# Integral equations and applications

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The goal of this Section is to formulate some of the basic results on the theory of integral equations and mention some of its applications. The literature of this subject is very large. Proofs are not given due to the space restriction. The results are taken from the works mentioned in the references.

## 1. Fredholm equations

*1.1. Fredholm alternative*

One of the most important results of the theory of integral equations is the Fredholm alternative. The results in this Subsection are taken from [5], [20],[21], [6].

Consider the equation

$$u = Ku + f, \qquad Ku = \int_D K(x,y)u(y)dy. \tag{1}$$

Here $f$ and $K(x,y)$ are given functions, $K(x,y)$ is called the kernel of the operator $K$. Equation (1) is considered usually in $H = L^2(D)$ or $C(D)$. The first is a Hilbert space with the norm $||u|| := \big(\int_D |u|^2dx\big)^{1/2}$ and the second is a Banach space with the norm $||u|| = \sup_{x\in D}|u(x)|$. If the domain $D \subset \mathbb{R}^n$ is assumed bounded, then $K$ is compact in $H$ if, for example, $\int_D\int_D |K(x,y)|^2dxdy < \infty$, and $K$ is compact in $C(D)$ if, for example, $K(x,y)$ is a continuous function, or, more generally, $sup_{x\in D}\int_D |K(x,y)|dy < \infty$.
For equation (1) with compact operators, the basic result is the Fredholm alternative. To formulate it one needs some notations. Consider equation

(1) with a parameter $\mu \in \mathbb{C}$:

$$u = \mu K u + f, \tag{2}$$

and its adjoint equation

$$v = \bar{\mu} K^* v + g, \qquad K^*(x, y) = \overline{K(y, x)}, \tag{3}$$

where the overbar stands for complex conjugate. By ($2_0$) and ($3_0$) the corresponding equations with $f = 0$, respectively, $g = 0$ are denoted. If $K$ is compact so is $K^*$. In $H$ one has $(Ku, v) = (u, K^* v)$, where $(u, v) = \int_D u \bar{v} dx$. If equation ($2_0$) has a non-trivial solution $u_0$, $u_0 \neq 0$, then the corresponding parameter $\mu$ is called the characteristic value of $K$ and the non-trivial solution $u_0$ is called an eigenfunction of $K$. The value $\lambda = \frac{1}{\mu}$ is then called an eigenvalue of $K$. Similar terminology is used for equation ($3_0$). Let us denote by $N(I - \mu K) = \{u : u = \mu K u\}$, $R(I - \mu K) = \{f : f = (I - \mu K) u\}$. Let us formulate the Fredholm alternative:

**Theorem 1.1.** *If* $N(I - \mu K) = \{0\}$*, then* $N(I - \bar{\mu} K^*) = \{0\}, R(I - \mu K) = H, R(I - \bar{\mu} K^*) = H$*.*
*If* $N(I - \mu K) \neq \{0\}$*, then* $\dim N(I - \mu K) = n < \infty, \dim N(I - \bar{\mu} K^*) = n, f \in R(I - \mu K)$ *if and only if* $f \perp N(I - \bar{\mu} K^*)$*, and* $g \in R(I - \bar{\mu} K^*)$ *if and only if* $g \perp N(I - \mu K)$*.*

*Remark* 1.2. Conditions $f \perp N(I - \bar{\mu} K^*)$ and $g \perp N(I - \mu K)$ are often written as $(f, v_{0j}) = 0, 1 \leq j \leq n$, and, respectively, $(g, u_{0j}) = 0, 1 \leq j \leq n$, where $\{v_{0j}\}_{j=1}^n$ is a basis of the subspace $N(I - \bar{\mu} K^*)$, and $\{u_{0j}\}_{j=1}^n$ is a basis of the subspace $N(I - \mu K)$.

*Remark* 1.3. Theorem 1.1 is very useful in applications. It says that equation (1) is solvable for any $f$ as long as equation ($2_0$) has only the trivial solution $v_0 = 0$. A simple proof of Theorem 1.1 can be found in [20] and [21].

*Remark* 1.4. Theorem 1.1 can be stated similarly for equations (2) and (3) in $C(D)$.

*Remark* 1.5. If $n > 0$ and $f \in R(I - \mu K)$, then the general solution to equation (2) is $u = \tilde{u} + \sum_{j=1}^{n} c_j u_{0j}$, where $c_j, 1 \leq j \leq n$, are arbitrary constants and $\tilde{u}$ is a particular solution to (2). Similar result holds for $v$.

*1.2. Degenerate kernels*

Suppose that

$$K(x,y) = \sum_{m=1}^{M} a_m(x)\overline{b_m(y)}, \tag{4}$$

where the functions $a_m$ and $b_m$ are linearly independent, $a_m, b_m \in H$. The operator $K$ with the degenerate kernel (4) is called a finite-rank operator. Equation (2) with degenerate kernel (4) can be reduced to a linear algebraic system. Denote

$$u_m = \int_D u\overline{b_m}dx. \tag{5}$$

Multiply equation (2) by $\overline{b_p}$, integrate over $D$, and get

$$u_p = \mu \sum_{m=1}^{M} a_{pm}u_m + f_p, \quad 1 \le p \le M, \tag{6}$$

where $a_{pm} := (a_m, b_p)$. There is a one-to-one correspondence between solutions to equation (2) and (6):

$$u(x) = \mu \sum_{m=1}^{M} a_m(x)u_m + f(x). \tag{7}$$

*Exercise 1.* Solve the equation $u(x) = \mu \int_0^\pi \sin(x-t)u(t)dt + 1$.

*Hint:* Use the formula $\sin(x-t) = \sin x \cos t - \cos x \sin t$. Find characteristic values and eigenfunctions of the operator with the kernel $\sin(x-t)$ on the interval $(0,\pi)$.

*Remark* 1.6. If $K$ is a compact integral operator in $L^2(D)$ then there is a sequence $K_n(x,y)$ of degenerate kernels such that $\lim_{n\to\infty} ||K - K_n|| = 0$, where $||K||$ is the norm of the operator $K$, $||K|| := \sup_{u\neq 0} \frac{||Ku||}{||u||}$.

*1.3. Volterra equations*

If $K(x,y) = 0$ for $y > x$ and $\dim D = 1$ then equation (2) is called Volterra equation. It is of the form

$$u(x) = \mu \int_a^x K(x,y)u(y)dy + f(x), \quad a \le x \le b. \tag{8}$$

This is a particular case of the Fredholm equations. But the Volterra equation (8) has special property.

**Theorem 1.7.** *Equation (8) is uniquely solvable for any* $\mu$ *and any* $f$ *in* $C(D)$*, if the function* $K(x, y)$ *is continuous in the region* $a \leq y \leq x, \quad a \leq x \leq b$*. The solution to (8) can be obtained by iterations:*

$$u_{n+1}(x) = \mu V u_n + f, \quad u_1 = f; \quad Vu := \int_a^x K(x, y)u(y)dy. \tag{9}$$

*1.4. Selfadjoint operators*

If $K^* = K$ then $K$ is called selfadjoint. In this case equation (2) has at least one characteristic value, all characteristic values are real numbers, and the corresponding eigenfunctions are orthogonal. One can solve equation (2) with selfadjoint operator $K$ by the formula

$$u = \sum_{j=1}^{\infty} \frac{\mu_j f_j}{\mu_j - \lambda} u_{0j}(x), f_j = (f, u_{0j}), \tag{10}$$

where $u_{0j} = \mu_j K u_{oj}, \; j = 1, 2, \ldots.$
For the eigenvalues of a selfadjoint compact operator $K$ the following result (minimax representation) holds. We write $K \geq 0$ if $(Ku, u) \geq 0, \forall u \in H$, and $A \leq B$ if $(Au, u) \leq (Bu, u), \forall u \in H$.

**Theorem 1.8.** *If* $K = K^* \geq 0$ *is compact in* $H$*,* $Ku_j = \lambda_j u_j, \; u_j \neq 0$ *then*

$$\lambda_{j+1} = \max_{u \perp L_j} \frac{(Ku, u)}{||u||^2}, \quad j = 0, 1, \ldots \tag{11}$$

*Here* $L_0 = \{0\}, L_j$ *is the subspace with the basis* $\{u_1, \ldots, u_j\}, \; ||u_j|| = 1, \; (u_i, u_m) = \delta_{im}$*. Also*

$$\lambda_{j+1} = \min_{M} \max_{u \perp M} \frac{(Ku, u)}{||u||^2}, \quad \dim M = j, \tag{12}$$

*where* $M$ *runs through the set of* $j$*-dimensional subspaces of* $H$*.*

One has $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. The set of all eigenfunctions of $K$, including the ones corresponding to the zero eigenvalue, if such an eigenvalue exists, forms an orthonormal basis of $H$.

*Remark* 1.9. If $0 \leq A \leq B$ are compact operators in $H$, then $\lambda_j(A) \leq \lambda_j(B)$.

*Remark* 1.10. If $A$ is a linear compact operator in $H$, then $A = U|A|$, where $|A| = (A^*A)^{1/2} \geq 0$ and $U$ is a partial isometry,

$$U : R(|A|) \to R(A), \ ||U|A|f|| = ||Af|| \ \forall f \in H, \ N(U) = N(A),$$

and $\overline{R(|A|)} = \overline{R(A^*)}$, where the overline denotes the closure in $H$. One has

$$Au = \sum_{j=1}^{r(A)} s_j(A)(u, u_j)v_j. \tag{13}$$

Here $r(A) = \dim R(|A|)$, $s_j \geq 0$ are the eigenvalues of $|A|$. They are called the $s$-values of $A$, $\{u_j\}$ and $\{v_j\}$ are orthonormal systems, $|A|u_j = s_j u_j$, $s_1 \geq s_2 \geq \dots$, $v_j = Uu_j$.

*Remark* 1.11. If $A$ is a linear compact operator in $H$ then

$$s_{j+1}(A) = \min_{K \in K_j} ||A - K||, \tag{14}$$

where $K_j$ is the set of finite rank operators of dimension $\leq j$, that is, $dim R(K_j) = j$.

*1.5. Equations of the first kind*

If $Ku = \lambda u + f$ and $\lambda = 0$, then one has

$$Ku = f. \tag{15}$$

If $K$ is compact in $H$, then equation (15) is called Fredholm equation of the first kind. The linear set $R(K)$ is not closed unless $K$ is a finite-rank operator. Therefore, small pertubation of $f \in R(K)$ may lead to an equation which has no solutions, or, if it has a solution, this solution may differ very much from the solution to equation (15). Such problems are called ill-posed. They are important in many applications, especially in inverse problems, [15], [4] . The usual statement of the problem of solving an ill-posed equation (15) consists of the following. It is assumed that equation (15) is solvable for the exact data $f$, possibly non-uniquely, that the exact data $f$ are not known but the “noisy” data $f_\delta$ are known, and $||f_\delta - f|| \leq \delta$, where $\delta > 0$ is known. Given $f_\delta$, one wants to find $u_\delta$ such that $\lim_{\delta \to 0} ||u - u_\delta|| = 0$.

There are many methods to solve the above problem: the DSM (Dynamical Systems Method) [17] [18], iterative methods, variational regularization, and their variations, see [4] and [15].

Examples of ill-posed problems of practical interest include stable numerical differentiation, stable summation of the Fourier series and integrals with perturbed coefficients, stable solution to linear algebraic systems with large condition numbers, solving Fredholm and Volterra integral equations of the first kind, various deconvolution problems, various inverse problems, the Cauchy problem for the Laplace equation and other elliptic equations, the backward heat problem, tomography, and many other problems, see [15].

*1.6. Numerical solution of Fredholm integral equations*

Let us describe the projection method for solving equation (2) with compact operator $K$. Let $L_n \subset H$ be a linear subspace of $H$, $\dim L_n = n$. Assume throughout that the sequence $L_n \subset L_{n+1}$ is limit dense in $H$, that is, $\lim_{a\to\infty} \rho(f, L_n) = 0$ for any $f \in H$, where $\rho(f, L_n) = \inf_{u\in L_n} ||f - u||$. Let $P_n$ be an orthogonal projection operator on $L_n$, that is, $P_n^2 = P_n$, $P_n f \in L_n$, $||f - P_n f|| \leq ||f - u||, \forall u \; in \; L_n$.

Assume that $N(I - K) = \{0\}$. Then $||(I - K)^{-1}|| \leq c$. Consider the projection method for solving equation (1):

$$P_n(I - K)P_n u_n = P_n f. \tag{16}$$

**Theorem 1.12.** *If $N(I - K) = \{0\}$, $K$ is compact, and $\{L_n\}$ is limit dense, then equation (16) is uniquely solvable for all $n$ sufficiently large, and $\lim_{n\to\infty} ||u_n - u|| = 0$, where $u$ is the solution to equation (1).*

*Remark* 1.13. Projection method is valid also for solving equations $Au = f$, where $A : X \to Y$ is a linear operator, $X$ and $Y$ are Hilbert spaces. Let $Y_n \subset Y$ and $X_n \subset X$ be $n$-dimensional subspaces, $P_n$ and $Q_n$ are projection operators on $X_n$ and $Y_n$ respectively. Assume that the sequences $\{X_n\}$ and $\{Y_n\}$ are limit dense and consider the projection equation

$$Q_n A P_n u_n = Q_n f, \quad u_n \in X_n. \tag{17}$$

**Theorem 1.14.** *Assume that $A$ is a bounded linear operator, $||A^{-1}|| \leq c$, and*

$$||Q_n A P_n u|| \geq c||P_n u||, \quad \forall u \in X, \tag{18}$$

*where $c > 0$ stands for various positive constants independent of $n$ and $u$. Then equation (17) is uniquely solvable for any $f \in Y$ for all sufficiently large $n$, and*

$$\lim_{n\to\infty} ||u - u_n|| = 0. \tag{19}$$

*Conversely, if equation (17) is uniquely solvable for any $f \in Y$ for all sufficiently large $n$, and (19) holds, then (18) holds.*

*Remark* 1.15. If (18) holds for $A$, then it will hold for $A + B$ if $||B|| > 0$ is sufficiently small. If (18) holds for $A$, then it holds for $A+B$ if $B$ is compact and $||(A + B)^{-1}|| \leq c$.

Iterative methods for solving integral equation (1) are popular. If $||K|| < 1$, then equation (1) has a unique solution for every $f \in H$, and this solution can be obtained by the iterative method

$$u_{n+1} = Ku_n + f, \quad \lim_{n\to\infty} ||u_n - u|| = 0, \tag{20}$$

where $u_1 \in H$ is arbitrary.

Define the spectral radius of a linear bounded operator $A$ in a Banach space by the formula $\rho(A) = \lim_{n\to\infty} ||A^n||^{1/n}$. It is known that this limit exists. If $a$ is a number, then $\rho(aA) = |a|\rho(A)$. The basic fact is:

*If $|\lambda| > \rho(A)$, then the equation $\lambda u = Au + f$ is uniquely solvable for any $f$, and the iterative process $\lambda u_{n+1} = Au_n + f$ converges to $u = (\lambda I - A)^{-1} f$ for any $f \in H$ and any initial approximation $u_1$.*

Given an equation $Au = f$, can one transform it into an equation $u = Bu + g$ with $\rho(B) < 1$, so that it can be solved by iterations? To do so, one may look for an operator $T$ such that $(A + T)u = Tu + f, (A + T)^{-1}$ is bounded, so $u = (A+T)^{-1}Tu+(A+T)^{-1}f$, and $\rho((A+T)^{-1}T) < 1$. Finding such an operator $T$ is, in general, not simple. If $A = A^*,\ 0 \leq m \leq A \leq M$, where $m$ and $M$ are constants, one may transform equation $Au = f$ to the equivalent form $u = u - \frac{2Au}{m+M} + \frac{2}{m+M}f$, and get $\rho(I - \frac{2A}{m+M}) \leq \frac{M-m}{M+m} < 1$. If $A \neq A^*$ then equation $A^*Au = A^*f$ is useful since $A^*A$ is a selfadjoint operator. The above results can be found, for example, in [6] and [27].

Equation $Au = f$ is often convenient to consider in a cone of a Banach space $X$, see [27]. A cone is a closed convex set $K \subset X$ which contains with a point $u$ all the points $tu, t \geq 0$, and such that $u \in K$ and $-u \in K$ imply $u = 0$. One writes $u \leq v$ if $v - u \in K$. This defines a semiorder in $X$. If $K$ is invariant with respect to $K$, that is, $u \in K$ implies $Au \in K$, then one can give conditions for the solvability of the equation $Au = f$ in $K$. For example, let $X = C(D), D \subset \mathbb{R}^m$ is a bounded domain, $K = \{u : u \geq 0\}$,

$$Au = \int_D A(x, y)u(y)dy, \quad A(x, y) \geq 0.$$

The following result is useful:

*Suppose that there exist functions $v$ and $w$ in $K$ such that $Av \geq v$, $Aw \leq w$, and $v \leq w$. Let us assume that every monotone bounded sequence in $K$, $u_1 \leq u_2 \leq \cdots \leq U$ has a limit: $\exists u \in K$ such that $\lim_{n\to\infty} ||u_n - u|| = 0$. Then there exists $u \in K$ such that $u = Au$, where $v \leq u \leq w$.*

A cone is called normal if $0 \leq u \leq v$ implies $||u|| \leq c||v||$, where $c > 0$ is constant independent of $u$ and $v$.

## 2. Integral equations with special kernels

*2.1. Equations with displacement kernel*

The results in this Subsection are taken from [3], [27].

Let us mention some equations with displacement kernels. We start with the equation

$$\lambda u(t) - \int_0^t K(t-s)u(s)ds = f(t), \quad t \geq 0.$$

This equation is solved by taking Laplace transform: $\lambda\bar{u} - \bar{K}\bar{u} = \bar{f}$, $\bar{u} = (\lambda - \bar{K})^{-1}\bar{f}$, where $\bar{u} := Lu := \int_0^\infty e^{-pt}u(t)dt$, and

$$u(t) = L^{-1}\bar{u} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{pt}\bar{u}(p)dp.$$

Equation

$$\lambda u(t) - \int_{-\infty}^{\infty} K(t-s)u(s)ds = f(t), \quad -\infty < t < \infty,$$

is solved by taking the Fourier transform: $\tilde{u}(\xi) := \int_{-\infty}^{\infty} u(t)e^{-i\xi t}dt$. One has $\tilde{u}(\xi) = (\lambda - \tilde{K}(\xi))^{-1}\tilde{f}$, and

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{u}(\xi)e^{i\xi t}d\xi.$$

In these formal calculations one assumes that $1 - \tilde{K}(\xi) \neq 0$, see [3], [10].

More complicated is Wiener-Hopf equation

$$u(t) - \int_0^\infty K(t-s)u(s)ds = f(t), \quad t \geq 0. \tag{21}$$

Assume that $K(t) \in L(-\infty, \infty)$ and $1 - \tilde{K}(\xi) \neq 0$, $\forall \xi \in (-\infty, \infty)$. Define the index: $\kappa := -\frac{1}{2\pi} \arg[1 - \tilde{K}(\xi)]|_{-\infty}^{\infty}$. The simplest result is:

*If and only if $\kappa = 0$ equation (21) is uniquely solvable in $L^p(-\infty, \infty)$.*

The solution can be obtained analytically by the following scheme. Denote $u_+(t) := \begin{cases} u(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$. Write equation (21) as

$$u_+(t) = \int_{-\infty}^{\infty} K(t-s) u_+(s) ds + u_-(t) + f_+, \quad -\infty < t < \infty,$$

where $u_-(t) = 0$ for $t \geq 0$ is an unknown function. Take the Fourier transform of this equation to get $[1 - \tilde{K}(\xi)]\tilde{u}_+ = \tilde{f}_+ + \tilde{u}_-$. Write $1 - \tilde{K}(\xi) = \tilde{K}_+(\xi)\tilde{K}_-(\xi)$, where $\tilde{K}_+$ and $\tilde{K}_-$ are analytic in the upper, respectively, lower half-plane functions. Then

$$\tilde{K}_+(\xi)\tilde{u}_+(\xi) = \tilde{f}_+(\xi)\tilde{K}_-^{-1}(\xi) + \tilde{u}_-(\xi)\tilde{K}_-(\xi).$$

Let $P_+$ be the projection operator on the space of functions analytic in the upper half-plane. Then

$$\tilde{u}_+ = \tilde{K}_+^{-1}(\xi) P_+(\tilde{f}_+(\xi)\tilde{K}_-^{-1}(\xi)).$$

If $\tilde{u}_+$ is known, then $u_+(t)$ is known, and the Wiener-Hopf equation is solved. The non-trivial part of this solution is the factorization of the function $1 - \tilde{K}(\xi)$.

### *2.2. Equations basic in random fields estimation theory*

In this Subsection the results from [16] are given.

Let $u(x) = s(x) + n(x), x \in \mathbb{R}^2$, $s(x)$ and $n(x)$ are random fields, $s(x)$ is a "useful signal" and $n(x)$ is "noise". Assume that the mean values $\overline{s(x)} = \overline{n(x)} = 0$, the overbar denotes mean value. Assume that covariance functions

$$\overline{u^*(x)u(y)} := R(x,y), \quad \overline{u^*(x)s(y)} := f(x,y),$$

are known. The star here denotes complex conjugate. Suppose that a linear estimate $Lu$ of the observations in a domain $D$ is

$$Lu = \int_D h(x,y)u(y)dy,$$

where $h(x,y)$ is, in general, a distributional kernel. Then the optimal estimate of $As$, where $A$ is a linear operator, is found by solving the optimization problem

$$\epsilon := \overline{(Lu - As)^2} = \min . \tag{22}$$

A necessary condition for the kernel (filter) $h(x,y)$ to solve this problem is the integral equation:

$$Rh := \int_D R(x,y)h(z,y)dy = f(x,z), \quad x,z \in \bar{D} := D \cup \Gamma, \tag{23}$$

where $\Gamma := \partial D$ is the boundary of $D$. Since $z$ enters into equation (23) as a parameter, one has to study the following integral equation

$$\int_D R(x,y)h(y)dy = f(x), \quad x \in \bar{D}. \tag{24}$$

The questions of interest are:

1) Under what assumptions equation (24) has a unique solution in some space of distributions?

2) What is the order of singularity of this solution?

3) Is this solution stable under small (in some sense) perturbations of $f$? Does it solve estimation problem (22)?

4) How does one calculate this solution analytically and numerically?

If a distribution $h = D^l h_1$, where $h_1 \in L^2_{\text{loc}}$, then the order of singularity of $h$, ordsing$h$, is $|l|$ if $h_1$ is not smoother than $L^2_{\text{loc}}$. By $|l|$ one denotes $l_1 + l_2 + \cdots + l_r$, $D^l = \dfrac{\partial^{|l|}}{\partial x_1^{l_1} \dots \partial x_r^{l_r}}$. For the distribution theory one can consult, for example, [2].

Let us define a class of kernels $R(x,y)$, or, which is the same, a class $\mathcal{R}$ of integral equations (24), for which the questions 1) - 4) can be answered.

Let $L$ be a selfadjoint elliptic operator of order $s$ in $L^2(\mathbb{R}^r)$, $\Lambda$, $d\rho(\lambda)$ and $\Phi(x,y,\lambda)$ are, respectively, its spectrum, spectral measure and spectral kernel. This means that the spectral function $E_\lambda$ of the operator $L$ has the kernel $E_\lambda(x,y) = \int_\Lambda \Phi(x,y,\lambda)d\rho(\lambda)$ and a function $\phi(L)$ has the kernel

$$\phi(L)(x,y) = \int_\Lambda \phi(\lambda)\Phi(x,y,\lambda)d\rho(\lambda),$$

where $\phi(\lambda)$ is an arbitrary function such that $\int_\Lambda |\phi(\lambda)|^2 d(E_\lambda f, f) < \infty$ for any $f \in D(\phi(L))$.

Define class $\mathcal{R}$ of the kernels $R(x, y)$ by the formula

$$R(x, y) = \int_\Lambda \frac{P(\lambda)}{Q(\lambda)} \Phi(x, y, \lambda) d\rho(\lambda), \tag{25}$$

where $P(\lambda) > 0$ and $Q(\lambda) > 0$ are polynomials of degree $p$ and, respectively, $q$, and $p \le q$. Let $\alpha := 0.5(q - p)s$. The number $\alpha \ge 0$ is an integer because $p$ and $q$ are even integers if the polynomials $P > 0$ and $Q > 0$ are positive for all values of $\lambda$. Denote by $H^\alpha = H^\alpha(D)$ the Sobolev space and by $\dot{H}^{-\alpha}$ its dual space with respect to the $L^2(D)$ inner product. Our basic results are formulated in the following theorems, obtained in [16].

**Theorem 2.1.** *If $R(x, y) \in \mathcal{R}$, then the operator $R$ in equation (24) is an isomorphism between the spaces $\dot{H}^{-\alpha}$ and $H^\alpha$. The solution to equation (24) of minimal order of singularity, ordsingh $\le \alpha$, exists, is unique, and can be calculated by the formula*

$$h(x) = Q(L)G, G(x) = \begin{cases} g(x) + v(x) & \text{in } D \\ u(x) & \text{in } D' := \mathbb{R}^r \setminus D \end{cases} \tag{26}$$

*where $g(x) \in H^{0.5s(p+q)}$ is an arbitrary fixed solution to the equation*

$$P(L)g = f \ \text{ in } D, \quad f \in H^\alpha, \tag{27}$$

*and the functions $u(x)$ and $v(x)$ are the unique solution to the problem*

$$Q(L)u = 0 \ \text{ in } D', \quad u(\infty) = 0, \tag{28}$$

$$P(L)v = 0 \ \text{ in } D, \tag{29}$$

$$\partial_N^j u = \partial_N^j (v + g) \ \text{ on } \partial D, \ \ 0 \le j \le 0.5s(p + q) - 1. \tag{30}$$

*Here $\partial_N u$ is the derivative of $u$ along the outer unit normal $N$.*

Theorem 2.1 gives answers to questions 1) - 4), except the question of numerical solution of equation (24). This question is discussed in [16].

*Remark* 2.2. If one considers the operator $R \in \mathcal{R}$ as an operator in $L^2(D)$, $R : L^2(D) \to L^2(D)$ and denotes by $\lambda_j = \lambda_j(D)$ its eigenvalues, $\lambda_1 \ge \lambda_2 \ge \cdots \ge 0$, then

$$\lambda_j = c j^{-(q-p)s/r} [1 + o(1)] \quad \text{as} \ \ j \to \infty, \tag{31}$$

where $c = \gamma^{(q-p)s/r}$, $\gamma := (2\pi)^{-r} \int_D \eta(x) dx$, and the $\eta(x)$ is defined in [16].

*Example.* Consider equation (24) with $D = [-1, 1]$, $R(x, y) = \exp(-|x - y|)$, $L = -i\frac{d}{dx}$, $r = 1$, $s = 1$, $P(\lambda) = 1$, $Q(\lambda) = (\lambda^2 + 1)/2$, $\Phi(x, y, \lambda) = (2\pi)^{-1} \exp\{i\lambda(x - y)\}$, and $d\rho(\lambda = d\lambda)$. Then

$$\int_{-1}^{1} \exp(-|x - y|)h(y)dy = f, \quad -1 \leq x \leq 1,$$

$\alpha = 1$, and Theorem 2.1 yields the following formula for the solution $h$:

$$h(x) = \frac{-f'' + f}{2} + \frac{f'(1) + f(1)}{2}\delta(x - 1) + \frac{-f'(-1) + f(-1)}{2}\delta(x + 1),$$

where $\delta$ is the delta-function.

## 3. Singular integral equations

*3.1. One-dimensional singular integral equations*

The results in this Section are taken from [11], [25] and [27].

Consider the equation

$$Au := a(t)u(t) + \frac{1}{i\pi} \int_L \frac{M(t, s)}{s - t} u(s)ds = f(t). \tag{32}$$

Here the functions $a(t)$, $M(t, s)$, and $f(t)$ are known. Assume that they satisfy the Hőlder condition. The contour $L$ is a closed smooth curve on the complex plane, $L$ is the boundary of a connected smooth domain $D$, and $D'$ is its complement on the complex plane. Since $M(t, s)$ is assumed Hőlder-continuous, one can transform equation (32) to the form

$$Au := a(t)u + \frac{b(t)}{i\pi} \int_L \frac{u(s)ds}{s - t} + \int_L K(t, s)u(s)ds = f(t), \tag{33}$$

where the operator with kernel $K(t, s)$ is of Fredholm type, $b(t) = M(t, t)$, $K(t, s) = \frac{1}{2\pi} \frac{M(t,s) - M(t,t)}{s - t}$. The operator $A$ in (33) is of the form $A = A_0 + K$, where $A_0 u = a(t)u + Su$, and $Su = \frac{1}{i\pi} \int_L \frac{u(s)ds}{s-t}$. The singular operator $S$ is defined as the limit $\lim_{\epsilon \to 0} \frac{1}{i\pi} \int_{|s-t|>\epsilon} \frac{u(s)ds}{s - t}$.

The operator $S$ maps $L^p(L)$ into itself and is bounded, $1 < p < \infty$. Let us denote the space of Hőlder-continuous functions on $L$ with exponent $\gamma$ by

$\mathrm{Lip}_\gamma(L)$. The operator $S$ maps the space $\mathrm{Lip}_\gamma(L)$ into itself and is bounded in this space. If $\Phi(z) = \frac{1}{2\pi i}\int_L \frac{u(s)ds}{s-z}$ and $u \in \mathrm{Lip}_\alpha(L)$, then

$$\Phi^{\pm}(t) = \pm\frac{u(t)}{2} + \frac{1}{2\pi i}\int_L \frac{u(s)ds}{s-t}, \tag{34}$$

where $+(-)$ denotes the limit $z \to t$, $z \in D\,(D')$.

If $u \in L^p(L), 1 < p < \infty$, then

$$\frac{d}{dt}\int_L u(s)\ln\frac{1}{|s-t|}ds = i\pi u(t) + \int_L \frac{u(s)ds}{s-t}. \tag{35}$$

One has $S^2 = I$, that is, for $1 < p < \infty$,

$$\frac{1}{(i\pi)^2}\int_L \frac{1}{\tau - t}\left(\int_L \frac{u(s)ds}{s-\tau}ds\right)d\tau = u(t), \quad \forall u \in L^p(L). \tag{36}$$

Let $n = \dim N(A)$ and $n^* := \dim N(A^*)$. Then index of the operator $A$ is defined by the formula $\mathrm{ind}A := n - n^*$ if at least one of the numbers $n$ or $n^*$ is finite.

If $A = B + K$, where $B$ is an isomorphism and $K$ is compact, then $A$ is Fredholm operator and $\mathrm{ind}A = 0$. If $A$ is a singular integral operator (33), then its index may be not zero,

$$\kappa := \mathrm{ind}A = \frac{1}{2\pi}\int_L d\arg\frac{a(t)-b(t)}{a(t)+b(t)}, \tag{37}$$

where $\arg f$ denotes the argument of the function $f$. The operator $A$ with non-zero index is called a Noether operator. Formula (37) was derived by F.Noether in 1921. Equation (33) with $K(x,y) = 0$ can be reduced to the Riemann problem. Namely, if $\Phi(z) = \frac{1}{2\pi i}\int_L \frac{u(s)}{s-z}ds$, then $u(t) = \Phi^+(t) - \Phi^-(t)$ and $\Phi^+(t) + \Phi^-(t) = Su$ by formula (34).

Thus, equation (33) with $K(x,y) = 0$ can be written as

$$a(t)(\Phi^+ - \Phi^-) + b(t)(\Phi^+ + \Phi^-) = f,$$

or as

$$\Phi^+(t) = \frac{a(t)-b(t)}{a(t)+b(t)}\Phi^- + \frac{f}{a(t)+b(t)}, \quad t \in L. \tag{38}$$

This problem is written usually as

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in L, \tag{39}$$

and is a Riemann problem for finding piece-wise analytic function $\Phi(z)$ from the boundary condition (39).

Assume that $a(t) \pm b(t) \neq 0, \forall t \in L$, and that $L$ is a closed smooth curve, the boundary of a simply connected bounded domain $D$.

If one solves problem (39) then the solution to equation (33) (with $K(x, y) = 0$) is given by the formula $u = \Phi^+ - \Phi^-$. Let $\kappa = \text{ind}G$.

**Theorem 3.1.** *If $\kappa \geq 0$ then problem* (39) *is solvable for any $g$ and its general solution is*

$$\Phi(z) = \frac{X(z)}{2\pi i} \int_L \frac{g(s)ds}{X^+(s)(s-z)} + X(z)P_\kappa(z), \tag{40}$$

*where*

$$X^+(z) = exp(\Gamma_+(z)), \quad X^-(z) = z^{-\kappa} exp(\Gamma_-(z)), \tag{41}$$

$$\Gamma(z) = \frac{1}{2\pi i} \int_L \frac{\ln[s^{-\kappa}G(s)]}{s-z} ds, \tag{42}$$

*where $P_\kappa(z)$ is a polynomial of degree $\kappa$ with arbitrary coefficients. If $\kappa = -1$, then problem* (39) *is uniquely solvable and $X^-(\infty) = 0$. If $\kappa < -1$ then problem* (39)*, in general, does not have a solution. For its solvability, it is necessary and sufficient that*

$$\int_L \frac{g(s)}{X^+(s)} s^{k-1} ds = 0, k = 1, 2, \ldots, -\kappa - 1. \tag{43}$$

*If these conditions hold, then the solution to problem* (39) *is given by formula* (40) *with $P_\kappa(z) \equiv 0$.*

This theorem is proved, for example, in [3] and [11]. It gives also an analytic formula for solving integral equation (33) with $k(x, y) = 0$.

If one consider equation (33) with $k(x, y) \neq 0$, then this equation can be transformed to a Fredholm-type equation by inverting the operator $A_0$. A detailed theory can be found in [3] and [10].

*3.2. Multidimensional singular integral equations*

Consider the singular integral

$$Au := \int_{\mathbf{R}^m} r^{-m} f(x, \theta) u(y) dy, \quad x, y \in \mathbf{R}^m, \quad r = |x - y|, \tag{44}$$

Assume that $u(x) = O(\frac{1}{|x|^k})$ for large $|x|$, that $k > 0$, and $f$ is a bounded function continuous, with respect to $\theta$. Let $\int_S f(x,\theta)dS = 0$, where $S$ stands in this section for the unit sphere in $\mathbf{R}^m$.

Suppose that $|u(x)-u(y)| \leq c_1 r^\alpha (1+|x|^2)^{-k/2}$ for $r \leq 1$, where $c_1 = const$, and $|u(x)| \leq c_2(1+|x|^2)^{-k/2}$, where $c_2 = const$. Denote this set of functions $u$ by $A_{\alpha,k}$. If the first inequality holds, but the second is replaced by the inequality $|u(x)| \leq c_3(1+|x|^2)^{-k/2}\ln(1+|x|^2)$, then this set of $u$ is denoted $A'_{\alpha,k}$.

**Theorem 3.2.** *The operator* (44) *maps* $A_{\alpha,k}$ *with* $k \leq m$ *into* $A'_{\alpha,k}$, *and* $A_{\alpha,k}$ *with* $k > m$ *into* $A_{\alpha,m}$.

Let $D \subset \mathbf{R}^m$ be a domain, possibly $D = \mathbf{R}^m$, the function $f(x,\theta)$ be continuously differentiable with respect to both variables, and $u$ satisfies the Dini condition, that is $\sup_{||x-y||<\delta; x,y\in D} |u(x)-u(y)| = w(u,t)$, where $\int_0^\delta \frac{w(u,t)}{t}dt < \infty, \quad \delta = const > 0$.

Then

$$\frac{\partial}{\partial x_j}\int_D \frac{f(x,\theta)}{r^{m-1}}u(y)dy = \int_D u(y)\frac{\partial}{\partial x_j}\left[\frac{f(x,\theta)}{r^{m-1}}\right]dy - u(x)\int_S f(x,\theta)\cos(r_{x_j})ds. \tag{45}$$

Assume that

$$\int_S |f(x,\theta)|^{p'}dS \leq C_0 = const, \quad \frac{1}{p}+\frac{1}{p'} = 1, \tag{46}$$

and let

$$A_0 u = \int_{\mathbf{R}^m} \frac{f(x,\theta)}{r^m}u(y)dy. \tag{47}$$

**Theorem 3.3.** *If* (46) *holds then*

$$||A_0||_{L^p} \leq c\sup_x ||f(x,\theta)||_{L^{p'}(S)}.$$

The operator (47) is a particular case of a pseudodifferential operator defined by the formula

$$Au = \int_{\mathbf{R}^m}\int_{\mathbf{R}^m} e^{i(x-y)\cdot\xi}\sigma_A(x,\xi)u(y)dyd\xi, \tag{48}$$

where the function $\sigma_A(x,\xi)$ is called the symbol of $A$. For the operator $Au = a(x)u(x) + A_0u$, where $A_0$ is defined in formula (47), the symbol of $A$

is defined as $\sigma(x,\xi) = a(x) + \widetilde{K}(x,\xi)$, where $K(x,x-y) = \frac{f(x,\theta)}{r^m}$, and $\widetilde{K}(x,\xi)$ is the Fourier transform of $K(x,z)$ with respect to $z$. One has

$$\sigma_{A+B}(x,\xi) = \sigma_A(x,\xi) + \sigma_B(x,\xi), \quad \sigma_{AB}(x,\xi) = \sigma_A(x,\xi)\sigma_B(x,\xi).$$

It is possible to estimate the norm of the operator (47) in terms of its symbol,

$$\sigma_{A_0}(x,\xi) = \int_{\mathbf{R}^m} \frac{f(x,y^0)}{|y|^m} e^{-iy\cdot z} dy, \quad y^0 = \frac{y}{|y|}, \quad \xi = \frac{z}{|z|}.$$

If $T$ is a compact operator in $L^2(\mathbf{R}^m)$ then its symbol is equal to zero. Let $Au = a(x)u + A_0 u$, and $\sigma_A(x,\xi) = a(x) + \sigma_{A_0}(x,\xi)$. If $\sigma_A(x,\xi)$ is sufficiently smooth and does not vanish, then the singular integral equation $Au + Tu = f$ is of Fredholm type and its index is zero. In general, index of a system of singular integral equations (and of more general systems) is calculated in the work [1]. Additional material about singular integral equations and pseudodifferential operators one finds in [11] and [25].

## 4. Nonlinear integral equations

The results from this Subsection are taken from [8], [27], [7], [5].

Let us call the operators

$$Uu = \int_D K(x,t)u(t)dt, \quad Hu = \int_D K(x,t)f(t,u(t))dt,$$

and $Fu := f(t,u(t))$, respectively, Urysohn, Hammerstein, and Nemytskii operators.

The operator $F$ acts from $L^p := L^p(D)$ into $L^q := L^q(D)$ if and only if

$$|f(t,u)| \leq a(t) + b|u|^{p/q}, \quad p < \infty, \quad a(t) \in L^q, \quad b = const.$$

If $p = \infty$, then $F : L^\infty \to L^q$ if and only if $|f(t,u)| \leq a_h(t)$, $a_h(t) \in L^q$, and $|u(t)| \leq h$, $0 \leq h < \infty$. The function $f(t,u)$, $t \in D$, $u \in (-\infty,\infty)$, is assumed to satisfy the Caratheodory conditions, that is, this function is continuous with respect to $u$ for almost all $t \in D$ and is measurable with respect to $t \in D$ for all $u \in (-\infty,\infty)$. If $F : L^p \to L^q$ and $f(t,u)$ satisfies the Caratheodory conditions, then $F$ is compact provided that one of the following conditions holds: 1) $q < \infty$, 2) $q = \infty$, $p < \infty$, $f(t,u) \equiv a(t)$, 3) $q = p = \infty$, $|f(t,u) - f(t,v)| \leq \phi_h(u-v)$ with respect to $z$ for any $h > 0$.

If $f$ satisfies the Caratheodory conditions, and $K(x,t)$ is a measurable function on $D \times D$, and $F : L^p \to L^r$, $K : L^r \to L^q$, then the operator $H = KF : L^p \to L^q$, $H$ is continuous if $r < \infty$, and $H$ is compact if $r < \infty$ and $K$ is compact. These results one can find, for example, in [27].

A nonlinear operator $A : X \to Y$ is Fréchet differentiable if

$$A(u+h) - A(u) = Bh + w(u,h),$$

where $B : X \to Y$ is a linear bounded operator and $\lim_{||h||_X \to 0} \frac{||w(u,h)||_Y}{||h||_X} = 0$.

Consider the equation

$$u(x) = \mu \int_D K(x,t,u(t))dt + f(x) := Au, \tag{49}$$

where $D \subset \mathbf{R}^m$ is a bounded closed set, meas$D > 0$, the functions $K(x,t,u)$ and $f(x)$ are given, the function $u$ is unknown, and $\mu$ is a number.

**Theorem 4.1.** *Let us assume that $A$ is a contraction on a set $M$, that is, $||A(u) - A(v) \leq q||u - v||$, where $u, v \in M$, $M$ is a subset of a Banach space $X$, $0 < q < 1$ is a number, and $A : M \to M$. Then equation* (49) *has a unique solution $u$ in $M$, $u = \lim_{n\to\infty} u_n$ , where $u_{n+1} = A(u_n), u_0 \in M$, and $||u_n - u|| \leq \frac{q^n}{1-q}||u_1 - u_0||, n = 1, 2, \ldots$.*

This result is known as the *contraction mapping principle.*

**Theorem 4.2.** *Assume that $A$ is a compact operator, $M \subset X$ is a bounded closed convex set, and $A : M \to M$. Then equation* (49) *has a solution.*

This result is called *the Schauder principle*. The solution is a fixed point of the mapping $A$, that is, $u = A(u)$. Uniqueness of the solution in Theorem 4.2 is not claimed: in general, there can be more than one solution.

*Remark* 4.3. A version of Theorem 4.2 can be formulated as follows:

**Theorem 4.4.** *Assume that $A$ is a continuous operator which maps a convex closed set $M$ into its compact subset. Then $A$ has a fixed point in $M$.*

Let us formulate *the Leray-Schauder* principle:

**Theorem 4.5.** *Let $A$ be a compact operator, $A : X \to X$, $X$ be a Banach space. Let all the solutions $u(\lambda)$ of the equation $u = A(u, \lambda)$, $0 \leq \lambda \leq 1$, satisfy the estimate* $\sup_{0\leq\lambda\leq1} ||u(\lambda)|| \leq a < \infty$. *Let $||A(u, 0)|| \leq b$ for $b > a$ and $||u|| = b$. Then the equation $u = A(u, 1)$ has a fixed point in the ball $||u|| \leq a$.*

Let $D \subset X$ be a bounded convex domain, $A^n u \subset \overline{D}$ if $u \in \overline{D}, n = 1, 2, \ldots$, $\overline{D}$ is the closure of $D$, $A^n u \neq u$ for $n > n_0$ and $u \in \partial D$, where $\partial D$ is the boundary of $D$. Finally, let us assume that $A$ is compact.

**Theorem 4.6.** *Under the above assumptions $A$ has a fixed point in $D$.*

This theorem is proved in [7]

Consider the equation $u = H(u), H : L^2(D) \to L^2(D)$, and assume that $K(x,t)$ is positive-definite, continuous kernel, $f(t,u)$ is continuous with respect to $t \in D$ and $u \in (-\infty, \infty)$ function, such that $\int_0^u f(t,s)ds \leq 0.5au^2 + b$, where $a < \Lambda$, and $\Lambda$ is the maximal eigenvalue of the linear operator $K$ in $L^2(D)$. Then there exists a fixed point of the operator $H$. This result can be found in [27].

## 5. Applications

*5.1. General remarks.*

The results in this Subsection are taken from [13], [22] and [24].

There are many applications of integral equations. In this section, there is no space to describe in detail applications to solving boundary value problems by integral equations involving potentials of single and double layers, applications of integral equations in the elasticity theory ([10], [9]), applications to acoustics electrodynamics, etc.

We will restrict this Subsection to some questions not considered by other authors. The first question deals with the possibility to express potentials of the single layer by potentials of the double layer and vice versa. The results are taken from [13]. The second application deals with the wave scattering by small bodies of an arbitrary shape. These results are taken from [22] and [24].

*5.2. Potentials of a single and double layers.*

$$V(x) = \int_S g(x,t)\sigma(t)dt, \quad g(x,t) = \frac{e^{ik|x-t|}}{4\pi|x-t|}, \tag{50}$$

where $S$ is a boundary of a smooth bounded domain $D \subset \mathbf{R}^3$ with the boundary $S$, $\sigma(t) \in Lip_\gamma(S),\ 0 < \gamma \leq 1$.

It is known that $V(x) \in C(\mathbf{R}^3), V(\infty) = 0$,

$$V_N^\pm(s) = \frac{A\sigma \pm \sigma(s)}{2}, \quad A\sigma := 2\int_S \frac{\partial g(s,t)}{\partial N_s}\sigma(t)dt, \tag{51}$$

where $N = N_s$ is the unit normal to $S$ at the point $s$ pointing out of $D$, $+(-)$ denotes the limiting value of the normal derivative of $V$ when $x \to s \in S$ and $x \in D$ $(D')$, where $D' := \mathbf{R}^3 \setminus D$.

Potential of double layer is defined as follows:

$$W(x) := \int_S \frac{\partial g(x,t)}{\partial N_t} \mu(t) dt.$$

One has the following properties of $W$:

$$W^{\pm}(s) = W(s) \mp \frac{\mu(t)}{2}, \quad W(s) := \int_S \frac{\partial g(s,t)}{\partial N_t} \mu(t) dt \tag{52}$$

$$W^{+}_{N_s} = W^{-}_{N_s}. \tag{53}$$

**Theorem 5.1.** *For any $V(W)$ there exists a $W(V)$ such that $W = V$ in $D$. The $V(W)$ is uniquely defined.*

**Theorem 5.2.** *A necessary and sufficient condition for $V(\sigma) = W(\mu)$ in $D'$ is*

$$\int_S V h_j dt = 0, \quad 1 \leq j \leq r', \quad (I + A) h_j = 0, \tag{54}$$

*where the set $\{h_j\}$ forms a basis of $N(I + A)$, and $A$ is defined in* (51).

*A necessary and sufficient condition for $W(\mu) = V(\sigma)$ in $D'$ is:*

$$\int_S W \sigma_j dt = 0, \quad 1 \leq j \leq r, \quad A\sigma_j - \sigma_j = 0, \tag{55}$$

*where the set $\{\sigma_j\}$ forms a basis of $N(I - A)$.*

Theorem 5.1 and 5.2 are proved in [13].

*5.3. Wave scattering by small bodies of an arbitrary shape.*

The results in this Subsection are taken from [22] and [24]. Consider the wave scattering problem:

$$(\nabla^2 + k^2) u = 0 \text{ in } D' := \mathbf{R}^3 \setminus D, \quad u_N = \zeta u \text{ on } S = \partial D, \tag{56}$$

$$u = u_0 + v, \quad u_0 = e^{ik\alpha \cdot x}, \quad \alpha \in S^2, \tag{57}$$

$$\frac{\partial v}{\partial r} - ikv = o\left(\frac{1}{r}\right), \quad r = |x| \to \infty. \tag{58}$$

Here $k = const > 0$, $S^2$ is the unit sphere in $\mathbf{R}^3, \zeta = const$ is a given parameter, the boundary impedance, $D$ is a small body, a particle, $S$ is its boundary, which we assume Hölder-continuous, and $N$ is the unit normal to $S$ pointing out of $D$. If $Im\zeta \le 0$ then problem (56)-(58) has exactly one solution, $v$ is the scattered field,

$$v = A(\beta, \alpha, \kappa)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad |x| = r \to \infty, \quad \beta = \frac{x}{r}, \tag{59}$$

$A(\beta, \alpha, \kappa)$ is called the scattering amplitude.

*Our basic assumption is the smallness of the body $D$: this body is small if $ka \ll 1$, where $a = 0.5 diam D$.*

In applications $a$ is called the characteristic size of $D$.

Let us look for the solution to problem (56)-(58) of the form

$$u(x) = u_0(x) + \int_S g(x,t)\sigma(t)dt, \quad g(x,t) = \frac{e^{ik|x-t|}}{4\pi|x-t|}, \tag{60}$$

and write

$$\int_S g(x,t)\sigma(t)dt = g(x,x_1)Q + \int_S [g(x,t) - g(x,x_1)]\sigma(t)dt, \tag{61}$$

where $x_1 \in D$ is an arbitrary fixed point inside $D$, and

$$Q := \int_S \sigma(t)dt. \tag{62}$$

If $ka \ll 1$ and $|x - x_1| := d \gg a$, then

$$\left|\int_S [g(x,t) - g(x,x_1)]\sigma(t)dt\right| \ll |g(x,x_1)|Q. \tag{63}$$

Therefore the scattering problem (56)-(58) has an approximate solution of the form:

$$u(x) = u_0(x) + g(x,x_1)Q, \quad ka \ll 1, \quad |x - x_1| \gg a. \tag{64}$$

The solution is reduced to finding just one number $Q$ in contrast with the usual methods, based on the boundary integral equation for the unknown function $\sigma(t)$.

To find the main term of the asymptotic of $Q$ as $a \to 0$, one uses the exact boundary integral equation:

$$u_{0N}(s) - \zeta u_0(s) + \frac{A\sigma - \sigma}{2} - \zeta \int_S g(s,t)\sigma(t)dt = 0, \quad s \in S, \tag{65}$$

where formula (51) was used.

We do not want to solve equation (65) which is only numerically possible, but want to derive an asymptotically exact analytic formula for $Q$ as $a \to 0$. Integrate both sides of formula (65) over $S$. The first term is equal to $\int_S u_{0N} ds = \int_D \nabla^2 u_0 dx \simeq O(a^3)$. The sign $\simeq$ stands for the equality up to the terms of higher order of smallness as $a \to 0$. The second term is equal to $-\zeta \int_S u_0 ds \simeq -\zeta u_0(x_1)|S|$, where $|S| = O(a^2)$ is the surface area of $S$. The third term is equal to $\frac{1}{2}\int_S A\sigma ds - \frac{1}{2}Q$. When $a \to 0$ one checks that $A\sigma \simeq A_0\sigma$, where $A_0 = A|_{k=0}$, and $\int_S A_0\sigma dt = -\int_S \sigma dt$. Therefore the third term is equal to $-Q$ up to the term of higher order of smallness. The fourth term is equal to $-\zeta \int_S dt\sigma(t) \int_S g(s,t)ds = o(Q)$, as $a \to 0$. Thus,

$$Q \simeq -\zeta |S| u_0(x_1), \quad a \to 0, \tag{66}$$

and $u_0(x_1) \simeq 1$,

$$A(\beta, \alpha, \kappa) = -\frac{\zeta |S| u_0(x_1)}{4\pi}. \tag{67}$$

For the scattering by a single small body one can choose $x_1 \in D$ to be the origin, and take $u_0(x_1) = 1$. But in the many-body scattering problem the role of $u_0$ is played by the effective field which depends on $x$. Formulas (64)-(**??**) solve the scattering problem (56)-(58) for one small body $D$ of an arbitrary shape if the impedance boundary condition (56) is imposed. The scattering in this case is isotropic and $A(\beta, \alpha, \kappa) = O(a^2\zeta)$.

If the boundary condition is the Dirichlet one, $u|_S = 0$, then one derives that

$$Q \simeq -Cu_0, \quad a \to 0; \quad u_0 \simeq 1, \tag{68}$$

and

$$A(\beta, \alpha, \kappa) = -\frac{C}{4\pi}, \;\; a \to 0, \quad u_0 \simeq 1, \quad C = O(a). \tag{69}$$

Here $C$ is the electrical capacitance of the perfect conductor with the shape $D$.

In this case the scattering is isotropic and $|A| = O(a), a \to 0$. Thus, the scattered field is much larger than for the impedance boundary condition.

For the Neumann boundary condition $u_N = 0$ on $S$ the scattering is anisotropic and $A(\beta, \alpha, \kappa) = O(a^3)$, which is much smaller than for impedance boundary condition.

One has for the Neumann boundary condition the following formula:

$$A(\beta, \alpha, \kappa) = \frac{D}{4\pi} \left( ik\beta_{pq} \frac{\partial u_0}{\partial x_q} \beta_p + \nabla^2 u_0 \right), \tag{70}$$

where $\beta_{pq}$ is some tensor, $|D|$ is the volume of $|D|$, $\beta_p = \lim_{|x| \to \infty} \frac{x_p}{x}$, $x_p := x \cdot e_p$ is the $p-$th coordinate of a vector $x \in \mathbf{R}^3$, $u_0$ and its derivatives are calculated at an arbitrary point inside $D$. This point can be chosen as the origin of the coordinate system. Since $D$ is small, the choice of this point does not influence the results. The tensor $\beta_{pq}$ is defined for a body with volume $V$ and boundary $S$ as follows:

$$\beta_{pq} = \frac{1}{V} \int_S t_p \sigma_q dt,$$

where $\sigma_q$ solves the equation $\sigma_q = A\sigma_q - 2N_q$, $N_q := N \cdot e_q$, and $\{e_q\}_{q=1}^3$ is a Cartesian basis of $\mathbf{R}^3$.

Finally, consider the many-body scattering problem. Its statement can be written also as (56)-(58) but now $D = \cup_{m=1}^M D_m$ is the union of many small bodies, $\zeta = \zeta_m$ on $S_m = \partial D_m$.

One looks for the solution of the form

$$u(x) = u_0(x) + \sum_{m=1}^{M} \int_{S_m} g(x,t)\sigma_m(t)dt. \tag{71}$$

Let us define the effective field in the medium by the formula

$$u_e(x) = u_0(x) + \sum_{m \neq j} g(x, x_m)Q_m, \quad |x - x_m| \sim a. \tag{72}$$

Here $x_m \in D_m$ are arbitrary fixed points, and $|x - x_m| \sim a$ means that $|x - x_m|$ is of the size of $a$.

Assume that these points are distributed by the formula:

$$\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_\Delta N(x)dx[1 + o(1)], \quad a \to 0. \tag{73}$$

Here $\mathcal{N}(\Delta) = \sum_{x_m \in \Delta} 1$ is the number of points in an arbitrary open set $\Delta$, $N(x) \geq 0$ is a given function, $0 \leq \kappa < 1$ is a parameter, and an experimentalist can choose $N(x)$ and $\kappa$ as he wishes. Let us assume that $\zeta_m = \frac{h(x_m)}{a^\kappa}$, where $h(x)$ is an arbitrary continuous in $D$ function such that $Imh(x) \leq 0$. This function can be chosen by an experimenter as he wishes.

Under these assumptions in [24] the following results are proved.

Assume that $|S_m| = ca^2$, where $c > 0$ is a constant depending on the shape of $D_m$. Then $Q_m \simeq -ch(x_m)u_e(x_m)a^{2-\kappa}$. Denote $u_e(x_m) := u_m, h(x_m) := h_m$. Then one can find the unknown numbers $u_m$ from the following linear algebraic system (LAS):

$$u_j = u_{0j} - c\sum_{m\neq j}^{M} g_{jm}h_m u_m a^{2-\kappa}, \quad 1 \leq j \leq M, \quad g_{jm} := \frac{e^{ik|x_j - x_m|}}{4\pi|x_j - x_m|}. \tag{74}$$

LAS (74) can be reduced to a LAS of much smaller order. Namely, denote by $\Omega$ a finite domain in which all the small bodies are located. Partition $\Omega$ into a union of $P$ cubes $\Delta_p$ with the side $b = b(a)$, the cubes are non-intersecting in the sense that they do not have common interior points, they can have only pieces of common boundary. Denote by $d = d(a)$ the smallest distance between neighboring bodies $D_m$. Assume that

$$a \ll d \ll b, \quad ka \ll 1. \tag{75}$$

Then system (74) can be reduced to the following LAS:

$$u_q = u_{0q} - c\sum_{p\neq q}^{P} g_{qp}h_p u_p N_p|\Delta_p|, \quad 1 \leq q \leq P, \tag{76}$$

where $N_p := N(x_p)$, $x_p \in \Delta_p$, is an arbitrary point, $N(x)$ is the function from formula (73), and $|\Delta_p|$ is the volume of the cube $\Delta_p$.

The effective field has a limit as $a \to 0$, and this limit solves the equation

$$u(x) = u_0(x) - c\int_\Omega g(x, y)h(y)N(y)u(y)dy, \tag{77}$$

where the constant $c$ is the constant in the definition $|S_m| = ca^2$. One may consider by the same method the case of small bodies of various sizes. In this case $c = c_m$, but we do not go into details. If the small bodies are spheres of radius $a$ then $c = 4\pi$.

Applying the operator $\nabla^2 + k^2$ to the equation (77) one gets

$$(\nabla^2 + k^2 n^2(x))u := (\nabla^2 + k^2 - cN(y)h(y))u = 0 \text{ in } \mathbf{R}^3. \tag{78}$$

Therefore, embedding of many small impedance particles, distributed according formula (73) leads to a medium with a zero refraction coefficient.

$$n^2(x) = 1 - k^{-2}cN(x)h(x). \tag{79}$$

Since $N(x)$ and $h(x)$ can be chosen as one wishes, with the only restrictions $N(x) \geq 0, Imh(x) \leq 0$, one can create a medium with a desired refraction coefficient by formula (79) if one chooses suitable $N(x)$ and $h(x)$.

A similar theory is developed for electromagnetic wave scattering by small impedance bodies in [23].

## References


[1] M. Atiyah, I. Singer, The index of elliptic oprators on compact manifolds, Bull. Amer. Math. Soc., 69, (1963), 422-433.

[2] I. Gel'fand, G. Shilov, *Generalized functions*, vol. 1, Acad. Press, New York, 1964.

[3] F. Gahov, *Boundary value problems,* Pergamon Press, New York, 1966.

[4] V.Ivanov et al, *Theory of linear ill-posed problems and its applications*, VSP, Utrecht, 2002.

[5] L. Kantorovich, G. Akilov, *Functional analysis*, Pergamon Press, New York, 1982.

[6] M. Krasnoselskii et al., *Approximate solution of operator equations,* Walters-Noordhoff, Groningen, 1972.

[7] M. Krasnoselskii, P. Zabreiko, *Geometric methods of nonlinear analysis*, Springer-Verlag, Berlin, 1984.

[8] M. Krasnoselskii et al., *Integral operators in the spaces of summable functions,* Noordhoff Internat., Leiden, 1976.

[9] V. D. Kupradze et al, *Three-dimensional problems of mathematical theory of elasticity and thermo-elasticity*, Nauka, Moscow, 1976. (in Russian)

[10] N.Muskhelishvili, *Singular integral equations. Boundary problems of functions theory and their applications to mathematical physics*, Wolters-Noordhoff Publishing, Groningen, 1972.

[11] S. Mikhlin, S. Prössdorf, *Singular integral operators*, Springer-Verlag, Berlin, 1980.

[12] G. Polya, G. Szegö, *Isoperimetric inequalities in mathematical physics*, Princeton University Press, Princeton, 1951.

[13] A.G. Ramm, *Scattering by obstacles*, D.Reidel, Dordrecht, 1986.

[14] A.G. Ramm, *Wave scattering by small bodies of arbitrary shapes*, World Sci. Publ., Singapore, 2005.

[15] A.G. Ramm, *Inverse problems*, Springer, New York, 2005.

[16] A.G. Ramm, *Random fields estimation,* World Sci. Publishers, Singapore, 2005.

[17] A.G. Ramm, *Dynamical systems method for solving operator equations*, Elsevier, Amsterdam, 2007.

[18] A.G. Ramm, N.S.Hoang, *Dynamical Systems Method and Applications. Theoretical Developments and Numerical Examples.* Wiley, Hoboken, 2012

[19] A.G. Ramm, Singularities of the inverses of Fredholm operators. Proc. of Roy. Soc. Edinburgh, 102A, (1986), 117-121.

[20] A.G. Ramm, A simple proof of the Fredholm alternative and a characterization of the Fredholm operators, Amer. Math. Monthly, 108, N 9, (2001), 855-860.

[21] A.G. Ramm, A characterization of unbounded Fredholm operators, Cubo a Mathem. Journ., 5, N3, (2003), 91-95.

[22] A.G. Ramm, Wave scattering by many small bodies and creating materials with a desired refraction coefficient, Afrika Mathematika, 22, N1, (2011), 33-55.

[23] A.G. Ramm, Electromagnetic wave scattering by small impedance particles of an arbitrary shape, J. of Appl. Math and Comput., (JAMC), DOI: 10.1007/s12190-013-0671-3

[24] A.G. Ramm, Many-body wave scattering problems in the case of small scatterers, J. of Appl. Math and Comput., (JAMC), 41, N1, (2013),473-500.

[25] M. Shubin, *Pseudodifferential operators and spectral theory*, Springer-Verlag, New York, 1987.

[26] E. Stein, *Harmonic analysis: real-variable methods, orthogonality and oscillatory integrals*, Princeton University Press, Princeton, 1993.

[27] P. Zabreiko et al, *Integral equations*, Reference book, Nauka, Moscow, 1968