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**INTEGRAL DOMAINS THAT ARE NOT EMBEDDABLE IN
DIVISION RINGS**

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A class of totally ordered rings V is constructed having the property $1 < \alpha \in V \Rightarrow 1/\alpha \in V$, but such that V cannot be embedded in any division ring.

1. Inverses in semigroup power series rings. This note has only one objective—to construct the above class of counterexamples (see [6]).

NOTATION 1.1. Throughout Γ will be a totally ordered cancellative semigroup with identity e ; R will denote any totally ordered division ring. If $\alpha: \Gamma \rightarrow R$ is any function, then the *support* of α is the set $\text{supp } \alpha = \{s \in \Gamma \mid \alpha(s) \neq 0\}$. The set $V = V(\Gamma, R)$ of all functions α such that $\text{supp } \alpha$ satisfies the a.c.c. (ascending chain condition) form a totally ordered abelian group. If Γ is cancellative, then under the usual power series multiplication (see [3]), V is a totally ordered ring.

1.2. Any $1 < \alpha \in V$ with $\alpha(s) = 0$ for $s > e$ may be written $\alpha = \alpha(e)(1 - \lambda)$, where $1 \leq \alpha(e)$ and $\lambda = \sum \{\lambda(a)a \mid a < e\}$. It will be shown that

$$(1 - \lambda)^{-1} = 1 + \lambda + \lambda^2 + \dots = 1 + \sum_s \sum' \lambda(a(1)\lambda(a(2)) \dots \lambda(a(n)),$$

where the finite sum Σ' is over all integers and over all distinct n -tuples of Γ^n satisfying $s = a(1)a(2) \dots a(n)$ with each $a(i) < e$; the sum Σ is over all $s < e$. To prove that $1/\alpha \in V$ it suffices to establish conditions (a) and (b) below.

(a) For each $s \in \Gamma$, there are only a finite number of n with $\lambda^n(s) \neq 0$;

(b) $\text{supp}(1 - \lambda)^{-1}$ satisfies the a.c.c.

Assuming (a) and (b), the main theorem follows at once. By adjoining an identity as in [8; p. 158] to the semigroup in [2] a semigroup that actually satisfies the hypothesis in (ii) below can be constructed.

MAIN THEOREM 1.3. *If Γ is a totally ordered cancellative semigroup with identity e and R any totally ordered division ring, then the power series ring $V = V(\Gamma, R)$ has the following properties:*

(i) $1 < \alpha \in V$ and $\alpha(s) = 0$ for $s > e \implies 1/\alpha \in V$.

(ii) *If in addition Γ cannot be embedded in a group, then V*

cannot be embedded in a division ring.

An already known result ([8; p. 135]) follows immediately from 1.3 (i).

COROLLARY 1.4. *If in addition Γ is a group, then $V(\Gamma, R)$ is a division ring.*

2. Proof of the main theorem. Assume 1.2 (a) or (b) fails. Then a lengthy but elementary argument shows there exists a doubly indexed matrix $\{a(i, j) \in \text{supp } \lambda \mid 1 \leq i < \infty; 1 \leq j \leq n(i)\}$ such that the products $u(i) = a(i, 1)a(i, 2) \cdots a(i, n(i))$ of the rows form an infinite properly ascending chain. Eventually a contradiction will be derived from this. Without loss of generality assume $\Gamma \leq e$.

DEFINITION 2.1. For any totally ordered semigroup Γ with identity e and any element $a \in \Gamma$ with $a \leq e$, define a semigroup by

$$\Gamma(a) = \{q \in \Gamma \mid \exists \text{ an integer } m > 0, q^m \leq a\}.$$

LEMMA 2.2. *With Γ as above, for any $a(1), \dots, a(m) \in \Gamma$ with each $a(j) \leq e$, set $u = a(1)a(2) \cdots a(m)$ and define*

$$a^* = \min \{a(1), \dots, a(m)\}.$$

Then $\Gamma(u) = \Gamma(a^)$.*

2.3. Consider a fixed subset $L \subseteq \Gamma$ all of whose elements satisfy $L \leq e$ and where L satisfies the a.c.c., e.g., $L = \text{supp } \lambda < e$. Consider an array of elements $A = \{a(i, j) \mid 1 \leq i < \infty, 1 \leq j \leq n(i)\} \subseteq L$, where repetitions in the $a(i, j)$ are allowed. Assume all $n(i) \geq 2$. Define $u(i) = u(i, A)$ by

$$u(i) = u(i, A) = a(i, 1)a(i, 2) \cdots a(i, n(i)).$$

Let \mathcal{X} be the set of all such $A = \{a(i, j)\}$ for which $u(1) < u(2) < \cdots < u(i) < \cdots$ is strictly ascending at each i . With each member $A = \{a(i, j)\} \in \mathcal{X}$, we next associate three objects

$$\{a(i)^* \mid 1 \leq i < \infty\}, m = m(A), \text{ and } G = G(A).$$

Define $a(i)^* \equiv \min \{a(i, j) \mid 1 \leq j \leq n(i)\}$. Note that $u(1) < u(2) < \cdots$ implies that $\Gamma(a(1)^*) \subseteq \Gamma(a(2)^*) \subseteq \Gamma(a(i)^*) \subseteq \cdots$. Thus since L satisfies the a.c.c., there is a unique smallest integer $m \equiv m(A)$ such that the semigroups $G \equiv \Gamma(a(m)^*) = \Gamma(a(m+1)^*) = \cdots$ are all equal. The following schematic diagram of all these quantities may be helpful.

$$\begin{aligned}
 \Gamma(a(1)^*) &= \Gamma(u(1)) & u(1) &= a(1, 1)a(1, 2) \cdots a(1)^* \cdots a(1, n(1)) \\
 &\quad \cap \\
 \Gamma(a(2)^*) &= \Gamma(u(2)) & u(2) &= a(2, 1)a(2, 2) \cdots a(2)^* \cdots a(2, n(2)) \\
 &\quad \cap \\
 \Gamma(a(m)^*) &= \Gamma(u(m)) & u(m) &= a(m, 1)a(m, 2) \cdots a(m)^* \cdots a(m, n(m)) \\
 &\quad \cap \\
 \mathbf{G} &= \Gamma(u(m + 1)) .
 \end{aligned}$$

2.4. Among the elements of \mathcal{H} , let $\mathcal{N} \subset \mathcal{H}$ be all those $A = \|a(i, j)\|$ such that this associated $\mathbf{G} = \mathbf{G}(A)$ is as big as possible and call this particular $\mathbf{G} \equiv \mathbf{M}$. If $\mathcal{H} \neq \emptyset$, also $\mathcal{N} \neq \emptyset$. Define $\bar{a} = \max \{a(m)^* \mid A \in \mathcal{H}, m = m(A)\}$. Pick an element $B = \|b(i, j)\| \in \mathcal{N}$. Then by our choice of \mathbf{M} , $\Gamma(\bar{a}) = \mathbf{M}$. Thus $\mathbf{M} = \mathbf{G}(B) = \Gamma(b(i)^*) = \Gamma(b(i, j)) = \Gamma(u(i)) = \Gamma(\bar{a})$ for $i \geq m(B) \equiv m$. Finally, with each element B of \mathcal{N} , we associate an integer $r = r(B)$. Since $\bar{a} \in \Gamma(u(m))$, there is a unique smallest integer $r \equiv r(B) \geq 1$ such that $\bar{a}^r \leq u(m) < \bar{a}^{r-1}$.

2.5. By omitting some of the rows of B and renumbering the remaining ones, it may be assumed as a consequence of the a.c.c. without loss of generality that $m = 1$, and also that $b(1)^* \geq b(2)^* \geq \cdots$ is not ascending. Each $u(i)$ is of one of the following three forms:

- (1) $u(i) = q(i)b(i)^*$,
- (2) $u(i) = b(i)^*w(i)$,
- (3) $u(i) = q(i)b(i)^*w(i)$,

where the $q(i), w(i)$ are certain products of the $b(i, j)$. If there are an infinite number of $u(i)$ of the forms (1) or (2), then since

$$\begin{aligned}
 u(i + 1) &= q(i + 1)b(i + 1)^* > u(i) = q(i)b(i)^*, \quad b(i + 1)^* \leq b(i)^* \\
 &\implies q(i + 1) > q(i) ;
 \end{aligned}$$

it follows (after omitting some rows and renumbering) that there is a properly infinite ascending chain:

Case 1. $q(1) < q(2) < \cdots$;

Case 2. $w(1) < w(2) < \cdots$.

If neither Case 1 nor Case 2 applies, then

$$\begin{aligned}
 u(i + 1) &= q(i + 1)b(i + 1)^*w(i + 1) > q(i)b(i)^*w(i) \\
 &\quad \text{and } b(i + 1)^* \leq b(i)^*
 \end{aligned}$$

implies that one of the inequalities $q(i + 1) > q(i)$ or $w(i + 1) > w(i)$

must necessarily hold. It is asserted that there is a subsequence $\{i(k) \mid k = 1, 2, \dots\}$ such that

Case 3. either (a): $q(i(1)) < q(i(2)) < \dots$
or (b): $w(i(1)) < w(i(2)) < \dots$.

For if not, then the a.c.c. must hold in both the sets $\{q(i)\}$ and $\{w(i)\}$. Then by omitting some rows and renumbering the remaining ones it may be assumed that we have an element B in \mathcal{N} with $q(1) \geq q(2) \geq \dots$ and $w(1) \geq w(2) \geq \dots$. However, then

$$q(1)b(1)^*w(1) \geq q(2)b(2)^*w(2) \geq \dots$$

gives a contradiction.

2.6. We may assume $q(1) < q(2) < \dots$ or $w(1) < w(2) < \dots$ are properly ascending, depending on which of the Cases 1, 2, 3(a) or 3(b) is applicable. Set $t = r(B)$, so that $\bar{a}^t \leq u(m) = u(1) \leq u(i)$.

2.7. It is next shown that either $q(i) \geq \bar{a}^{t-1}$ or $w(i) \geq \bar{a}^{t-1}$ holds for all i . Suppose that the following holds.

Case 1. $q(1)b(1)^* < q(2)b(2)^* < \dots$;
 $q(1) < q(2) < \dots$;
 $b(1)^* \geq b(2)^* \geq \dots$.

Then $\bar{a}^t \leq u(1) \leq u(i) = q(i)b(i)^*$, and $\bar{a} \geq b(i)^*$ implies that

$$\bar{a}^{t-1} \leq q(1) \leq q(i).$$

(For if $\bar{a}^{t-1} > q(i)$, then $\bar{a} \geq b(i)^*$ implies that $\bar{a}^t > q(i)b(i)^*$.) (If $t = 1$, then $\bar{a}^0 = e$.) Similarly, in Case 2 also $\bar{a}^{t-1} \leq w(1) \leq w(i)$.

Only Case 3(b) will be proved, since 3(a) is entirely parallel.

Case 3(b). $q(1)b(1)^*w(1) < q(2)b(2)^*w(2) < \dots$;
 $w(1) < w(2) < \dots$;
 $b(1)^* \geq b(2)^* \geq \dots$.

Then again $\bar{a}^t \leq u(1) \leq u(i) = q(i)b(i)^*w(i)$ and $\bar{a} \geq b(i)^* \geq q(i)b(i)^*$ imply that $\bar{a}^{t-1} \leq w(1) \leq w(i)$. (Otherwise, if $\bar{a}^{t-1} > w(i)$, then $\bar{a}^t > q(i)b(i)^*w(i)$.)

The basic idea motivating the proof is that for $B \in \mathcal{N}$, a new $C \in \mathcal{N}$ can be constructed with $r(C) \leq r(B) - 1$.

2.8. Thus either $q(1) < q(2) < \dots$ and all $q(i) \geq \bar{a}^{t-1}$; or $w(1) < w(2) < \dots$ and all $w(i) \geq \bar{a}^{t-1}$. Assume the latter. Let

$$C = \|c(i, j)\| \in \mathcal{N}$$

be defined by taking as its i -th row all the $b(i, j)$ appearing in $w(i)$. (In view of $w(1) < w(2) < \dots$, there does not exist an infinite number of rows of C containing only one element. By omitting a finite number of rows it may be assumed that all rows of C contain two or more elements of L .) Define $c(i)^* \equiv \inf \{c(i, j) \mid j \geq 1\}$. Since $b(i)^* \leq c(i)^* \leq \bar{a}$, it follows that

$$M = \Gamma(b(i)^*) \subseteq \Gamma(c(i)^*) \subseteq \Gamma(\bar{a}) = M.$$

Consequently, $G(C) = M$ and $C \in \mathcal{N}$. Since $w(1) \geq \bar{a}^{t-1}$, $r(C) \leq t - 1$. By repetition of this process, we may reduce the r to one so that finally $\bar{a}^r = \bar{a} \leq w(1) < w(2) \dots$. Since all $c(i, j) \in L$ satisfy $c(i, j) \leq e$ and since $w(i)$ is a product of these, it follows that $\bar{a} \geq c(i)^* \geq w(i)$. Thus $\bar{a} = w(1) = w(2) = \dots$ gives a contradiction. Thus $\mathcal{N} = \emptyset$ and the main theorem has been proved.

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