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INTEGRAL EQUATIONS FOR TRANSMISSION PROBLEMS IN LINEAR ELASTICITY

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ABSTRACT. The scattering of elastic waves by a penetrable homogeneous object is described by a system of integral equations for the field and its traction on the boundary of the scatterer. The system contains the operators of the single and double layer potentials, of the traction of the single layer potential, and of the traction of the double layer potential. It is a strongly elliptic system of pseudodifferential equations. Therefore every Galerkin scheme for its approximate solution is convergent. For Lipschitz boundaries we show strong ellipticity of the system of boundary integral operators. This implies existence and uniqueness of the solution and quasioptimal error estimates for its Galerkin solutions.

1. Introduction. A classical tool for the analysis of transmission problems is the reduction to boundary integral equations (see [**11**]). The property of *strong ellipticity* of boundary integral equations is known to be useful in several respects: It yields existence proofs as well as convergence results for approximate solutions. In the case of *boundary value problems*, this strong ellipticity has been analyzed extensively (see [**3**, **7**, **12**, **17**] and the literature quoted there). It turned out that there exist methods for constructing strongly elliptic boundary integral equations for general strongly elliptic boundary value problems and even for non-smooth (Lipschitz) boundaries in the case of second order systems. For *transmission problems*, in [**4**] the case of the Helmholtz equation was analyzed and a proof for strong ellipticity of the boundary integral equations on smooth boundaries and on plane polygonal boundaries was given. In the paper [**5**], the scattering of electromagnetic waves was studied and a general principle was found that allows proof of strong ellipticity for the system of integral equations which is obtained by application of the "direct method" to transmission problems. This principle yields strongly elliptic boundary integral equations for general combinations of boundary and transmission conditions [**13**].

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Here we apply this principle to the equations of time-harmonic elastodynamics and show that it works in the general setting of Lipschitz boundaries. We consider solutions of the transmission problem with locally finite energy and represent them by boundary potentials whose densities are the jumps of the displacements and of the tractions across the transmission boundary Γ. These densities are the solution of our system of integral equations of the first kind on the boundary manifold. This system contains integral operators with weakly singular, strongly singular and hypersingular kernels. We prove that this system satisfies a Garding inequality in the energy norm, i.e., it defines a bilinear form which is positive definite up to a compact perturbation. This is the property of strong ellipticity.

Strong ellipticity implies on one hand that the integral operator is Fredholm of index zero. Thus one obtains existence under the assumption that the transmission problem has no eigensolutions. On the other hand, it follows that every Galerkin approximation method converges quasioptimally in the energy norm [**10**, **16**]. Thus convergence is guaranteed for any choice of finite dimensional subspaces of the Sobolev spaces $H^{1/2}(\Gamma)$ for the displacements and of $H^{-1/2}(\Gamma)$ for the tractions, respectively, as defined in standard boundary element methods, for example the h–, p–, and h–p versions of boundary elements [**15**, **17**] or spectral methods. Our system of boundary integral equations therefore provides a constructive solution procedure for the transmission problem.

The same boundary integral operators can be used for screen or crack scattering problems [**14**] and also for the scattering from an inhomogeneous body which leads to a coupled finite element/boundary element procedure [**3**, **6**].

2. Strong ellipticity of the boundary integral equations. In the following we present a boundary integral equation method to solve the scattering of time-harmonic elastic waves at a homogeneous, isotropic scatterer given by a bounded domain Ω_1 with unbounded exterior $\Omega_2 = \mathbf{R}^3\backslash\bar{\Omega}_1$. We assume that the boundary $\Gamma = \partial\Omega_1$ of Ω_1 is Lipschitz continuous.

The transmission problem in steady state elastodynamics, under consideration here, reads: *For given vector fields u_0 and t_0 on the* *boundary* Γ *find vector fields* u_j *in* $\Omega_j, j = 1, 2$, *satisfying the equations of linear elasticity,*

$$P_j u_j - \rho_j \omega^2 u_j = 0 \quad \textit{in } \Omega_j, \qquad j = 1, 2, \tag{2.1}$$

and the transmission conditions

$$u_1 = u_2 + u_0, \qquad t_1 = t_2 + t_0 \quad \textit{on } \Gamma. \tag{2.2}$$

Here the differential operators P_j are given by

$$P_j u \equiv -(\mu_j \Delta u + (\lambda_j + \mu_j) \operatorname{grad} \operatorname{div} u).$$

$\rho_j > 0$ is the density of the medium Ω_j, and $\omega > 0$ is the frequency of the incident wave. u_1, u_2 and u_0 denote the displacement of the refracted, scattered and incident wave, respectively. The corresponding tractions t_1, t_2, t_0 are given by $t_j = T_j(u_j)|_\Gamma$, where

$$T_j(u) = 2\mu_j \partial_n u + \lambda_j n \operatorname{div} u + \mu_j n \times \operatorname{curl} u$$

with Lamé constants $\mu_j > 0, 3\lambda_j + 2\mu_j > 0$. Here $\partial_n u$ is the derivative with respect to the outer normal n on Γ.

In addition to (2.1), (2.2) we need a regularity condition at infinity [**11**] for the displacement vector u_2 which we resolve into a sum of an irrotational (lamellar) vector u^L and a solenoidal vector u^T. For $u_2^* = u_2^L$ and $u_2^* = u_2^T$, respectively, we require

$$\frac{\partial u_2^*}{\partial |x|} - i k_2^* u_2^* = O\Big(\frac{1}{|x|}\Big), \qquad u_2^* = o(1), \qquad \text{as } |x| \to \infty. \tag{2.3}$$

If $u_2^* = u_2^L$ then we take for k_2^* the longitudinal (dilational) wave number $k_2^L = \omega \rho_2^{1/2} (\lambda_2 + 2\mu_2)^{-1/2}$, whereas, if $u_2^* = u_2^T$ we take the transverse (shear) wave number $k_2^T = \omega \rho_2^{1/2} \mu_2^{-1/2}$.

We are interested in solutions u_j of (2.1), (2.2) which belong to $H^1_{\mathrm{loc}}(\Omega_j)$, i.e., in solutions which have local finite energy. We define

(2.4)

$$\mathcal{L}_1 = \{u_1 \in H^1(\Omega_1) : P_1 u_1 = \rho_1 \omega^2 u_1 \text{ in } \Omega_1\}$$

$$\mathcal{L}_2 = \{u_2 \in H^1_{\mathrm{loc}}(\bar{\Omega}_2) : P_2 u_2 = \rho_2 \omega^2 u_2 \text{ in } \Omega_2 \text{ and } u_2 \text{ satisfies (2.3)}\}$$

We note that, due to the trace lemma, $u_j|_\Gamma \in H^{1/2}(\Gamma)$ for $u_j \in \mathcal{L}_j$. Further we observe that the corresponding tractions t_j are defined in the usual way as distributions in $H^{-1/2}(\Gamma)$ via the First Green formula (see [**2**, **9**]).

LEMMA 2.1. *Let* $u \in H^1_{\text{loc}}(\bar{\Omega}_j)$ *with* $\operatorname{supp} u$ *compact satisfy* $P_j u \in L^2_{\text{loc}}(\bar{\Omega}_j)$ *and let* $v \in H^1(\Omega_j)$ *with bounded support. Then* $T_j u|_\Gamma \in H^{-1/2}(\Gamma)$ *is defined by*

$$\int_{\Omega_j} P_j u \cdot v\, dx = (-1)^j \langle T_j u, v\rangle + \Phi_j(u, v) \tag{2.5}$$

with

$$\Phi_j(u,v) = \int_{\Omega_j} \sum_{i,h,k,l=1}^{3} a^j_{ihkl} \varepsilon_{kl}(u) \varepsilon_{ih}(v)\, dx.$$

Here we have

$$a^j_{ihkl} = \lambda_j \delta_{ih}\delta_{kl} + \mu_j(\delta_{ik}\delta_{hl} + \delta_{il}\delta_{hk})$$

with $\delta_{ik} = 1$ for $i = k$ and $\delta_{ik} = 0$ for $i \neq k$.

The brackets $\langle\,\cdot\,,\,\cdot\,\rangle$ denote the duality between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ which gives

$$\langle f, g\rangle = \int_\Gamma f \cdot g\, ds$$

for smooth vector fields f, g. The bilinear form $\Phi_j(u,v)$ represents the local displacement work due to the strain tensor

$$\varepsilon_{kl}(u) = \frac{1}{2}(\partial_k u_l + \partial_l u_k).$$

For the transmission problem (2.1), (2.2) there holds the following general uniqueness result (see [**11**; Chapter III, §2.15]). (In Kupradze's terminology, we are considering the "basic contact problem for steady elastic oscillations.")

LEMMA 2.2. *The only solution* $u_j \in \mathcal{L}_j, j = 1, 2$, *of the homogeneous problem corresponding to* (2.1)-(2.3) *is the trivial solution* $u_1 \equiv 0, u_2 \equiv 0$.

From (2.5) one obtains, with the symmetry of Φ_j (which holds due to the symmetry properties of the coefficients a^j_{ihkl}), the Second Green formula

$$(2.6) \quad \int_{\Omega_j} (P_j u \cdot v - u \cdot P_j v)\, dx = (-1)^j \int_\Gamma (v \cdot T_j(u) - u \cdot T_j(v))\, ds.$$

This gives, in Ω_j, with the fundamental solution $G_j(x, y, \omega)$ of $(P_j - \rho_j \omega^2) u_j = 0$, the Somigliana representation formula for $x \in \Omega_j$:

$$(2.7) \quad u_j(x) = (-1)^j \int_\Gamma \{T_j(x, y, \omega) v_j(y) - G_j(x, y, \omega) \phi_j(y)\}\, ds(y),$$

where $v_j = u_j, \phi_j = T_j(u_j) = t_j$ on Γ. Here G_j is the 3×3 matrix function

$$(G_j)_{ik} := \frac{-1}{4\pi\mu_j} \Big\{ \frac{1}{r} e^{ik_j^T r} \delta_{ik} + (k_j^T)^{-2} \partial_i \partial_k [(e^{ik_j^T r} - e^{ik_j^L r}) r^{-1}] \Big\}$$

with $r := |x - y|$ and

$$T_j(x, y, \omega) = T_{j,y}(G_j(x, y, \omega))^T.$$

From the analysis in [**2**] follows the validity of (2.7) for a Lipschitz boundary Γ.

LEMMA 2.3. *Let* $u_j \in \mathcal{L}_j$. *Then the representation formula* (2.7) *holds for* u_j *in* Ω_j. *For any* $v_j \in H^{1/2}(\Gamma)$ *and any* $\phi_j \in H^{-1/2}(\Gamma)$ *the formula* (2.7) *defines a vector field* $u_j \in \mathcal{L}_j$.

Taking Cauchy data in (2.7) one finds the relations on Γ,

$$(2.8) \quad \begin{pmatrix} v_j \\ \phi_j \end{pmatrix} = \mathcal{C}_j \begin{pmatrix} v_j \\ \phi_j \end{pmatrix},$$

where the Calderón projector

$$\mathcal{C}_j = \begin{pmatrix} \frac{1}{2} + (-1)^j \Lambda_j & -(-1)^j V_j \\ -(-1)^j D_j & \frac{1}{2} - (-1)^j \Lambda_j' \end{pmatrix} \tag{2.9}$$

is defined via the boundary integral operators

$$V_j v(x) = \int_\Gamma G_j(x,y,\omega) v(y)\, ds(y), \quad \Lambda_j v(x) = \int_\Gamma T_j(x,y,\omega) v(y)\, ds(y),$$

$$D_j v(x) = -T_{j,x} \int_\Gamma T_j(x,y,\omega) v(y)\, ds(y), \tag{2.10}$$

$$\Lambda_j' v(x) = \int_\Gamma T_j(y,x,\omega)^T v(y)\, ds(y).$$

We remark that, due to [**1**, **2**], the Calderón projector $\mathcal{C}_j$ is well-defined for boundary data (v_j, ϕ_j) belonging to the space $\mathcal{H} = H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$. The mapping properties of the Calderón projector $\mathcal{C}_j$ are described as follows.

LEMMA 2.4. (a) *The statements* (i) *and* (ii) *on* $(v, \psi) \in \mathcal{H}$ *are equivalent*:

(i) (v, ψ)*are Cauchy data of some* $u_j \in \mathcal{L}_j$;

(ii) $(I - \mathcal{C}_j)\begin{pmatrix} v \\ \psi \end{pmatrix} = 0$.

(b) *The operators* $\mathcal{C}_j$ *are projection operators mapping* $\mathcal{H}$ *onto its subspace of Cauchy data of weak solutions in* $\mathcal{L}_j$.

The proof is an immediate consequence of the representation formula (2.7) due to Lemma 2.3 and the definition of the Calderón projector $\mathcal{C}_j$.

Thus we can write the transmission problem (2.1), (2.2) in the equivalent form

$$(I - \mathcal{C}_1)\begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} = 0 \tag{2.11}$$

$$(I - \mathcal{C}_2)\begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix} = 0 \tag{2.12}$$

$$\begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} - \begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix} \tag{2.13}$$

This is a system of 6 vector equations for 4 vector unknowns. From this system we can extract a square subsystem by inserting $\begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix}$ from (2.13) into (2.12) and subtracting (2.11) from the resulting equation. We obtain the boundary integral equation

$$A\begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} = (I - \mathcal{C}_2)\begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix} \quad \text{with} \quad A := \mathcal{C}_1 - \mathcal{C}_2. \tag{2.14}$$

We have the following equivalence theorem.

THEOREM 2.5. *Let* $\begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix} \in \mathcal{H} = H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ *be given. Then there holds*

(i) *If* $u_j \in \mathcal{L}_j$ *solve the transmission problem* (2.1)–(2.3), *then*

$$\begin{pmatrix} v \\ \phi \end{pmatrix} := \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} u_1|_\Gamma \\ T(u_1|)|_\Gamma \end{pmatrix} \in \mathcal{H}$$

solves the boundary integral equation (2.14).

(ii) *If* $\begin{pmatrix} v \\ \phi \end{pmatrix} \in \mathcal{H}$ *solves* (2.14), *then, with*

$$\begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} := \mathcal{C}_1\begin{pmatrix} v \\ \phi \end{pmatrix} \text{ and } \begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix} := \mathcal{C}_2\left(\begin{pmatrix} v \\ \phi \end{pmatrix} - \begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix}\right)$$

and u_j *defined by* (2.7), $u_j \in \mathcal{L}_j$ *solve the problem* (2.1)–(2.3).

PROOF. (i) follows from the derivation of (2.14) above.

(ii). From the definition of $\begin{pmatrix} v_j \\ \phi_j \end{pmatrix}$ and the projection property of $\mathcal{C}_j$ follows

$$(I - \mathcal{C}_j)\begin{pmatrix} v_j \\ \phi_j \end{pmatrix} = 0,$$

hence $\binom{v_j}{\phi_j}$ are Cauchy data of certain $u_j \in \mathcal{L}_j$ which are then given by the representation formula (2.7). It remains to show that the transmission condition (2.2) is satisfied:

$$\begin{aligned}\begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix} - \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} &= (\mathcal{C}_2 - \mathcal{C}_1)\begin{pmatrix} v \\ \phi \end{pmatrix} - \mathcal{C}_2\begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix} = -A\begin{pmatrix} v \\ \phi \end{pmatrix} - \mathcal{C}_2\begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix} \\ &= -(I - \mathcal{C}_2)\begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix} - \mathcal{C}_2\begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix} = -\begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix}. \ \square\end{aligned}$$

REMARK. If $\binom{v}{\phi}$ is a solution of the boundary integral equation (2.14), then the solution u_j of the transmission problem is given by the representation formula (2.7) with $\binom{v}{\phi}$ as source term, but it does not immediately follow that $\binom{v}{\phi} = \binom{v_1}{\phi_1}$ holds. Therefore the uniqueness of the transmission problem (Lemma 2.2) does not immediately carry over to the boundary integral equations. However, consider the difference

$$\begin{pmatrix} w \\ \psi \end{pmatrix} := \begin{pmatrix} v \\ \phi \end{pmatrix} - \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix}.$$

There holds

$$\begin{pmatrix} w \\ \psi \end{pmatrix} = (I - \mathcal{C}_1)\begin{pmatrix} v \\ \phi \end{pmatrix}$$

and

$$\begin{aligned}\begin{pmatrix} w \\ \psi \end{pmatrix} &= (I - \mathcal{C}_2 - A)\begin{pmatrix} v \\ \phi \end{pmatrix} = (I - \mathcal{C}_2)\begin{pmatrix} v \\ \phi \end{pmatrix} - (I - \mathcal{C}_2)\begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix} \\ &= (I - \mathcal{C}_2)\left(\begin{pmatrix} v \\ \phi \end{pmatrix} - \begin{pmatrix} v_0 \\ \phi_0 \end{pmatrix}\right).\end{aligned}$$

Therefore $\mathcal{C}_1\binom{w}{\psi} = 0 = \mathcal{C}_2\binom{w}{\psi}$, and this shows that $\binom{w}{\psi}$ are Cauchy data of a homogeneous transmission problem for which the roles of P_1 and P_2 are interchanged:

$$\begin{pmatrix} w \\ \psi \end{pmatrix} = \begin{pmatrix} w_1 \\ T_2(w_1) \end{pmatrix} = \begin{pmatrix} w_2 \\ T_1(w_2) \end{pmatrix} \quad \text{on} \quad \Gamma,$$

where $w_j \in \mathcal{L}_j$ satisfy

$$P_2 w_1 = \rho_2 \omega^2 w_1 \ \text{ on } \Omega_1; \qquad P_1 w_2 = \rho_1 \omega^2 w_2 \ \text{ in } \Omega_2.$$

From Lemma 2.2 we infer that $w_1 = w_2 = 0$, hence $\binom{w}{\psi} = 0$. Finally this implies $\binom{v}{\phi} = \binom{v_1}{\phi_1}$ and hence also the uniqueness of the solution of the integral equation (2.14). Thus, for the uniqueness of the latter equation, we need uniqueness for both the original transmission problem and the above "adjoint" transmission problem.

THEOREM 2.6. *The operator A is strongly elliptic: There exist $\gamma > 0$ and a compact operator $T : \mathcal{H} \to \mathcal{H}$ with*

$$\text{(2.15)} \quad \mathrm{Re}\left\langle (A+T)\begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} v \\ \phi \end{pmatrix} \right\rangle \geq \gamma\Big(\|v\|^2_{H^{1/2}(\Gamma)} + \|\phi\|^2_{H^{-1/2}(\Gamma)}\Big)$$

for all $\begin{pmatrix} v \\ \phi \end{pmatrix} \in \mathcal{H}$. Here the brackets denote the natural (anti-) duality of $\mathcal{H}$ with itself:

$$\left\langle \begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} w \\ \psi \end{pmatrix} \right\rangle := \int_\Gamma (\bar{v}\psi + w\bar{\phi})\, ds \quad \textit{for} \quad \begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} w \\ \psi \end{pmatrix} \in \mathcal{H}.$$

PROOF. We write

$$A = A_1 + A_2 \quad \text{with} \quad A_j = (-1)^j\Big(\frac{1}{2} - \mathcal{C}_j\Big) = \begin{pmatrix} -\Lambda_j & V_j \\ D_j & \Lambda_j' \end{pmatrix}.$$

Since the sum of two strongly elliptic operators is strongly elliptic, it suffices to show the strong ellipticity of the operators A_1 and A_2. We give the proof for A_1. Due to density arguments, one needs to show the Garding inequality (2.15) only for smooth (v, ϕ). Let then u_j, $j = 1, 2$, be defined by

$$u_j(x) = \chi(x) \int_\Gamma \{T_1(x,y)v(y) - G_1(x,y)\phi(y)\}\, ds(y), \quad x \in \Omega_j.$$

Here we choose $\chi \in C_0^\infty(\mathbf{R}^3)$ satisfying $\chi \equiv 1$ in a neighborhood of $\bar{\Omega}_1$. Then, by definition of the Calderón projectors (i.e., the classical jump relations for the elastic potentials), the Cauchy data

$$v_j := u_j|_\Gamma \quad \text{and} \quad \phi_j := T(u_j)$$

satisfy

$$\begin{pmatrix} v_j \\ \phi_j \end{pmatrix} = (-1)^j \Big(\frac{1}{2} - (-1)^j A_1\Big) \begin{pmatrix} v \\ \phi \end{pmatrix}.$$

By adding and subtracting these two equations, we find

$$A_1 \begin{pmatrix} v \\ \phi \end{pmatrix} = -\begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} - \begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix},$$
$$\begin{pmatrix} v \\ \phi \end{pmatrix} = -\begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} + \begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix}.$$

Thus the bilinear form defined by A_1 is given by

$$\begin{aligned}
&2\left\langle A_1 \begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} v \\ \phi \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} + \begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} - \begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix}, \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix} \right\rangle \\
&\quad + \left\langle \begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix}, \begin{pmatrix} v_2 \\ \phi_2 \end{pmatrix} \right\rangle \\
&= 2\,\mathrm{Re} \int_\Gamma (\bar{v}_1\phi_1 - \bar{v}_2\phi_2)\, ds + 2i\,\mathrm{Im} \int_\Gamma (\bar{v}_2\phi_1 - \bar{v}_1\phi_2)\, ds.
\end{aligned}$$

Hence

$$\text{(2.16)} \qquad \mathrm{Re} \left\langle A_1 \begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} v \\ \phi \end{pmatrix} \right\rangle = \mathrm{Re} \int_\Gamma (\bar{v}_1\phi_1 - \bar{v}_2\phi_2)\, ds.$$

Now we need the first Green formulas for P_1 in Ω_1 and Ω_2. This leads to

$$\text{(2.17)} \qquad \widetilde{\Phi}_j(u_j, u_j) - \int_{\Omega_j} \bar{u}_j \cdot (P_1 - \rho_1\omega^2) u_j \, dx = -(-1)^j \int_\Gamma \bar{v}_j \phi_j \, ds,$$

where

$$\widetilde{\Phi}_j(u_j, u_j) = \int_{\Omega_j} (\Sigma a^1_{ihkl} \overline{\varepsilon_{kl}(u_j)} \varepsilon_{ih}(u_j) - \rho_1^2 \omega |u_j|^2)\, dx.$$

Now $P_1 u_1 - \rho_1^2 \omega u_1 = 0$ and $P_1 u_2 - \rho_1^2 \omega u_2 = f_2$, where $f_2 \in C_0^\infty(\Omega_2)$. ($f_2 \equiv 0$ whenever $\chi \equiv 1$ or $\chi \equiv 0$ holds.) From (2.16) and (2.17) together we find

(2.18)

$$\mathrm{Re}\left\langle A_1 \begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} v \\ \phi \end{pmatrix} \right\rangle = \mathrm{Re}\,\{\widetilde{\Phi}_1(u_1, u_1) + \widetilde{\Phi}_2(u_2, u_2) - \int_{\Omega_2} \overline{u}_2 \cdot f_2 \, dx\}.$$

As the support of f_2 is disjoint from Γ, there is a compact operator T_1 on $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ such that

$$\left| \int_{\Omega_2} \overline{u}_2 \cdot f_2 \, dx \right| \leq \left\langle T_1 \begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} v \\ \phi \end{pmatrix} \right\rangle.$$

From Korn's inequality and the trace lemma we find that there exist compact quadratic forms k_j on $H^1(\Omega_j)$ and hence a compact operator T_2 on $\mathcal{H} = H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ such that

$$\begin{aligned} \widetilde{\Phi}_1(u_1, u_1) + \widetilde{\Phi}_2(u_2, u_2) &\geq \gamma_1 \left(\|u_1\|^2_{H^1(\Omega_1)} + \|u_2\|^2_{H^1(\Omega_2)} \right) \\ &\quad - k_1(u_1) - k_2(u_2) \\ &\geq \gamma_2 \left(\|v\|^2_{H^{1/2}(\Gamma)} + \|\phi\|^2_{H^{-1/2}(\Gamma)} \right) \\ &\quad - \left\langle T_2 \begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} v \\ \phi \end{pmatrix} \right\rangle. \end{aligned}$$

From (2.18) and (2.19) together we get the desired result:

$$\mathrm{Re}\left\langle (A_1 + T_1 + T_2) \begin{pmatrix} v \\ \phi \end{pmatrix}, \begin{pmatrix} v \\ \phi \end{pmatrix} \right\rangle \geq \gamma_2 \left(\|v\|^2_{H^{1/2}(\Gamma)} + \|\phi\|^2_{H^{-1/2}(\Gamma)} \right). \square$$

By Garding's inequality (2.15) the operator A is a Fredholm operator of index zero from $\mathcal{H}$ into itself. Therefore, we obtain existence of a solution of the integral equation (2.14) as soon as we know its uniqueness, and Theorem 2.5 then implies the existence of a solution of the transmission problem. But under the assumptions of Lemma 2.2 the possible solution of the transmission problem is unique. Therefore, from the equivalence between the transmission problem (2.1), (2.2) and the integral equation (2.14) stated in Theorem 2.5, follows the existence of a unique solution in $\mathcal{H}$ of the integral equation. We summarize these results as follows.

COROLLARY 2.7. *For given* $\binom{v_0}{\phi_0} \in \mathcal{H}$, *there exists exactly one solution* $\binom{v_1}{\phi_1} \in \mathcal{H}$ *of the integral equation* (2.14) *yielding exactly one solution* u *with* $u_j \in \mathcal{L}_j$ *of the transmission problem* (2.1)–(2.3) *via the representation formula* (2.7).

As a final remark, we want to emphasize that the system of integral equations analyzed in this paper is well suited for practical computations. Parts of the system, including the most "nonclassical" hypersingular integral operator D, have already been numerically tested [**8**].

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