Figure 5.3** Circle pattern.

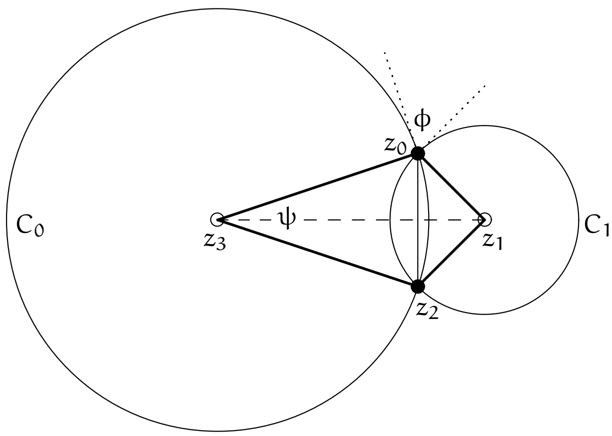


**Figure 5.4** Two intersecting circles.

circles $C_0$, $C_1$ of the pattern. Let $z_0$, $z_2$ be the intersection points, and let $z_1$, $z_3$ be the centers of the circles. See Figure 5.4.

**Lemma 5.10.** If the intersection angle of $C_0$, $C_1$ is equal to $\phi$, then

$$q(z_0, z_1, z_2, z_3) = \exp(2i\phi). \tag{5.47}$$

□

Proof. The claim is obvious for the Euclidean centers. These can be mapped onto conformal centers by an appropriate Möbius transformation which preserves the cross-ratios. ■

**Proposition 5.11.** Let $\mathcal{G}$ be a cellular decomposition of an open topological disc, and consider a circle pattern with the combinatorics of $\mathcal{G}$. Let $z : V(\mathcal{D}) \mapsto \widehat{\mathbb{C}}$ come from a circle pattern, that is, $\{z(v) : v \in V(\mathcal{G})\}$ consists of intersection points of the circles, and

$\{z(v) : v \in V(\mathcal{G}^*)\}$ consists of their conformal centers. Suppose that the intersection angles of the circles satisfy the following condition:

- For each circle of the pattern the sum of its intersection angles $\phi_k$, $k = 1, 2, \ldots, m$ with all neighboring circles of the pattern satisfies $2\sum_{k=1}^{m} \phi_k \equiv 0 \pmod{2\pi}$.

Then there exists a labelling $\alpha : E(\mathcal{D}) \mapsto \mathbb{S}^1 = \{\theta \in \mathbb{C} : |\theta| = 1\}$ depending only on the intersection angles, such that $z$ is a solution of the corresponding cross-ratio system. □

Proof. According to Corollary 5.3, the necessary and sufficient condition for the claim of the proposition to hold is given by (5.5). The second of these two formulas is exactly the condition formulated in the proposition. The first one is equivalent to the following condition: for an arbitrary common intersection point of $n$ circles the sum of their pairwise intersection angles $\phi_1, \ldots, \phi_n$ satisfies the similar condition $2\sum_{j=1}^{n} \phi_j \equiv 0 \pmod{2\pi}$. This condition is satisfied automatically for geometrical reasons (its left-hand side is equal to $2\pi$). ■

**Corollary 5.12.** Under the condition of Proposition 5.11, the restriction of the function $z$ to $V(\mathcal{G})$ (i.e., the intersection points of the circles) satisfies the additive rational Toda system on $\mathcal{G}$, while the restriction of the function $z$ to $V(\mathcal{G}^*)$ (i.e., the conformal centers of the circles) satisfies the additive rational Toda system on $\mathcal{G}^*$. □

## 6 Shifted-ratio system and multiplicative rational Toda system

We now define a system on a quad-graph $\mathcal{D}$ which can be considered as a deformation of the cross-ratio system.

Definition 6.1. Given a labelling of $E(\mathcal{D})$, the *shifted-ratio system* on $\mathcal{D}$ is the collection of equations for the function $z : V(\mathcal{D}) \mapsto \mathbb{C}$, one equation per face of $\mathcal{D}$, which read, in notations as in Figure 4.1 and $z_k = z(v_k)$

$$\frac{(z_0 - z_1 + \alpha_1)(z_2 - z_3 + \alpha_1)}{(z_1 - z_2 - \alpha_2)(z_3 - z_0 - \alpha_2)} = \frac{\alpha_1}{\alpha_2}, \tag{6.1}$$

or, equivalently (as one checks directly),

$$\frac{(z_0 - z_1 - \alpha_1)(z_2 - z_3 - \alpha_1)}{(z_1 - z_2 + \alpha_2)(z_3 - z_0 + \alpha_2)} = \frac{\alpha_1}{\alpha_2}. \tag{6.2}$$

The cross-ratio system of Section 5 may be obtained from the shifted-ratio system by the following limit: replace in (6.1) $\alpha_k$ by $\epsilon\alpha_k$ and then send $\epsilon \to 0$. The theory of the shifted-ratio system is parallel to that of the cross-ratio system, therefore our presentation here will be more tense.

**Proposition 6.2.** The shifted-ratio system on $\mathcal{D}$ admits a zero curvature representation in the sense of Section 2, with the transition matrices for $\mathfrak{e} = (v_0, v_1)$

$$L(\mathfrak{e},\lambda) = L(z_1, z_0, \alpha, \lambda) = (1-\lambda\alpha)^{-1/2}\begin{pmatrix} 1 & z_0 - z_1 + \alpha \\ \dfrac{\lambda\alpha}{z_0 - z_1 - \alpha} & \dfrac{z_0 - z_1 + \alpha}{z_0 - z_1 - \alpha}\end{pmatrix}. \tag{6.3}$$

Here $\alpha = \alpha(\mathfrak{e})$, and $z_k = z(v_k)$. □

Proof. As in the proof of Proposition 5.4, we have to verify (4.2) and (4.3). This is done again by a direct computation. In particular, one checks that the entry 11 and the $\lambda$-term of the entry 22 of the matrix equation

$$L(z_2, z_1, \alpha_2, \lambda)L(z_1, z_0, \alpha_1, \lambda) = L(z_2, z_3, \alpha_1, \lambda)L(z_3, z_0, \alpha_2, \lambda). \tag{6.4}$$

are equivalent to (6.2) and (6.1), respectively. The $\lambda^0$-term of the entry 22 reads

$$\frac{z_1 - z_2 + \alpha_2}{z_1 - z_2 - \alpha_2}\cdot\frac{z_0 - z_1 + \alpha_1}{z_0 - z_1 - \alpha_1} = \frac{z_3 - z_2 + \alpha_1}{z_3 - z_2 - \alpha_1}\cdot\frac{z_0 - z_3 + \alpha_2}{z_0 - z_3 - \alpha_2}, \tag{6.5}$$

which is a plain consequence of (6.1) and (6.2). Further, the entry 12 reads

$$\begin{aligned} &z_0 - z_1 + \alpha_1 + (z_1 - z_2 + \alpha_2)\frac{z_0 - z_1 + \alpha_1}{z_0 - z_1 - \alpha_1} \\ &\quad = z_0 - z_3 + \alpha_2 + (z_3 - z_2 + \alpha_1)\frac{z_0 - z_3 + \alpha_2}{z_0 - z_3 - \alpha_2}, \end{aligned} \tag{6.6}$$

which is equivalent to

$$\frac{z_0 - z_1 + \alpha_1}{z_0 - z_1 - \alpha_1}\cdot\frac{z_0 - z_3 - \alpha_2}{z_0 - z_3 + \alpha_2} = \frac{z_0 - z_2 + \alpha_1 - \alpha_2}{z_0 - z_2 - \alpha_1 + \alpha_2}. \tag{6.7}$$

The latter equation (after taking logarithm) is nothing but the *three-leg form* of the shifted-ratio equation. Indeed, straightforward algebraic manipulations show that it is equivalent to (6.1) (or to (6.2)). Finally, consider the entry 21 of the above matrix equation

$$\begin{aligned} &\frac{\alpha_2}{z_1 - z_2 - \alpha_2} + \frac{\alpha_1}{z_0 - z_1 - \alpha_1}\cdot\frac{z_1 - z_2 + \alpha_2}{z_1 - z_2 - \alpha_2} \\ &\quad = \frac{\alpha_1}{z_3 - z_2 - \alpha_1} + \frac{\alpha_2}{z_0 - z_3 - \alpha_2}\cdot\frac{z_3 - z_2 + \alpha_1}{z_3 - z_2 - \alpha_1}. \end{aligned} \tag{6.8}$$

With the help of (6.2) this may be rewritten as

$$\frac{\alpha_2}{z_1 - z_2 - \alpha_2}\left(1 + \frac{z_2 - z_3 - \alpha_1}{z_3 - z_0 + \alpha_2}\right) = \frac{\alpha_1}{z_3 - z_2 - \alpha_1}\left(1 + \frac{z_1 - z_2 + \alpha_2}{z_0 - z_1 - \alpha_1}\right), \tag{6.9}$$

and this, in turn, is rewritten with the help of (6.1) as (6.7). ■

The shifted-ratio system, as well as its generalization from Section 7 appeared in [19] (of course, only in the simplest possible situation: regular square lattice, labelling constant along each of the two lattice directions). They denoted this equation as *the discrete Krichever-Novikov system*. The zero curvature representation was given later in [18]. Actually, this zero curvature representation can be *derived* along the same lines as for the cross-ratio system, based on the *three-dimensional consistency*, which remains to hold true also in the present situation.

**Proposition 6.3.** The shifted-ratio system is consistent on the three-dimensional quad-graph $\mathbf{D}$. □

Proof. The proof is again a matter of straightforward computations. ■

To derive a zero curvature representation from this fact, we proceed as in Section 5: assign the label $\mu$ to all vertical edges $(v, v')$ of $\mathbf{D}$, and consider the shifted-ratio equation on the vertical face $(v_0, v_1, v_1', v_0')$

$$\frac{(z_1' - z_0' + \alpha)(z_0 - z_1 + \alpha)}{(z_0' - z_0 - \mu)(z_1 - z_1' - \mu)} = \frac{\alpha}{\mu}. \tag{6.10}$$

Denote $\lambda = \mu^{-1}$, and solve for $z_1'$ in terms of $z_0'$

$$z_1' = \widetilde{L}(z_1, z_0, \alpha, \lambda)\,[z_0'], \tag{6.11}$$

where

$$\widetilde{L}(z_1, z_0, \alpha, \lambda) = \begin{pmatrix} z_0 - z_1 + \lambda\alpha z_1 & \lambda^{-1}\alpha - \alpha^2 - \lambda\alpha z_0 z_1 \\ \lambda\alpha & z_0 - z_1 - \lambda\alpha z_0 \end{pmatrix}. \tag{6.12}$$

(Recall that the limit to the formulas of Section 5 is the following one: $\alpha \mapsto \epsilon\alpha$, $\mu \mapsto \epsilon\mu$, so that $\lambda \mapsto \epsilon^{-1}\lambda$, and then $\epsilon \to 0$.) To recover the matrices of Proposition 6.2, perform the gauge transformation

$$\Psi(v) \longmapsto C\big(z(v), \lambda\big)\Psi(v), \tag{6.13}$$

where

$$C(z,\lambda)=\begin{pmatrix}1 & z+\lambda^{-1}\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & \lambda^{-1}\\ 0 & 1\end{pmatrix}A(z). \tag{6.14}$$

An easy computation shows that

$$\begin{aligned}L(z_1,z_0,\alpha,\lambda)&=C^{-1}(z_1,\lambda)\widetilde{L}(z_1,z_0,\alpha,\lambda)C(z_0,\lambda)\\ &=\begin{pmatrix}z_0-z_1-\alpha & (z_0-z_1)^2-\alpha^2\\ \lambda\alpha & z_0-z_1+\alpha\end{pmatrix},\end{aligned} \tag{6.15}$$

which coincides with (6.3) up to a scalar factor (inessential when considering action of $L$ as a Möbius transformation).

Since the shifted-ratio equation possesses the three-leg form, there holds the analog of Proposition 4.2. The corresponding systems of the Toda type on $\mathcal{G}$, $\mathcal{G}^*$ will be called the *multiplicative rational Toda systems*. Their equations (in notations of Figures 4.2 and 4.3) read

$$\prod_{k=1}^{n}\frac{z_0-z_{2k}-\alpha_k+\alpha_{k+1}}{z_0-z_{2k}+\alpha_k-\alpha_{k+1}}=1. \tag{6.16}$$

As in Section 5, we can find the transition matrix across the edge $\mathfrak{e}_k=(v_0,v_{2k})\in E(\mathcal{G})$, that is, along the edge $\mathfrak{e}_k^*=(v_{2k-1},v_{2k+1})\in E(\mathcal{G}^*)$. This time we start with the untilded matrices (6.3)

$$\mathcal{L}_k(\lambda)=L(z_{2k+1},z_0,\alpha_{k+1},\lambda)L(z_0,z_{2k-1},\alpha_k,\lambda). \tag{6.17}$$

**Proposition 6.4.** Under the inverse gauge transformation (5.16), (5.17), and up to the scalar determinant normalizing factor, the matrix (6.17) turns into

$$\begin{aligned}\widetilde{\mathcal{L}}(z_{2k},z_0,\lambda)&=A(z_{2k+1})\mathcal{L}_k(\lambda)A^{-1}(z_{2k-1})\\ &=\begin{pmatrix}1 & (\alpha_{k+1}-\alpha_k)\dfrac{z_0+z_{2k}-\alpha_k-\alpha_{k+1}}{z_0-z_{2k}+\alpha_k-\alpha_{k+1}}\\ 0 & \dfrac{z_0-z_{2k}-\alpha_k+\alpha_{k+1}}{z_0-z_{2k}+\alpha_k-\alpha_{k+1}}\end{pmatrix}\\ &\quad+\frac{\lambda}{z_0-z_{2k}+\alpha_k-\alpha_{k+1}}\begin{pmatrix}\alpha_{k+1}z_{2k}-\alpha_k z_0 & (\alpha_k-\alpha_{k+1})(z_0z_{2k}-\alpha_k\alpha_{k+1})\\ \alpha_{k+1}-\alpha_k & \alpha_k z_{2k}-\alpha_{k+1}z_0\end{pmatrix}.\end{aligned} \tag{6.18}$$

□

Proof. The calculation of the $\lambda^0$-component of the matrix (6.18) is immediate, one has only to use the three-leg equation (6.7), that is,

$$\frac{z_0 - z_{2k+1} + \alpha_{k+1}}{z_0 - z_{2k+1} - \alpha_{k+1}} \cdot \frac{z_0 - z_{2k-1} - \alpha_k}{z_0 - z_{2k-1} + \alpha_k} = \frac{z_0 - z_{2k} - \alpha_k + \alpha_{k+1}}{z_0 - z_{2k} + \alpha_k - \alpha_{k+1}}. \tag{6.19}$$

The calculation of the $\lambda$-component is also straightforward but somewhat more involved. The main ingredient of this calculation is obtaining the following expression for the entry 21 (which is used actually in all four entries):

$$\frac{\alpha_k}{z_{2k-1} - z_0 - \alpha_k} \cdot \frac{z_0 - z_{2k+1} + \alpha_{k+1}}{z_0 - z_{2k+1} - \alpha_{k+1}} + \frac{\alpha_{k+1}}{z_0 - z_{2k+1} - \alpha_{k+1}} = \frac{\alpha_{k+1} - \alpha_k}{z_0 - z_{2k} + \alpha_k - \alpha_{k+1}}. \tag{6.20}$$

This is done as follows: one starts with the shifted-ratio equation

$$\alpha_k \frac{z_{2k} - z_{2k-1} - \alpha_{k+1}}{z_{2k-1} - z_0 + \alpha_k} - \alpha_{k+1} \frac{z_{2k+1} - z_{2k} + \alpha_k}{z_0 - z_{2k+1} - \alpha_{k+1}} = 0. \tag{6.21}$$

This is rewritten as

$$\alpha_k \frac{z_0 - z_{2k} - \alpha_k + \alpha_{k+1}}{z_{2k-1} - z_0 + \alpha_k} + \alpha_{k+1} \frac{z_0 - z_{2k} + \alpha_k - \alpha_{k+1}}{z_0 - z_{2k+1} - \alpha_{k+1}} = \alpha_{k+1} - \alpha_k. \tag{6.22}$$

But the latter equation is equivalent to (6.20) due to the three-leg equation (6.19). ■

The transition matrix for the multiplicative rational Toda system found in [3] coincides with (6.18) up to a similarity transformation with a constant matrix $\left(\begin{smallmatrix} 1 & \lambda^{-1} \\ 0 & 1 \end{smallmatrix}\right)$.

Finally, we briefly discuss the dual to the shifted-ratio system. It appears, if one notices that (6.5) yields the existence of the function $Z : V(\mathcal{D}) \mapsto \mathbb{C}$ such that on each face of $\mathcal{D}$ there holds

$$\begin{aligned} Z_1 - Z_0 &= \frac{1}{2}\log\frac{z_0 - z_1 + \alpha_1}{z_0 - z_1 - \alpha_1}, \qquad & Z_2 - Z_1 &= \frac{1}{2}\log\frac{z_1 - z_2 + \alpha_2}{z_1 - z_2 - \alpha_2}, \\ Z_3 - Z_2 &= \frac{1}{2}\log\frac{z_2 - z_3 + \alpha_1}{z_2 - z_3 - \alpha_1}, \qquad & Z_0 - Z_3 &= \frac{1}{2}\log\frac{z_3 - z_0 + \alpha_2}{z_3 - z_0 - \alpha_2}. \end{aligned} \tag{6.23}$$

**Proposition 6.5.** For any solution $z : V(\mathcal{D}) \mapsto \mathbb{C}$ of the shifted-ratio system on $\mathcal{D}$, the formula

$$Z(v_2) - Z(v_1) = \frac{1}{2}\log\frac{z(v_2) - z(v_1) + \alpha(\mathfrak{e})}{z(v_2) - z(v_1) - \alpha(\mathfrak{e})} \tag{6.24}$$

(for any $\mathfrak{e} = (v_1, v_2) \in E(\mathcal{D})$) correctly defines a function $Z : V(\mathcal{D}) \mapsto \mathbb{C}$ which is a solution of the hyperbolic cross-ratio system

$$\frac{\sinh(Z_0 - Z_1)\sinh(Z_2 - Z_3)}{\sinh(Z_1 - Z_2)\sinh(Z_3 - Z_0)} = \frac{\alpha_1}{\alpha_2}. \tag{6.25}$$

□

Remark 6.6. The hyperbolic cross-ratio system is equivalent to the usual cross-ratio system

$$\frac{(X_0 - X_1)(X_2 - X_3)}{(X_1 - X_2)(X_3 - X_0)} = \frac{\alpha_1}{\alpha_2} \tag{6.26}$$

upon the change of variables $X_k = \exp(2Z_k)$. Restrictions of solutions of the hyperbolic cross-ratio system to $V(\mathcal{G})$, $V(\mathcal{G}^*)$ satisfy the *additive hyperbolic Toda systems*, whose equations are

$$\sum_{k=1}^{n} (\alpha_k - \alpha_{k+1}) \coth(z_0 - z_{2k}) = 0. \tag{6.27}$$

This system is also equivalent to the additive rational Toda system

$$\sum_{k=1}^{n} \frac{\alpha_k - \alpha_{k+1}}{x_0 - x_{2k}} = 0 \tag{6.28}$$

under the change of variables $x_k = \exp(2z_k)$.

## 7 Hyperbolic shifted-ratio system and multiplicative hyperbolic Toda system

Finally, we turn to the analog of the cross-ratio system which is the most general one.

Definition 7.1. Given a labelling of $E(\mathcal{D})$, the *hyperbolic shifted-ratio system* on $\mathcal{D}$ is the collection of equations for the function $z : V(\mathcal{D}) \mapsto \mathbb{C}$, one equation per face of $\mathcal{D}$, which read, in notations as in Figure 4.1 and $z_k = z(v_k)$

$$\frac{\sinh(z_0 - z_1 + \alpha_1)\sinh(z_2 - z_3 + \alpha_1)}{\sinh(z_1 - z_2 - \alpha_2)\sinh(z_3 - z_0 - \alpha_2)} = \frac{\sinh(2\alpha_1)}{\sinh(2\alpha_2)}, \tag{7.1}$$

or, equivalently,

$$\frac{\sinh(z_0 - z_1 - \alpha_1)\sinh(z_2 - z_3 - \alpha_1)}{\sinh(z_1 - z_2 + \alpha_2)\sinh(z_3 - z_0 + \alpha_2)} = \frac{\sinh(2\alpha_1)}{\sinh(2\alpha_2)}. \tag{7.2}$$

The hyperbolic shifted-ratio system may be considered as a deformation of the rational one, the corresponding limit from the former to the latter is achieved by setting replace $z_k \mapsto \epsilon z_k$ and $\alpha_k \to \epsilon\alpha_k$ and then sending $\epsilon \to 0$.

We will systematically use in this section the notations $x_k = \exp(2z_k)$, $c_k = \exp(2\alpha_k)$. In these notations (7.1) and (7.2) take the following rational form:

$$\frac{(c_1x_0 - x_1)(c_1x_2 - x_3)}{(x_1 - c_2x_2)(x_3 - c_2x_0)} = \frac{1-c_1^2}{1-c_2^2}, \qquad \frac{(x_0 - c_1x_1)(x_2 - c_1x_3)}{(c_2x_1 - x_2)(c_2x_3 - x_0)} = \frac{1-c_1^2}{1-c_2^2}. \tag{7.3}$$

**Proposition 7.2.** The hyperbolic shifted-ratio system on $\mathcal{D}$ admits a zero curvature representation in the sense of Section 2, with the transition matrices for $\mathfrak{e} = (v_0, v_1)$

$$L(\mathfrak{e}, \lambda) = L(x_1, x_0, c, \lambda) = \big(1 - \lambda(1-c^2)\big)^{-1/2} \begin{pmatrix} 1 & x_0 - cx_1 \\ \dfrac{\lambda(1-c^2)}{cx_0 - x_1} & \dfrac{x_0 - cx_1}{cx_0 - x_1} \end{pmatrix}. \tag{7.4}$$

Here $c = c(\mathfrak{e})$, and $x_k = x(v_k)$. □

Proof. The proof is absolutely parallel to the proof of Proposition 6.2. ■

Again, this statement may be derived from the *three-dimensional consistency* of the shifted hyperbolic ratio system.

**Proposition 7.3.** The hyperbolic shifted-ratio system is consistent on the three-dimensional quad-graph $\mathbf{D}$. □

Proof. The proof is a direct computation. ■

The derivation of a zero curvature representation from this fact is not different from the previous two cases: we assign the label $\mu$ to all vertical edges $(v, v')$ of $\mathbf{D}$, set $\nu = \exp(2\mu)$, and consider the hyperbolic shifted-ratio equation on the vertical face $(v_0, v_1, v_1', v_0')$

$$\frac{(cx_0 - x_1)(cx_1' - x_0')}{(x_1 - \nu x_1')(x_0' - \nu x_0)} = \frac{1-c^2}{1-\nu^2}. \tag{7.5}$$

Solving for $x_1'$ in terms of $x_0'$, we find

$$x_1' = \widetilde{L}(x_1, x_0, c, \nu)\,[x_0'], \tag{7.6}$$

where

$$\widetilde{L}(x_1, x_0, c, \nu) = \begin{pmatrix} cx_0 - x_1 + \dfrac{1-c^2}{1-\nu^2}x_1 & -\dfrac{\nu(1-c^2)}{1-\nu^2}x_0x_1 \\ \dfrac{\nu(1-c^2)}{1-\nu^2} & x_0 - cx_1 - \dfrac{1-c^2}{1-\nu^2}x_0 \end{pmatrix}. \tag{7.7}$$

Perform a gauge transformation

$$\Psi(\nu) \longmapsto D\big(x(\nu), \nu\big)\Psi(\nu), \tag{7.8}$$

where

$$D(x, \nu) = \begin{pmatrix} \nu^{-1} & \nu^{-1}x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \nu^{-1} & 0 \\ 0 & 1 \end{pmatrix} A(x). \tag{7.9}$$

It leads to the following form of the transition matrices:

$$\begin{aligned} L(x_1, x_0, c, \lambda) &= D^{-1}(x_1, \nu)\widetilde{L}(x_1, x_0, c, \nu)D(x_0, \nu) \\ &= \begin{pmatrix} cx_0 - x_1 & (cx_0 - x_1)(x_0 - cx_1) \\ \lambda(1-c^2) & x_0 - cx_1 \end{pmatrix}, \end{aligned} \tag{7.10}$$

where $\lambda = (1-\nu^2)^{-1}$. This coincides with (7.4) up to an inessential scalar factor.

Next, we turn to the derivation of the Toda type system related to the hyperbolic shifted-ratio system. As usual, the *three-leg form* of the latter system is fundamentally important in this respect. It is not difficult to check that (7.1) is equivalent to the following equation:

$$\frac{\sinh(z_0 - z_1 + \alpha_1)}{\sinh(z_0 - z_1 - \alpha_1)} \cdot \frac{\sinh(z_0 - z_3 - \alpha_2)}{\sinh(z_0 - z_3 + \alpha_2)} = \frac{\sinh(z_0 - z_2 + \alpha_1 - \alpha_2)}{\sinh(z_0 - z_2 - \alpha_1 + \alpha_2)}. \tag{7.11}$$

The rational form of the three-leg equation reads

$$\frac{(c_1x_0 - x_1)}{(x_0 - c_1x_1)} \cdot \frac{(x_0 - c_2x_3)}{(c_2x_0 - x_3)} = \frac{(c_1x_0 - c_2x_2)}{(c_2x_0 - c_1x_2)}. \tag{7.12}$$

The three-leg form of the equations immediately yields the analog of Proposition 4.2. The corresponding Toda type systems on $\mathcal{G}$, respectively $\mathcal{G}^*$, read (in notations of Figures 4.2 and 4.3)

$$\prod_{k=1}^{n} \frac{\sinh(z_0 - z_{2k} + \alpha_k - \alpha_{k+1})}{\sinh(z_0 - z_{2k} - \alpha_k + \alpha_{k+1})} = 1, \tag{7.13}$$

or in the rational form

$$\prod_{k=1}^{n} \frac{c_k x_0 - c_{k+1} x_{2k}}{c_{k+1} x_0 - c_k x_{2k}} = 1. \tag{7.14}$$

They will be called *multiplicative hyperbolic Toda systems*.

Turning to the zero curvature formulation of the multiplicative hyperbolic Toda system on $\mathcal{G}$, we find, as in Section 6, the transition matrix across the edge $\mathfrak{e}_k = (v_0, v_{2k}) \in E(\mathcal{G})$, that is, along the edge $\mathfrak{e}_k^* = (v_{2k-1}, v_{2k+1}) \in E(\mathcal{G}^*)$:

$$\mathcal{L}_k(\lambda) = L\big(x_{2k+1}, x_0, c_{k+1}, \lambda\big) L\big(x_0, x_{2k-1}, c_k, \lambda\big). \tag{7.15}$$

**Proposition 7.4.** Under the inverse gauge transformation (5.16) and (5.17), and up to the scalar determinant normalizing factor, the matrix (7.15) turns into

$$\begin{aligned}\widetilde{\mathcal{L}}\big(x_{2k}, x_0, \lambda\big) &= A\big(x_{2k+1}\big)\mathcal{L}_k(\lambda) A^{-1}\big(x_{2k-1}\big)\\ &= \begin{pmatrix} (1-\lambda) + \lambda c_k c_{k+1} \dfrac{c_k x_0 - c_{k+1} x_{2k}}{c_{k+1} x_0 - c_k x_{2k}} & (1-\lambda) \dfrac{(c_{k+1}^2 - c_k^2) x_0 x_{2k}}{c_{k+1} x_0 - c_k x_{2k}} \\ \lambda \dfrac{c_{k+1}^2 - c_k^2}{c_{k+1} x_0 - c_k x_{2k}} & (1-\lambda) \dfrac{c_k x_0 - c_{k+1} x_{2k}}{c_{k+1} x_0 - c_k x_{2k}} + \lambda c_k c_{k+1} \end{pmatrix}.\end{aligned} \tag{7.16}$$

□

Proof. By a direct calculation. The key formula in this calculation is

$$\frac{1 - c_k^2}{x_0 - c_k x_{2k-1}} \cdot \frac{x_0 - c_{k+1} x_{2k+1}}{c_{k+1} x_0 - x_{2k+1}} - \frac{1 - c_{k+1}^2}{c_{k+1} x_0 - x_{2k+1}} = \frac{c_{k+1}^2 - c_k^2}{c_{k+1} x_0 - c_k x_{2k}} \tag{7.17}$$

which is proved similarly to (6.20), based on the three-leg formula (7.12). ■

Again, the transition matrices for the multiplicative hyperbolic Toda systems essentially equivalent to (7.16) were found in [3].

The multiplicative hyperbolic Toda lattice can be given a geometrical interpretation. For this aim consider, as at the end of Section 5.6, a circle pattern with the combinatorics of a cellular decomposition $\mathcal{G}$, and extend it by the Euclidean centers of the circles. So, the Euclidean radii of the circles $r : V(\mathcal{G}^*) \mapsto \mathbb{R}_+$ are defined. For any two neighboring circles $C_0, C_1$ of the pattern we use the notations as in Figure 5.4, including the notation $\psi$ for the angle $z_0 z_3 z_2$. It is not difficult to express the angle $\psi$ in terms of the radii $r_0, r_1$ of the circles $C_0, C_1$ and their intersection angle $\phi$:

$$\exp(i\psi) = \frac{r_0 + r_1 \exp(-i\phi)}{r_0 + r_1 \exp(i\phi)}. \tag{7.18}$$

For any circle $C_0$ of the pattern, denote by $C_k$, $k = 1, 2, \ldots, m$ its $m$ neighboring circles. Let $r_k$ be the radius of the circle $C_k$, denote by $\psi_k$, $k = 1, 2, \ldots, m$ the angles $\psi$ corresponding to all pairs $(C_0, C_k)$, and denote by $\phi_k$, $k = 1, 2, \ldots, m$ the intersection angles of $C_0$ with $C_k$. Obviously, for geometrical reasons we have

$$\prod_{k=1}^{m} \exp\left(i\psi_k\right) = 1, \tag{7.19}$$

therefore

$$\prod_{k=1}^{m} \frac{r_0 + r_k \exp(-i\phi_k)}{r_0 + r_k \exp(i\phi_k)} = 1. \tag{7.20}$$

Now from Lemmas 5.2 and 5.10 we have the following proposition.

**Proposition 7.5.** Let $\mathcal{G}$ be a cellular decomposition of an open topological disc, and consider a circle pattern with the combinatorics of $\mathcal{G}$. Suppose that the intersection angles of the circles satisfy the condition from Proposition 5.11, namely,

- For each circle of the pattern the sum of its intersection angles $\phi_k$, $k = 1, 2, \ldots, m$ with all neighboring circles of the pattern satisfies $2\sum_{k=1}^{m} \phi_k \equiv 0 \pmod{2\pi}$.

Then there exists a labelling $c : E(\mathcal{D}) \mapsto \mathbb{S}^1 = \{\theta \in \mathbb{C} : |\theta| = 1\}$ depending only on the intersection angles, such that the Euclidean radii of the circles $r : V(\mathcal{G}^*) \mapsto \mathbb{R}_+$ satisfy the corresponding multiplicative hyperbolic Toda system on $\mathcal{G}^*$

$$\prod_{k=1}^{m} \frac{c_k r_0 - c_{k+1} r_k}{c_{k+1} r_0 - c_k r_k} = 1. \tag{7.21}$$

□

Proof. Intersection angles are naturally assigned to the edges from $E(\mathcal{G})$. So, in Figure 5.4 $\phi = \phi(v_0, v_2)$. Comparing (7.20) with (7.21), we see that the problem in question is finding a labelling $c : E(\mathcal{D}) \mapsto \mathbb{S}^1$ such that, in the notations of Figure 4.1,

$$-\exp\left(i\phi(v_0, v_2)\right) = \exp\left(i\pi - i\phi(v_0, v_2)\right) = \frac{c_1}{c_2}. \tag{7.22}$$

The end of the proof is exactly the same as that of Proposition 5.11. ■

Finally, we discuss the duality for the hyperbolic shifted-ratio system. As a basis for the duality there serves the formula

$$\frac{\sinh(z_2 - z_1 + \alpha_2)}{\sinh(z_2 - z_1 - \alpha_2)} \cdot \frac{\sinh(z_1 - z_0 + \alpha_1)}{\sinh(z_1 - z_0 - \alpha_1)} = \frac{\sinh(z_2 - z_3 + \alpha_1)}{\sinh(z_2 - z_3 - \alpha_1)} \cdot \frac{\sinh(z_3 - z_0 + \alpha_2)}{\sinh(z_3 - z_0 - \alpha_2)}, \tag{7.23}$$

which is a simple consequence of (7.1) and (7.2). As usual, the latter equation allows us to introduce the dual system:

**Proposition 7.6.** For any solution $z : V(\mathcal{D}) \mapsto \mathbb{C}$ of the hyperbolic shifted-ratio system on $\mathcal{D}$, the formula

$$Z(v_2) - Z(v_1) = \frac{1}{2} \log \frac{\sinh\left(z(v_1) - z(v_2) + \alpha(\mathfrak{e})\right)}{\sinh\left(z(v_1) - z(v_2) - \alpha(\mathfrak{e})\right)} \tag{7.24}$$

(for any $\mathfrak{e} = (v_1, v_2) \in E(\mathcal{D})$) correctly defines a function $Z : V(\mathcal{D}) \mapsto \mathbb{C}$ which is another solution of the same hyperbolic shifted-ratio system. □

In the notation $X(v) = \exp(2Z(v))$, the latter formula reads

$$\frac{X(v_2)}{X(v_1)} = \frac{c(\mathfrak{e})x(v_1) - x(v_2)}{x(v_1) - c(\mathfrak{e})x(v_2)}. \tag{7.25}$$

In particular, on each face of $\mathcal{D}$ we have

$$\frac{X_1}{X_0} = \frac{c_1x_0 - x_1}{x_0 - c_1x_1}, \quad \frac{X_2}{X_1} = \frac{c_2x_1 - x_2}{x_1 - c_2x_2}, \quad \frac{X_3}{X_2} = \frac{c_1x_2 - x_3}{x_2 - c_1x_3}, \quad \frac{X_0}{X_3} = \frac{c_2x_3 - x_0}{x_3 - c_2x_0}. \tag{7.26}$$

The first and the fourth of these equations together with formula (7.12) yield

$$\frac{X_1}{X_3} = \frac{c_1x_0 - c_2x_2}{c_2x_0 - c_1x_2}. \tag{7.27}$$

This shows that, like for the additive rational Toda systems, a solution of the multiplicative hyperbolic Toda system on $\mathcal{G}$ determines the dual solution of the multiplicative hyperbolic Toda system on $\mathcal{G}^*$ almost uniquely (up to a constant factor, in terms of the variables $x, X$, or up to an additive constant, in terms of the variables $z, Z$).

One can also consider the function $w : V(\mathcal{D}) \mapsto \widehat{\mathbb{C}}$ which coincides with $x$ on $V(\mathcal{G})$ and with $X$ on $V(\mathcal{G}^*)$. Then formula (7.27) defines a system of equations on the diamond, reading on the face $(v_0, v_1, v_2, v_3)$ as

$$\frac{w_1}{w_3} = \frac{c_1w_0 - c_2w_2}{c_2w_0 - c_1w_2}. \tag{7.28}$$

This is a well-defined equation on the quad-graph, which in the simplest possible situation (square lattice, constant labelling) was introduced in [12] and therefore will be called the *Hirota equation on* $\mathcal{D}$. It appeared also in [19] under the name of *discrete MKdV*. Actually, when considered on a single face, this equation represents nothing but the famous result of Bianchi on the permutability of Bäcklund transformations for the sine-Gordon equation. See a discussion of the role of this equation (on a square lattice) for the geometrical and analytic discretization of surfaces of a constant negative Gaussian curvature in [8].

Equation (7.28) is in the three-leg form, and restrictions of its solutions to $V(\mathcal{G})$ and to $V(\mathcal{G}^*)$ are solutions of the multiplicative hyperbolic Toda systems on the corresponding graphs. Also this equation possesses the property of the three-dimensional consistency, which yields integrability: assign the label $\mu$ to all vertical edges $(v, v')$ of $\mathbf{D}$, and consider the Hirota equation on the vertical face $(v_0, v_1, v_1', v_0')$

$$\frac{w_0'}{w_1} = \frac{c w_1' - \mu w_0}{\mu w_1' - c w_0}. \tag{7.29}$$

Solving for $w_1'$ in terms of $w_0'$, we find

$$w_1' = \frac{c w_0 w_0' - \mu w_0 w_1}{\mu w_0' - c w_1} = \begin{pmatrix} c w_0 & -\mu w_0 w_1 \\ \mu & -c w_1 \end{pmatrix} [w_0'] = L(w_1, w_0, c, \mu)[w_0'], \tag{7.30}$$

where (after normalizing the determinant)

$$L(w_1, w_0, c, \mu) = (\mu^2 - c^2)^{-1/2} \begin{pmatrix} c\left(\dfrac{w_0}{w_1}\right)^{1/2} & -\mu (w_0 w_1)^{1/2} \\ \mu (w_0 w_1)^{-1/2} & -c\left(\dfrac{w_1}{w_0}\right)^{1/2} \end{pmatrix}. \tag{7.31}$$

This is the well-known transition matrix for the Hirota equation [10, 12].

We remark also that there holds the diagram like that of Figure 5.2, with the replacement of the cross-ratio system by the hyperbolic shifted-ratio system, of the additive rational Toda systems by the multiplicative hyperbolic Toda systems, and of the discrete KdV equation by the Hirota equation.

## 8 Conclusions

The present paper is devoted to the study of discrete integrable systems on arbitrary graphs. We considered several fundamental examples supporting our viewpoint that integrable systems on quad-graphs play here a prominent role. In particular, the recently introduced systems of the Toda type or arbitrary graphs find their natural and simple explanation on this way. We indicate several points that were left open in this paper, and some directions for the further progress.

(1) We restricted ourselves here only to *local problems* of the theory of integrable systems. Therefore, we considered only systems on graphs having the combinatorics of

a cellular decomposition of a topological disc. We intend to address global problems for integrable systems on graphs embedded in arbitrary Riemann surfaces in our future work. This has important applications to the discrete differential geometry and the discrete complex analysis.

(2) Although we have unveiled several mysteries of integrable systems here (in particular, we mention the derivation of zero curvature representation from the three-dimensional consistency of discrete systems), there remain several phenomena which still wait for their explanation. For example, a deeper reasons for the existence of the *three-leg form* for all our equations are still unclear. Also the role of the duality and its interrelations with integrability calls for a better understanding.

(3) In all our examples the basic equation on a face of the quad-graph may be written as (cf. Figure 4.1)

$$\Phi(z_0, z_1, z_2, z_3, \alpha_1, \alpha_2) = 0, \tag{8.1}$$

where the function $\Phi$ is a polynomial (of degree $\leq 2$), linear in each of the arguments $z_k$. As a result, this equation may be solved uniquely for each of $z_k$, the result being a linear-fractional (Möbius) transformation with respect to any of the remaining variables. As we have seen, the three-dimensional permutability of Möbius transformations of this kind lies in the heart of the integrability. A further important class of integrable four-point equations, with functions $\Phi$ being polynomials of degree $\leq 4$, linear in each of the arguments, was found in [1] and identified as a superposition formula for Bäcklund transformations of the Krichever-Novikov equation. Thus, these integrable systems are related to an elliptic curve, where the spectral parameter of the Krichever-Novikov equation naturally lives [15]. In a subsequent publication [4] it will be shown that the corresponding equations on quad-graphs still possess the three-leg form (after parametrizing variables by elliptic $\wp$-functions, $z_k = \wp(u_k)$)

$$\begin{aligned}\frac{\sigma(u_1 - u_0 + \alpha_1)\sigma(u_1 + u_0 - \alpha_1)}{\sigma(u_1 - u_0 - \alpha_1)\sigma(u_1 + u_0 + \alpha_1)} \cdot \frac{\sigma(u_3 - u_0 - \alpha_2)\sigma(u_3 + u_0 + \alpha_2)}{\sigma(u_3 - u_0 + \alpha_2)\sigma(u_3 + u_0 - \alpha_2)}&\\ = \frac{\sigma(u_2 - u_0 + \alpha_1 - \alpha_2)\sigma(u_2 + u_0 - \alpha_1 + \alpha_2)}{\sigma(u_2 - u_0 - \alpha_1 + \alpha_2)\sigma(u_2 + u_0 + \alpha_1 - \alpha_2)}&.\end{aligned} \tag{8.2}$$

As one of the consequences, we will find the *elliptic Toda system* on an arbitrary cellular decomposition $\mathcal{G}$. In the notations of Figure 4.3 it reads

$$\prod_{k=1}^{n} \frac{\sigma(u_{2k} - u_0 + \alpha_k - \alpha_{k+1})\sigma(u_{2k} + u_0 - \alpha_k + \alpha_{k+1})}{\sigma(u_{2k} - u_0 - \alpha_k + \alpha_{k+1})\sigma(u_{2k} + u_0 + \alpha_k - \alpha_{k+1})} = 1. \tag{8.3}$$

We remark that in the degenerate case when $\sigma$-functions are replaced by sinh, the latter equation is again related to circle patterns with the combinatorics of $\mathcal{G}$: the radii of the circles calculated in the non-Euclidean (hyperbolic) geometry always deliver a solution to such a system on $\mathcal{G}^*$.

As the simplest particular case of this system (regular square lattice, constant labelling in each of the two directions), we will find [4]

$$\begin{aligned}&\frac{\sigma(\widetilde{x}_j - x_j + ha)}{\sigma(\widetilde{x}_j - x_j - ha)} \cdot \frac{\sigma(x_j + \widetilde{x}_j - ha)}{\sigma(x_j + \widetilde{x}_j + ha)} \cdot \frac{\sigma(x_j - \underset{\sim}{x}_j - ha)}{\sigma(x_j - \underset{\sim}{x}_j + ha)} \cdot \frac{\sigma(x_j + \underset{\sim}{x}_j - ha)}{\sigma(x_j + \underset{\sim}{x}_j + ha)} \\ &\quad \cdot \frac{\sigma(x_j - x_{j-1} + ha)}{\sigma(x_j - x_{j-1} - ha)} \cdot \frac{\sigma(x_j + x_{j-1} + ha)}{\sigma(x_j + x_{j-1} - ha)} \cdot \frac{\sigma(x_{j+1} - x_j - ha)}{\sigma(x_{j+1} - x_j + ha)} \\ &\quad \cdot \frac{\sigma(x_{j+1} + x_j + ha)}{\sigma(x_{j+1} + x_j - ha)} = 1.\end{aligned} \tag{8.4}$$

Here the index $j$ corresponds to one lattice direction of the regular square lattice, and for the second lattice direction we use an index-free notation, so that the variables $x_j$ correspond to one and the same value of the second lattice coordinate $t \in h\mathbb{Z}$, the tilded variables $\widetilde{x}_j$ correspond to the value $t + h$, and subtilded variables $\underset{\sim}{x}_j$ correspond to the value $t - h$. In this interpretation, the above equation serves as a time discretization of the so-called *elliptic Toda lattice* [13],

$$\begin{aligned}\ddot{x}_j = \left(\dot{x}_j^2 - a^2\right)\bigl(\zeta(x_j - x_{j-1}) + \zeta(x_j + x_{j-1}) \\ - \zeta(x_{j+1} - x_j) + \zeta(x_{j+1} + x_j) - 2\zeta(2x_j)\bigr).\end{aligned} \tag{8.5}$$

Similarly, if the cellular decomposition $\mathcal{G}$ is generated by the regular triangular lattice, the resulting discrete system may be considered as a time discretization of a novel integrable lattice, which is a sort of a relativistic generalization of the elliptic Toda lattice (cf. [2]). So, we will see that the root of integrability of all these discrete and continuous systems still lies in the three-dimensional permutability of certain Möbius transformations.

(4) In the light of the above discussion, there appears a meaningful problem of finding and classifying *all* equations $\Phi(z_0, z_1, z_2, z_3, \alpha_1, \alpha_2) = 0$ with analogous properties, that is, uniquely solvable with respect to each argument, and leading to transformations with the three-dimensional compatibility. We plan to address this problem in our future research.

(5) Finally, a very intriguing problem is a generalization of the results described above to the case of spectral parameter belonging to higher genus algebraic curves, for example, finding the discrete counterparts of the Hitchin systems in Krichever's formulation [14].

**Note added in proof.** The method of derivation of the zero curvature representation from the three-dimensional consistency, described in Section 5.3, was discovered simultaneously and independently in: F. W. Nijhoff. Lax pair for the Adler (lattice Krichever-Novikov) System. Preprint nlin.SI/0110027 at http://arXiv.org.

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## References

[1] V. E. Adler, *Bäcklund transformation for the Krichever-Novikov equation*, Internat. Math. Res. Notices **1998** (1998), no. 1, 1–4.

[2] ———, *Legendre transforms on a triangular lattice*, Funct. Anal. Appl. **34** (2000), no. 1, 1–19, translated from Funkts. Anal. Prilozh. **34** (2000), no. 1, 1–11.

[3] ———, *Discrete equations on planar graphs*, J. Phys. A: Math. Gen. **34** (2001), no. 48, 10453–10460.

[4] V. E. Adler and Yu. B. Suris, *Elliptic Toda systems on planar graphs*, in preparation.

[5] A. Bobenko and T. Hoffmann, *Hexagonal circle patterns and integrable systems. Patterns with constant angles*, preprint, http://front.math.ucdavis.edu/math.CV/0109018, 2001.

[6] A. Bobenko, T. Hoffmann, and Yu. B. Suris, *Hexagonal circle patterns and integrable systems: patterns with the multi-ratio property and Lax equations on the regular triangular lattice*, Internat. Math. Res. Notices **2002** (2002), no. 3, 111–164.

[7] A. Bobenko and U. Pinkall, *Discrete isothermic surfaces*, J. Reine Angew. Math. **475** (1996), 187–208.

[8] ———, *Discrete surfaces with constant negative Gaussian curvature and the Hirota equation*, J. Differential Geom. **43** (1996), no. 3, 527–611.

[9] ———, *Discretization of surfaces and integrable systems*, Discrete Integrable Geometry and Physics, Oxf. Lect. Ser. Math. Appl., vol. 16, Oxford University Press, New York, 1999, pp. 3–58.

[10] L. Faddeev and A. Yu. Volkov, *Hirota equation as an example of an integrable symplectic map*, Lett. Math. Phys. **32** (1994), no. 2, 125–135.

[11] R. Hirota, *Nonlinear partial difference equations. I. A difference analogue of the Korteweg-de Vries equation*, J. Phys. Soc. Japan **43** (1977), no. 4, 1424–1433.

[12] ———, *Nonlinear partial difference equations. III. Discrete sine-Gordon equation*, J. Phys. Soc. Japan **43** (1977), no. 6, 2079–2086.

[13] I. Krichever, *Elliptic analog of the Toda lattice*, Internat. Math. Res. Notices **2000** (2000), no. 8, 383–412.

[14] ———, *Vector bundles and Lax equations on algebraic curves*, preprint, http://aps.arxiv.org/abs/hep-th/0108110, 2001.

[15] I. M. Krichever and S. P. Novikov, *Holomorphic bundles over algebraic curves, and nonlinear equations*, Russ. Math. Surv. **35** (1980), no. 6, 53–79, translated from Uspekhi Mat. Nauk **35** (1980), no. 6, 47–68.

[16] ———, *Trivalent graphs and solitons*, Russ. Math. Surv. **54** (1999), no. 6, 1248–1249, translated from Usp. Mat. Nauk **54** (1999), no. 6, 149–150.

[17] Ch. Mercat, *Discrete Riemann surfaces and the Ising model*, Comm. Math. Phys. **218** (2001), no. 1, 177–216.

[18] F. W. Nijhoff and H. W. Capel, *The discrete Korteweg-de Vries equation*, Acta Appl. Math. **39** (1995), no. 1-3, 133–158.

[19] F. W. Nijhoff, G. R. W. Quispel, and H. W. Capel, *Direct linearization of nonlinear difference-difference equations*, Phys. Lett. A **97** (1983), no. 4, 125–128.

[20] S. P. Novikov and I. A. Dynnikov, *Discrete spectral symmetries of small-dimensional differential operators and difference operators on regular lattices and two-dimensional manifolds*, Russ. Math. Surv. **52** (1997), no. 5, 1057–1116, translated from Usp. Mat. Nauk **52** (1997), no. 5, 175–234.

[21] S. P. Novikov and A. S. Shvarts, *Discrete Lagrangian systems on graphs. Symplecto-topological properties*, Russ. Math. Surv. **54** (1999), no. 1, 258–259, translated from Usp. Mat. Nauk **54** (1999), no. 1, 257–258.

[22] A. A. Oblomkov and A. V. Penskoi, *Two-dimensional algebro-geometric difference operators*, J. Phys. A **33** (2000), no. 50, 9255–9264.

[23] Yu. B. Suris, *New integrable systems related to the relativistic Toda lattice*, J. Phys. A **30** (1997), no. 5, 1745–1761.

[24] P. Wynn, *On a device for computing the $e_m(S_n)$ tranformation*, Math. Tables Aids Comput. **10** (1956), 91–96.



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