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INTEGRAL FORMULAS FOR SUBMANIFOLDS AND THEIR APPLICATIONS

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Various integral formulas for hypersurfaces have been established and applied to the study of closed hypersurfaces with constant mean curvature. For the literature, see [12]. Integral formulas for submanifolds of arbitrary codimensions have been obtained by Chen [4], [5], [13], Katsurada [7], [8], [9], Kôjyô [8], Nagai [9], Okumura [15], Tani [16] and Yano [5], [10], [11], [13], [15], [16]. In the present paper, we first obtain the most general integral formulas for closed submanifolds in an m-dimensional euclidean space, and then apply those formulas to obtain some characterizations of spherical submanifolds.

1. Preliminaries[1]

Let M^n be an n-dimensional manifold with an immersion $x\colon M^n \to E^m$ of M^n into a euclidean space E^m of dimension m. Let $F(M^n)$ and $F(E^m)$ be respectively the bundles of orthonormal frames of M^n and E^m. Let B be the set of elements $b = (p, e_1, \cdots, e_n, e_{n+1}, \cdots, e_m)$ such that $(p, e_1, \cdots, e_n) \in F(M^n)$ and $(x(p), e_1, \cdots, e_m) \in F(E^m)$ whose orientation is coherent with that of E^m, by identifying e_i with $dx(e_i), i = 1, \cdots, n$. Define $\tilde{x}\colon B \to F(E^m)$ by $\tilde{x}(b) = (x(p), e_1, \cdots, e_m)$.

Throughout this paper, we shall agree, unless otherwise stated, on the indices of the following ranges:

$$1 \leq i, j, \cdots \leq n;\ 1 \leq A, B, \cdots \leq m;\ n+1 \leq r, s, \cdots \leq m\ .$$

The structure equations of E^m are given by

$$\begin{aligned} &dx = \Sigma\omega'_A e_A\ , \qquad && de_A = \Sigma\omega'_{AB} e_B\ , \\ (1)\qquad &d\omega'_A = \Sigma\omega'_B \wedge \omega'_{BA}\ , && d\omega'_{AB} = \Sigma\omega'_{AC} \wedge \omega'_{CB}\ , \\ &\omega'_{AB} + \omega'_{BA} = 0\ , \end{aligned}$$

where ω'_A, ω'_{AB} are differential 1-forms on $F(E^m)$. Let ω_A, ω_{AB} be the induced 1-forms on B from ω'_A, ω'_{AB} by the mapping $\tilde{x}$. Then we have

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[1] Manifolds, mappings, functions, ... are assumed to be differentiable and of class C^∞, and we shall restrict ourselves only to connected submanifolds of dimension $n>1$.

$$(2)\qquad \omega_r = 0\ , \qquad \omega_{ir} = \Sigma A_{rij}\omega_j\ , \qquad A_{tij} = A_{tji}\ .$$

From (2) we can define the *mean curvature vector* H by

$$(3)\qquad H = (1/n)\Sigma A_{rii}e_r\ .$$

H is a well-defined normal vector, and its length is called the *mean curvature.*

For each unit normal vector $e = \Sigma \cos\theta_r e_r$, the second fundamental form $A_{(e)} = (A_{ij}(e))$ at e is a linear transformation and is given by

$$A_{(e)}(e_i) = \Sigma \cos\theta_r A_{rij}e_j\ , \qquad j = 1, \cdots, n\ .$$

If N is a nonzero normal vector, then the *second fundamental form* at N is defined as the second fundamental form at the unit direction of N. The *principal curvatures* at e are defined as the eigenvalues of the second fundamental form $A_{(e)}$ at e. Furthermore, the pth *mean curvature* $M_p(e)$ at e is given by the pth elementary symmetric function, i.e.,

$$(4)\qquad \binom{n}{p} M_p(e) = \Sigma k_1 \cdots k_p\ ,$$

where $k_1, \cdots, k_n$ are the principal curvatures at e, and $\binom{n}{p} = n!/[p!(n-p)!]$. If all the principal curvatures at e are the same, i.e., $k_1 = \cdots = k_n$, everywhere, then M^n is said to be *umbilical with respect to e*. If the mean curvature vector $H \neq 0$ everywhere and M^n is umbilical with respect to H, then M^n is called a *pseudo-umbilical submanifold.*

2. Some integral formulas

Let $[\underbrace{\cdot\ \cdot\ \cdot}_{m-1 \text{ terms}}]$ denote the combined operation of exterior product and vector product, and (,) the combined operation of exterior product and scalar product in E^m. For simple cases we have

$$\begin{aligned} &[e_1, \cdots, \hat{e}_A, \cdots, e_m] = (-1)^{m+A}e_A\ ,\\ &(v, [v_1, \cdots, v_{m-1}]) = (-1)^{m-1}\det(v, v_1, \cdots, v_{m-1})\ ,\\ (5)\qquad &[\underbrace{dX, \cdots, dX}_{n-i}, \underbrace{de_{n+1}, \cdots, de_{n+1}}_{i}, e_{n+2}, \cdots, e_m]\\ &\qquad = n(-1)^{m+n+1}e_{n+1}M_i(e_{n+1})dV\ ,\end{aligned}$$

where ^ denotes the omitted term, $dV = \omega_1 \wedge \cdots \wedge \omega_n$ the volume element of M^n, and X the *position vector field* of M^n in E^m with respect to the origin of E^m.

The position vector field X can be decomposed into two parts:

$$X = X_t + X_n \ , \tag{6}$$

where X_t is tangent to $x(M^n)$, and X_n normal to $x(M^n)$. Let e be a unit normal vector field over M^n, and $\bar{e}$ a unit normal vector field perpendicular to e and in the direction of $X_n - (X_n \cdot e)e$, i.e.,

$$X_n = (X_n \cdot e)e + (X_n \cdot \bar{e})\bar{e} \ . \tag{7}$$

Throughout this paper, we always choose $e_1, \cdots, e_n$ in the principal directions with respect to e. Thus, if we denote the principal curvatures at e by $k_1, \cdots, k_n$, then we have

$$\omega_{i,n+1} = k_i \omega_i \ , \qquad i = 1, \cdots, n \ .$$

Define n functions $F_i(e)$, $i = 1, \cdots, n$, by

$$F_i(e) = \frac{(n-i)!}{n!}(X \cdot \bar{e})\Sigma k_{j_1} \cdots k_{j_{i-1}} A_{j_i j_i}(\bar{e}) \ , \tag{8}$$

where the summation is taken over all distinct $j_1, \cdots, j_i = 1, \cdots, n$.

Suppose that the unit normal vector field e is *parallel in the normal bundle*, i.e., by the definition, de is tangent to $x(M^n)$ everywhere. Then, by using (5), (6) and choosing $e_{n+1} = e$ and $e_{n+2} = \bar{e}$ everywhere, we have

$$\begin{aligned}
d(X, [&\underbrace{dX, \cdots, dX}_{n-i}, \underbrace{de, \cdots, de}_{i-1}, e_{n+1}, \cdots, e_m]) \\
&= (dX, [\underbrace{dX, \cdots, dX}_{n-i}, \underbrace{de, \cdots, de}_{i-1}, e_{n+1}, \cdots, e_m]) \\
&\quad + (-1)^{n-1}(X, [\underbrace{dX, \cdots, dX}_{n-1}, \underbrace{de, \cdots, de}_{i}, e_{n+2}, \cdots, e_m]) \\
&\quad + (-1)^{n-1} \sum_{s=n+2}^{m} (X, [\underbrace{dX, \cdots, dX}_{n-i}, \underbrace{de, \cdots, de}_{i-1}, \\
&\qquad\qquad e_{n+1}, \cdots, e_{s-1}, de_s, e_{s+1}, \cdots, e_m]) \\
&= (-1)^{m+i} n!(M_{i-1}(e) + (X \cdot e)M_i(e))dV \\
&\quad + (-1)^{n-1}(X, [\underbrace{dX, \cdots, dX}_{n-i}, \underbrace{de, \cdots, de}_{i-1}, e_{n+1}, d\bar{e}, e_{n+3}, \cdots, e_m]) \ .
\end{aligned} \tag{9}$$

and

$$
\begin{aligned}
&[\underbrace{dX, \cdots, dX}_{n-i}, \underbrace{de, \cdots, de}_{i-1}, e_{n+1}, d\bar{e}, e_{n+3}, \cdots, e_m] \\
&= \sum \omega_{j_1} \wedge \cdots \wedge \omega_{j_{n-i}} \wedge \omega_{n+1, j_{n-i+1}} \wedge \cdots \wedge \omega_{n+1, j_{n-1}} \wedge \omega_{n+2, j_n} \\
&\qquad \cdot [e_{j_1}, \cdots, e_{j_{n-i}}, \cdots, e_{j_{n-1}}, e_{n+1}, e_{j_n}, e_{n+3}, \cdots, e_m] \\
&= (-1)^{i-1} \sum k_{j_{n-i+1}} \cdots k_{j_{n-1}} \omega_{j_1} \wedge \cdots \wedge \omega_{j_{n-i}} \wedge \cdots \wedge \omega_{j_{n-1}} \\
&\qquad \wedge \omega_{n+2, j_n} [e_{j_1}, \cdots, e_{j_{n-1}}, e_{n+1}, e_{j_n}, e_{n+3}, \cdots, e_m] \\
&= (-1)^{i+1} \sum k_{j_1} \cdots k_{j_{i-1}} A_{j_i j_i}(\bar{e}) \omega_{j_1} \wedge \cdots \wedge \omega_{j_n} \\
&\qquad \cdot [e_{j_1}, \cdots, e_{j_n}, e_{n+1}, e_{n+3}, \cdots, e_m] \\
&= (n-i)!(-1)^{m+n+i+1} (\sum k_{j_1} \cdots k_{j_{i-1}} A_{j_i j_i}(\bar{e})) \bar{e} dV .
\end{aligned} \tag{10}
$$

By (8), (9) and (10) we thus get

$$
\begin{aligned}
&d(X, [\underbrace{dX, \cdots, dX}_{n-i}, \underbrace{de, \cdots, de}_{i-1}, e_{n+1}, \cdots, e_m]) \\
&\quad = (-1)^{m+i} n!(M_{i-1}(e) + (X \cdot e) M_i(e) + F_i(e)) dV , \qquad i = 1, \cdots, n .
\end{aligned}
$$

Hence we know that $F_i(e)$ are well-defined functions defined on the whole manifold M^n. By integrating both sides of the above equation and applying Stokes' theorem, we have

Proposition 2.1. *Let $x \colon M^n \to E^m$ be an immersion of an oriented closed manifold M^n into E^m. If e is a unit normal vector field over M^n and is parallel in the normal bundle, then*

$$
\int_{M^n} (M_{i-1}(e) + (X \cdot e) M_i(e)) dV = - \int_{M^n} F_i(e) dV , \qquad i = 1, \cdots, n , \tag{11}
$$

where $M_i(e)$ dentoe the ith mean curvature at e.

In particular, we have

Theorem 2.2. *Let $x \colon M^n \to E^m$ be an immersion of an oriented closed manifold M^n into E^m. If e is a unit normal vector field parallel in the normal bundle, and $F_i(e) = 0$ for some i, $1 \leq i \leq n$, then*

$$
\int_{M^n} M_{i-1}(e) dV + \int_{M^n} (X \cdot e) M_i(e) dV = 0 . \tag{12}
$$

Remark 2.1. If the codimension $m - n = 1$, then the assumption $F_i(e) = 0$ in Theorem 2.2 are automatically satisfied. In this case, the integral formulas (12) are called the Minkowski formulas [6].

Remark 2.2. If the codimension is greater than 1, then in order to get some generalized Minkowski's formulas, various authors have set various assumptions. For examples:

1. In [7], [8], [9], Katsurada, Kôjyô and Nagai assumed that the position vector field X is parallel to the mean curvature vector field H everywhere, and H is parallel in the normal bundle. In this case, we can choose e in the direction of H so that $X \cdot \bar{e} = 0$ by (7). Thus from (8) it follows that $F_1(e) = \cdots = F_n(e) = 0$ automatically.

2. In [10], [11], Yano assumed that the mean curvature vector field H is parallel in the normal bundle and the second fundamental form at $X_n - (X_n \cdot H)H/(H \cdot H)$ vanishes. In this case, if we choose e in the direction of H, then $A_{ij}(\bar{e}) = 0$ for all $i, j = 1, \cdots, n$. Thus by (8) we know that $F_1(e) = \cdots = F_n(e) = 0$ automatically.

Remark 2.3. Let e be a unit normal vector field. If the second mean curvature $M_2(e)$ is equal to an αth scalar curvature [3], then e is called a Frenet direction [3], [5]. From the definition of $F_i(e)$ it follows that $F_2(e) = 0$ if e is a Frenet direction. For the integral formulas in the Frent directions see Chen-Yano [5]. If e is parallel to X_n, then $F_1(e) = \cdots = F_n(e) = 0$ automatically.

Remark 2.4. In Proposition 2.1, the condition of the parallelism of e in the normal bundle can be replaced by the condition that M^n is immersed in a hypersphere of E^m centered at the origin of E^m.

Theorem 2.3. *Let $x: M^n \to E^m$ be an immersion of an oriented closed manifold M^n into E^m. If e is a unit normal vector field and is parallel in the normal bundle, and $(X \cdot \bar{e})A_{(\bar{e})} = 0$, then we have $F_1(e) = \cdots = F_n(e) = 0$ and*

$$\int_{M^n} M_{i-1}(e)dV + \int_{M^n} (X \cdot e)M_i(e)dV = 0 , \qquad i = 1, \cdots, n , \tag{13}$$

where $A_{(\bar{e})}$ denotes the second fundamental form at $\bar{e}$.

Since $(X \cdot \bar{e})A_{(\bar{e})} = 0$, we have either $X \cdot \bar{e} = 0$ or $A_{ij}(\bar{e}) = 0$, for all $i, j = 1, \cdots, n$. Thus $F_1(e) = \cdots = F_n(e) = 0$ by (8), and hence we get (13) by Theorem 2.2.

Remark 2.5. If e is in the direction of the mean curvature vector field H, then Theorem 2.3 was proved by Yano [10], [11] for $i = 2$.

Let f be a function on M^n. By grad f or ∇f we mean $\nabla f = \Sigma f_i e_i$, where f_i are given by $df = \Sigma f_i \omega_i$.

Theorem 2.4. *Let $x: M^n \to E^m$ be an immersion of an oriented closed manifold M^n into E^m. If e is a unit normal vector field over M^n, then we have*

$$\int_{M^n} (X \cdot \nabla M_i(e))dV + n \int_{M^n} (M_i(e) + (X \cdot H)M_i(e))dV = 0 , \tag{14}$$

for $i = 1, \cdots, n$, where H denotes the mean curvature vector field.

Proof. Let

$$\sigma = \Sigma(-)^{j-1}(X \cdot e_j)\omega_1 \wedge \cdots \wedge \hat{\omega}_j \wedge \cdots \wedge \omega_n . \tag{15}$$

Then

$$d(M_i(e)\sigma) = (dM_i(e)) \wedge \sigma + nM_i(e)(1 + (X\cdot H))dV \;, \tag{16}$$

which together with $df \wedge \sigma = (X\cdot \nabla f)dV$ implies

$$d(M_i(e)\sigma) = (X\cdot \nabla M_i(e))dV + nM_i(e)(1 + (X\cdot H))dV \;. \tag{17}$$

Hence by integrating both sides of (17) and applying Stokes' theorem, we get (14).

Remark 2.6. For the integral formulas for hypersurfaces consisting of $\nabla M_i(e)$, see Amur [1] and Chen [2].

3. Some characterizations of spherical submanifolds

The purpose of this section is to use the integral formulas in § 2 to get some characterizations of spherical submanifolds. The following lemmas are well-known.

Lemma 3.1. *Let $M_i(e), i = 1, \cdots, n$, be given by* (4), *and let $M_0(e) = 1$. Then*

$$M_i(e)^2 - M_{i-1}(e)M_{i+1}(e) \geq 0 \;, \tag{18}$$

for $i = 1, \cdots, n-1$. Moreover, if $M_1(e), \cdots, M_j(e)$ are positive, then

$$M_1(e) \geq (M_2(e))^{1/2} \geq \cdots \geq (M_j(e))^{1/j} \;, \tag{19}$$

where the equality at any stage of (18) *and* (19) *implies that M^n is umbilical with respect to e, i.e., $k_1 = \cdots = k_n$.*

Lemma 3.2. *If $M_{s-i}(e), M_{s-i-1}(e), \cdots, M_s(e)$ are positive, then*

$$M_{s-1}(e)/M_s(e) \geq M_{s-2}(e)/M_{s-1}(e) \geq \cdots \geq M_{s-i-1}(e)/M_{s-i}(e) \;, \tag{20}$$

where the equality at any stage implies that M^n is umbilical with respect to e, i.e., $k_1 = \cdots = k_n$.

If $x\colon M^n \to E^m$ is an immersion of M^n into E^m such that M^n is immersed into a hypersphere of E^m centered at c, then M^n is called a *spherical submanifold* of E^m, or simply M^n is *spherical*. The *radius vector field* R is defined by $X - c$.

Theorem 3.3. *Let $x\colon M^n \to E^m$ be an immersion of M^n into E^m. Then there exists a normal vector field $e \neq 0$ over M^n such that* (1) *e is parallel in the normal bundle, and* (2) *M^n is umbilical with respect to e, when and only when M^n is spherical and e is parallel to the radius vector field R.*

Proof. Suppose that there exists a normal vector field $e \neq 0$ over M^n such that e is parallel in the normal bundle and M^n is umbilical with respect to e.

Then we can easily verify that e has constant length $|e|$, and therefore we can choose the first unit normal $e_{n+1} = e/|e|$ so that

(21) $$\omega_{i,n+1} = h\omega_i \ , \qquad i = 1, \cdots, n \ ,$$

(22) $$\omega_{n+1,r} = 0 \ , \qquad r = n + 1, \cdots, m \ .$$

By taking the exterior derivative of (21) and applying (22), we get

$$dh \wedge \omega_i = 0 \ , \qquad i = 1, \cdots, n \ ,$$

which imply that h is constant. Now consider the mapping $Q\colon M^n \to E^m$ defined by $Q(p) = x(p) + e_{n+1}/h$. Then use of (21) and (22) yields $dQ(p) = 0$, which shows that

$$x(p) + e_{n+1}/h = c = \text{constant} \ .$$

Hence M^n is immersed into a hypersphere of E^m centered at c, and e is parallel to the radius vector field $X - c$.

Corollary 3.1. *Let $x\colon M^n \to E^{n+2}$ be an immersion of M^n into E^{n+2}. Then there exists a unit normal vector field e such that M^n is umbilical with respect to e and the first mean curvature $M_1(e)$ at e is constant when and only when M^n is spherical and e is parallel to the radius vector field R.*

Proof. Suppose that e is a unit normal vector field such that M^n is umbilical with respect to e, and the first mean curvature $M_1(e)$ at e is constant. Then by choosing $e_{n+1} = e$ we have

$$\omega_{i,n+1} = M_1(e)\omega_i \ , \qquad i = 1, \cdots, n \ .$$

By taking the exterior derivative of the above equations we get

$$dM_1(e) \wedge \omega_i = \omega_{i,n+2} \wedge \omega_{n+2,n+1} = 0 \ .$$

Thus

$$\omega_{i,n+2} = 0 \ , \qquad i = 1, \cdots, n \ ,$$

on the set $U = \{p \in M^n\colon \omega_{n+1,n+2} \neq 0 \text{ at } p\}$. On the open set U, by taking exterior derivative of the above equations, we have

$$\omega_i \wedge \omega_{n+1,n+2} = 0 \ , \qquad i = 1, \cdots, n \ ,$$

which imply that $\omega_{n+1,n+2} = 0$ on U. Therefore U is an empty set. This shows that e is parallel in the normal bundle. Hence, by using Theorem 3.3, we know that M^n is spherical and e is parallel to the radius vector field R.

Throughout the remainder of this paper, we always assume that $x\colon M^n \to E^m$ is an immersion of an oriented closed manifold M^n into E^m.

Theorem 3.4. *If there exist a unit normal vector field e parallel in the normal bundle and an integer i, $2 \leq i \leq n$, such that*

(i) $M_i(e) > 0$,

(ii) $X \cdot e \leq -M_{i-1}(e)/M_i(e)$, (or $X \cdot e \geq -M_{i-1}(e)/M_i(e)$),

(iii) $F_i(e) = F_{i-1}(e) = 0$,

then M^n is spherical and the radius vector field R is parallel to e.

Proof. Suppose that there exist a unit normal vector field e parallel in the normal bundle and an integer i, $1 < i \leq n$, such that (i), (ii) and (iii) hold. Then by (ii) and Theorem 2.2 we get

$$X \cdot e = -M_{i-1}(e)/M_i(e) ,$$

$$\int_{M^n} (M_{i-2}(e) + (X \cdot e)M_{i-1}(e))dV = 0 ,$$

and threfore

$$\int_{M^n} (1/M_i(e))(M_{i-1}(e)^2 - M_{i-2}(e)M_i(e))dV = 0 .$$

Thus by Lemma 3.1 we have $M_{i-1}(e)^2 - M_{i-2}(e)M_i(e) = 0$, and M^n is umbilical with respect to e. Hence Theorem 3.4 follows immediately from Theorem 3.3.

Theorem 3.5. *If there exist a unit normal vector field e parallel in the normal bundle and an integer i, $1 < i < n$, such that*

(i) $M_{i+1}(e) > 0$,

(ii) $X \cdot e \geq -M_{i-1}(e)/M_i(e)$,

(iii) $F_{i+1}(e) = 0$,

then M^n is spherial and e is parallel to the radius vector field.

Proof. By (ii) and Lemma 3.2 we have

$$X \cdot e \geq -M_{i-1}(e)/M_i(e) \geq -M_i(e)/M_{i+1}(e) ,$$

which, together with

$$\int_{M^n} (M_i(e) + (X \cdot e)M_{i+1}(e))dV = 0$$

by Theorem 2.2, implies

$$X \cdot e \geq -M_{i-1}(e)/M_i(e) \geq -M_i(e)/M_{i+1}(e) = X \cdot e .$$

Therefore $M_i(e)^2 = M_{i-1}(e)M_{i+1}(e)$, and hence by Theorem 3.3, M^n is spherical and e is parallel to the radius vector field.

Theorem 3.6. *If there exist a unit normal vector field e parallel in the normal bundle and two integers s and i,* $1 \leq i < s \leq n$, *such that*

(i) $M_s(e), \cdots, M_i(e) > 0$,

(ii) $M_s(e) = \sum_{j=i}^{s-1} c_j M_j(e)$, *for some constants* c_j, $i \leq j \leq s-1$,

(iii) $F_j(e) = 0$, $j = i, \cdots, s-1$,

then M^n *is spherical and e is parallel to the radius vector field.*

Proof. By Lemma 3.2 we have

$$\frac{M_j(e)}{M_s(e)} - \frac{M_{j-1}(e)}{M_{s-1}(e)} = \left(\frac{M_j(e)}{M_{s-1}(e)}\right)\left(\frac{M_{s-1}(e)}{M_s(e)} - \frac{M_{j-1}(e)}{M_j(e)}\right) \geq 0 ,$$

for $i \leq j \leq s-1$, so that

$$1 = \sum c_j M_j(e)/M_s(e) \geq \sum c_j M_{j-1}(e)/M_{s-1}(e) ,$$

or

$$M_{s-1}(e) - \sum c_j M_{j-1}(e) \geq 0 ,$$

where equality holds only if M^n is umbilical with respect to e. Thus by using (ii) and Theorem 2.2 we have

$$\begin{aligned} &\int_{M^n} \left(M_{s-1}(e) - \sum_{j=i}^{s-1} c_j M_{j-1}(e)\right) dV \\ &= - \int_{M^n} (X \cdot e)\left(M_s(e) - \sum_{j=i}^{s-1} c_j M_j(e)\right) dV = 0 , \end{aligned} \tag{23}$$

and therefore $M_{s-1} = \sum c_j M_{j-1}(e)$, which implies that M^n is umbilical with respect to e. Hence by Theorem 3.3, M^n is spherical and e is parallel to the radius vector field.

Similarly, by using (23), we have

Theorem 3.7. *If there exist a unit vector field e parallel in the normal bundle and two integers s and i,* $0 \leq i < s < n$, *such that*

(i) $M_{s+1}(e), \cdots, M_{i+1}(e) > 0$,

(ii) $M_s(e) = \sum_{j=i}^{s-1} c_j M_j(e)$ *for some constants* c_j, $i \leq j \leq s-1$,

(iii) $X \cdot e > 0$ *or* $X \cdot e < 0$,

(iv) $F_{s-1}(e) = \cdots = F_i(e) = 0$,

then M^n *is spherical and e is parallel to the radius vector field.*

Theorem 3.8. *If there exist a unit normal vector field e parallel in the normal bundle and an integer i,* $1 \leq i \leq n$, *such that*

(i) $M_i(e) > 0$,
(ii) $M_i(e) = cM_{i-1}(e)$, *for a constant* c,
(iii) $F_{i-1}(e) = F_i(e) = 0$,
then M^n *is spherical and* e *is parallel to the radius vector field.*

Proof. Since $M_i(e) > 0, c$ cannot be zero and $M_{i-1}(e)$ must be of fixed sign. By (ii) and Lemma 3.1, we have

$$M_{i-1}(e)(M_{i-1}(e) - cM_{i-2}(e)) = M_{i-1}(e)^2 - M_i(e)M_{i-2}(e) \geq 0 \,,$$

from which, together with (ii) again and Theorem 2.2, it follows

$$\int_{M^n} (M_{i-1}(e) - cM_{i-2}(e))dV = \int_{M^n} (cM_{i-1}(e) - M_i(e))(X \cdot e)dV = 0 \,,$$

so that $M_{s-1}(e) = cM_{s-2}(e)$. Thus by Theorem 3.6, M^n is umbilical with respect to e, and hence M^n is spherical and e is parallel to the radius vector field.

Theorem 3.9. *If there exists a unit normal vector field* e *parallel in the normal bundle such that*
(i) $M_n > 0$,
(ii) *the sum of the principal radii of curvature is constant at* e, *i.e.*, $\sum_{i=1}^{n} (1/k_i) =$ *constant*,
(iii) $F_n(e) = F_{n-1}(e) = 0$,
then M^n *is spherical and* e *is parallel to the radius vector field.*

Proof. Since

$$\sum_{i=1}^{n} (1/k_i) = nM_{n-1}(e)/M_n(e) = \text{constant} \,, \tag{24}$$

by Theorem 3.8 we know that M^n is spherical and e is parallel to the radius vector field.

Remark 3.1. If we replace e by a unit vector field parallel to the mean curvature vector field H, then the conclusion in Theorems 3.3 to 3.9 and Corollary 3.1 should be read as "M^n is a minimal submanifold of a hypersphere of E^m", since the only closed submanifolds of a euclidean space such that the position vector field parallel to the mean curvature vector field H everywhere are the minimal submanifolds of a hypersphere.

Remark 3.2. If M^n is spherical and e is a unit vector field parallel to the radius vector field R, then we have (i) e is parallel in the normal bundle, (ii) $M_i(e) =$ constant, for all $i = 1, \cdots, n$, (iii) $R \cdot e = \pm M_i(e)/M_{i-1}(e) =$ constant, for all $i = 1, \cdots, n$, and (iv) $F_i(e) = \cdots = F_n(e) = 0$.

Remark 3.3. If the ambient space is replaced by an m-dimensional Riemannian space of constant sectional curvature and the position vector field X replaced by a concurrent vector field over M^n, then we can get the same results for all theorems in §§ 2 and 3 except the statement "M^n is spherical and e is parallel to the radius vector field" should be replaced by the existence of a concurrent normal vector field (see Yano-Chen [14]).

Remark 3.4. Let N be a vector field in E^m over M^n. If there exist a function f and a 1-form λ such that $d(x + fN) = \lambda N$, then the vector field N is called a *torse-forming*, because if we develop the vector field N along a curve in the manifold M^n, we obtain a field of vectors along the curve whose prolongations are tangent to another curve. From Theorem 3.3 it follows that a normal vector field N in E^m over M^n is a concurrent vector field if it is a torse-forming.

References

[1] K. Amur, *Vector forms and integral formulas for hypersurfaces in Euclidean space,* J. Differential Geometry **3** (1969) 111–123.

[2] B. Y. Chen, *Some integral formulas for the hypersurfaces in euclidean space,* Nagoya Math. J. **43** (1971).

[3] ——, *On the scalar curvature of immersed manifolds,* Math. J. Okayama Univ. **15** (1971).

[4] ——, *Pseudo-umbilical surfaces in euclidean spaces,* Kōdai Math. Sem. Rep., to appear.

[5] B. Y. Chen & K. Yano, *Pseudo umbilical submanifolds in a Riemannian manifold of constant curvature,* to appear.

[6] C. C. Hsiung, *Some integral formulas for closed hypersurfaces,* Math. Scand. **2** (1954) 286–294.

[7] Y. Katsurada, *Closed submanifolds with constant v-th mean curvature related with a vector field in a Riemannian manifold,* J. Fac. Sci. Hokkaido Univ. (I) **20** (1969) 171–181.

[8] Y. Katsurada & H. Kôjyô, *Some integral formulas for closed submanifolds in a Riemann space,* J. Fac. Sci. Hokkaido Univ. (I) **20** (1968) 90–100.

[9] Y. Katsurada & T. Nagai, *On some properties of a submanifold with constant mean curvature in a Riemann space,* J. Fac. Sci. Hokkaido Univ. (I) **20** (1968) 79–89.

[10] K. Yano, *Integral formulas for submanifolds and their applications,* Canad. J. Math. **22** (1970) 376–388.

[11] ——, *Submanifolds with parallel mean curvature vector of a euclidean space or a sphere,* Kōdai Math. Sem. Rep., to appear.

[12] ——, *Integral formulas in Riemannian geometry,* Marcel Dekker, New York, 1970.

[13] K. Yano & B. Y. Chen, *Minimal submanifolds of a higher dimensional sphere,* Tensor, to appear.

[14] ——, *On the concurrent vector fields of immersed manifolds,* Kōdai Math. Sem. Rep., to appear.

[15] K. Yano & M. Okumura, *Integral formulas for submanifolds of codimension* 2 *and their applications,* Kōdai Math. Sem. Rep. **21** (1969) 463–471.

[16] K. Yano & M. Tani, *Submanifolds of codimension* 2 *in a euclidean space,* to appear.

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