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INTEGRAL EQUATIONS IN REFLEXIVE BANACH SPACES AND WEAK TOPOLOGIES

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ABSTRACT. The Schauder Tychonoff theorem in a locally convex topological space is used to establish existence results for Volterra-Hammerstein and Hammerstein integral equations in a reflexive Banach space.

1. INTRODUCTION

This paper studies integral equations in a reflexive Banach space relative to the weak topology. In particular, in §2 we establish the existence of a weak solution (described in §2) to the Volterra-Hammerstein integral equation

$$y(t) = h(t) + \int_0^t k(t,s) f(s, y(s))\, ds, \qquad t \in [0,T],$$

where $T > 0$ is fixed. Here y takes values in a reflexive Banach space B. The Schauder Tychonoff theorem in locally convex spaces is used to establish existence. Our method has the added advantage in that it discusses automatically the interval of existence $[0,T]$. We note as well that no compactness condition will be assumed on the nonlinearity f; this will be due to the fact that a subset of a reflexive Banach space is weakly compact iff it is weakly closed and norm bounded. The results of this section complement related work in the literature; see [2, 5, 13, 14]. For example in [5, 13, 14] some very interesting results for the differential system $y' = f(t,y)$ (which is a particular case of the Volterra-Hammerstein equation) are presented. The basic idea in these papers is to use a "successive approximation" type of approach to show "local" existence. The interval of existence from a "construction" point of view is only briefly discussed. Section 3 discusses the Hammerstein integral equation

$$y(t) = h(t) + \int_0^1 k(t,s) f(s, y(s))\, ds, \qquad t \in [0,1],$$

with y taking values in B.

For the remainder of this section we gather together some results which will be used throughout this paper. B will always be a reflexive Banach space with norm $\|\cdot\|$. B^* will denote the dual of B. We will let B_w denote the space B when endowed with the weak topology generated by the continuous linear functionals on B (the family of seminorms $\{\rho_h : h \in B^*\}$ is defined by $\rho_h(x) = |h(x)|$ for all $x \in B$).

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We recall, for convenience [7, 11, 12, 15], the following: Let $y(t)$ be a function from $[a,b]$ into B. Then

(i) $y(t)$ is said to be weakly continuous at $t_0 \in [a,b]$ if for every $\phi \in B^*$ we have $\phi(y(t))$ continuous at t_0.

(ii) $y(t)$ is weakly Riemann integrable on $[a,b]$ if for any partition $\{t_0, \dots, t_n\}$ of $[a,b]$ and any choice of points $\tau_i, t_{i-1} \le \tau_i \le t_i$, $i = 1, \dots, n$, the sums $\sum_{i=1}^n y(\tau_i)(t_i - t_{i-1})$ converge weakly to some element $y_0 \in B$, provided

$$\max_{1 \le i \le n} |t_i - t_{i-1}| \to 0,$$

i.e. there exists an element $y_0 \in B$ such that

$$\phi(y_0) = \int_a^b \phi(y(s))\, ds \quad \text{for all } \phi \in B^*.$$

Similarly we can define the Bochner (respectively Pettis) integral of a strongly (respectively weakly) measurable function $y : [a,b] \to B$; see [7, 11, 12]. In particular if y is weakly continuous on $[a,b]$, then y is strongly measurable [12]; also if B is reflexive, y is weakly Riemann integrable [11, 13].

Now $C([a,b], B_w)$ denotes the family of weakly continuous functions on $[a,b]$ (the family of seminorms $\{\eta_h\}$ is defined by $\eta_h(g) = \sup_{x \in [a,b]} \rho_h(g(x))$ for all $g \in C([a,b], B_w)$). $C([a,b], B_w)$ is a locally convex topological space; see [6, 8].

Next we recall the following results from the literature on functional analysis [1, 2, 4, 7, 13, 15].

Theorem 1.1 (Schauder Tychonoff)**.** *Let K be a closed convex subset of a locally convex (Hausdorff) space E. Assume that $f : K \to K$ is continuous and that $f(K)$ is relatively compact in E. Then f has at least one fixed point in K.*

Theorem 1.2 (Arzela Ascoli)**.** *Let F be a weakly equicontinuous family of functions from $I = [a,b]$ into B, and let $\{x_n(t)\}$ be a sequence in F such that for each $t \in I$, the set $\{x_n(t), n \ge 1\}$ is weakly relatively compact. Then there exists a subsequence $\{x_{n_k}(t)\}$ which converges weakly uniformly on I to a weakly continuous function.*

Remark. (i) A family $F = \{f_i, i \in J\}$, J some index set, is said to be weakly equicontinuous if given $\varepsilon > 0$, $\phi \in B^*$ there exists $\delta > 0$ such that, for $t, s \in [a,b]$, if $|t - s| < \delta$, then

$$|\phi(f_i(t) - f_i(s))| < \varepsilon \quad \text{for all } i \in J.$$

(ii) $\{x_n(t)\}_{n=1}^\infty$ converges weakly uniformly on I to a function $x(t)$ if for all $\varepsilon > 0$, $\phi \in B^*$ there exists an integer N so that $n > N$ implies

$$|\phi(x_n(t) - x(t))| < \varepsilon \quad \text{for all } t \in I.$$

Theorem 1.3 (Eberlein Šmulian)**.** *Suppose K is weakly closed in a Banach space E. Then the following are equivalent*:

(i) *K is weakly compact.*

(ii) *K is weakly sequentially compact, i.e. any sequence in K has a subsequence which converges weakly.*

Theorem 1.4. *A subset of a reflexive Banach space is weakly compact iff it is closed in the weak topology and bounded in the norm topology.*

Theorem 1.5. *A convex subset of a normed space X is closed iff it is weakly closed.*

Finally we state a result which is an immediate consequence of the Hahn Banach theorem.

Theorem 1.6. *Let X be a normed space with $0 \neq x_0 \in X$. Then there exists a $\phi \in X^*$ with $\|\phi\| = 1$ and $\phi(x_0) = \|x_0\|$.*

2. Volterra integral equations in reflexive Banach spaces

Throughout this section B will be a reflexive Banach space. We will study the Volterra-Hammerstein integral equation

$$y(t) = h(t) + \int_0^t k(t,s) f(s, y(s))\, ds, \qquad t \in [0,T], \tag{2.1}$$

where $T > 0$ is fixed. Assume

$$f : [0,T] \times B \to B \text{ is weakly-weakly continuous,} \tag{2.2}$$

$$h : [0,T] \to B \text{ is weakly continuous,} \tag{2.3}$$

and

$$\begin{cases} k(t,s) \in L^1([0,T], \mathbf{R}) \text{ for each } t \in [0,T] \text{ and the map} \\ t \to k(t,s) \text{ is continuous from } [0,T] \text{ to } L^1([0,T], \mathbf{R}); \\ \text{also there exists } v \in L^1[0,T] \text{ and constants } \alpha > 0, \\ \beta > 0 \text{ such that for } x < t \text{ in } [0,T] \text{ we have} \\ \displaystyle\int_x^t |k(t,s)|\, ds \leq \beta \left(\int_x^t v(s)\, ds \right)^{\alpha} \end{cases} \tag{2.4}$$

are satisfied.

Remark. Let $g : [a,b] \times B \to B$. Then $g(t,u)$ is said to be weakly-weakly continuous at (t_0, u_0) if given $\varepsilon > 0$, $\phi \in B^*$ there exists $\delta > 0$ and a weakly open set U containing u_0 such that

$$|\phi(g(t,u) - g(t_0,u_0))| < \varepsilon \quad \text{whenever } |t - t_0| < \delta \text{ and } u \in U.$$

Theorem 2.1. *Suppose f satisfies (2.2). Let $\mu > 0$ be given and define*

$$Q = \{(t,u) : 0 \leq t \leq T, \|u\| \leq \mu\} \subseteq [0,T] \times B.$$

Then there exists a constant $K_\mu > 0$ such that $\| f(t,u)\| \leq K_\mu$ for all $(t,u) \in Q$.

Proof. Let $V = \{u : \|u\| \leq \mu\} \subseteq B$. Now V is weakly compact from Theorems 1.4 and 1.5. Consequently Tychonoff's theorem implies that Q is compact in the product (i.e. real $\times$ weak) topology. Since f is weakly-weakly continuous on $[0,T] \times B$, we have that the range $f(Q)$ is weakly compact. Consequently Theorem 1.4 implies that $f(Q)$ is bounded in the norm topology. □

By a solution to (2.1) we mean a function $y \in C([0,T], B_w)$ which satisfies the integral equation in (2.1). This is equivalent (consequence of the Hahn Banach theorem) to finding a function $y \in C([0,T], B_w)$ with

$$\phi(y(t)) = \phi\left(h(t) + \int_0^t k(t,s) f(s, y(s))\, ds \right), \qquad t \in [0,T] \text{ for all } \phi \in B^*.$$

Theorem 2.2. *Suppose* (2.2), (2.3), *and* (2.4) *hold. In addition assume*

$$\text{(2.5)}\quad \begin{cases} \text{there exist a nondecreasing continuous (independent of } \phi\text{) function} \\ \psi : [0,\infty) \to (0,\infty) \text{ and a constant } \sigma \ge 1 \text{ with} \\ \left|\phi\left(\int_0^t k(t,s) f(s,y(s))\,ds\right)\right| \le \psi\left(\int_0^t \|y(s)\|^\sigma\,ds\right) \\ \text{for any (norm continuous) } y \in C([0,T],B_w) \text{ and any} \\ \phi \in B^* \text{ with } \|\phi\| = 1 \end{cases}$$

and

$$\text{(2.6)}\qquad T < \int_0^\infty \frac{du}{(\psi(u)+h_0)^\sigma} \quad \text{where } h_0 = \sup_{[0,T]} \|h(t)\|$$

are satisfied. Then (2.1) *has a solution* $y \in C([0,T],B_w)$; *in fact the solution we produce will be norm* (*strongly*) *continuous.*

Proof. Let

$$J(z) = \int_0^z \frac{du}{(\psi(u)+h_0)^\sigma},$$

so $J : [0,\infty) \to [0,\infty)$ is a strictly increasing function. Also let

$$K = \{y \in C([0,T],B_w) : y \text{ is norm continuous with } \textstyle\int_0^t \|y(s)\|^\sigma\,ds \le a(t) \text{ and } \|y\|_0 \le M_0\}$$

where

$$\text{(2.7)}\qquad a(t) = J^{-1}(t)$$

and

$$\text{(2.8)}\qquad M_0 = h_0 + \psi(a(T)).$$

Remark. For notational purposes $\|y\|_0 = \sup_{[0,T]} \|y(t)\|$.

First notice that K is convex and norm closed. Hence K is weakly closed by Theorem 1.5. Define an operator N by

$$\text{(2.9)}\qquad Ny(t) = h(t) + \int_0^t k(t,s) f(s,y(s))\,ds.$$

We **claim** that $N : K \to K$ is weakly continuous and $N(K)$ is weakly relatively compact. Once the claim is established, then Theorem 1.1 with $E = C([0,T],B_w)$ guarantees a fixed point of N in K, and hence (2.1) has a solution in $C([0,T],B_w)$.

We begin by showing that $N : K \to K$. To see this, take $y \in K$ and consider $Ny(s)$ for $s \in [0,T]$. Without loss of generality assume $Ny(s) \ne 0$ for all $s \in [0,T]$. Then Theorem 1.6 implies that there exists $\phi_s \in B^*$ with $\|\phi_s\| = 1$ and

$\phi_s(Ny(s)) = \|Ny(s)\|$. Thus

$$\begin{aligned}\int_0^t \|Ny(s)\|^\sigma\, ds &= \int_0^t \left(\phi_s\left(h(s) + \int_0^s k(s,x) f(x,y(s))\, dx\right)\right)^\sigma ds \\ &\le \int_0^t \left(|\phi_s(h(s))| + \int_0^s |\phi_s(k(s,x)f(x,y(x)))|\, dx\right)^\sigma ds \\ &\le \int_0^t \left(h_0 + \psi\left(\int_0^s \|y(x)\|^\sigma\, dx\right)\right)^\sigma ds \\ &\le \int_0^t (h_0 + \psi(a(s)))^\sigma\, ds = \int_0^t a'(s)\, ds = a(t)\end{aligned}$$

since

$$\int_0^{a(s)} \frac{dx}{(\psi(x)+h_0)^\sigma} = s.$$

Next, we show $\|Ny\|_0 = \sup_{[0,T]} \|Ny(t)\| \le M_0$ for any $y \in K$. To see this, look at $Ny(t)$ for $t \in [0,T]$. Without loss of generality assume $Ny(t) \ne 0$ for all $t \in [0,T]$. Then Theorem 1.6 implies that there exists $\phi_t \in B^*$ with $\|\phi_t\| = 1$ and $\phi_t(Ny(t)) = \|Ny(t)\|$. Thus

$$\begin{aligned}\|Ny(t)\| &= \phi_t\left(h(t) + \int_0^t k(t,s) f(s,y(s))\, ds\right) \\ &\le h_0 + \int_0^t |\phi_t(k(t,s)f(s,y(s)))|\, ds \\ &\le h_0 + \psi\left(\int_0^t \|y(x)\|^\sigma\, dx\right) \le h_0 + \psi(a(t)) \le h_0 + \psi(a(T)) = M_0.\end{aligned}$$

It remains to show Ny is norm continuous for any $y \in K$. To see this, let $t, x \in [0,T]$ with $t > x$, and without loss of generality assume $Ny(t) - Ny(x) \ne 0$. Then Theorem 1.6 implies that there exists $\phi \in B^*$ with $\|\phi\| = 1$ and $\phi(Ny(t) - Ny(x)) = \|Ny(t) - Ny(x)\|$. Notice also since $\|y\|_0 \le M_0$, then Theorem 2.1 guarantees the existence of a constant K_1 (independent of the chosen y) with

$$\| f(s,y(s))\| \le K_1 \quad \text{for all } s \in [0,T] \text{ and for all } y \in K. \tag{2.10}$$

Thus

$$\begin{aligned}\|Ny(t) - Ny(x)\| &= \phi(Ny(t) - Ny(x)) \\ &\le \|h(t) - h(x)\| + K_1 \int_0^T |k(t,s) - k(x,s)|\, ds \\ &\quad + K_1 \int_x^t |k(t,s)|\, ds,\end{aligned}$$

so Ny is norm continuous. Hence $N : K \to K$. Also $N : K \to K$ is weakly continuous. To see this, notice if $y_n \rightharpoonup y$ in K (here $\rightharpoonup$ denotes weak convergence and (y_n) is a net in K), i.e. y_n converges weakly uniformly to y on $[0,T]$, then since f satisfies (2.2) we have immediately that Ny_n converges weakly uniformly to Ny on $[0,T]$, so N is weakly continuous.

Next we show that $N(K)$ is weakly relatively compact. To see this, we apply both the Arzela Ascoli and the Eberlein Šmulian theorem. Choose a sequence $y_n \in$

$K, n \geq 1$. Our aim is to show first that for each $t \in [0,T]$ the set $\{Ny_n(t) : n \geq 1\}$ is weakly relatively compact. This follows immediately from Theorem 1.4 once we show that for each $t \in [0,T]$ the set $\{Ny_n(t) : n \geq 1\}$ is norm bounded. For fixed $t \in [0,T]$ we have $\|Ny_n(t)\| \leq M_0$, so the set $\{Ny_n(t) : n \geq 1\}$ is weakly relatively compact by Theorem 1.4. Next we show that $N(K)$ is weakly equicontinuous. Let $y \in K$ be arbitrary, and let $t, x \in [0,T]$ with $t > x$. Without loss of generality assume $Ny(t) - Ny(x) \neq 0$. Then Theorem 1.6 implies that there exists $\phi \in B^*$ with $\|\phi\| = 1$ and $\phi(Ny(t) - Ny(x)) = \|Ny(t) - Ny(x)\|$. Also (2.10) is true so

$$\begin{aligned}\|Ny(t) - Ny(x)\| = \phi\Bigg(h(t) - h(x) &+ \int_0^x [k(t,s) - k(x,s)] f(s, y(s))\, ds \\ &+ \int_x^t k(t,s) f(s,y)\, ds\Bigg) \\ \leq \|h(t) - h(x)\| &+ K_1 \int_0^T |k(t,s) - k(x,s)|\, ds \\ &+ K_1 \int_x^t |k(t,s)|\, ds,\end{aligned}$$

and this together with (2.3) and (2.4) implies that $N(K)$ is weakly equicontinuous. Theorem 1.2 guarantees that the weak closure of $N(K)$ is weakly sequentially compact, and this together with Theorem 1.3 implies that the weak closure of $N(K)$ is weakly compact, i.e. $N(K)$ is weakly relatively compact. Theorem 1.1 now guarantees that (2.1) has a solution $y \in K$. □

Remarks. (i) Another existence result for (2.1) will be established in §3.

(ii) Of course (2.5) could be replaced by other growth conditions and existence will again be guaranteed (provided (2.6) is appropriately adjusted).

3. Hammerstein integral equations in reflexive Banach spaces

Let B be a reflexive Banach space, and consider the Hammerstein integral equation

$$y(t) = h(t) + \int_0^1 k(t,s) f(s, y(s))\, ds, \qquad t \in [0,1], \tag{3.1}$$

with

$$f : [0,1] \times B \to B \text{ is weakly-weakly continuous}, \tag{3.2}$$

$$h : [0,1] \to B \text{ is weakly continuous}, \tag{3.3}$$

and

$$\begin{cases} k(t,s) \in L^1([0,1], \mathbf{R}) \text{ for each } t \in [0,1] \text{ and the map} \\ t \to k(t,s) \text{ is continuous from } [0,1] \text{ to } L^1([0,1], \mathbf{R}) \end{cases} \tag{3.4}$$

holding.

Theorem 3.1. *Suppose* (3.2)*,* (3.3)*, and* (3.4) *hold. In addition assume*

$$\begin{cases} \textit{there exists a nondecreasing continuous function } \Omega : [0,\infty) \to (0,\infty) \\ \textit{with } \|f(s,u)\| \leq \Omega(\|u\|) \textit{ for } t \in [0,1] \end{cases} \tag{3.5}$$

and

$$A \equiv \left(\sup_{t\in[0,1]} \int_0^1 |k(t,s)|\, ds\right) \limsup_{x\to\infty} \frac{\Omega(x)}{x} < 1 \tag{3.6}$$

are satisfied. Then (3.1) *has a solution* $y \in C([0,1], B_w)$.

Proof. Consider the set S of real numbers $x \geq 0$ which satisfy the inequality

$$x \leq \sup_{[0,1]} \|h(t)\| + \Omega(x) \left(\sup_{t\in[0,1]} \int_0^1 |k(t,s)|\, ds\right).$$

Then S is bounded above, i.e. there exists a constant M_1 with

$$x \leq M_1 \quad \text{for all } x \in S. \tag{3.7}$$

To see this, suppose (3.7) is not true. Then there exists a sequence $0 \neq x_n \in S$ with $x_n \to \infty$ as $n \to \infty$ and

$$1 \leq \frac{\sup_{[0,1]} \|h(t)\|}{x_n} + \frac{\Omega(x_n)}{x_n} \left(\sup_{t\in[0,1]} \int_0^1 |k(t,s)|\, ds\right).$$

Since $\limsup(s_n + t_n) \leq \limsup(s_n) + \limsup(t_n)$ for any sequences $s_n \geq 0$, $t_n \geq 0$, we have $1 \leq A$. This contradicts (3.6). Choose $M_0 > M_1$. Then

$$\sup_{[0,1]} \|h(t)\| + \Omega(M_0) \left(\sup_{t\in[0,1]} \int_0^1 |k(t,s)|\, ds\right) < M_0, \tag{3.8}$$

for otherwise $M_0 \in S$, and this would contradict (3.7). Let

$$K = \left\{ y \in C([0,T], B_w) : y \text{ is norm continuous and } \|y\|_0 = \sup_{[0,1]} \|y(t)\| \leq M_0 \right\}.$$

Define an operator N by

$$Ny(t) = h(t) + \int_0^1 k(t,s) f(s, y(s))\, ds.$$

We claim $N : K \to K$. To see this, let $y \in K$, and consider $Ny(t)$ for $t \in [0,1]$. Without loss of generality assume $Ny(t) \neq 0$ for $t \in [0,1]$. Then there exists $\phi_t \in B^*$ with $\|\phi_t\| = 1$ and $\phi_t(Ny(t)) = \|Ny(t)\|$. Now notice (3.5) and (3.8) imply for each $t \in [0,1]$ that

$$\begin{aligned}
\|Ny(t)\| &\leq |\phi_t(h(t))| + \int_0^1 |k(t,s)\phi_t(f(s,y(s)))|\, ds \\
&\leq \|h(t)\| + \int_0^1 |k(t,s)| \Omega(\|(y(s))\|)\, ds \\
&\leq \|h\|_0 + \Omega(\|y\|_0) \left(\sup_{[0,1]} \int_0^1 |k(t,s)|\, ds\right) \\
&\leq \|h\|_0 + \Omega(M_0) \left(\sup_{[0,1]} \int_0^1 |k(t,s)|\, ds\right) < M_0.
\end{aligned}$$

Thus $N : K \to K$. Also as in Theorem 2.2 N is weakly continuous and $N(K)$ is relatively weakly compact. Theorem 1.1 now guarantees that (3.1) has a solution $y \in K$. □

Remark. The ideas in Theorem 3.1 immediately yield an extra existence result for the Volterra integral equation (2.1), namely: suppose (2.2), (2.3), (2.4) with

$$\begin{cases} \text{there exists a nondecreasing continuous function } \Omega : [0,\infty) \to (0,\infty) \\ \text{with } \| f(s,u)\| \le \Omega(\|u\|) \text{ for } t \in [0,T] \end{cases} \tag{3.9}$$

and

$$\left(\sup_{t\in[0,T]} \int_0^t |k(t,s)|\, ds\right) \limsup_{x\to\infty} \frac{\Omega(x)}{x} < 1 \tag{3.10}$$

satisfied. Then (2.1) has a solution $y \in C([0,T], B_w)$.

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