Chapter 3
Integral Boundary Points of Convex Polyhedra

Alan J. Hoffman and Joseph B. Kruskal

Introduction by *Alan J. Hoffman* and *Joseph B. Kruskal*

Here is the story of how this paper was written.

(a) Independently, Alan and Joe discovered this easy theorem: if the “right hand side” consists of integers, and if the matrix is “totally unimodular”, then the vertices of the polyhedron defined by the linear inequalities will all be integral. This is easy to prove and useful. As far as we know, this is the only part of our theorem that anyone has ever used.

(b) But this was so easy, we each wanted to generalize it. Independently we worked hard to understand the cases where there are no vertices, i.e., the lowest dimensional faces of the polyhedron are 1-dimensional or higher. This was hard to write and hard to read.

(c) At this point, Alan benefitted greatly from simplifications suggested by David Gale and anonymous referees, but it was still not so simple.

(d) Independently, we both wondered: If the vertices were integral for every integral right hand side, did this mean the matrix was totally unimodular? This is discussed in our paper, and also in References [1] and [2], especially the latter.

Harold Kuhn and Al Tucker saw drafts from both of us and realized that we were working on the same problem, so they suggested that we start working together, and that Alan should send his latest draft to Joe to take the next step. For Joe, this turned out to be the most exciting collaboration he had ever experienced; and he still feels that way today.

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We must have met casually before the collaboration, but we never saw each other during it nor for a long time afterwards. Joe knew nothing about Alan's work. However Alan, who was working for the Navy, knew something about Joe's work through a Navy report on a real operations research project on which Joe and Bob Aumann had gotten impressive results. (Many years later, in 2005, Bob won the Nobel Prize for Economics.)

As it turned out, Joe merged our two papers—but did much more; Alan's ideas were very stimulating. When Alan got that version, he was also stimulated and made substantial improvements. Then Joe made further improvements, and finally Alan did the same. We could probably have made much more progress, but the deadline for publication cut off further work.

(e) One of our discoveries when collaborating was a new general class of totally unimodular matrices ... but several years later we were chagrined to learn from Jack Edmonds that in the 1800's Gustav Kirchoff (who was the inventor of Kirchoff's Laws) had constructed a class of totally unimodular matrices of which ours was only a special case.

(f) The term "totally unimodular" is due to Claude Berge, and far superior to our wishywashy phrase "matrices with the unimodular property". Claude had a flair for language.

(g) We had no thought about computational questions, practical or theoretical, that could be influenced by our work. We also did not imagine the host of interesting concepts, like total dual integrality, lattice polyhedral, etc. that would emerge, extending our idea. And we never dreamed that totally unimodular matrices could be completely described, see [3], because we didn't anticipate that a mathematician with the great talent of Paul Seymour would get interested in these concepts.

After we wrote the paper, we met once in a while (a theater in London, a meeting in Washington), but our interests diverged and we never got together again professionally. The last time we met was almost 15 years ago, when Vašek Chvátal organized at Rutgers a surprise 70th birthday party cum symposium for Alan. Joe spoke about this paper and read some of the letters we wrote each other, including reciprocal requests that each of us made begging the partner to forgive his stupidity.

References

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INTEGRAL BOUNDARY POINTS OF CONVEX POLYHEDRA

A. J. Hoffman and J. B. Kruskal

INTRODUCTION

Suppose every vertex of a (convex) polyhedron in n-space has (all) integral coordinates. Then this polyhedron has the integral property (i.p.). Sections 1, 2, and 3 of this paper are concerned with such polyhedra.

Define two polyhedra[1]:

$$P(b) = \{x \mid Ax \geqq b\} ,$$

$$Q(b, c) = \{x \mid Ax \geqq b, x \geqq c\} ,$$

where A, b, and c are integral and A is fixed. Theorem 1 states that $P(b)$ has the i.p. for every (integral) b if and only if the minors of A satisfy certain conditions. Theorem 2 states that $Q(b, c)$ has the i.p. for every (integral) b and c if and only if every minor of A equals 0, $+1$, or -1. Section 1 contains the exact statement of Theorems 1 and 2, and Sections 2 and 3 contain proofs.

A matrix A is said to have the unimodular property (u.p.) if it satisfies the condition of Theorem 2, namely if every minor determinant equals 0, + 1, or - 1. In Section 4 we give Theorem 3, a simple sufficient condition for a matrix to have the u.p. which is interesting in itself and necessary to the proof of Theorem 4. In Section 5 we state and prove — at length — Theorem 4, a very general sufficient condition for a matrix to have the u.p. Finally, in Section 6 we discuss how to recognize the unimodular property, and give two theorems, based on Theorem 4, for this purpose.

Our results include all situations known to the authors in which the polyhedron has the integral property independently of the "right-hand sides" of the inequalities (given that the "right-hand sides" are integral of course). In particular, the well-known "integrality" of transportation

[1] Unless otherwise stated, we assume throughout this paper that the inequalities defining polyhedra are consistent.

type linear programs and their duals follows immediately from Theorems 2 and 4 as a special case.

1. DEFINITIONS AND THEOREMS

A point of n-space is an <u>integral point</u> if every coordinate is an integer. A (convex) polyhedron in n-space is said to have the <u>integral property</u> (i.p.) if <u>every</u> face (of every dimension) contains an integral point. Of course, this is true if and only if every <u>minimal</u> face contains an integral point. If the minimal faces happen to be vertices[2] (that is, of dimension 0), then the integral property simply means that the vertices of P are themselves all integral points.

Let A be an m by n matrix of integers; let b and b' be m-tuples (vectors), and c and c' be n-tuples (vectors), whose components are integers or $\pm\infty$. We will let $\infty(-\infty)$ also represent a vector all of whose components are $\infty(-\infty)$; this should cause no confusion. The vector inequality $b < b'$ means that strict inequality holds at every component. Let $P(b; b')$ and $Q(b; b'; c; c')$ be the polyhedra in n-space defined by

$$P(b; b') = \{x \mid b \leqq Ax \leqq b'\},$$

$$Q(b; b'; c; c') = \{x \mid b \leqq Ax \leqq b' \quad \text{and} \quad c \leqq x \leqq c'\}.$$

Of course $Q(b; b'; -\infty, +\infty) = P(b; b')$. If S is any set of rows of A, then define

$$\gcd(S) = \begin{cases} 0, & \text{if each minor determinant in } S \text{ which has as many rows as } S \text{ equals } 0, \\ \text{greatest common divisor (g.c.d.) of all those minor determinants in } S \text{ which have as many rows as } S, & \text{otherwise.} \end{cases}$$

THEOREM 1. The following conditions are equivalent:

(1.1) $P(b; b')$ has the i.p. for every b, b';

(1.2) $P(b; \infty)$ has the i.p. for every b;

(1.2') $P(-\infty; b')$ has the i.p. for every b';

(1.3) if r is the rank of A, then for every set S of r linearly independent rows of A, $\gcd(S) = 1$;

[2] It is well known (and, incidentally, is a by-product of our Lemma 1) that all minimal faces of a convex polyhedron have the same dimension.

(1.4) for every set S of rows of A, gcd(S) = 1 or 0.

The main value of this theorem lies in the fact that condition (1.3) implies condition (1.1). However the converse implication is of esthetic interest. If it is believed that (1.3) does not hold, (1.4) often offers the easiest way to verify this, for it may suffice to examine small sets of rows.

A matrix (of integers) is said to have the unimodular property (u.p.) if every minor determinant equals 0, + 1, or - 1. We see immediately that the entries in a matrix with the u.p. can only be 0, + 1, or - 1.

THEOREM 2. The following conditions are equivalent:

(1.5) Q(b; b'; c; c') has the i.p. for every b, b', c, c';

(1.6) for some fixed c such that $-\infty < c < +\infty$, Q(b, ∞; c; ∞) has the i.p. for every b;

(1.6') for some fixed c such that $-\infty < c < \infty$, Q(- ∞; b'; c; ∞) has the i.p. for every b';

(1.6'') for some fixed c' such that $-\infty < c' < \infty$, Q(b; ∞; - ∞; c') has the i.p. for every b;

(1.6''') for some fixed c' such that $-\infty < c' < \infty$, Q(- ∞; b'; - ∞; c') has the i.p. for every b';

(1.7) the matrix A has the unimodular property (u.p.).

The main value of this theorem for applications lies in the fact that condition (1.7) implies condition (1.5), a fact which can be proved directly (with the aid of Cramer's rule) without difficulty. However the converse implication is also of esthetic interest. The relationship between Theorems 1 and 2 is that Theorem 2 asserts the equivalence of stronger properties while Theorem 1 asserts the equivalence of weaker ones. Condition (1.5) is clearly stronger than condition (1.1), and condition (1.7) is clearly stronger than condition (1.3).

For A to have the unimodular property is the same thing as for A transpose to have the unimodular property. Therefore if a linear program has the matrix A with the u.p., both the "primal" and the dual programs lead to polyhedra with the i.p. This can be very valuable when applying the duality theorem to combinatorial problems (for examples, see several other papers in this volume).

2. PROOF OF THEOREM 1

We note that (1.1) $\Longrightarrow$ (1.2) and (1.2') trivially. Likewise (1.4) $\Longrightarrow$ (1.3) trivially. To see that (1.3) $\Longrightarrow$ (1.4), let S and S' be sets of rows of A. If $S \subset S'$, then the relevant determinants of S' are integral combinations of the relevant determinants of S. Hence $\gcd(S)$ divides $\gcd(S')$. From this we easily see that (1.3) $\Longrightarrow$ (1.4).

As (1.2) and (1.2') are completely parallel, we shall only treat the former in our proofs.

Let the rows of A be $A_1, \ldots, A_m$ and the components of b and b' be $b_1, \ldots, b_m$ and $b_1', \ldots, b_m'$. Suppose that we know that (1.3) for any matrix A_* implies (1.2) for the corresponding polyhedra $P_*(b; \infty)$. Also, suppose that (1.3) holds for the particular matrix A. Then setting

$$A_* = \begin{bmatrix} A \\ \\ -A \end{bmatrix},$$

we see immediately that (1.3) holds for A_*. Consequently

$$P_*(b_1, \ldots, b_m, -b_1', \ldots, -b_m'; \infty)$$

has the i.p. But it is easy to see that this polyhedron is identical with $P(b; b')$; hence the latter also has the i.p. Therefore if for every matrix (1.3) implies (1.2), then (1.3) implies (1.1) for every matrix.

Let $P(b) = P(b; \infty)$ for convenience.

It only remains to prove that (1.2) is equivalent to (1.3)[3]. If S is any set of rows A_i of A, we define

$$F_S = F_S(b) = \{x \mid Ax \geq b \quad \text{and} \quad A_i x = b_i \quad \text{if} \quad A_i \quad \text{in} \quad S\},$$

$$G_S = \text{the subspace of } n\text{-space spanned by the rows } A_i \text{ in } S.$$

If $F_S(b)$ is not empty, it is the face of $P(b)$ corresponding to S. (We do not consider the empty set to be a face of a polyhedron.) We easily see that $F_S(b)$, if non-empty, corresponds to the usual notion of a face. Of course $F_\emptyset(b) = P(b)$, where $\emptyset$ is the empty set. We shall use the letter A to stand for the set of <u>all</u> rows of the matrix A. In general we will use the same letter to denote a set of rows and to denote the matrix formed by these rows. (This double meaning should cause no confusion.)

[3] The authors are indebted to Professor David Gale for this proof, which is much simpler than the original proof.

LEMMA 1. If $S \subset S'$, and if $F_S(b)$ and $F_{S'}(b)$ are faces (that is, not empty), then $F_{S'}(b)$ is a subface of $F_S(b)$. If $F_S(b)$ is a face, then it is a minimal face if and only if $G_S = G_A$, that is, if and only if S has rank r, where r is the rank of A.

PROOF. The first sentence of the lemma follows directly from the definitions. To prove the rest of the lemma, let S' be all rows of A which are in G_S. Then $G_S = G_{S'}$, and A_j is a linear combination of the A_i in S if and only if A_j is in S'. Clearly $G_S = G_A$ if and only if $S' = A$.

If $S' \neq A$, there is at least one row A_k in $A - S'$. Then there is a vector y such that $A_i y = 0$ for A_i in S, $A_k y < 0$. Let x be in F_S. As $A_k x \geqq b_k$, there is a number $\lambda_k \geqq 0$ for which $A_k(x + \lambda_k y) = b_k$. For every A_j in $A - S'$ such that $A_j y < 0$, the equation $A_j(x + \lambda y) = b_j$ has a non-negative solution. Let λ_j be that solution. Define $\lambda = \text{minimum } \lambda_j$, and let j' be a value such that $\lambda = \lambda_{j'}$. As λ_k exists, there is at least one λ_j, so λ exists. By the definition of λ,

$$\begin{aligned} A(x + \lambda y) &\geqq b \ , \\ A_i(x + \lambda y) &= b_i \qquad \text{for } A_i \text{ in } S, \\ A_{j'}(x + \lambda y) &= b_{j'} \ . \end{aligned}$$

Thus $F_{S \cup A_{j'}}$ is not empty, and is therefore a subface of F_S. Furthermore as $A_{j'}$ is not a linear combination of the A_i in S, $F_{S \cup A_{j'}}$ is a proper subface of F_S. Therefore F_S is not minimal.

On the other hand, if F_S is not minimal it has some proper subface $F_{S \cup A_k}$. Then there must be x_1 and x_2 in F_S such that $A_k x_1 = b_k$ and $A_k x_2 > b_k$. Therefore $A_k x$ varies as x ranges over F_S. But for A_i in S, $A_i x = b_i$ is constant as x varies over F_S. Hence A_k cannot be a linear combination of the A_i in S, so A_k is in $A - S'$. Hence $S' \neq A$. This proves the lemma.

If b, as usual, is an m-tuple and S is a set of r rows of A, then b_S is the "sub-vector" consisting of the r components of b which correspond to the rows of S. Let $\tilde{b}$ always represent an (integral) r-tuple. The components of $\tilde{b}$ and b_S will be indexed by the indices used for the rows of S, not by the integers from 1 to r. Let

$$L_S(\tilde{b}) = \{x \mid Sx = \tilde{b}\} \ .$$

LEMMA 2. Suppose S is a set of r linearly independent rows of A. Then for any $\tilde{b}$ there is a b such that

$$b_S = \tilde{b} \; ; \tag{2.1}$$

$$F_S(b) \text{ is a minimal face of } P(b). \tag{2.2}$$

PROOF. As S is a set of linearly independent rows, the equation $Sx = \tilde{b}$ has at least one solution: call it y. Define b as follows:

$$b_i = \begin{cases} \tilde{b}_i & \text{if } A_i \text{ in } S, \\ [A_i y] & \text{if } A_i \text{ not in } S. \end{cases}$$

Clearly $b_S = \tilde{b}$, so (2.1) is satisfied. Obviously b is integral. Furthermore y is seen to be in $F_S(b)$, so $F_S(b)$ is not empty, and hence is a face of $P(b)$. By Lemma 1, $F_S(b)$ is a minimal face, so (2.2) is satisfied.

LEMMA 3. Suppose S' is a set of rows of A of rank r, and $S \subset S'$ is a set of r linearly independent rows. For any b such that $F_{S'}(b)$ is a face (that is, not empty),

$$F_{S'}(b) = L_S(b_S).$$

PROOF. Let y be a fixed element in $F_{S'}(b)$, and let x be any element of $L_S(b)$. As $F_{S'}(b) \subset L_S(b_S)$ is trivial, we only need show the reverse inclusion. Thus it suffices to prove that x is in $F_{S'}(b)$.

As S has rank r, any row A_k in A can be expressed as a linear combination of the rows A_i in S:

$$A_k = \Sigma \alpha_{ki} A_i \; .$$

Then

$$A_i x = b_i = A_i y$$

for A_i in S, so

$$A_k x = \Sigma \alpha_{ki} A_i x = \Sigma \alpha_{ki} A_i y = A_k y \; .$$

Then as y is in $F_{S'}(b)$, x must be also. This completes the proof of the lemma.

LEMMA 4. Any minimal face of $P(b)$ can be expressed in the form $F_S(b)$ where S is a set of r linearly independent rows of A.

PROOF. Suppose the face is $F_{S'}(b)$. By Lemma 1, S' must have rank r. Let S be a set of r linearly independent rows of S'. Then by applying Lemma 3 to both $F_{S'}(b)$ and $F_S(b)$, we see that

$$F_{S'}(b) = L_S(b_S) = F_S(b).$$

This proves the lemma.

LEMMA 5. If S is a set of r linearly independent rows of A, then the following two conditions are equivalent:

(2.3) $L_S(\tilde{b})$ contains an integral point for every (integral) $\tilde{b}$;

(2.4) $$\gcd(S) = 1.$$

PROOF. We use a basic theorem of linear algebra, namely that any integral matrix S which is r by n can be put into the form

$$S = UDV$$

where D is a (non-negative integral) diagonal matrix, and U and V are (integral) unimodular matrices. (Of course U is r by r, V is n by n, and D is r by n.) As U and V are unimodular, they have integral inverses. Furthermore $\gcd(S) = \gcd(D)$. (For proofs of these facts, see for example [3].)

Let the diagonal elements of D be d_{ii}. Clearly $\gcd(D) = d_{11}d_{22} \cdots d_{rr}$. Therefore condition (2.4) is equivalent to the condition that every $d_{ii} = 1$. Now we show that (2.3) is also equivalent to this same condition.

Suppose that some diagonal element of D is greater than 1. For convenience we may suppose that this element is $d_{11} = k > 1$. Let $\tilde{e}$ be the r-tuple $(1, 0, \ldots, 0)$, and let $\tilde{b} = U\tilde{e}$. Then $L_S(\tilde{b})$ contains no integral point. To see this, let x be in $L_S(\tilde{b})$. Then

$$Sx = UDVx = \tilde{b} = U\tilde{e} ,$$

so $DVx = \tilde{e}$. Clearly the first component of $y = Vx$ is $1/k$, so y is not integral. Hence x cannot be integral. This shows that (2.3) cannot hold if some d_{ii} is greater than 1.

Suppose every $d_{ii} = 1$. Let x be in $L_S(\tilde{b})$ and set

$$Vx = (y_1, \ldots, y_r, y_{r+1}, \ldots, y_n).$$

Then

$$U^{-1}\tilde{b} = DVx = (y_1, \ldots, y_r),$$

and so $y_1, \ldots, y_r$ are integral. Let $y = (y_1, \ldots, y_r, 0, \ldots, 0)$. Then $V^{-1}y$ is integral, and since $Dy = DVx$,

$$S(V^{-1}y) = UDV(V^{-1}y) = UDy = UDVx = \tilde{b}\ .$$

Thus $V^{-1}y$ is in $L_S(\tilde{b})$. This shows that (2.3) does hold if every $d_{ii} = 1$, and completes the proof of the lemma.

Now it is easy to prove that (1.2) $\Longleftrightarrow$ (1.3). First we prove $\Longrightarrow$. Let S be any set of r linearly independent rows of A. Let $\tilde{b}$ be any (integral) r-tuple. Choose a b which satisfies (2.1) and (2.2). By (1.2), $F_S(b)$ must contain an integral point x. By Lemma 3 and (2.1),

$$F_S(b) = L_S(b_S) = L_S(\tilde{b})\ .$$

Hence $L_S(\tilde{b})$ contains x. Therefore (2.3) is satisfied, so by Lemma 5, $\gcd(S) = 1$. This proves $\Longrightarrow$.

To prove $\Longleftarrow$, let $F_{S'}(b)$ be some minimal face of $P(b)$. By Lemma 4 this face can be expressed as $F_S(b)$ where S consists of r linearly independent rows of A. By Lemma 3, $F_S(b) = L_S(b_S)$. By (1.3), $\gcd(S) = 1$, and by Lemma 5 $L_S(b_S)$ must contain an integral point x. Hence $F_{S'}(b)$ contains the integral point x. Therefore every minimal face of $P(b)$ contains an integral point, and hence also every face. This proves $\Longleftarrow$, and completes the proof of Theorem 1.

3. PROOF OF THEOREM 2

The role of (1.6) and its primed analogues are exactly similar, so we treat only the former in our proofs. For convenience we let

$$Q(b;\ c) = Q(b;\ \infty;\ c;\ \infty).$$

It is not hard to see that (1.7) $\Longrightarrow$ (1.5). For suppose that A has the u.p. (that is, satisfies (1.7)). Then

$$A_* = \begin{bmatrix} A \\ -A \\ I \\ -I \end{bmatrix}$$

satisfies (1.3). By Theorem 1, the associated polyhedron

$$P_*(b_1, \ldots, b_m, -b'_1, \ldots, -b'_m, c_1, \ldots, c_n, -c'_1, \ldots, -c'_n)$$

has the i.p. But it is easy to see that this polyhedron is identical with Q(b; b'; c; c'). Therefore the latter has the i.p., so (1.7) $\Longrightarrow$ (1.5). (An alternate proof of this can easily be constructed using Cramer's Rule.)

Clearly (1.5) $\Longrightarrow$ (1.6). Hence it only remains to prove that (1.6) $\Longrightarrow$ (1.7). We shall prove[4] this by applying Theorem 1 to the matrix

$$A^* = \begin{bmatrix} I \\ A \end{bmatrix}.$$

Let d be any (integral) (n+m)-tuple, and let

$$c \cup b = (c_1, \ldots, c_n, b_1, \ldots, b_m) .$$

Then $P^*(c \cup b) = Q(b, c)$.

To verify condition (1.2) for A^*, we need to show that $P^*(d)$ has the i.p. for every d. Condition (1.6) yields only the fact that $P^*(d)$ has the i.p. for every d such that $d_I = c$. To fill this gap, note that A^* has rank n as it contains the n by n identity matrix, and let $F^*_{S'}(d)$ be any face of $P^*(d)$. This face contains some minimal face, which by Lemma 4 can be expressed as $F^*_S(d)$ where S consists of n linearly independent rows of A^*. By Lemma 3,

$$F^*_S(d) = L^*_S(d_S) \equiv \{x \mid Sx = d_S\} .$$

As S is an n by n matrix of rank n, $F^*_S(d)$ consists only of a single point. Call this point x. We shall show that x is integral.

Let I_1 be the rows of I in S, I_2 the rows of I not in S, A_1 the rows of A in S, and A_2 the rows of A not in S. We wish to pick an integral vector q such that

[4] The authors are indebted to Professor David Gale for this proof, which is much simpler than the original proof.

(3.1) $$x + q \geqq c,$$

(3.2) $$(x + q)_{I_1} = c_{I_1} .$$

Let $q = c - d_I$. Then q satisfies these requirements, for

$$x_i + q_i \left\{ \begin{matrix} = \\ \\ \geqq \end{matrix} \right\} d_i + (c_i - d_i) = c_i \left\{ \begin{array}{l} \text{if the i-th row of} \\ \quad I \text{ is in } I_1, \\ \text{otherwise.} \end{array} \right.$$

Define $d' = c \cup (d_A + Aq)$. Then d' is integral, and $d'_I = c$, so by (1.6) the polyhedron $P^*(d')$ has the i.p.

Now $F^*_S(d')$ is not empty because it contains $(x + q)$, as we may easily verify:

$$A^*(x + q) = I(x + q) \cup A(x + q) \geqq c \cup (d_A + Aq) = d' ,$$

$$S(x + q) = I_1(x + q) \cup A_1(x + q) = c_{I_1} \cup (d_{A_1} + A_{1q}) = d'_S .$$

Therefore $F^*_S(d')$ must contain an integral point. However $F^*_S(d')$ can contain only a single point for the same reasons that applied to $F^*_S(d)$. Hence $x + q$ must be that single point, so $x + q$ must itself be integral. As q is integral, x must be integral also. Thus $F^*_S(d)$, and a fortiori $F^*_{S'}(d)$, contains the integral point x. This verifies condition (1.2) for A^*.

By Theorem 1, (1.3) holds for A^*. As the rank of A^* is n, $\gcd(S) = |S| = 1$ for every set S of n linearly independent rows of A^*. From this we wish to show that A has the u.p. Suppose E is any non-singular square submatrix of A. Let the order of E be s. By choosing S to consist of the rows of A^* which contain E together with the proper set of $(n - s)$ rows of I, and by rearranging columns, we can easily insure that

$$S = \begin{bmatrix} I & 0 \\ F & E \end{bmatrix}$$

where I is the identity matrix of order $(n - s)$, F is some s by $(n - s)$ matrix, and 0 is the $(n - s)$ by s matrix of zeros. Then $|S| = |E| \neq 0$, so S is non-singular. Therefore S consists of n linearly independent rows, so

$$|E| = |S| = \gcd(S) = 1.$$

This completes the proof of Theorem 2.

4. A THEOREM BY HELLER AND TOMPKINS

In this and the remaining sections we give various sufficient conditions for a matrix to have the unimodular property.

THEOREM 3. (Heller and Tompkins). Let A be an m by n matrix whose rows can be partitioned into two disjoint sets, T_1 and T_2, such that A, T_1, and T_2 have the following properties:

(4.1) every entry in A is 0, $+1$, or -1;

(4.2) every column contains at most two non-zero entries;

(4.3) if a column of A contains two non-zero entries, and both have the same sign, then one is in T_1 and one is in T_2;

(4.4) if a column of A contains two non-zero entries, and they are of opposite sign, then both are in T_1 or both in T_2.

Then A has the unimodular property.

This theorem is closely related to the central result of the paper by Heller and Tompkins in this Study. The theorem, as stated above, is given an independent proof in an appendix to their paper.

COROLLARY.[5] If A is the incidence matrix of the vertices versus the edges of an ordinary linear graph G, then in order that A have the unimodular property it is necessary and sufficient that G have no loops with an odd number of vertices.

PROOF. To prove the sufficiency, recall the following. The condition that G have no odd loops is well-known to be equivalent to the property that the vertices of G can be partitioned into two classes so that each edge of G has one vertex in each class. If we partition the rows of A correspondingly, it is easy to verify the conditions (4.1)-(4.4). Therefore A has the u.p.

If A has an odd loop, let A' be the submatrix contained in the rows and columns corresponding to the vertices and edges of the loop. Then

[5] The authors are indebted to the referee for this result.

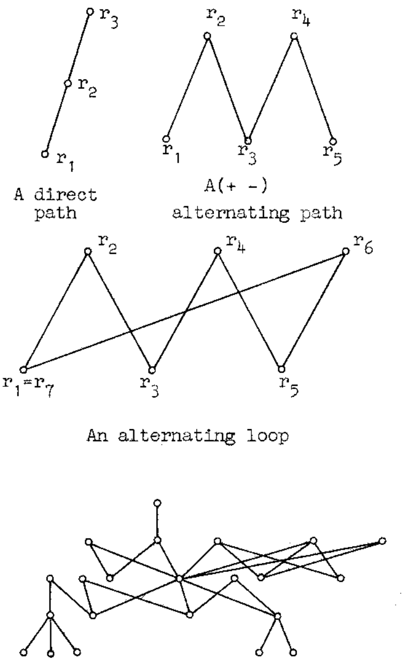
it is not hard to see that $|A'| = \pm 2$. This proves the necessity.

5. A SUFFICIENT CONDITION FOR THE UNIMODULAR PROPERTY

We shall consider oriented graphs. For our purposes an oriented graph G is a graph (a) which has no circular edges, (b) which has at most one edge between any two given vertices, and (c) in which each edge has an orientation. Let V denote the set of vertices of G, and E the set of edges. If (r, s) is in E (that is, if (r, s) is an edge of G), then we shall call (s, r) an <u>inverse edge</u>. (Note that by (b), and inverse edge cannot be in E; thus an inverse edge cannot be an edge. This slight ambiguity in terminology should cause no confusion.) We shall often use the phrase <u>direct edge</u> to denote an ordinary edge.

A <u>path</u> is a sequence of distinct vertices $r_1, \ldots, r_k$, such that for each i, from 1 to $k - 1$, (r_i, r_{i+1}) is either a direct or an inverse edge. A path is directed if every edge is oriented forward, that is, if every edge (r_i, r_{i+1}) in the path is a direct edge. A path is <u>alternating</u> if successive edges are oppositely oriented. More precisely, a path is alternating if its edges are alternately direct and inverse. An alternating path may be described as being $(++)$, $(+-)$, $(-+)$, or $(--)$. The first sign indicates the orientation of the first edge of the path, the second sign the orientation of the last edge of the path. A $+$ indicates a direct edge; a $-$ indicates an inverse edge. A <u>loop</u> is a path which closes back on itself. More precisely, a loop is a sequence of vertices $r_1, \ldots, r_k$ in which $r_1 = r_k$ but which are otherwise distinct, and such that for each i (r_i, r_{i+1}) is either a direct or an inverse edge. A loop is alternating if successive edges are

Diagram 1



A direct path

A(+ -) alternating path

An alternating loop

An alternating graph (arrows omitted - all should be upward) oppositely oriented and if the first and last edges are oppositely oriented. An alternating loop must obviously contain an even number of edges.

A graph is <u>alternating</u> if every loop in it is alternating. Let $V = \{v_1, \ldots, v_m\}$ be the vertices of G, and let $P = \{p_1, \ldots, p_n\}$ be some set of directed paths in G. Then the incidence matrix $A = \|a_{ij}\|$ of G versus P is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is in } p_j, \\ 0 & \text{if } v_i \text{ is not in } p_j. \end{cases}$$

We let A_v represent the row of A corresponding to the vertex v and A^p represent the column of A corresponding to the path p. We often write a_{vp} instead of a_{ij} for the entry common to A_v and A^p.

THEOREM 4. Suppose G is an oriented graph, P is some set of directed paths in G, and A is the incidence matrix of G versus P. Then for A to have the unimodular property it is sufficient that G be alternating. If P consists of the set of <u>all</u> directed paths of G, then for A to have the unimodular property it is necessary and sufficient that G be alternating.

This theorem does not state that every matrix of zeros and ones with the u.p. can be obtained as the incidence matrix of an alternating graph versus a set of directed paths. Nor does it give necessary and sufficient conditions for a matrix of zeros and ones to have the unimodular property. (Such conditions would be very interesting.) However it does provide a very general sufficient condition. For example, the coefficient matrix of the i by j transportation problem (or its transpose, depending on which way you write the matrix) is the incidence matrix of the alternating graph versus the set of all directed paths. Hence this matrix has the u.p., from which by Theorem 2 follows the well-known i.p. of transportation problems and their duals. The extent to which alternating graphs can be more general than the graph shown to the left is a measure of how general Theorem 4 is.

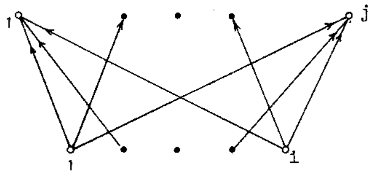


Diagram 2

So that the reader may follow our arguments more easily, we describe here what alternating graphs look like.

(As logically we do not need these facts and as the proofs are tedious, we omit them.) An integral <u>height</u> function $h(v)$ may be defined in such a way that (r, s) is a direct edge only when (but not necessarily when) $h(r) + 1 = h(s)$. If we define $r \leqq s$ to mean that there is a directed path from r to s, then $\leqq$ is a partial order. Then (r, s) is a direct edge if and only if both $r < s$ and there is no element t such that $r < t < s$.

PROOF OF NECESSITY. We consider here the case in which P is the set of <u>all</u> directed paths in G, and we prove that for A to have the u.p. it is necessary that G be alternating. It is easy to verify that the matrix (shown below) of odd order which has ones down the main diagonal and sub-diagonal and in the upper right-hand corner, and zeros elsewhere, has determinant $+2$.

$$\begin{bmatrix} 1 & & & & 1 \\ 1 & 1 & & & \\ & 1 & \ddots & & \\ & & \ddots & 1 & \\ & & & 1 & 1 \end{bmatrix}$$

We shall show that if G is not alternating then it contains this matrix, perhaps with rows and columns permuted, as a submatrix.

Let ℓ be a non-alternating loop in G. If ℓ has an odd number of distinct vertices, consider the rows in A which correspond to these vertices, and consider the columns in A which correspond to the one-edge directed paths which correspond to the edges in ℓ. The submatrix contained in these rows and columns is clearly the matrix shown above, up to row and column permutations. Hence in this case A does not have the u.p. If ℓ has an even number of distinct vertices, then find in it three successive vertices r, s, t such that (r, s) and (s, t) are both direct (or both inverse) edges. (To find r, s, t it may be necessary to let s be the initial-terminal vertex of ℓ, in which case r, s, t are successive only in a cyclic sense.) Consider the rows of A which correspond to all the vertices of ℓ except s. Consider the columns of A which correspond to the following directed paths: the two-edge path r, s, t (or t, s, r) and the one-edge paths using the other edges in ℓ. The submatrix contained in these rows and columns is the square matrix of odd order shown above, up to row and column permutations. Hence in this case also A does not have the u.p. This completes the proof of necessity.

The proof of the sufficiency condition, when P may be any set of directed paths in G, occupies the rest of this section. As this proof is long and complicated, it has been broken up into lemmas.

If $r_1, \ldots, r_k$ is a loop, then $r_1, \ldots, r_k, r_2, \ldots, r_1$ is called a cyclic permutation of the loop. Clearly a loop is alternating if and only if any cyclic permutation is alternating.

LEMMA 6. Suppose A is the incidence matrix of an alternating graph G versus some set of directed paths P in G. For any submatrix A' of A, there is an alternating graph G' and a set of directed paths P' in G' such that A' is the incidence matrix of G' versus P'.

PROOF. Any submatrix can be obtained by a sequence of row and column deletions. Hence it suffices to consider the two cases in which A' is formed from A by deleting a single column or a single row. If A' is formed from A by deleting the column A^p, let G' = G, and P' = P - {p}. Then A' is clearly the incidence matrix of G' versus P', and G' is indeed an alternating graph.

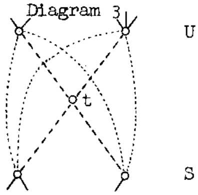
Suppose now that A' is formed from A by deleting row A_t. Define

$$V' = V - \{t\},$$
$$E' = \{(v, w) \mid v, w \text{ in } V' \text{ and either } (v, w) \text{ in } E \text{ or } (v, t) \text{ and } (t, w) \text{ in } E\},$$
$$G' = \text{the graph with vertices } V' \text{ and edges } E',$$
$$P' = \{p - \{t\} \mid p \text{ in } P\} .$$

Clearly A' is the incidence matrix of G' versus P'. We shall prove (a) that P' is a collection of directed paths and (b) that G' is alternating.

The proof of (a) is quite simple. Suppose v, w are successive vertices of p' = p - {t} in P'. It may or may not happen that p contains t. In either case, however, if v, w are successive vertices in p, then (v, w) is a

Diagram 3



Solid edges - G and G'
Dashed edges - G only
Dotted edges - G' only

direct edge in G, so (v, w) is a direct edge in G'. If v, w are not successive vertices in p, then necessarily v, t, w are successive vertices in p. In this case (v, t) and (t, w) are direct edges in G, so (v, w) is a direct edge in G'.

The proof of (b) is more extended. Define

$$S = \{s \mid (s, t) \text{ in } E\}$$
$$U = \{u \mid (t, u) \text{ in } E\} .$$

Then each "new" edge in E', that is, each edge of E' - E, is of the form (s, u) with s in S and u in U. Let ℓ be any loop in G'. If ℓ contains no new edge, then ℓ is also a loop in G and hence alternating. If ℓ contains a new edge, it contains at least two vertices of $S \cup U$. Hence the vertices of $S \cup U$ break ℓ up into pieces which are paths of the form

$$p = v, r_1, \ldots, r_k, v'$$

where v and v' are in $S \cup U$ and the r's are not.

CASE (U, U): both v and v' belong to U. In this case

$$t, v, r_1, \ldots, r_k, v', t$$

is a loop in G, hence alternating. Therefore p is an alternating path. As (t, v) is a direct edge and (v', t) is an inverse edge in G, p must be a (- +) alternating path in G'.

CASE (S, S): both v and v' belong to S. In this case dual argument to the above proves that p must be a (+ -) alternating path in G'.

CASE (U, S): v belongs to U and v' belongs to S. In this case p must be exactly the one-edge path v, v'. For if not, p consists solely of edges in E, so the loop which we may represent symbolically v', t, p is a loop in G. But as (v', t) and (t, v) are both direct edges in G this loop is not alternating, which is impossible. As (v, v') is an inverse edge in G', p is a (- -) alternating path in G'.

CASE (S, U): v belongs to S and v' belongs to U. In this case dual argument to the above proves that p must be exactly the one-edge path v, v' and hence a (+ +) alternating path in G'.

Using these four cases, we easily see that the pieces of ℓ are alternating and fit together in such a way that ℓ itself is alternating - except for one technical difficulty, namely the requirement that the initial and terminal edges of ℓ must have opposite orientations. However, if we form a cyclic permutation of ℓ and apply the reasoning above to this new loop, we obtain the necessary information to complete our proof that ℓ is alternating. This completes the proof of (b) and Lemma 6.

In view of Lemma 6, the sufficiency condition of Theorem 4 will be proved if we prove that every square incidence matrix of an alternating graph versus a set of directed paths has determinant, 0, + 1, or - 1. We prove this by a kind of induction on two new variables, c(G) and d(G), which we shall now define:

c(G) = the number of unordered pairs {st} of distinct vertices of G which satisfy

(5.1) there is a vertex u such that (s, u) and (t, u) are direct edges of G;

d(G) = the number of unordered pairs {st} of distinct vertices of G which satisfy

(5.2) there is no directed path from s to t nor any directed path from t to s.

Though not logically necessary the following information may help orient the reader to the significance of these two variables. Assume G is alternating. Then using the partial-order $\leqq$ introduced informally earlier, d(G) is the number of pairs of vertices which are incomparable under $\leqq$. Any pair {st} which satisfies (5.1) also satisfies (5.2), so $c(G) \leqq d(G)$. If c(G) = 0, then each vertex of G has at most one "predecessor", and G consists of a set of trees, each springing from a single vertex and oriented outward from that vertex. If d(G) = 0, then G is even more special: it consists of a single directed path.

LEMMA 7. If G is alternating, and {st} satisfies (5.1), then it also satisfies (5.2). Hence $c(G) \leqq d(G)$.

PROOF. Let u be a vertex such that (s, u) and (t, u) are direct edges of G. Suppose there is a directed path

$$s, r_1, \dots, r_k, t .$$

If none of the r's is u, then

$$s, r_1, \dots, r_k, t, u, s$$

is a loop, hence alternating. As (t, u) is a direct edge, (r_k, t) is an inverse edge, so the path is not directed, a contradiction. If one of the r's is u, take the piece from u to t. By renaming, we may call this directed path

$$u, r_1, \ldots, r_k, t .$$

Then

$$t, u, r_1, \ldots, r_k, t$$

is a loop, hence alternating. As (t, u) is a direct edge, (u, r_1) must be an inverse edge, so the path is not directed, a contradiction. Therefore, there can be no directed path from s to t. By symmetrical argument, there can be no directed path from t to s. Therefore $\{st\}$ satisfies (5.2). It follows trivially that $c(G) \leq d(G)$. This completes the proof of Lemma 7.

The induction proceeds in a slightly unusual manner. The "initial case" consists of all graphs G for which $c(G) = 0$. The inductive step consists of showing that the truth of the assertion for a graph G such that $c(G) > 0$ follows from the truth of the assertion for a graph G' for which $d(G') < d(G)$. It is easy to see that by using the inductive step repeatedly, we may reduce to a graph G^* for which either $c(G^*)$ or $d(G^*)$ is 0. But as $d(G^*) = 0$ implies $c(G^*) = 0$ by the inequality between c and d, we are down to the initial case either way.

We now treat the initial case.

LEMMA 8. Let A be the incidence matrix of an alternating graph G versus some set of directed paths P. Suppose that P contains as many directed paths as G contains vertices, so A is square. Suppose that $c(G) = 0$. Then $|A| = 0, +1$, or -1.

PROOF. If (r, s) is a direct edge of G, we call r a <u>predecessor</u> of s and s a <u>successor</u> of r. The fact that $c(G) = 0$ means that each vertex of G has at most one predecessor. If V' is a subset of V, and r is in V' but has no predecessor in V', then r is called an <u>initial vertex</u> of V'.

Every non-empty subset V' of V has at least one initial vertex. For if V' has none, then we can form in V' a sequence $r_1, r_2, \ldots$ of vertices such that for every i, r_{i+1} is a predecessor of r_i. Let r_j be the first term in the sequence which is the same as a vertex picked earlier, and let r_i be the earlier name for this vertex. Then r_i, r_{i+1}, ..., r_j is a loop all of whose edges are inverse. As G is alternating, this is impossible.

Let $U(r) = \{s \mid s$ is a successor of $r\}$. Let r_1 be an initial vertex in V. Recursively, let r_i be an initial vertex of $V - \{r_1, r_2, \ldots, r_{i-1}\}$. Then define matrices $B(i)$ recursively:

$$B(0) = A,$$
$$B(i) = B(i - 1)$$

with the row $B_{r_i}(i - 1)$ replaced by

$$B_{r_i}(i - 1) - \sum_{s \text{ in } U(r_i)} B_s(i - 1) .$$

Let B be the final $B(i)$. We see immediately that $|A| = |B(1)| = \ldots = |B|$. Thus we only need show that $|B| = 0, + 1,$ or $- 1$.

We claim that each column B^p of B consists of zeros with either one or two exceptions: if w is the final vertex of the directed path p, then $b_{wp} = 1$, and if v is the unique predecessor to the initial vertex of p, then $b_{vp} = - 1$. As the initial vertex of p may have no predecessor at all, the $- 1$ may not occur.

We shall not prove in detail the assertions of the preceding paragraph. We content ourselves with considering the column corresponding to a fixed path p during the transition from $B(i - 1)$ to $B(i)$. Only one entry is altered, namely $b_{r_ip}(i - 1)$. There are four possible cases.

CASE (i): neither r_i nor any of its successors is in p.

CASE (ii): r_i is not in p but one of its successors is in p.

CASE (iii): both r_i and one of its successors is in p.

CASE (iv): r_i is in p but none of its successors is in p.

At most one successor of a vertex can be in a directed path because G is alternating, so these cases cover every possibility. In case (i), the entry we are considering starts as 0 and ends as 0. In case (ii), it starts as 0 and ends as $- 1$. In case (iii), it starts as 1 and ends as 0. In case (iv), it starts as 1 and ends as 1. From these facts, it is not hard to see that B satisfies our assertions.

From our assertions about B it is trivial to check that B satisfies the hypotheses of Theorem 3. It is only necessary to partition the rows of B into two classes, one class being empty and the other class containing every row. Then by Theorem 3, B has the u.p. Therefore, $|B| = 0, +1,$ or -1. As $|A| = |B|$, this completes the proof of Lemma 8.

We now prove the inductive step.

LEMMA 9. Suppose that A is the square incidence matrix of an alternating graph G versus a set of directed paths P. Suppose that $c(G) > 0$. Then there is a square matrix A' such that $|A'| = |A|$ and such that A' is the square incidence matrix of an alternating graph G' versus a set of directed paths P', where $d(G') < d(G)$.

PROOF. As $c(G) > 0$, G contains a vertex u which has at least two distinct predecessors, s and t. Define

$$A' = A \text{ with row } A_t \text{ replaced by } A_s + A_t.$$

Clearly $|A'| = |A|$. Define

$$V' = V,$$

$$E_s = \{(s, w) \mid (s, w) \text{ in } E\},$$

$$E_t = \{(t, w) \mid (s, w) \text{ in } E\},$$

$$E' = E \cup E_t \cup \{(s, t)\} - E_s,$$

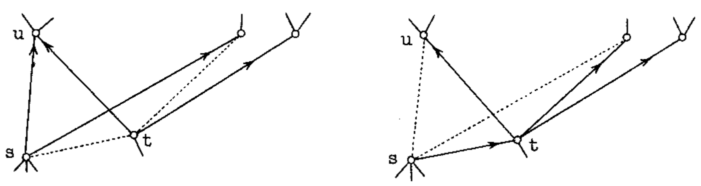
G' = the graph with vertices V' and edges E',

$$p' = \begin{cases} p \text{ if } p \text{ does not contain } s, \\ p \text{ with } t \text{ inserted after } s \text{ if } p \text{ does contain } s, \end{cases}$$

$$P' = \{p' \mid p \text{ in } P\}.$$

We shall prove (a) that G' is alternating, (b) that P' is a set of directed paths of G', (c) that $d(G') < d(G)$, and (d) that A' is the incidence matrix of G' versus P'.

Diagram 4



Graph G Graph G'

The proof of (b) is simple. If p does not contain s, then every edge in $p' = p$ is in E', so p' is a directed path in G'. If p does contain s, write p thus:

$$r_1, \ldots, r_i, s, r_{i+1}, \ldots, r_j .$$

Then p' is

$$r_1, \ldots, r_i, s, t, r_{i+1}, \ldots, r_j .$$

Each edge of p except (s, r_{i+1}) is also in E'. Hence to show that p' is a directed path in G', we only need show that (s, t) and (t, r_{i+1}) are in E'. The former is in E' by definition, and the latter is in E_t because (s, r_{i+1}) must be in E. This proves (b).

To prove (c), let r_1 and r_2 be any pair of vertices such that there is a directed path p from one to the other in G. Then p' is a directed path from one to the other in G'. Hence every pair of vertices which satisfies (5.2) in G' also satisfies (5.2) in G. Furthermore, $\{st\}$ does not satisfy (5.2) in G' because (s, t) is in E', while $\{st\}$ does satisfy (5.2) in G by Lemma 7. This proves that $d(G') < d(G)$.

To prove (d), we first show that A' consists entirely of zeros and ones. The only way in which this could fail to happen is if A_s and A_t both contained ones in the same column. But if this were the case, then the directed path corresponding to this column would contain both s and t, which cannot happen by Lemma 7. To see that A' is the desired incidence matrix, consider how P' differs from P. Each directed path which did not contain s remains unchanged; each directed path which did contain s has t inserted in it. Thus the change from A to A' should be the following. Each column which has a zero in row A_s should remain unchanged; each column which has a one in row A_s should have the zero in row A_t changed to a one. But adding row A_s to A_t accomplishes exactly this. Therefore (d) is true.

The proof of (a) is more complicated.[6] Define S' to be the set of successors of s in G which are not also successors of t. Note that every edge in G' which is not in G terminates either in t, or in a vertex of S'. Let ℓ be any loop of G'. If ℓ is already a loop of G, then it is alternating. If not, it must contain either the edge (s, t) or an edge (t, s') with s' in S'. (Of course, ℓ might contain the inverse of one of these edges instead. If so, reversing the order of ℓ brings us to the situation above.) Ignoring trivial loops, that is,

[6] We are indebted to the referee for this proof, which replaces a considerably more complicated one.

loops of the form aba (which are alternating trivially), ℓ must have the form

$$str_1 \cdots r_k s$$

or

$$ts'r_1 \cdots r_k t, \quad \text{with} \quad s' \text{ in } S'.$$

The first form is impossible. To prove this, first suppose that no r_i is in $S' \cup \{u\}$. Then $sutr_1 \cdots r_k s$ is a loop of G, hence alternating. Thus (r_k, s) is an inverse edge and belongs to both G and G', which is impossible. Now suppose that some r_i is in $S' \cup \{u\}$, and let r_j be the last such r_i. Then $sr_j \cdots r_k s$ is a loop of G, hence alternating. Hence (r_k, s) is inverse, which is impossible as before.

We may now assume that ℓ is $ts'r_1 \cdots r_k t$. No r_i can be s. Clearly r_1 cannot be s, and if $r_j = s$, $j > 1$, then $ss'r_1 \cdots r_{j-1}s$ is a loop of G, hence alternating, so (r_{j-1}, s) is inverse and belongs to both G and G', which is impossible. Thus $r_1, \ldots, r_k$ are distinct from s. Suppose that r_k is in S'. Then $ss'r_1 \cdots r_k s$ is a loop of G, hence alternating. Consequently, so is ℓ. Suppose that r_k is not in S' and that no r_i is u. Then $ss'r_1 \cdots r_k tus$ is a loop of G, hence alternating. Thus $s'r_1 \cdots r_k t$ is a (- -) alternating path in G and also in G'. Hence ℓ is alternating. Finally, suppose that r_k is not in S' and that r_j is u. Then $ss'r_1 \cdots r_{j-1}us$ and $tur_{j+1} \cdots r_k t$ are loops of G, hence alternating. Thus $s'r_1 \cdots r_{j-1}u$ is a (- +) alternating path, and $ur_{j+1} \cdots r_k t$ is a (- -) alternating path. Fitting these paths together and adjoining t at the beginning, we see that ℓ is alternating. This completes the proof of (a), of Lemma 9, and of the sufficiency condition of Theorem 4.

6. HOW TO RECOGNIZE THE UNIMODULAR PROPERTY

To apply Theorem 3 is easy, although even there one point is important. To say that A has the unimodular property is the same thing as to say that A^T, the transpose of A, has the unimodular property. However the hypotheses of Theorem 3 or 4 may quite easily be satisfied for A^T but not for A. Consequently it is desirable to examine both A and A^T when using these theorems.

To apply Theorem 4 is not so easy: how shall we recognize whether matrix A (or matrix A^T) is the incidence matrix of an alternating graph versus some set of directed paths? We point out that in actual applications the graph G generally lies close at hand. For example, it was pointed out in Section 5 that the coefficient matrix A of the i by j transportation problem is the incidence matrix of the alternating graph shown in Diagram 2 (at the beginning of Section 5) versus all its directed paths. This graph is no strange object - it portrays the i "producing points", the j "consuming points", and the transportation routes between them.

In a given linear programming problem there will often be one (or several) graphs which are naturally associated with the coefficient matrix. Whenever the problem can be stated in terms of producers, consumers, and intermediate handlers, this is the case. It may well be possible in this situation to identify the matrix as a suitable incidence matrix.

However it is still useful to have criteria available which can be applied directly to the matrix A and which guarantee that A can be obtained as a suitable incidence matrix. The two following theorems give such conditions. Each corresponds to a very special case of Theorem 4. Theorem 5, historically, derives from the integrality of transportation-type problems, and finds application in [2]; Theorem 6 from the integrality of certain caterer-type problems (see [1]).

We shall write $A_i \geqq A_j$ to indicate that row A_i is componentwise $\geqq$ row A_j.

THEOREM 5. Suppose A is a matrix of 0's and 1's, and suppose that the rows of A can be partitioned into two disjoint classes V_1 and V_2 with this property: if A_i and A_j are both in V_1 or both in V_2, and if there is a column A^k in which both A_i and A_j have a 1, then either $A_i \leqq A_j$ or $A_i \geqq A_j$. Then A has the unimodular property.

This theorem corresponds to a generalized transportation situation, in which each upper vertex of the transportation graph has attached an outward flowing tree and each lower vertex has attached an inward flowing tree. Only directed paths which have at least one vertex in the original transportation graph can be represented as columns of the matrix A.

PROOF. Briefly the proof is this: Let vertices v_i in V correspond to the rows A_i of A. Define a partial-order $\leqq$ on the vertices:

$$\begin{aligned} v_i \leqq v_j \quad & \text{if } A_i \text{ in } V_1 \text{ and } A_j \text{ in } V_2 \\ & \text{or } A_i, A_j \text{ in } V_1 \text{ and } A_i \leqq A_j \\ & \text{or } A_i, A_j \text{ in } V_2 \text{ and } A_i \geqq A_j. \end{aligned}$$

Let G be the graph naturally associated with this partially-ordered set. We leave to the reader verification of the fact that G is alternating, and that the columns of A represent directed paths in P.

Say that two column vectors of the same size consisting of 0's and 1's are <u>in accord</u> if the portions of them between (in the inclusive sense) their lowest common 1 and the lower of their highest separate 1's are identical.

THEOREM 6. Suppose A is a matrix of 0's and 1's, and suppose that the rows of A can be rearranged in such a way that every pair of columns is in accord. Then A has the unimodular property.

This theorem corresponds to a situation in which $c(G) = 0$, that is, every vertex has at most one predecessor (or to the dual situation in which every vertex has at most one successor). The columns of A may represent any directed paths in the graph.

PROOF. Let vertices v_i in V correspond to the rows A_i in A. Assume that the rows are already arranged as described above. Define E as follows:

> (v_i, v_j) is in E if $i > j$ and if there is a column A^k of A such that a_{ik} and a_{jk} are both 1 while all intervening entries are 0's.

Let G be the graph with vertices V and edges E. We leave to the reader verification of the fact that G is an alternating graph in which every vertex has at most one successor, and that the columns of A represent directed paths in G.

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