Integral Distance Graphs

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Abstract: Suppose D is a subset of all positive integers. The distance graph $G(Z, D)$ with distance set D is the graph with vertex set Z, and two vertices x and y are adjacent if and only if $|x - y| \in D$. This paper studies the chromatic number $\chi(Z, D)$ of $G(Z, D)$. In particular, we prove that $\chi(Z, D) \leq |D| + 1$ when $|D|$ is finite. Exact values of $\chi(G, D)$ are also determined for some D with $|D| = 3$. © 1997 John Wiley & Sons, Inc. J Graph Theory **25:** 287–294, 1997

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1. INTRODUCTION

What is the least number of colors needed to color all points of the Euclidean plane such that no two points of distance one have the same color? It is well known that four colors are necessary (see [14]) and seven colors are sufficient (see [12]). The exact number of colors needed remains to be determined, however. For more results on variations of this problem, see Section G10 of [6]. For good surveys, see [3, 13]. Motivated by the graph coloring problem, Eggleton [8] considered the following general problem. We refer [1] for general notation and terminology in graphs.

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Let S be any subset of a metric space with metric d. Denote by dist(S) the set of all distances realized by pairs of distinct points in S, i.e.,

$$\text{dist}(S) = \{d(x, y)\colon\ x \neq y \text{ in } S\}.$$

For each subset D of dist(S), the *distance graph* $G(S, D)$ with *distance set* D is the graph with vertex set S and edge set $\{\{x, y\}\colon\ d(x, y) \in D\}$.

A (*vertext*) *coloring* of a graph G is a function f from $V(G)$ to a set whose elements are called the *colors*. A *k-coloring* is a coloring that uses k colors. A *proper k-coloring* is a coloring f such that $f(x) \neq f(y)$ whenever x is adjacent to y. The *chromatic number* $\chi(G)$ of G is the minimum number k such that there exists a proper k-coloring of G. We use $\chi(S, D)$ as an abbreviation for $\chi(G(S, D))$.

When the metric space is the n-dimensional Euclidean space R^n, it is usually hard to determine $\chi(R^n, D)$ or $\chi(Z^n, D)$, see [2, 4, 5] for recent results. For the case of the real line, [10] studied $\chi(R, D)$ when D is the union of intervals or a finite set. For the case where D is a set of positive integers with $g =$ gcd(D), each component of $G(R, D)$ is isomorphic to $G(Z, D')$, where $D' = \{d'\colon\ gd' \in D\}$. So the authors of [10] also studied $\chi(Z, D)$ with gcd$(D) = 1$. In particular, they determined $\chi(Z, D)$ for $|D| = 2$. They also studied the case where D is a subset of all prime numbers (see [11]). Note that $\chi(Z, P) = 4$, where P is the set of all primes.

The purpose of this paper is to study $\chi(Z, D)$ for a general subset D of Z. In particular, we prove that $\chi(Z, D) \leq |D|+1$ when $|D|$ is finite. Exact values of $\chi(Z, D)$ for some special D with $|D| = 3$ are also determined. At the end of this paper, we make a conjecture for necessary and sufficient conditions on $|D|$ such that $|D| = 3$ and $\chi(Z, D) = 4$. After the paper was submitted, we were informed that the conjecture was also made by [15]. The conjecture was then studied by Deuber and Zhu [7, 17] and finally proved by Zhu [18].

2. $\chi(Z, D)$ WHEN D IS FINITE

Theorem 1. *For any finite set $D, \chi(Z, D) \leq |D| + 1$.*

Proof. We color the vertices in $G(Z, D)$ recursively as follows. First, $f(0) = 1$. When $f(j)$ are defined for $-i \leq j \leq i$, let $f(i + 1)$ be the minimum positive integer not in $A = \{f(j)\colon -i \leq j \leq i \text{ and } i + 1 - j \in D\}$ and $f(-i - 1)$ the minimum positive integer not in $B = \{f(j)\colon -i \leq j \leq i + 1 \text{ and } j + i + 1 \in D\}$. f is clearly a proper coloring for $G(Z, D)$. Note that $|A| \leq |D|$ and $|B| \leq |D|$ imply that $f(i + 1) \leq |D| + 1$ and $f(-i - 1) \leq |D| + 1$. So f is a proper $(|D| + 1)$-coloring. ∎

The upper bound in Theorem 1 is sharp, for it is easy to see that $\chi(Z, \{1, \ldots, k\}) = k + 1$.

Lemma 2. *If D contains no multiple of a fixed positive integer k, then $\chi(Z, D) \leq k$.*

Proof. Color each vertex x in $G(Z, D)$ by x mod k. Since x and y have the same color if and only if $|x - y|$ is a multiple of k, but D contains no multiple of k, this is a proper k-coloring. ∎

For simplicity, we assume that gcd$(D) = 1$ in the rest of this section. If $|D| = 1$, then it is not hard to see that $\chi(Z, D) = 2$. For the case of $|D| = 2$, $D = \{a, b\}$, where a and b are relatively prime, and so either they are both odd or they are of opposite parity.

Theorem 3. *If a and b are relatively prime positive odd integers, then $\chi(Z, \{a, b\}) = 2$.*

Proof. The theorem follows immediately from Lemma 2 and the fact that $\chi(Z, D) \geq 2$ for any nonempty D. ∎

Theorem 4 [10]. *If a and b are relatively prime positive integers of opposite parity, then* $\chi(Z, \{a, b\}) = 3$.

Proof. The theorem follows from Theorem 1 and the fact that $0, a, 2a, \ldots, ba, ba - b, ba - 2b, \ldots, 0$ is a cycle of length $a + b$ in $G(Z, \{a, b\})$. ∎

We now consider the case of $D = \{a, b, c\}$, where $0 < a < b < c$ are such that $\gcd\{a, b, c\} = 1$. For the case where a, b, c are odd integers, by Lemma 2, $\chi(Z, D) = 2$. On the other hand, if D contains at least one even number, then, by Theorems 1 and 4, $3 \leq \chi(Z, D) \leq 4$. It is generally not easy to determine the exact value of $\chi(Z, D)$.

Lemma 5. *Suppose $1 \leq a < b$ and $\{a, b, a + b\} \subseteq D$. If $G(Z, D)$ has a proper 3-coloring f, then $f(k) = f(k + g)$ for all $k \in Z$, where $g = \gcd\{b - a, 3a\}$.*

Proof. For any $k \in Z$, the four vertices $k, k - a, k + b, k + b - a$ are pairwise adjacent except possibly that k is not adjacent to $k + b - a$. Since f is a proper 3-coloring of $G(Z, D)$, we have

$$f(k) = f(k + b - a) \qquad \text{for all } k \in Z. \tag{2.1}$$

For any $k \in Z$, the four vertices $k, k + a, k + a + b, k + 2a + b$ are pairwise adjacent except that k is not adjacent to $k + 2a + b$. Again,

$$f(k) = f(k + 2a + b) \qquad \text{for all } k \in Z. \tag{2.2}$$

Now $g = \gcd\{b - a, 3a\} = \gcd\{b - a, 2a + b\}$. So there exist two integers x and y such that $g = x(b - a) + y(2a + b)$. The lemma follows from repeatedly applying (2.1) and (2.2). ∎

Theorem 6. *If $D = \{a, b, a + b\}$, where $1 \leq a < b$ and $\gcd\{a, b\} = 1$, then*

$$\chi(Z, D) = \begin{cases} 3, & \text{if } a \equiv b \pmod 3, \\ 4, & \text{if } a \not\equiv b \pmod 3. \end{cases}$$

Proof. Since D contains at least one even number and $\gcd\{a, b\} = \gcd\{a, a + b\} = \gcd\{b, a + b\} = 1$, by Theorem 4, $\chi(Z, D) \geq 3$. For the case where $a \equiv b \pmod 3$, D contains no multiple of 3. The theorem then follows from Lemma 2. For the case where $a \not\equiv b \pmod 3$, $\gcd\{b - a, 3a\} = \gcd\{b - a, a\} = \gcd\{b, a\} = 1$. Suppose $\chi(Z, D) = 3$, i.e., $G(Z, D)$ has a proper 3-coloring f. By Lemma 5, $f(k) = f(k + 1)$ for all $k \in Z$, which is impossible. So, $\chi(Z, D) = 4$ if $a \not\equiv b \pmod 3$. ∎

Theorem 1 of [11] is a special case of Theorem 7 below. Also see Theorems 15(e) and 17(b) of [10].

Theorem 7. *If n is an odd number of at least 5, then* $\chi(Z, \{2, 3, n, n + 2\}) = 4$.

Proof. Since $\{2, 3, n, n + 2\}$ contains no multiple of 4, $\chi(Z, \{2, 3, n, n + 2\}) \leq 4$. Suppose $G(Z, \{2, 3, n, n + 2\})$ has a proper 3-coloring f. Since $\{2, n, n + 2\} \subseteq \{2, 3, n, n + 2\}$, by Lemma 5, $f(k) = f(k + g)$ for all $k \in Z$ where $g = \gcd\{n - 2, 3 \times 2\} = 1$ or 3, which is impossible. ∎

Theorem 8. *If $c \geq 3$, then*

$$\chi(Z, \{1, 2, c\}) = \begin{cases} 4, & \text{if } c \equiv 0 \pmod 3, \\ 3, & \text{if } c \not\equiv 0 \pmod 3. \end{cases}$$

Proof. By Theorems 1 and 4, $3 \le \chi(Z,D) \le 4$. For the case where $c \equiv 0 \pmod 3$, suppose there exists a proper 3-coloring f. Since $\{k, k+1, k+2\}$ is a clique for any $k \in Z, f(k), f(k+1), f(k+2)$ are distinct. Consequently, $f(k) = f(k+3)$ for all $k \in Z$. This is impossible, because $f(0) = f(c)$ but $|0-c| \in D$. So $\chi(Z,D) = 4$ when $c \equiv 0 \pmod 3$. For the case where $c \not\equiv 0 \pmod 3$, $\chi(Z,D) \le 3$, and so $\chi(Z,D) = 3$, follows from Lemma 2. ∎

Lemma 9. *If x, y, a are integers and $a \ge 1$, then $\lfloor \frac{x}{a} \rfloor + \lfloor \frac{y}{a} \rfloor = \lfloor \frac{x+y}{a} \rfloor - \alpha = \lfloor \frac{x+y+1}{a} \rfloor - \beta$, where α and β are in $\{0,1\}$.*

Proof. Let $\lfloor \frac{x}{a} \rfloor = z$ and $\lfloor \frac{y}{a} \rfloor = w$. By definition,

$$az \le x \le az + a - 1 \quad \text{and} \quad aw \le y \le aw + a - 1.$$

Then

$$a(z+w) \le x + y < x + y + 1 \le a(z+w+1) + a - 1$$

and so

$$z + w \le \left\lfloor \frac{x+y}{a} \right\rfloor \le \left\lfloor \frac{x+y+1}{a} \right\rfloor \le z + w + 1.$$

The lemma then follows. ∎

Theorem 10. *If $a \ge 2$ and $c \ge a+2$, then*

$$\chi(Z, \{a, a+1, c\}) = \begin{cases} 4, & \text{if } c = 2a+1, \\ 3, & \text{if } c \ne 2a+1. \end{cases}$$

Proof. The case of $c = 2a+1$ follows from Theorem 6. For the other cases, $3 \le \chi(Z, \{a, a+1, c\})$ follows from Theorem 4. The rest of this proof presents a proper 3-coloring function f for $G(Z, \{a, a+1, c\})$.

Case 1. $\lfloor \frac{a+c-1}{a} \rfloor \not\equiv 0 \pmod 3$.

For any integer k, write $k = qc + r$ with $0 \le r < c$ and define $f(k) = q\lfloor \frac{a+c-1}{a} \rfloor + \lfloor \frac{x}{a} \rfloor \pmod 3$. We shall check whether f is a proper 3-coloring. First,

$$\begin{aligned} f(k+c) &= f((q+1)c + r) \\ &= (q+1)\left\lfloor \frac{a+c-1}{a} \right\rfloor + \left\lfloor \frac{r}{a} \right\rfloor \pmod 3 \\ &= f(k) + \left\lfloor \frac{a+c-1}{a} \right\rfloor \pmod 3 \\ &\ne f(k) \pmod 3. \end{aligned}$$

Next, for $a' = a$ or $a+1$, let $a'' = a' - a \in \{0,1\}$. For the case where $0 \le r + a' < c$,

$$\begin{aligned} f(k+a') &= f(qc + (r+a')) \\ &= q\left\lfloor \frac{a+c-1}{a} \right\rfloor + \left\lfloor \frac{r+a'}{a} \right\rfloor \pmod 3 \\ &= f(k) + 1 \text{ or } f(k) + 2 \pmod 3 \\ &\quad (\text{since } r + a \le r + a' < r + 2a) \\ &\ne f(k) \pmod 3. \end{aligned}$$

For the case where $r + a' \ge c, 0 \le r + a' - c < a' < c$ and so

$$\begin{aligned} f(k+a') &= f((q+1)c+(r+a'-c)) \\ &= (q+1)\left\lfloor \frac{a+c-1}{a} \right\rfloor + \left\lfloor \frac{r+a'-c}{a} \right\rfloor (\mathrm{mod}\ 3). \end{aligned}$$

By Lemma 9,

$$\begin{aligned} \left\lfloor \frac{a+c-1}{a} \right\rfloor + \left\lfloor \frac{r+a'-c}{a} \right\rfloor &= \left\lfloor \frac{(a+c-1)+(r+a'-c)+(1-a'')}{a} \right\rfloor - \alpha \\ &= \left\lfloor \frac{r}{a} \right\rfloor + 2 - \alpha, \end{aligned}$$

where $\alpha \in \{0,1\}$. So

$$\begin{aligned} f(k+a') &= q\left\lfloor \frac{a+c-1}{a} \right\rfloor + \left\lfloor \frac{r}{a} \right\rfloor + 2 - \alpha\,(\mathrm{mod}\ 3) \\ &= f(k)+1 \text{ or } f(k)+2\,(\mathrm{mod}\ 3) \\ &\neq f(k)\,(\mathrm{mod}\ 3). \end{aligned}$$

Case 2. $\lfloor \frac{a+c-1}{a} \rfloor \equiv \lfloor \frac{a+c-2}{a} \rfloor \equiv 0\,(\mathrm{mod}\ 3)$.
For any integer k, write $k = qc + r$ with $0 \leq r < c$ and define

$$f(k) = \begin{cases} q + \lfloor \frac{r}{a} \rfloor \,(\mathrm{mod}\ 3), & \text{if } r \leq a, \\ q + \lfloor \frac{r+a-1}{a} \rfloor \,(\mathrm{mod}\ 3), & \text{if } r \geq a. \end{cases}$$

Note that the function is well defined as $\lfloor \frac{r}{a} \rfloor = \lfloor \frac{r+a-1}{a} \rfloor$ for $r = a$. We shall check whether f is a proper 3-coloring. First,

$$\begin{aligned} f(k+c) &= f((q+1)c+r) \\ &= \begin{cases} (q+1) + \lfloor \frac{r}{a} \rfloor \,(\mathrm{mod}\ 3), & \text{if } r \leq a, \\ (q+1) + \lfloor \frac{r+a-1}{a} \rfloor \,(\mathrm{mod}\ 3), & \text{if } r \geq a, \end{cases} \\ &= f(k)+1\,(\mathrm{mod}\ 3) \\ &\neq f(k)\,(\mathrm{mod}\ 3). \end{aligned}$$

Next, for $a' = a$ or $a+1$, let $a'' = a' - a \in \{0,1\}$. For the case where $r + a' < c$,

$$\begin{aligned} f(k+a') &= f(qc+(r+a')) \\ &= q + \left\lfloor \frac{(r+a')+a-1}{a} \right\rfloor (\mathrm{mod}\ 3) \\ &\quad (\text{since } r+a' \geq a) \\ &= f(k)+1 \text{ or } f(k)+2\,(\mathrm{mod}\ 3) \\ &\quad (\text{since } r+2a-1 \leq (r+a')+a-1 \leq r+2a) \\ &\neq f(k)\,(\mathrm{mod}\ 3). \end{aligned}$$

For the case where $r + a' \geq c$,

$$
\begin{aligned}
f(k+a') &= f((q+1)c+(r+a'-c)) \\
&= q+1+\left\lfloor \frac{r+a'-c}{a} \right\rfloor \text{(mod 3)} \\
&\quad \text{(since } 0 \le r+a'-c \le a\text{)} \\
&= q+1+\left\lfloor \frac{r+a'-c}{a} \right\rfloor + \left\lfloor \frac{a+c-1-a''}{a} \right\rfloor \text{(mod 3)} \\
&\quad \text{(by the assumption that } \left\lfloor \frac{a+c-1}{a} \right\rfloor \equiv \left\lfloor \frac{a+c-2}{a} \right\rfloor \equiv 0 \,\text{(mod 3))} \\
&= q+1+\left\lfloor \frac{r+2a-1}{a} \right\rfloor - \alpha \text{ (mod 3)} \\
&\quad \text{(by Lemma 9, where } \alpha \in \{0,1\}\text{)} \\
&= q+\left\lfloor \frac{r+a-1}{a} \right\rfloor + 2-\alpha \,\text{(mod 3)} \\
&= f(k)+1 \text{ or } f(k)+2 \,\text{(mod 3)} \\
&\quad \text{(since } c \ge 2a+1 \text{ and so } r \ge c-a' \ge a\text{)} \\
&\ne f(k) \,\text{(mod 3)}.
\end{aligned}
$$

Case 3. $\lfloor \frac{a+c-1}{a} \rfloor \equiv 0 \,\text{(mod 3)}$ and $\lfloor \frac{a+c-2}{a} \rfloor \not\equiv 0 \,\text{(mod 3)}$.
For any integer k, write $k = qc + r$ with $0 \le r < c$ and define

$$
f(k) = \begin{cases} q + \lfloor \frac{r}{a} \rfloor \text{ (mod 3)}, & \text{if } r \le a, \\ q + \lfloor \frac{r+a-1}{a} \rfloor \text{ (mod 3)}, & \text{if } a \le r \le 3a+1, \\ q + \lfloor \frac{r+2a-2}{a} \rfloor \text{ (mod 3)}, & \text{if } r \ge 3a+1. \end{cases}
$$

Note that the function is well defined as $\lfloor \frac{r}{a} \rfloor = \lfloor \frac{r+a-1}{a} \rfloor$ for $r = a$ and $\lfloor \frac{r+a-1}{a} \rfloor = \lfloor \frac{r+2a-2}{a} \rfloor$ for $r = 3a+1$. We shall check whether f is a proper 3-coloring. First,

$$
\begin{aligned}
f(k+c) &= f((q+1)c+r) \\
&= \begin{cases} (q+1) + \lfloor \frac{r}{a} \rfloor \text{ (mod 3)}, & \text{if } r \le a, \\ (q+1) + \lfloor \frac{r+a-1}{a} \rfloor \text{ (mod 3)}, & \text{if } a \le r \le 3a+1, \\ (q+1) + \lfloor \frac{r+2a-2}{a} \rfloor \text{ (mod 3)}, & \text{if } r \ge 3a+1, \end{cases} \\
&= f(k)+1 \text{(mod 3)} \\
&\ne f(k) \text{(mod 3)}.
\end{aligned}
$$

Next, for $a' = a$ or $a+1$, let $a'' = a' - a \in \{0,1\}$. For the case where $r + a' < c$, we shall discuss two sub-cases: $a \le r + a' \le 3a+1$ and $r + a' \ge 3a+1$. If $a \le r+a' \le 3a+1$, then $0 \le r \le 3a+1$ and so

$$
\begin{aligned}
f(k+a') &= f(qc+(r+a')) \\
&= q + \left\lfloor \frac{(r+a')+a-1}{a} \right\rfloor \text{(mod 3)}
\end{aligned}
$$

$$\begin{aligned} &= f(k)+1 \text{ or } f(k)+2\,(\text{mod } 3) \\ &\quad (\text{since } r+2a-1 \le (r+a')+a-1 \le r+2a) \\ &\ne f(k)\,(\text{mod } 3). \end{aligned}$$

If $r + a' \ge 3a + 1$, then $r \ge a$ and so

$$\begin{aligned} f(k+a') &= f(qc+(r+a')) \\ &= q + \left\lfloor \frac{(r+a')+2a-2}{a} \right\rfloor (\text{mod } 3) \\ &= f(k)+1 \text{ or } f(k)+2\,(\text{mod } 3) \\ &\quad (\text{since } r+3a-2 \le (r+a')+2a-2 \le r+3a-1) \\ &\ne f(k)\,(\text{mod } 3). \end{aligned}$$

For the case where $r + a' \ge c, 0 \le r + a' - c \le a$ and so

$$\begin{aligned} f(k+a') &= f((q+1)c+(r+a'-c)) \\ &= q+1+\left\lfloor \frac{r+a'-c}{a} \right\rfloor (\text{mod } 3). \end{aligned}$$

Note that $\lfloor \frac{a+c-1}{a} \rfloor \equiv 0\,(\text{mod } 3)$ and $\lfloor \frac{a+c-2}{a} \rfloor \not\equiv 0\,(\text{mod } 3)$ imply that $a + c \equiv 1 \pmod{3a}$ and then $\lfloor \frac{a+c-2}{a} \rfloor \equiv \lfloor \frac{a+c-3}{a} \rfloor \equiv 2\,(\text{mod } 3)$. Thus

$$\begin{aligned} f(k+a') &= q+1+\left\lfloor \frac{r+a'-c}{a} \right\rfloor + \left\lfloor \frac{a+c-2-a''}{a} \right\rfloor + 1\,(\text{mod } 3) \\ &= q+1+\left\lfloor \frac{r+2a-2}{a} \right\rfloor - \alpha + 1\,(\text{mod } 3) \\ &\quad (\text{by Lemma 9, where } \alpha \in \{0,1\}). \end{aligned}$$

Note that $a + c \equiv 1 \pmod{3a}$ and $c \ne 2a + 1$ imply $c \ge 5a + 1$ and so $r \ge c - a' \ge 3a + 1$. Then

$$\begin{aligned} f(k+a') &= f(k)+1 \text{ or } f(k)+2\,(\text{mod } 3) \\ &\ne f(k)\,(\text{mod } 3). \end{aligned}$$

∎

The following corollary generalizes Theorem 16 of [10].

Corollary 11. *For any* $n \ge 6, \chi(Z, \{2, 3, n\}) = 3.$

3. CONCLUDING REMARKS

This paper first proves that $\chi(Z, D) \le |D| + 1$ when $|D|$ is finite. We then determine the exact values of $\chi(Z, D)$ for some special D. In particular, we consider the case where $D = \{a, b, c\}$ with $0 < a < b < c$ and $\gcd\{a, b, c\} = 1$. We prove that $\chi(Z, D) = 2$ if a, b, c are all odd. $\chi(Z, D) = 3$ if a, b, c are not all odd and one of (*i*)–(*iii*) holds:

(*i*) $c = a + b$ and $a \equiv b\,(\text{mod } 3)$.

(*ii*) $a = 1, b = 2$, and $c \not\equiv 0\,(\text{mod } 3)$.

(*iii*) $a \ge 2, b = a + 1$, and $c \ne 2a + 1$.

$\chi(Z,D)=4$ if a,b,c are not all odd and one of (*iv*)–(*v*) holds:
(*iv*) $c=a+b$ and $a\not\equiv b\,(\text{mod }3)$.
(*v*) $a=1, b=2$, and $c\equiv 0\,(\text{mod }3)$.

We close this paper with the following conjecture:

Conjecture. For $D=\{a,b,c\}$, where $0<a<b<c$, $\gcd\{a,b,c\}=1$, and a,b,c are not all odd, $\chi(Z,D)=4$ if and only if (*iv*) or (*v*) holds.

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