INTEGRAL GEOMETRY OF TENSOR FIELDS

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Preface

I am not sure that this book will gain a wide readership. A pure mathematician would most likely consider it rather old-fashioned that the book is abundant in cumbersome calculations performed in coordinate form with use made of multilevel indices. Or such a person would probably find the discussion of details of optical polarization measurements to be dull. On the other hand, an applied mathematician will likely be confused by vector bundles, Riemannian connections and Sobolev's spaces. In any case both of them will not admire my Russian English. Nevertheless, I hope that a young mathematician (physicist, engineer or computer scientist), skipping through this book, would wish to get acquaintance with some relevant field of physics (mathematics). In such a case the aim of the book will be achieved. As for me, I like my subject just for its diversity and unpredictability. Dealing with integral geometry, one never knows what it will require next time; today it may be perusing a textbook on commutative algebra and tomorrow, technical description of an electronic microscope.

What is integral geometry? Since the famous paper by I. Radon in 1917, it has been agreed that integral geometry problems consist in determining some function or a more general quantity (cohomology class, tensor field, etc.), which is defined on a manifold, given its integrals over submanifolds of a prescribed class. In this book we only consider integral geometry problems for which the above-mentioned submanifolds are one-dimensional. Strictly speaking, the latter are always geodesics of a given Riemannian metric and, in particular, straight lines in Euclidean space.

Stimulated by internal demands of mathematics, in recent years integral geometry has gain a powerful impetus from computer tomography. Now integral geometry serves as the mathematical background for tomography which in turn provides most of the problems for the former.

The book deals with integral geometry of symmetric tensor fields. This section of integral geometry can be viewed as mathematical basis for tomography of anisotropic media whose interaction with sounding radiation depends essentially on the direction in which the latter propagates.

As is seen from the table of contents, the main mathematical objects tackled in the book are termed "ray transform" with various adjectives. I feel it obligatory to explain the origin of the term, since I am afraid some of readers will not recognize their old crony. But first I would like to tell a funny story that is popular among scientific translators.

A French poet tried to render Pushkin's verses. Later another poet, having not recognized Pushkin, find the verses to be worthy of translation into Russian. As result of such double translation, the famous string from Tatyana's letter "I am writing to you, what can surpass it?" becomes: "I am here, and wholly yours!"

Of course, the reader understands that it is only a colorless English copy of the Russian original, since I am neither A. S. Pushkin nor even one of the heroes of the story.

Something of the kind has happened with our crucial term. It is the term "X-ray transform" that is widely spread in western literature on mathematical tomography. In particular, this term is employed in the book [48] by S. Helgason. Rendering the last book in Russian, the translators encountered the following problem. By the Russian physical tradition, the term "X-rays" is translated as " " ("the Röntgen rays" in reverse translation word by word). So the translators had to choose between the two versions: " " (the Röntgen transform) or " " (the ray transform). The second was preferred and so the term " " (the ray transform) had gain wide acceptance in the Russian literature on tomography. But the friendship of this term with X-rays was lost as a result of the choice. I have found this circumstance very fortunate for the purposes of my book. Indeed, as the reader will see below, the ray transforms, which are considered in the book, relates to the optical and seismic rays rather than to X-rays. Therefore, I also prefer to use the term "the ray transform" without the prefix X- in the English version of the book.

In the course of the first four chapters one and the same operator I is investigated which is simply called the ray transform without any adjective. In the subsequent chapters, after introducing other kinds of the ray transform, we refer to the operator I as the longitudinal ray transform.

I will not retell the contents of the book here. Every chapter is provided with a little introductory section presenting the posed problems and the results of the chapter. Now I will only give a remark on the interdependency of chapters.

The first chapter is included with the purpose to motivate formulations of the problems considered in the remainder of the book. This chapter is oriented to a well-qualified reader. Therein we use some notions of tensor analysis and Riemannian geometry without providing definitions in detail. If the reader is not so skilled, then it is possible to start reading with Chapter 2 and return to Chapter 1 after gaining acquaintance with the main definitions of Chapter 3. It is also note-worthy that Chapters 4–8 can be read independently of Chapter 2 to which they are related only in a few episodes.

Some words now are in order about the applied problems treated in the book. Chapters 5–7 deal with some aspects of the theory of propagation of electromagnetic and elastic waves in slightly anisotropic media. The respective considerations first pursue of the goal motivating the mathematical objects to be further introduced; no concrete applied problem is considered here. Section 2.16 is of a quite different character where an application of the ray transform to some inverse problem of photoelasticity is considered. Photoelasticity is an interesting branch of experimental physics at the interface between optics and elasticity theory. In Section 2.16 we consider one of the methods of the branch, the method of integral photoelasticity, which seems to become a new prospective field of optical tomography in the nearest future. In this section our investigation is carried out up to some concrete algorithm which, in the author's opinion, is suitable for use in real optical measurements.

Perhaps, the applications of integral geometry, which are considered in the book, are not those most successful or important. I believe that tensor integral geometry will find its true applications in such fields as lightconductor technology, plasma physics, tomography of liquid crystals or, probably, in the problem of earthquake prediction. My modest physical knowledge does not allow me to treat such problems by myself. I would be glad if one of my readers will address them.

Finally, I wish to express my sincere gratitude to Professor S. S. Kutateladze, the editor of the book, who took pains to make my Russian English comprehensible if not readable. I hope that he succeeded.

February, 1994, Novosibirsk.

Contents

Chapter 1

Introduction

In the first section we formulate the problem of determining a Riemannian metric on a compact manifold with boundary from known distances in this metric between the boundary points. This geometrical problem is interesting from the theoretical and applied points of view. Here it is considered as an example of a question leading to an integral geometry problem for a tensor field. In fact, by linearization of the problem we arrive at the question of finding a symmetric tensor field of degree 2 from its integrals over all geodesics of a given Riemannian metric. The operator sending a tensor field into the family of its integrals over all geodesics is called the ray transform. The principal difference between scalar and tensor integral geometry is that in the last case the operators under consideration have, as a rule, nontrivial kernels. It is essential that in the process of linearization there arises a conjecture on the kernel of the ray transform.

Integral geometry is well known to be closely related to inverse problems for kinetic and transport equations. In Section 1.2 we introduce the kinetic equation on a Riemannian manifold and show that the integral geometry problem for a tensor field is equivalent to an inverse problem of determining the source, in the kinetic equation, which depends polynomially on a direction.

Section 1.3 contains a survey of some results that are related to the questions under consideration but are not mentioned in the main part of the book.

1.1 The problem of determining a metric by its hodograph and a linearization of the problem

A Riemannian metric g on a compact manifold M with boundary ∂M is called *simple* if every pair of points $p, q \in M$ can be joint by the unique geodesic γ_{pq} of this metric whose all points, with the possible exception of its endpoints, belong to $M \setminus \partial M$ and such that γ_{pq} depends smoothly on p and q. For $p, q \in \partial M$ we denote by $\Gamma_g(p, q)$ the length of γ_{pq} in the metric g. The function $\Gamma_g : \partial M \times \partial M \to \mathbf{R}$ is called the *hodograph* of the metric g (the term is taken from geophysics). The problem of determining a metric by its hodograph is formulated as follows: for a given function $\Gamma : \partial M \times \partial M \to \mathbf{R}$, one has to determine whether it is the hodograph of a simple metric and find all such metrics. The next question of stability in this problem seems to be important as well: are two metrics close (in some sense) to each other in the case when their hodographs are close?

We shall discuss only questions that are related to uniqueness of a solution to the posed problem. The following nonuniqueness of a solution is evident. Let φ be a diffeomorphism of M onto itself which is identical on ∂M. It transforms a simple metric g^0 to a simple metric $g^1 = \varphi^* g^0$ (the last equality means that for a point $x \in M$ and every pair of vectors ξ, η of the space $T_x M$ tangent to M at the point x the equality $\langle \xi, \eta \rangle^1_x = \langle (d_x\varphi)\xi, (d_x\varphi)\eta \rangle^0_{\varphi(x)}$ is valid where $d_x\varphi : T_x M \to T_{\varphi(x)} M$ is the differential of φ and $\langle , \rangle^\alpha_x$ is the scalar product on $T_x M$ in the metric g^α). These two metrics have different families of geodesics and the same hodograph. The question arises: is the nonuniqueness of the posed problem settled by the above mentioned construction? In other words: is it true that a simple metric is determined by its hodograph up to an isometry identical on the boundary? Let us formulate the precise statement.

Problem 1.1.1 (the problem of determining a metric by its hodograph) *Let g^0, g^1 be two simple metrics on a compact manifold M with boundary. Does the equality $\Gamma_{g^0} = \Gamma_{g^1}$ imply existence of a diffeomorphism $\varphi : M \to M$, such that $\varphi|_{\partial M} = \mathrm{Id}$ and $\varphi^* g^0 = g^1$?*

Until now a positive answer to this question is obtained for rather narrow classes of metrics (there is a survey of such results in Section 1.3; some new results in this direction are obtained in Chapters 4 and 8). On the other hand, the author is not aware of any pair of metrics for which the answer is negative.

Let us linearize Problem 1.1.1. For this we suppose g^τ to be a family, of simple metrics on M, smoothly depending on $\tau \in (-\varepsilon, \varepsilon)$. Let us fix $p, q \in \partial M$, $p \neq q$, and put $a = \Gamma_{g^0}(p, q)$. Let $\gamma^\tau : [0, a] \to M$ be a geodesic of the metric g^τ, for which $\gamma^\tau(0) = p$, $\gamma^\tau(a) = q$. Let $\gamma^\tau = (\gamma^1(t, \tau), \dots, \gamma^n(t, \tau))$ be the coordinate representation of γ^τ in a local coordinate system, $g^\tau = (g^\tau_{ij})$. Simplicity of γ^τ implies smoothness for the functions $\gamma^i(t, \tau)$. The equality

$$\frac{1}{a}[\Gamma_{g^\tau}(p, q)]^2 = \int\limits_0^a g^\tau_{ij}(\gamma^\tau(t))\dot{\gamma}^i(t, \tau)\dot{\gamma}^j(t, \tau)dt \tag{1.1.1}$$

is valid in which the dot denotes differentiation with respect to t. In (1.1.1) and through what following the next rule is used: on repeating sub- and super-indices in a monomial the summation from 1 to n is assumed. Differentiating (1.1.1) with respect to τ and putting $\tau = 0$, we get

$$\frac{1}{a}\frac{\partial}{\partial \tau}\bigg|_{\tau=0} [\Gamma_{g^\tau}(p, q)]^2 = \int\limits_0^a f_{ij}(\gamma^0(t))\dot{\gamma}^i(t, 0)\dot{\gamma}^j(t, 0)\, dt +$$

$$+ \int\limits_0^a \left[\frac{\partial g^0_{ij}}{\partial x^k}(\gamma^0(t))\dot{\gamma}^i(t, 0)\dot{\gamma}^j(t, 0)\frac{\partial \gamma^k}{\partial \tau}(t, 0) + 2g^0_{ij}(\gamma^0(t))\dot{\gamma}^i(t, 0)\frac{\partial \dot{\gamma}^j}{\partial \tau}(t, 0)\right] dt \tag{1.1.2}$$

where

$$f_{ij} = \frac{\partial}{\partial \tau}\bigg|_{\tau=0} g^\tau_{ij}. \tag{1.1.3}$$

The second integral on the right-hand side of (1.1.2) is equal to zero since $\frac{\partial \gamma^i}{\partial \tau}(0,0) = \frac{\partial \gamma^i}{\partial \tau}(a,0) = 0$ and the geodesic γ^0 is an extremal of the functional

$$E_0(\gamma) = \int_0^a g^0_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)\,dt.$$

(One can also verify the vanishing of this integral by transforming the second term in brackets with the help of differentiation by parts and use made of the equation of geodesics). Thus we come to the equality

$$\frac{1}{a}\,\frac{\partial}{\partial \tau}\bigg|_{\tau=0} [\Gamma_{g^\tau}(p,q)]^2 = If(\gamma_{pq}) \equiv \int\limits_{\gamma_{pq}} f_{ij}(x)\dot x^i \dot x^j dt \tag{1.1.4}$$

in which γ_{pq} is a geodesic of the metric g^0 and t is the length of this geodesic in the metric g^0.

If the hodograph Γ_{g^τ} does not depend on τ then the left-hand side of (1.1.4) is equal to zero. On the other hand, if Problem 1.1.1 has a positive answer for the family g^τ then there exists a one-parameter group of diffeomorphisms $\varphi^\tau : M \to M$ such that $\varphi^\tau|_{\partial M} = \mathrm{Id}$ and $g^\tau = (\varphi^\tau)^* g^0$. Written in coordinate form, the last equality gives

$$g^\tau_{ij} = (g^0_{kl} \circ \varphi^\tau)\frac{\partial \varphi^k(x,\tau)}{\partial x^i}\frac{\partial \varphi^l(x,\tau)}{\partial x^j}$$

where $\varphi^\tau(x) = (\varphi^1(x,\tau), ..., \varphi^n(x,\tau))$. Differentiating this relation with respect to τ and putting $\tau = 0$, we get the equation

$$(dv)_{ij} \equiv \frac{1}{2}(v_{i\,;\,j} + v_{j\,;\,i}) = \frac{1}{2} f_{ij}, \tag{1.1.5}$$

for the vector field v generating the group φ^τ where $v_{i\,;\,j}$ are covariant derivatives of the field v in the metric g^0. The condition $\varphi^\tau|_{\partial M} = \mathrm{Id}$ implies that $v|_{\partial M} = 0$. Thus we come to the following question which is a linearization of Problem 1.1.1: to what extent is a symmetric tensor field $f = (f_{ij})$ on a simple Riemannian manifold (M, g^0) determined by the family of integrals (1.1.4) which are known for all $p, q \in \partial M$? In particular, is it true that the equality $If(\gamma_{pq}) = 0$ for all $p, q \in \partial M$ implies existence of a vector field v such that $v|_{\partial M} = 0$ and $dv = f$?

Let us generalize this linear problem to tensor fields of arbitrary degree. Given a Riemannian manifold (M, g), let $\tau_M = (TM, p, M)$ and $\tau'_M = (T'M, p', M)$ denote the tangent and cotangent bundles respectively. Let $S^m\tau'_M$ be the m-th symmetric power of the bundle τ'_M and $C^\infty(S^m\tau'_M)$ be the space of smooth sections of this bundle. Elements of this space are smooth symmetric covariant tensor fields of degree m on M. The operator $d : C^\infty(S^m\tau'_M) \to C^\infty(S^{m+1}\tau'_M)$, determined by the equality $d = \sigma\nabla$ will be called *the operator of inner differentiation* with ∇ standing for the covariant differentiation and σ, for the symmetrization. Thus for $v \in C^\infty(S^m\tau'_M)$ dv is the symmetric part of the covariant derivative of the field v (compare with (1.1.5)).

Problem 1.1.2 (the integral geometry problem for tensor fields) *Let (M,g) be a simple Riemannian manifold. To what extent is a symmetric tensor field $f \in C^\infty(S^m\tau'_M)$ determined by the set of the integrals*

$$If(\gamma_{pq}) = \int\limits_{\gamma_{pq}} f_{i_1\dots i_m}(x)\dot{x}^{i_1}\dots\dot{x}^{i_m}\,dt \tag{1.1.6}$$

that are known for all $p,q \in \partial M$? Here γ_{pq} is the geodesic with endpoints p,q and t is the arc length on this geodesic. In particular, does the equality $If(\gamma_{pq}) = 0$ for all $p,q \in \partial M$ imply existence of a field $v \in C^\infty(S^{m-1}\tau'_M)$, such that $v|_{\partial M} = 0$ and $dv = f$?

By the *ray transform* of the field f we will mean the function If that is determined by formula (1.1.6) on the set of geodesics joining boundary points. In Chapter 4 this problem will be generalized to a wider class of metrics.

Let us formulate one more version of the problem under consideration which does not include boundary conditions. An open (i.., without boundary and noncompact) Riemannian manifold (M,g) is called *dissipative* provided every geodesic leaves any compact set if continued in each of the two directions. The geodesic $\gamma : (a,b) \to M$ $(-\infty \le a < b \le \infty)$ is called *maximal*, if it does not extrapolate onto an interval $(a-\varepsilon_1, b+\varepsilon_2)$, where $\varepsilon_1 \ge 0,\ \varepsilon_2 \ge 0,\ \varepsilon_1+\varepsilon_2 > 0$. For a compactly-supported field $f \in C^\infty(S^m\tau'_M)$ the ray transform If is defined by formula (1.1.6) on the set of all maximal geodesics.

Problem 1.1.3 *Let (M,g) be an open dissipative Riemannian manifold. To what extent is a compactly-supported field $f \in C^\infty(S^m\tau'_M)$ determined by its ray transform If which is known on the set of all maximal geodesics? In particular, does the equality $If = 0$ imply existence of a compactly-supported field $v \in C^\infty(S^{m-1}\tau'_M)$, such that $f = dv$?*

1.2 The kinetic equation on a Riemannian manifold

We denote the points of the space TM of the tangent bundle to a manifold M by the pairs (x,ξ) where $x \in M,\ \xi \in T_xM$. Let $T^0M = \{(x,\xi) \in TM \mid \xi \ne 0\}$ be the manifold of nonzero tangent vectors. We recall that by the (*contra-variant*) *coordinates* of the vector $\xi \in T_xM$ with respect to a local coordinate system $(x^1,\dots,x^n)$ in a neighborhood $U \subset M$ of a point x are meant the coefficients of the decomposition $\xi = \xi^i\dfrac{\partial}{\partial x^i}(x)$. The family of functions $(x^1,\dots,x^n,\xi^1,\dots,\xi^n)$ constitutes a local coordinate system on the domain $p^{-1}(U) \subset TM$, where p is the projection of the tangent bundle. We will call it *associated* with the system $(x^1,\dots,x^n)$. From now on we will consider only such coordinate systems on TM.

Let (M,g) be a simple compact Riemannian manifold, $f \in C^\infty(S^m\tau'_M)$. For $(x,\xi) \in T^0M$, let $\gamma_{x,\xi} : [\tau_-(x,\xi), \tau_+(x,\xi)] \to M$ denotes the maximal geodesic determined by the initial conditions $\gamma_{x,\xi}(0) = x,\ \dot\gamma_{x,\xi}(0) = \xi$. We define a function u on T^0M by the equality

$$u(x,\xi) = \int\limits_{\tau_-(x,\xi)}^{0} f_{i_1\dots i_m}(\gamma_{x,\xi}(t))\dot\gamma^{i_1}_{x,\xi}(t)\dots\dot\gamma^{i_m}_{x,\xi}(t)\,dt. \tag{1.2.1}$$

This function is easily seen to be positively homogeneous in its second argument:

$$u(x,\lambda\xi) = \lambda^{m-1}u(x,\xi) \qquad (\lambda > 0) \tag{1.2.2}$$

and, consequently, is uniquely determined by its restriction to the manifold $\Omega M = \{(x,\xi) \in TM \mid |\xi|^2 = g_{ij}(x)\xi^i\xi^j = 1\}$ of unit vectors. For compactness it is more convenient then T^0M. We divide the boundary $\partial(\Omega M)$ of this manifold into two submanifolds

$$\partial_\pm \Omega M = \{(x,\xi) \in \Omega M \mid x \in \partial M, \pm\langle \xi,\, \nu(x)\rangle \geq 0\},$$

where $\nu(x)$ is the unit vector of the outer normal to the boundary ∂M. Comparing (1.1.6) and (1.2.1), we see that u satisfies the boundary conditions

$$u(x,\xi)|_{x\in\partial M} = \begin{cases} 0, & \text{if } (x,\xi) \in \partial_-\Omega M, \\ If(\gamma_{x,\xi}), & \text{if } (x,\xi) \in \partial_+\Omega M. \end{cases} \tag{1.2.3}$$

The equation

$$Hu \equiv \xi^i \frac{\partial u}{\partial x^i} - \Gamma^i_{jk}(x)\xi^j\xi^k \frac{\partial u}{\partial \xi^i} = f_{i_1\dots i_m}(x)\xi^{i_1}\dots\xi^{i_m} \tag{1.2.4}$$

is valid on T^0M where Γ^i_{jk} are the Christoffel symbols of the metric g. Indeed, let us fix $(x,\xi) \in T^0M$ and put $\gamma = \gamma_{x,\xi}$. By the evident equality $\gamma_{\gamma(t_0),\dot\gamma(t_0)}(t) = \gamma(t+t_0)$ the relation (1.2.1) implies

$$u(\gamma(t_0),\dot\gamma(t_0)) = \int\limits_{\tau_-(x,\xi)}^{t_0} f_{i_1\dots i_m}(\gamma(t))\dot\gamma^{i_1}(t)\dots\dot\gamma^{i_m}(t)\,dt.$$

Differentiating this equality with respect to t_0 and putting $t_0 = 0$, we have

$$\frac{\partial u}{\partial x^i}\dot\gamma^i + \frac{\partial u}{\partial \xi^i}\ddot\gamma^i = f_{i_1\dots i_m}(x)\xi^{i_1}\dots\xi^{i_m}.$$

By using the equation of geodesics

$$\ddot\gamma^i + \Gamma^i_{jk}\dot\gamma^j\dot\gamma^k = 0 \tag{1.2.5}$$

and the relation $\dot\gamma(0) = \xi$, we arrive at (1.2.4).

The differential operator

$$H = \xi^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk}\xi^j\xi^k \frac{\partial}{\partial \xi^i} \tag{1.2.6}$$

participating in (1.2.4) is the vector field on the manifold TM. H is called the *geodesic vector field* and the one-parameter group of diffeomorphisms G^t of the manifold TM generated by H is called the *geodesic flow* or *geodesic pulverization*. These notions are widely used in differential geometry (see, for instance, [13]). The geodesic flow has a simple geometric meaning: G^t is the translation along geodesics in time t, i.., $G^t(x,\xi) = (\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))$ in our notation. In particular, this implies that the field H is tangent to ΩM at all points of the manifold ΩM and, consequently, equation (1.2.4) can be considered on ΩM.

The operator H is related to the inner differentiation operator d by the following equality:

$$H\left(v_{i_1\dots i_{m-1}}(x)\xi^{i_1}\dots\xi^{i_{m-1}}\right) = (dv)_{i_1\dots i_m}(x)\xi^{i_1}\dots\xi^{i_m}, \tag{1.2.7}$$

which can be proved by an easy calculation in coordinates. Thus we arrive at the following question which is equivalent to Problem 1.1.2.

Problem 1.2.1 *Let (M, g) be a simple compact Riemannian manifold. To what extent is the right-hand side of equation* (1.2.4)*, considered on ΩM, determined by the boundary values $u|_{\partial\Omega M}$ of the solution? In particular, does the equality $u|_{\partial\Omega M} = 0$ imply that $u(x, \xi)$ is a homogeneous polynomial of degree $m - 1$ in ξ?*

Similarly, the second question of Problem 1.1.3 can be formulated in the following equivalent form.

Problem 1.2.2 *Let (M, g) be an open dissipative Riemannian manifold. Is it true that any compactly-supported solution to equation* (1.2.4) *on ΩM is a homogeneous polynomial of degree $m - 1$ in ξ?*

The equation

$$Hu = F(x, \xi) \tag{1.2.8}$$

on ΩM with the right-hand side depending arbitrarily on ξ is called (*stationary, unit-velocity*) *kinetic equation of the metric g*. It has a simple physical sense. Let us imagine a stationary distribution of particles moving in M. Every particle moves along a geodesic of the metric g with unit speed, the particles do not influence one another and the medium. Assume that there are also sources of particles in M. By $u(x, \xi)$ and $F(x, \xi)$ we mean the densities of particles and sources with respect to the volume form $dV^{2n} = \det(g_{ij})d\xi^1 \wedge \ldots \wedge d\xi^n \wedge dx^1 \wedge \ldots \wedge dx^n$ (which corresponds to the standard symplectic volume form under isomorphism between the tangent and cotangent bundles induced by the metric g). Then equation (1.2.8) is valid. We omit its proof which can be done in exact analogy with the proof of the Liouville theorem well-known in statistical physics [135].

If the source $F(x, \xi)$ is known then, to get a unique solution u of equation (1.2.8), one has to set the incoming flow $u|_{\partial_-\Omega M}$. In particular, the first of the boundary conditions (1.2.3) means the absence of the incoming flow. The second of the boundary conditions (1.2.3), i.e., the outcoming flow $u|_{\partial_+\Omega M}$, must be used for the inverse problem of determining the source. This inverse problem has the very essential (and not although quite physical) requirement on the source to depend polynomially on the direction ξ. The operator d gives us the next means of constructing sources which are invisible from outside and polynomial in ξ: if $v \in C^\infty(S^{m-1}\tau'_M)$ and $v|_{\partial M} = 0$, then the source $F(x, \xi) = (dv)_{i_1 \ldots i_m}\xi^{i_1} \ldots \xi^{i_m}$ is invisible from outside. Does this construction exhaust all sources that are invisible from outside and polynomial in ξ? It is the physical interpretation of the problem 1.2.1.

1.3 Some remarks

In geophysics the so-called inverse kinematic problem of seismics is well known which is formulated as follows. In a domain $D \subset \mathbf{R}^n$ there is a simple Riemannian metric of the type $g = n(x)g_e$, where g_e is the Euclidean metric. One has to determine the function $n(x)$ by the hodograph Γ_g. In geophysics the metrics of such type are called isotropic. Linearization of this problem in the class of isotropic metrics leads to Problem 1.1.2 in the case $m = 0$. Due to their significance for practice, these problems attract attention of geophysicists and mathematicians for a long time. We refer the reader to the book [109] that contains a discussion of these problems as well as an extensive bibliography.

Questions of the following type seems to be of interest. Is it possible to determine by the hodograph of a metric whether the latter has various geometrical properties? For instance, is it flat, conformally flat, of constant-sign curvature, does it decompose into a product of two metrics and so on? Clearly, each of these questions is correct only if Problem 1.1.1 has a positive answer for the corresponding class of metrics. For example, if we are interested in how to determine by the hodograph whether a metric is flat, the first question we have to answer is: may the hodographs of two metrics coincide provided that one of these metrics is flat and another is not flat? In other words, we have to solve Problem 1.1.1 under the complementary condition of flatness of g^0. The author has thus arrived at Problem 1.1.1. Independently this problem has been formulated by R. Michel [80]. We will now list the known results on Problem 1.1.1. In [8] Yu. E. Anikonov has proved an assertion that amounts to the following: a simple Riemannian metric on a compact two-dimensional manifold M is flat if and only if any geodesic triangle with vertices on ∂M has the sum of angles equal to π. As is easily seen these angles can be expressed by the hodograph. Thus, this result answers the question: how to determine whether a metric is flat given the hodograph? A similar result was obtained by M. L. Gerver and N. S. Nadirashvili [37]. R. Michel obtained a positive answer to Problem 1.1.1 in the two-dimensional case when g^0 has the constant Gauss curvature [80]. A positive answer to Problem 1.1.1 for a rather wide class of two-dimensional metrics has been obtained in [38]. We see that all above mentioned results are dealing with the two-dimensional case. The first and, as the author knows, the only result in the multidimensional case has been obtained by M. Gromov [42]. He has found a positive answer to Problem 1.1.1 under the assumption of flatness of one of the two metrics.

In the case $m = 0$ a solution to the linear problem 1.1.2 for simple metrics was found by R. G. Mukhometov [83, 86], I. N. Bernstein and M. L. Gerver [12], and in the case $m = 1$, by Yu. E. Anikonov and V. G. Romanov [7, 10]. In [87] R. G. Mukhometov generalized these results to metrics whose geodesics form a typical caustics. For $m \geq 2$ no result like these has been obtained until now.

The kinetic equation on Riemannian manifolds and related questions of theoretical photometry are discussed in the articles by V. R. Kireytov [58, 59]. The inverse problems for the kinetic equation were investigated by A. Kh. Amirov [3, 4].

The first idea arising in any attempt at constructing a counter-example to Problem 1.2.2 is as follows. Is it possible that the equation (1.2.4) has a finite solution $u(x, \xi)$ which is polynomial in ξ of degree strictly greater than $m - 1$. It turns out that such counter-example does not exist. This statement was proved by the author [114] for any connected Riemannian manifold (M, g), in which the requirement of compactness of the support can be replaced by the assumption that in the case $n = \dim M \geq 3$ a few first derivatives of the solution $u(x, \xi)$ are equal to zero at a point $x_0 \in M$ and in the case $n = 2$ that all derivatives vanish at the point.

Chapter 2

The ray transform of symmetric tensor fields on Euclidean space

The chapter is devoted to the theory of the ray transform I in the case when the manifold M under consideration coincides with $\mathbf{R}^n$ and the metric of $\mathbf{R}^n$ is Euclidean. In other words, in this chapter I is the integration of a symmetric tensor field along straight lines. The theory of the ray transform on $\mathbf{R}^n$, developed below, has much in common with the classical theory of the Radon transform [35, 48]. However, there is an essential difference between these theories stipulated by the next fact: if considered on the space of symmetric tensor fields of degree m, the operator I has a nontrivial kernel in the case $m > 0$. This circumstance plays an important role through the whole theory.

Section 2.1 contains the definitions of the ray transform and some differential operators that are needed for treating it. Then some relationship between I and the Fourier transform is established.

Sections 2.2–2.5 are devoted to investigation of the kernel of I on the space of compactly-supported tensor fields. In this investigation a differential operator W arises which plays an essential role in the theory of the ray transform as well as in its applications. We call W the Saint Venant operator for the reason explained at the end of Section 2.2.

It is well known that a vector field can be represented as a sum of a potential and solenoidal fields. A similar assertion is valid for a symmetric tensor field of arbitrary degree. This fact plays an important role in the theory of the ray transform. Two versions of such representation are given in Section 2.6.

Sections 2.7–2.10 are devoted to the description of the range of the ray transform on the space of the smooth rapidly decreasing fields. Here the main role is played by the differential equations obtained at first by F. John [56] in the case $m = 0, n = 3$. In the articles by I. . Gelfand, S. G. Gindikin, . I. Graev and Z. Ja. Shapiro [31, 34] these equations were generalized to the operator of integration of a function on p-dimensional planes ($p \le n - 2$), their geometrical as well as topological meaning was explicated. Our proof of sufficiency of the John conditions begins by the same scheme as in [34]; however, in this way we have soon arrived at new aspects of the problem which are stipulated by nontriviality of the kernel of I. We have grouped these aspects in two theorems: a theorem on the tangent component and a theorem on conjugate tensor fields on a sphere, which are also of some interest by themselves. Proving the last theorem, the author ran into an algebraic problem which was kindly solved by I. V. L'vov at the author's request. This algebraic theorem is presented in Section 2.9.

Sections 2.11–2.14 are devoted to the inversion formulas for the ray transform. We present two formulas; the first recovers the solenoidal part of a field f from If and the second, the value Wf of the Saint Venant operator. The components of Wf form a full family of local linear functionals that can be determined by the integrals If.

The classic Plancherel formula asserts that the Fourier transform is an isometry of the space $L_2(\mathbf{R}^n)$. Yu. G. Reshetnyak obtained a formula that expresses $\|f\|_{L_2(\mathbf{R}^n)}$ through the norm $\|\mathcal{R}f\|_H$ of the Radon transform of the function f, where $\|\cdot\|_H$ is some special norm on the space of functions defined on the set of all hyperplanes. Later this formula was called the Plancherel formula for the Radon transform by the authors of the books [35, 48]. It allows one to extend $\mathcal{R}$ to $L_2(\mathbf{R}^n)$. In Section 2.15 a similar result is obtained for the ray transform.

In Section 2.16 an application of the ray transform to an inverse problem of photoelasticity is presented.

Section 2.17 contains bibliographical references as well as the formulations of some results that are not included into the main exposition.

2.1 The ray transform and its relationship to the Fourier transform

First of all we will agree upon the terminology and notations related to the tensor algebra. Then we will introduce various spaces of tensor fields and define the differential operators that are needed for investigating the ray transform: the operator of inner differentiation, the divergence and the Saint Venant operator. After this we define the ray transform and establish its relationship to the Fourier transform.

We consider $\mathbf{R}^n$ as a Euclidean vector space with scalar product $\langle x, y\rangle$ and norm $|x|$. From now on we always assume that $n \geq 2$. We put $\mathbf{R}^n_0 = \mathbf{R}^n \setminus \{0\}$. If $e_1, \dots, e_n$ is a basis for $\mathbf{R}^n$, then we assign $g_{ij} = \langle e_i, e_j\rangle$. Let (g^{ij}) be the inverse matrix to (g_{ij}). We restrict ourselves to using only affine coordinate systems on $\mathbf{R}^n$.

Given an integer $m \geq 0$, by $T^m = T^m(\mathbf{R}^n)$ we mean the complex vector space of all functions $\mathbf{R}^n \times \ldots \times \mathbf{R}^n \to \mathbf{C}$ (there are m factors to the left of the arrow, $\mathbf{C}$ is the field of complex numbers) that are $\mathbf{R}$-linear in each of its argument, and by $S^m = S^m(\mathbf{R}^n)$ the subspace of T^m that consists of the functions symmetric in all arguments. The elements of T^m (S^m) are called *tensors* (*symmetric tensors*) *of degree* m on $\mathbf{R}^n$. Let $\sigma : T^m \to S^m$ be the *canonical projection* (*symmetrization*) defined by the equality

$$\sigma u(x_1, \dots, x_m) = \frac{1}{m!} \sum_{\pi \in \Pi_m} u(x_{\pi(1)}, \dots, x_{\pi(m)}) \tag{2.1.1}$$

where the summation is taken over the group Π_m of all permutations of the set $\{1, \dots, m\}$. We identify the space $T^1 = S^1$ with $\mathbf{C}^n$ by the equality $(x + iy)(z) = \langle x, z\rangle + i\langle y, z\rangle$ for $x, y, z \in \mathbf{R}^n$. We also assume that $T^m = S^m = 0$ for $m < 0$.

If $e_1, \dots, e_n$ is a basis for $\mathbf{R}^n$, then the numbers $u_{i_1 \dots i_m} = u(e_{i_1}, \dots, e_{i_m})$ are called the *covariant coordinates* or *components* of the tensor $u \in T^m$ relative to the given basis. Assuming the choice of the basis to be clear from the context, we shall denote this fact by the record $u = (u_{i_1 \dots i_m})$. Alongside the covariant coordinates the *contravariant coordinates*

$$u^{i_1 \dots i_m} = g^{i_1 j_1} \dots g^{i_m j_m} u_{j_1 \dots j_m} \tag{2.1.2}$$

are useful too. We recall the summation rule: on repeating sub- and super-indices in a monomial the summation from 1 to n is understood. It is evident that the *scalar product* on T^m defined by the formula

$$\langle u, v \rangle = u^{i_1 \dots i_m} \overline{v}_{i_1 \dots i_m} \tag{2.1.3}$$

(the bar denotes complex conjugation) does not depend on the choice of the basis.

Together with operator (2.1.1) of full symmetrization, we will make use of the *symmetrization with respect to a part of indices* that is in coordinate form defined by the equality

$$\sigma(i_1 \dots i_p) u_{i_1 \dots i_m} = \frac{1}{p!} \sum_{\pi \in \Pi_p} u_{i_{\pi(1)} \dots i_{\pi(p)} i_{p+1} \dots i_m}$$

We shall also need the *alternation* with respect to two indices

$$\alpha(i_1 i_2) u_{i_1 i_2 j_1 \dots j_p} = \frac{1}{2}(u_{i_1 i_2 j_1 \dots j_p} - u_{i_2 i_1 j_1 \dots j_p}).$$

Symmetrization with respect to all indices will be denoted without indicating arguments, i. e., by σ. A record of the next type

$$\operatorname{sym} u_{i_1 \dots i_k j_1 \dots j_l} : (i_1 \dots i_{k-1}) i_k (j_1 \dots j_{l-1}) j_l \tag{2.1.4}$$

is convenient for notation of the partial symmetry of the tensor u. Namely, (2.1.4) denotes the fact that the tensor u is symmetric with respect to each group of indices in parentheses.

We recall that the *tensor product* $u \otimes v \in T^{k+m}$ is defined for $u \in T^k,\ v \in T^m$ by the equality

$$u \otimes v(x_1, \dots, x_{k+m}) = u(x_1, \dots, x_k) v(x_{k+1}, \dots, x_{k+m}).$$

The *symmetric tensor product* defined by the equality $uv = \sigma(u \otimes v)$ turns $S^* = \bigoplus\limits_{m=0}^{\infty} S^m$ to the commutative graded algebra. By $i_u : S^* \to S^*$ we mean the operator of symmetric multiplication by u and by j_u, the operator dual to i_u. In coordinate form they are written as follows:

$$\begin{aligned} (i_u v)_{i_1 \dots i_{k+m}} &= \sigma(u_{i_1 \dots i_k} v_{i_{k+1} \dots i_{k+m}}), \\ (j_u v)_{i_1 \dots i_{m-k}} &= v_{i_1 \dots i_m} u^{i_{m-k+1} \dots i_m} \end{aligned} \tag{2.1.5}$$

for $u \in S^k,\ v \in S^m$. The tensor $j_u v$ will also be denoted by v/u. A particular role will be played by the operators i_g and j_g, where $g = (g_{ij})$ is the metric tensor. We will therefore distinguish them by putting $i = i_g,\ j = j_g$. Let us extend the scalar product (2.1.3) to S^*, by declaring S^k and S^l orthogonal to each other in the case of $k \neq l$.

For $f \in S^m$, $\xi \in \mathbf{R}^n \subset \mathbf{C}^n = S^1$, the equality

$$\langle f, \xi^m \rangle = f_{i_1 \dots i_m} \xi^{i_1} \dots \xi^{i_m}$$

is valid which we will make use of abbreviating various formulas.

For an open $U \subset \mathbf{R}^n$, let $\mathcal{D}'(U)$ be the space of all distributions (generalized functions) on U. For any space $\mathcal{A} \subset \mathcal{D}'(U)$ of distributions on U, we put $\mathcal{A}(S^m; U) = \mathcal{A} \otimes_{\mathbf{C}} S^m$. If $\mathcal{A}$ is considered with a topology, then we endow $\mathcal{A}(S^m; U)$ with the topology of the tensor product (there is no problem with its definition because S^m has a finite dimension). The notation $\mathcal{A}(S^m; \mathbf{R}^n)$ will usually be abbreviated to $\mathcal{A}(S^m)$. For $T \in \mathcal{A}(S^m; U)$, the components $T_{i_1 \dots i_m} \in \mathcal{A}$ are defined; we will denote this fact by the same record $T = (T_{i_1 \dots i_m})$ as before. If $\mathcal{A}$ consists of ordinary functions, i.e., $\mathcal{A} \subset L_{1,\mathrm{loc}}(U)$ ($L_{1,\mathrm{loc}}(U)$ is the space of functions locally summable on U), then a field $u \in \mathcal{A}(S^m; U)$ can be considered as a function on U with values in S^m. In accord with this scheme the next spaces are introduced in particular:

$C^l(S^m; U)$ — the space of l times continuously differentiable symmetric tensor fields on U;

$\mathcal{E}(S^m; U)$ — the space of smooth (infinitely differentiable) fields;

$\mathcal{D}(S^m; U)$ — the space of compactly-supported smooth fields;

$L_{1,\mathrm{loc}}(S^m; U)$ — the space of locally summable or ordinary fields;

$\mathcal{D}'(S^m; U)$ — the space of symmetric tensor field-distributions (or the space of generalized symmetric tensor fields);

$\mathcal{E}'(S^m; U)$ — the space of compactly-supported tensor field-distributions;

$\mathcal{S}(S^m)$ — the space of smooth fields rapidly decreasing with all derivatives on $\mathbf{R}^n$;

$L_2(S^m)$ — the space of square integrable fields.

Each of these spaces is furnished with the corresponding topology. Considered without any topology, the space $\mathcal{E}(S^m; U)$ will also be denoted by $C^\infty(S^m; U)$. The scalar product on $L_2(S^m)$ is defined by the formula

$$(u, v)_{L_2(S^m)} = \int\limits_{\mathbf{R}^n} \langle u(x), v(x) \rangle \, dV^n(x) \tag{2.1.6}$$

in which $dV^n(x) = [\det(g_{ij})]^{1/2} dx^1 \wedge \dots \wedge dx^n$ is the Lebesgue measure. The lower index will sometimes be omitted in (2.1.6) if such abbreviation is not misleading.

If L is a linear topological space and L' is its dual space, then by $\langle T, \varphi \rangle$ we mean the value of the functional $T \in L'$ at the element $\varphi \in L$. The space L' will usually be considered with the strong topology. We identify the space $\mathcal{D}'(S^m)$ with the dual of $\mathcal{D}(S^m)$, and the space $\mathcal{E}'(S^m)$ with the space dual of $\mathcal{E}(S^m)$, by the formula

$$\langle T, \varphi \rangle = \langle T_{i_1 \dots i_m}, \varphi^{i_1 \dots i_m} \rangle.$$

For an ordinary field $u \in L_{1,\mathrm{loc}}(S^m; U)$, this equality takes the next form

$$\langle u, \varphi \rangle = \int\limits_U \langle u(x), \overline{\varphi(x)} \rangle \, dV^n(x) \qquad (\varphi \in \mathcal{D}(S^m; U)). \tag{2.1.7}$$

Note that on the left-hand side of this equality the angle brackets mean the value of a distribution while on the right-hand side they denote the scalar product. The author hopes that the coincidence of these notations will not confuse.

Given an affine coordinate system $(x^1, \dots, x^n)$, we define the operator of (*covariant*) *differentiation* $\nabla : C^\infty(T^m) \to C^\infty(T^{m+1})$ by the equalities $\nabla u = (u_{i_1 \dots i_m\,;\,j})$, $u_{i_1 \dots i_m\,;\,j} = \partial u_{i_1 \dots i_m} / \partial x^j$. Evidently, it does not depend on the choice of a coordinate system. The following circumstance is rather essential: the iterated derivative $\nabla^k u = (u_{i_1\dots i_m\,;\,j_1\dots j_k})$ is symmetric in $j_1,\dots,j_k$. The operator $d = \sigma\nabla : C^\infty(S^m) \to C^\infty(S^{m+1})$ is called the *inner differentiation*, the *divergence operator* $\delta : C^\infty(S^m) \to C^\infty(S^{m-1})$ in coordinate form is defined by the equality $(\delta u)_{i_1\dots i_{m-1}} = u_{i_1\dots i_m\,;\,i_{m+1}} g^{i_m i_{m+1}}$. The operators d and $-\delta$ are formally dual to each other with respect to the scalar product (2.1.6). Moreover, for a compact domain $G \subset \mathbf{R}^n$ bounded by a piece-smooth hypersurface ∂G, the next Green's formula is valid:

$$\int\limits_G [\langle du, v\rangle + \langle u, \delta v\rangle]\, dV^n = \int\limits_{\partial G} \langle i_\nu u, v\rangle\, dV^{n-1} \tag{2.1.8}$$

where dV^{n-1} is the $(n-1)$-dimensional area on ∂G and ν is the unit vector of the outer normal to ∂G. To prove (2.1.8), it suffices to write down the identity

$$\langle du, v\rangle = u_{i_1\dots i_m\,;\,i_{m+1}} \overline{v}^{i_1\dots i_{m+1}} = (u_{i_1\dots i_m} \overline{v}^{i_1\dots i_{m+1}})_{\,;\,i_{m+1}} - \langle u, \delta v\rangle,$$

to introduce the vector field ξ by the equality $\xi^j = u_{i_1\dots i_m} \overline{v}^{i_1\dots i_m j}$ and to apply the Gauss-Ostrogradskiĭ formula to ξ.

The *Saint Venant operator* $W : C^\infty(S^m) \to C^\infty(S^m \otimes S^m)$ is defined by the equality

$$(Wu)_{i_1\dots i_m j_1\dots j_m} = \sigma(i_1\dots i_m)\sigma(j_1\dots j_m) \sum_{p=0}^{m} (-1)^p \binom{m}{p} u_{i_1\dots i_{m-p} j_1\dots j_p\,;\,j_{p+1}\dots j_m i_{m-p+1}\dots i_m} \tag{2.1.9}$$

where $\binom{m}{p} = \dfrac{m!}{p!(m-p)!}$ are the binomial coefficients.

By the *ray transform* of a field $f \in C^\infty(S^m)$ we shall mean the function If defined on $\mathbf{R}^n \times \mathbf{R}^n_0$ by the equality

$$If(x,\xi) = \int\limits_{-\infty}^{\infty} \langle f(x+t\xi), \xi^m\rangle\, dt \tag{2.1.10}$$

under the condition that the integral converges. From this definition, we immediately derive that the function $\psi(x,\xi) = If$ has the following homogeneity in its second argument:

$$\psi(x,t\xi) = \frac{t^m}{|t|}\psi(x,\xi) \qquad (0 \ne t \in \mathbf{R}) \tag{2.1.11}$$

and satisfies the identity

$$\psi(x+t\xi,\xi) = \psi(x,\xi) \qquad (t \in \mathbf{R}) \tag{2.1.12}$$

which shows that $\psi(x,\xi) = If$ is constant with respect to its first argument on the straight line passing through the point x in the direction ξ.

Let $T\Omega = \{(x,\xi) \in \mathbf{R}^n \times \mathbf{R}^n \mid \langle x,\xi\rangle = 0,\ |\xi| = 1\}$ be the space of the tangent bundle of the unit sphere $\Omega = \Omega^{n-1} = \{\xi \in \mathbf{R}^n \mid |\xi| = 1\}$. Equalities (2.1.11) and (2.1.12) imply that the function ψ can be recovered from its trace $\varphi = \psi|_{T\Omega}$ on $T\Omega$ by the formula

$$\psi(x,\xi) = |\xi|^{m-1}\varphi\left(x - \frac{\langle x,\xi\rangle}{|\xi|^2}\xi,\ \frac{\xi}{|\xi|}\right). \tag{2.1.13}$$

In various sections of this chapter the ray transform If of a field f will be understood as follows: either as a function on $\mathbf{R}^n \times \mathbf{R}^n_0$ or as a function on $T\Omega$. It will be clear from the context which of these ways is taken. By (2.1.13), the two ways are equivalent.

Let us observe that the operator I has a nontrivial kernel. Indeed, if the field $v \in C^\infty(S^{m-1})$ satisfies the condition $v(x) \to 0$ as $|x| \to \infty$, then

$$(Idv)(x,\xi) = \lim_{a\to\infty} \int\limits_{-a}^{a} \frac{d}{dt} \langle v(x+t\xi), \xi^{m-1}\rangle \, dt = 0.$$

Under some conditions on the behavior of the field $f(x)$ at infinity, the converse assertion is valid: the equality $If = 0$ implies existence of a field v such that $f = dv$. In Section 2.4 this statement will be proved for compactly-supported f and in Section 2.6, for $f \in L_2(S^m)$.

We shall use the *Fourier transform* on $\mathbf{R}^n$ in the following form:

$$F[f] = \hat{f}(y) = (2\pi)^{-n/2} \int\limits_{\mathbf{R}^n} f(x)\, e^{-i\langle y,x\rangle}\, dV^n(x).$$

We define the Fourier transform $F : \mathcal{S}(S^m) \to \mathcal{S}(S^m)$ by components, i.e., $(F[f])_{i_1\dots i_m} = F[f_{i_1\dots i_m}]$. By the Fourier-Plancherel theorem it is extendible to an isometry of the space $L_2(S^m)$.

Let $\mathcal{S}(T\Omega)$ be the space of smooth functions $\varphi(x,\xi)$ on $T\Omega$ such that all their derivatives decrease rapidly in the first argument. The *Fourier transform* $F : \mathcal{S}(T\Omega) \to \mathcal{S}(T\Omega)$ is defined by the formula

$$\hat{\varphi}(y,\xi) = (2\pi)^{(1-n)/2} \int\limits_{\xi^\perp} \varphi(x,\xi)\, e^{-i\langle y,x\rangle}\, dV^{n-1}(x) \qquad (2.1.14)$$

where $\xi^\perp = \{x \in \mathbf{R}^n \mid \langle x,\xi\rangle = 0\}$, and $dV^{n-1}(x)$ is the $(n-1)$-dimensional Lebesgue measure on $\xi^\perp$. Thus, it is the ordinary Fourier transform in the $(n-1)$-dimensional variable x, where ξ stands as a parameter.

It is easily seen that $If \in \mathcal{S}(T\Omega)$ for $f \in \mathcal{S}(S^m)$, and the operator $I : \mathcal{S}(S^m) \to \mathcal{S}(T\Omega)$ is bounded. For $f \in \mathcal{S}(S^m)$, the relationship between the ray transform and the Fourier transform is expressed by the equality

$$\widehat{If}(y,\xi) = (2\pi)^{1/2} \langle \hat{f}(y), \xi^m\rangle \qquad ((y,\xi) \in T\Omega). \qquad (2.1.15)$$

Indeed, let $|\xi| = 1$. Substituting $x = x' + t\xi,\ x' \in \xi^\perp$ into the integral

$$\langle \hat{f}(y), \xi^m\rangle = (2\pi)^{-n/2} \int\limits_{\mathbf{R}^n} \langle f(x), \xi^m\rangle\, e^{-i\langle y,x\rangle}\, dV^n(x),$$

we obtain

$$\langle \hat{f}(y), \xi^m\rangle = (2\pi)^{-n/2} \int\limits_{\xi^\perp} \left[\int\limits_{-\infty}^{\infty} \langle f(x'+t\xi), \xi^m\rangle\, e^{-it\langle y,\xi\rangle}\, dt \right] e^{-i\langle y,x'\rangle}\, dV^{n-1}(x').$$

If $\langle y,\xi\rangle = 0$, i.e., $(y,\xi) \in T\Omega$, then the previous equality gives

$$\langle \hat{f}(y), \xi^m\rangle = (2\pi)^{-n/2} \int\limits_{\xi^\perp} If(x,\xi)\, e^{-i\langle y,x\rangle}\, dV^{n-1}(x)$$

which coincides with (2.1.15).

2.2 Description of the kernel of the ray transform in the smooth case

Theorem 2.2.1 *Let $n \geq 2$ and $m \geq 0$ be integers, $l = \max\{m, 2\}$. For a compactly-supported field $f \in C^l(S^m; \mathbf{R}^n)$ the following statements are equivalent:*

(1) $If = 0$;

(2) *there exists a compactly-supported field $v \in C^{l+1}(S^{m-1}; \mathbf{R}^n)$ such that its support is contained in the convex hull of the support of f and*

$$dv = f; \tag{2.2.1}$$

(3) *the equality*

$$Wf = 0 \tag{2.2.2}$$

is valid in $\mathbf{R}^n$.

This theorem answers to the question: to what extent is a compactly-supported field $h \in C^l(S^m)$ determined by its ray transform Ih? According to the claims (2) and (3) of the theorem, the answer is given in two forms. First, Ih determines h up to a summand of the type $f = dv$ where v is an arbitrary compactly-supported field of degree $m-1$. Second, there is a system of local linear functionals of the field h which is uniquely determined by the integral information Ih. Namely, the family of the components of the field Wh is such a system. Moreover, this family is a full system of the local linear functionals which can be restored by Ih.

The scheme of the proof of theorem 2.2.1 is as follows. We start with proving equivalence of claims (1) and (2) in the case $n = 2$. Then we prove the equivalence of (2) and (3) for an arbitrary n. After this equivalence of (1) and (2) for an arbitrary n follows from the next two observations :

(a) I and W commute with the operator that restricts tensor fields onto affine subspaces of $\mathbf{R}^n$;

b) for $f \in C^l(S^m; \mathbf{R}^n)$, if the restriction of the field Wf on each two-dimensional plane is equal to zero, then $Wf = 0$.

The last claim follows from the fact that the field Wf has the next symmetry

$$\mathrm{sym}\,(Wf)_{i_1\dots i_m j_1\dots j_m} : (i_1 \dots i_m)(j_1 \dots j_m)$$

as one can see from definition (2.1.9).

Equation (2.2.1) can be considered not only on the whole $\mathbf{R}^n$ but also on a domain $U \subset \mathbf{R}^n$. If U is a simply-connected domain, then condition (2.2.2) remains sufficient for solvability of equation (2.2.1) as the next theorem shows.

Theorem 2.2.2 *Let U be a domain in $\mathbf{R}^n$, $x_0 \in U$, $m \geq 1$ be an integer, $f \in C^m(S^m; U)$, and $v^p \in T^{m+p-1}$ $(0 \leq p \leq m-1)$. Equation (2.2.1) has at most one solution $v \in C^{m+1}(S^{m-1}; U)$ that satisfies the initial conditions*

$$\nabla^p v(x_0) = v^p \qquad (p = 0, 1, \dots, m-1). \tag{2.2.3}$$

For existence of a solution $v \in C^{m+1}(S^{m-1}; U)$ *to problem* (2.2.1), (2.2.3), *it is necessary that the right-hand side of equation* (2.2.1) *satisfies condition* (2.2.2), *and the tensors* v^p *have the symmetries*

$$\mathrm{sym}\,(v^p_{i_1\dots i_{m+p-1}}) : (i_1 \dots i_{m-1})(i_m \dots i_{m+p-1}) \qquad (p = 0, 1, \dots, m-1) \tag{2.2.4}$$

and satisfy the relations

$$\sigma(i_1 \dots i_m)v^p_{i_1\dots i_{m+p-1}} = f_{i_1\dots i_m\,;\,i_{m+1}\dots i_{m+p-1}}(x_0) \qquad (p = 1, \dots, m-1). \tag{2.2.5}$$

If U *is a simply-connected domain, then conditions* (2.2.2), (2.2.4), (2.2.5) *are sufficient for existence a solution* $v \in C^{m+1}(S^{m-1}; U)$ *to problem* (2.2.1), (2.2.3).

Let us note that v is a function in the case $m = 1$ and equation (2.2.1) looks like: $dv = f_i dx^i$. In this case equality (2.2.2) is the well-known condition of integrability for a Pfaff form: $\partial f_i/\partial x^j = \partial f_j/\partial x^i$. In the case $m = 2$ Theorem 2.2.2 is well known in deformation theory [91]. In this case equation (2.2.1) looks like: $v_{i\,;\,j} + v_{j\,;\,i} = 2f_{ij}$; and, by Lemma 2.4.2 below, equation (2.2.2) takes the form $f_{ij\,;\,kl} + f_{kl\,;\,ij} - f_{il\,;\,jk} - f_{jk\,;\,il} = 0$ and is called the deformations compatibility condition. It was obtained by Saint Venant and therefore the operator W is named after him.

2.3 Equivalence of the first two statements of Theorem 2.2.1 in the case $n = 2$

If claim (2) of Theorem 2.2.1 holds, then

$$If(x,\xi) = \int\limits_{-\infty}^{\infty} \frac{d}{dt}\langle v(x+\xi t), \xi^{m-1}\rangle\, dt = 0.$$

Thus (2) $\to$ (1) for arbitrary n.

In the proof of the implication (1) $\to$ (2) in the case $n = 2$ we will use Fourier series in the form that differs slightly from the ordinary one and is described as follows.

Lemma 2.3.1 *For an integer* $p \ge 0$, *the system of* $2p+1$ *functions*

$$\cos^{p-i-1}\theta\,\sin^i\theta \;\; (0 \le i \le p-1), \qquad \cos^{p-i}\theta\,\sin^i\theta \;\; (0 \le i \le p)$$

is linearly independent over $(-\pi, \pi)$. *If* V_p *is the linear span of this system and* U_p *is the closure of the linear span of the system*

$$\cos k\theta,\ \sin k\theta \qquad (k = p+1, p+2, \dots)$$

in $L_2(-\pi,\pi)$, *then* $L_2(-\pi,\pi) = V_p \bigoplus U_p$ *and the summands are orthogonal to each other.*

To verify this, it is enough to notice the coincidence of V_p with the linear span of the system

$$\cos k\theta,\ \sin k\theta \qquad (0 \le k \le p).$$

Let us begin proving the implication (1) → (2) for $n = 2$. Let $(x, y) = (x^1, x^2)$ be a Cartesian coordinate system in $\mathbf{R}^2$. In $\mathbf{R}^2$ a symmetric tensor of degree m has only $m+1$ different components. Therefore it is convenient to introduce another notation for components by putting $\tilde{f}_i = f_{1\dots12\dots2}$ $(0 \le i \le m)$; index 1 is repeated i times, and 2 is repeated $m-i$ times. We also assume that $\tilde{f}_i = 0$ when either $i < 0$ or $i > m$. Equality (2.2.1) can be rewritten in this notation as follows:

$$\tilde{f}_i = \frac{1}{m}\left[i\frac{\partial \tilde{v}_{i-1}}{\partial x} + (m-i)\frac{\partial \tilde{v}_i}{\partial y}\right] \qquad (i = 0, \dots, m), \tag{2.3.1}$$

and claim (1) of theorem 2.2.1 looks like:

$$\int_{-\infty}^{\infty} \sum_{i=0}^{m} \binom{m}{i} \tilde{f}_i(x + t\cos\theta, y + t\sin\theta) \cos^i\theta \, \sin^{m-i}\theta \, dt = 0. \tag{2.3.2}$$

This equality holds identically on x, y, θ. For $x, y, \theta \in \mathbf{R}$, we put

$$w(x, y, \theta) = \int_{-\infty}^{0} \sum_{i=0}^{m} \binom{m}{i} \tilde{f}_i(x + t\cos\theta, y + t\sin\theta) \cos^i\theta \, \sin^{m-i}\theta \, dt. \tag{2.3.3}$$

The function $w(x, y, \theta)$ is of class C^2 with respect to the set of all its arguments and, by (2.3.2), satisfies the identity

$$w(x, y, \theta + \pi) = (-1)^{m+1} w(x, y, \theta). \tag{2.3.4}$$

It follows from (2.3.3) that w satisfies the equation

$$\cos\theta \, w_x + \sin\theta \, w_y = \sum_{i=0}^{m} \binom{m}{i} \tilde{f}_i(x, y) \cos^i\theta \, \sin^{m-i}\theta \tag{2.3.5}$$

which is the simplest form of the kinetic equation (compare with (1.2.4)). Here the notations $w_x = \partial w/\partial x$ and $w_y = \partial w/\partial y$ are used. By Lemma 2.3.1, w can be uniquely represented by the series

$$w(x, y, \theta) = \sum_{i=0}^{m-2} \beta^i \cos^{m-i-2}\theta \, \sin^i\theta + \sum_{i=0}^{m-1} \alpha^i \cos^{m-i-1}\theta \, \sin^i\theta +$$

$$+ \sum_{k=m}^{\infty} (\tilde{a}^k \cos k\theta + \tilde{b}^k \sin k\theta) \tag{2.3.6}$$

whose coefficients $\alpha^i, \beta^i, \tilde{a}^k, \tilde{b}^k$ are compactly-supported functions of the variables x, y with continuous derivatives of the first and second order. The series

$$\sum_{k=m}^{\infty} \left[(\tilde{a}^k_x)^2 + (\tilde{a}^k_y)^2 + (\tilde{b}^k_x)^2 + (\tilde{b}^k_y)^2\right]$$

converges uniformly in $(x, y) \in \mathbf{R}^2$ (we assume here that the components of the field f are real; it is sufficient to prove theorem 2.2.1 in this case). From (2.3.4), we see that

$\beta^1 = \ldots = \beta^m = \tilde a^m = \tilde b^m = \tilde a^{m+2} = \tilde b^{m+2} = \ldots = 0$, and (2.3.6) can be rewritten as follows:

$$w = \sum_{i=0}^{m-1} \alpha^i \cos^{m-i-1}\theta \, \sin^i\theta + \sum_{k=0}^{\infty} \Big[a^k \cos(m+2k+1)\theta + b^k \sin(m+2k+1)\theta\Big]. \quad (2.3.7)$$

Series (2.3.7) is differentiable termwise with respect to x, y because $w \in C^2$. We substitute the so-find values of w_x and w_y into (2.3.5) and then transform the left-hand sides of this equation to a series of type (2.3.6) (increasing m by 1). Comparing the coefficients of the series on the left- and right-hand sides of the so-obtained equality, we arrive at the next infinite system:

$$\alpha^i_x + \alpha^{i-1}_y + \gamma_i(a^0_x + b^0_y) + \delta_i(b^0_x - a^0_y) = \binom{m}{i} \tilde f_{m-i} \qquad (i = 0, \ldots, m), \qquad (2.3.8)$$

$$\left.\begin{aligned} a^k_x - b^k_y &= -(a^{k+1}_x + b^{k+1}_y), \\ a^k_y + b^k_x &= -(-a^{k+1}_y + b^{k+1}_x) \end{aligned}\right\} \qquad (k = 0, 1, \ldots) \qquad (2.3.9)$$

where it is assumed that $\alpha^{-1} = \alpha^m = 0$, and the numbers γ_i, δ_i are the coefficients of the expansions

$$\cos m\theta = \sum_{i=0}^{m} \gamma_i \cos^{m-i}\theta \, \sin^i\theta, \qquad \sin m\theta = \sum_{i=0}^{m} \delta_i \cos^{m-i}\theta \, \sin^i\theta.$$

Let us show that (2.3.9) implies

$$a^k \equiv b^k \equiv 0 \qquad (k = 0, 1, \ldots) \qquad (2.3.10)$$

Indeed, squaring each of the equalities (2.3.9) and summing them, we obtain

$$(a^k_x)^2 + (a^k_y)^2 + (b^k_x)^2 + (b^k_y)^2 + 2\left[\frac{\partial}{\partial x}(a^k_y b^k) - \frac{\partial}{\partial y}(a^k_x b^k)\right] =$$

$$= (a^{k+1}_x)^2 + (a^{k+1}_y)^2 + (b^{k+1}_x)^2 + (b^{k+1}_y)^2 + 2\left[\frac{\partial}{\partial x}(a^{k+1}_y b^{k+1}) - \frac{\partial}{\partial y}(a^{k+1}_x b^{k+1})\right].$$

The expressions in brackets have a divergence form and, consequently, vanish after integration over $\mathbf{R}^2$. We thus obtain

$$c_k \equiv \iint\limits_{\mathbf{R}^2} \Big[(a^k_x)^2 + (a^k_y)^2 + (b^k_x)^2 + (b^k_y)^2\Big] \, dxdy = c_{k+1} \qquad (k = 0, 1, \ldots).$$

The series $\sum_{k=0}^{\infty} c_k$ converges as we have seen before. Therefore all $c_k = 0$, (2.3.10) is valid.

Equalities (2.3.8) now look like:

$$\binom{m}{i} \tilde f_{m-i} = \alpha^i_x + \alpha^{i-1}_y \qquad (i = 0, \ldots, m).$$

Defining $\tilde v_i$ by the equality $(m-i)\binom{m}{i} \tilde v_i = m\alpha^{m-i-1}$ $(0 \le i \le m-1)$, we arrive at (2.3.1) which is equivalent to the relation $f = dv$.

By (2.3.3), $\operatorname{supp}_{x,y} w$ is contained in the convex hull of the support of f. Consequently, the same is true for $\operatorname{supp} v$. Examining the above proof, one can see that smoothness of the field v is not less than that of f, i.e., $v \in C^l(S^{m-1})$ if $f \in C^l(S^m)$. Nevertheless, (2.2.1) implies $v \in C^{l+1}(S^{m-1})$ as will be shown later in the proof of Theorem 2.2.2.

2.4 Proof of Theorem 2.2.2

First of all, the part of Theorem 2.2.1 which remains unproved, namely, the equivalence of claims (2) and (3) in this theorem, is a consequence of Theorem 2.2.2. The proof of Theorem 2.2.2 given below is based on the following observation: being differentiated $(m-1)$ times, equation (2.2.1) can be resolved with respect to $\nabla^m v$.

At first we give a few purely algebraic propositions.

Lemma 2.4.1 *For $1 \le l \le m$, let a tensor $y \in T^m$ have the symmetry*

$$\operatorname{sym} y_{i_1\dots i_m} : (i_1 \dots i_l)(i_{l+1} \dots i_m). \tag{2.4.1}$$

Then the next equality holds:

$$\sigma(i_1 \dots i_m) y_{i_1\dots i_m} = \frac{1}{m}\sigma(i_1 \dots i_{m-1}) \Big[(m-l) y_{i_1\dots i_m} + l y_{i_m i_1 \dots i_{m-1}}\Big]. \tag{2.4.2}$$

P r o o f. By definition,

$$\sigma(i_1 \dots i_m) y_{i_1\dots i_m} = \frac{1}{m!}\sum_{\pi} y_{i_{\pi(1)}\dots i_{\pi(m)}}.$$

We combine the summands of this sum according to the position of the index i_m. We then get

$$\sigma(i_1 \dots i_m) y_{i_1\dots i_m} = \frac{1}{m}\sigma(i_1 \dots i_{m-1}) \Big[y_{i_m i_1 \dots i_{m-1}} + y_{i_1 i_m i_2 \dots i_{m-1}} + \dots + y_{i_1 \dots i_m}\Big].$$

By (2.4.1), the first l summands in the brackets coincide; in the other summands the index i_m can be moved to the final position. The lemma is proved.

We call (2.4.2) the formula of decomposition of the symmetrization $\sigma(i_1 \dots i_m)$ with respect to the index i_m. Of course, similar formulas are valid for other indices of the tensor y possessing a symmetry of the type (2.4.1).

We define the operator $V : C^\infty(S^m) \to C^\infty(T^{2m})$ by the equality

$$(Vf)_{i_1\dots i_m j_1 \dots j_m} = \alpha(i_m j_m)\, \sigma(i_1 \dots i_{m-1})\, \sigma(j_1 \dots j_{m-1}) \times$$

$$\times \sum_{p=0}^{m-1} (-1)^p \binom{m-1}{p} f_{i_1 \dots i_{m-p-1} i_m j_1 \dots j_p\, ;\, j_{p+1} \dots j_m i_{m-p} \dots i_{m-1}}. \tag{2.4.3}$$

Lemma 2.4.2 *For $f \in C^\infty(S^m)$, the next relations are valid:*

$$\sigma(i_1 \dots i_m)\, \sigma(j_1 \dots j_m)\, (Vf)_{i_1\dots i_m j_1 \dots j_m} = \frac{1}{2}(Wf)_{i_1\dots i_m j_1 \dots j_m}, \tag{2.4.4}$$

$$\alpha(i_m j_m)\, (Wf)_{i_1\dots i_m j_1 \dots j_m} = \frac{m+1}{m}(Vf)_{i_1\dots i_m j_1 \dots j_m}. \tag{2.4.5}$$

P r o o f. From the definition of V and the evident equality

$$\sigma(i_1\dots i_m)\,\sigma(j_1\dots j_m)\ \alpha(i_mj_m)\,\sigma(i_1\dots i_{m-1})\,\sigma(j_1\dots j_{m-1}) =$$

$$= \sigma(i_1\dots i_m)\,\sigma(j_1\dots j_m)\,\alpha(i_mj_m),$$

we obtain

$$\sigma(i_1\dots i_m)\,\sigma(j_1\dots j_m)\,(Vf)_{i_1\dots i_mj_1\dots j_m} = \frac{1}{2}\sigma(i_1\dots i_m)\,\sigma(j_1\dots j_m)\sum_{p=0}^{m-1}(-1)^p\binom{m-1}{p}\times$$

$$\times\Big[f_{i_1\dots i_{m-p}j_1\dots j_p\,;\,j_{p+1}\dots j_mi_{m-p+1}\dots i_m} - f_{i_1\dots i_{m-p-1}j_1\dots j_{p+1}\,;\,j_{p+2}\dots j_mi_{m-p}\dots i_m}\Big].$$

Combining the second summand in the brackets of the p-th term of the sum with the first summand of the $(p+1)$-st term, we arrive at (2.4.4).

Decomposing the symmetrizations $\sigma(i_1\dots i_m)$ and $\sigma(j_1\dots j_m)$ in the definition (2.1.9) of the operator W with respect to indices i_m and j_m, we obtain

$$(Wf)_{i_1\dots i_mj_1\dots j_m} = \sigma(i_1\dots i_{m-1})\,\sigma(j_1\dots j_{m-1})\sum_{p=0}^{m}(-1)^p\binom{m}{p}\frac{1}{m^2}\Big[h+$$

$$+\ p^2 f_{i_1\dots i_{m-p}j_1\dots j_{p-1}j_m\,;\,j_p\dots j_{m-1}i_{m-p+1}\dots i_m} + (m-p)^2 f_{i_1\dots i_{m-p-1}i_mj_1\dots j_p\,;\,j_{p+1}\dots j_mi_{m-p}\dots i_{m-1}}\Big]$$

where h denotes some tensor symmetric in i_m, j_m. Applying the operator $\alpha(i_mj_m)$ to this equality and taking it into account that the last operator commutes with $\sigma(i_1\dots i_{m-1})$ and $\sigma(j_1\dots j_{m-1})$, we have

$$\alpha(i_mj_m)\,(Wf)_{i_1\dots i_mj_1\dots j_m} = \alpha(i_mj_m)\,\sigma(i_1\dots i_{m-1})\,\sigma(j_1\dots j_{m-1})\sum_{p=0}^{m}(-1)^p\binom{m}{p}\frac{1}{m^2}\times$$

$$\times\Big[(m-p)^2 f_{i_1\dots i_{m-p-1}i_mj_1\dots j_p\,;\,j_{p+1}\dots j_mi_{m-p}\dots i_{m-1}}\ -$$

$$-\ p^2 f_{i_1\dots i_{m-p}i_mj_1\dots j_{p-1}\,;\,j_p\dots j_mi_{m-p+1}\dots i_{m-1}}\Big].$$

Combining the first summand in the brackets of the p-th term of the sum with the second summand of the $(p+1)$-st term, we get (2.4.5). The lemma is proved.

For $f\in C^\infty(S^m)$, we define $Rf\in C^\infty(T^{2m})$ by the equality

$$(Rf)_{i_1j_1\dots i_mj_m} = \alpha(i_1j_1)\dots\alpha(i_mj_m)f_{i_1\dots i_m\,;\,j_1\dots j_m}.$$

This tensor is skew-symmetric with respect to each pair of indices $(i_1,j_1),\dots,(i_m,j_m)$ and symmetric with respect to these pairs. Consequently, $R: C^\infty(S^m)\to C^\infty(S^m(\Lambda^2))$, where $\Lambda^2=\Lambda^2\mathbf{R}^n$ is the exterior square of the space $\mathbf{R}^n$. The next equalities can be proved in the same way as Lemma 2.4.2:

$$(Wf)_{i_1\dots i_mj_1\dots j_m} = \sigma(i_1\dots i_m)\,\sigma(j_1\dots j_m)\,(Rf)_{i_1j_1\dots i_mj_m}, \tag{2.4.6}$$

$$(Rf)_{i_1j_1\dots i_mj_m} = (m+1)\,\alpha(i_1j_1)\dots\alpha(i_mj_m)\,(Wf)_{i_1\dots i_mj_1\dots j_m}. \tag{2.4.7}$$

Relations (2.4.4)–(2.4.7) show that the equalities $Wf=0$, $Vf=0$ and $Rf=0$ are equivalent. Thus V and R can be considered as instances of the Sent Venant operator.

Each of these three operators has its own advantages. For instance, the operator V is more directly related to the integrability conditions for equation (2.2.1), as we shall see later. The operator R has the most of symmetries (which are similar to the symmetries of the curvature tensor [25]) and, consequently, is more appropriate for answering the question about a number of linearly independent equations in the system $Wf = 0$. We have preferred W because it can be generalized more directly to the case of the equation $d^p v = f$ (see Section 2.17).

Lemma 2.4.3 *For every tensor $f \in T^{2m-1}$ $(m \geq 1)$ possessing the symmetry*

$$\operatorname{sym} f_{i_1\dots i_m j_1\dots j_{m-1}} : (i_1 \dots i_m)(j_1 \dots j_{m-1}), \tag{2.4.8}$$

there exists a unique solution to the equation

$$\sigma(i_1 \dots i_m) u_{i_1\dots i_m j_1\dots j_{m-1}} = f_{i_1\dots i_m j_1\dots j_{m-1}} \tag{2.4.9}$$

possessing the symmetry

$$\operatorname{sym} u_{i_1\dots i_m j_1\dots j_{m-1}} : (i_1 \dots i_{m-1})(i_m j_1 \dots j_{m-1}). \tag{2.4.10}$$

This solution is given by the formula

$$u_{i_1\dots i_m j_1\dots j_{m-1}} = \sigma(i_1 \dots i_{m-1})\,\sigma(i_m j_1 \dots j_{m-1}) \sum_{p=1}^{m} (-1)^{p-1} \binom{m}{p} \times$$

$$\times f_{i_1\dots i_{m-p} i_m j_1 \dots j_{m-1} i_{m-p+1}\dots i_{m-1}}. \tag{2.4.11}$$

P r o o f. The dimension of the space formed by tensors possessing the symmetry (2.4.8) is equal to the dimension of the space formed by tensors possessing the symmetry (2.4.10). Consequently, the lemma will be proved if we verify that the tensor u defined by formula (2.4.11) is a solution to equation (2.4.9).

Decomposing the symmetrization $\sigma(i_m j_1 \dots j_{m-1})$ in (2.4.11) with respect to the index i_m and using the equality $\sigma(i_1 \dots i_m)\,\sigma(i_1 \dots i_{m-1}) = \sigma(i_1 \dots i_m)$, we obtain

$$\sigma(i_1 \dots i_m) u_{i_1\dots i_m j_1\dots j_{m-1}} = \frac{1}{m} \sigma(i_1 \dots i_m)\,\sigma(j_1 \dots j_{m-1}) \sum_{p=1}^{m} (-1)^{p-1} \binom{m}{p} \times$$

$$\times \left[p f_{i_1\dots i_{m-p} i_m j_1\dots j_{m-1} i_{m-p+1}\dots i_{m-1}} + (m-p) f_{i_1\dots i_{m-p} j_1\dots j_{m-1} i_m i_{m-p+1}\dots i_{m-1}} \right].$$

Combining the second summand of the brackets of the p-th term of the sum with the first summand of the $(p+1)$-st term and taking (2.4.8) into account, we arrive at the equality

$$\sigma(i_1 \dots i_m) u_{i_1\dots i_m j_1\dots j_{m-1}} = f_{i_1\dots i_m j_1\dots j_{m-1}} + \frac{1}{m} \sigma(j_1 \dots j_{m-1})\,\sigma(i_1 \dots i_m) \sum_{p=2}^{m} (-1)^{p-1} \times$$

$$\times \binom{m}{p} p \left[f_{i_1\dots i_{m-p} i_m j_1\dots j_{m-1} i_{m-p+1}\dots i_{m-1}} - f_{i_1\dots i_{m-p+1} j_1\dots j_{m-1} i_m i_{m-p+2}\dots i_{m-1}} \right].$$

Tensor in the brackets is skew-symmetric with respect to the indices i_{m-p+1}, i_m and, consequently, is nullified by the operator $\sigma(i_1 \dots i_m)$. Deleting it, we get (2.4.9). The lemma is proved.

To prove theorem 2.2.2 we need the next easy

Lemma 2.4.4 *Assume that U is a domain in $\mathbf{R}^n$; $x_0 \in U$; $p \geq 0$ and $q > 0$ are integers; $y \in C^1(T^{p+q}; U)$ has the symmetry*

$$\operatorname{sym} y_{i_1 \dots i_p j_1 \dots j_q} : i_1 \dots i_p (j_1 \dots j_q).$$

For every tensors $z^k = (z^k_{i_1 \dots i_p j_1 \dots j_k}) \in T^{p+k}$ $(k = 0, \dots, q-1)$ the equation

$$\nabla^q z = y \tag{2.4.12}$$

has at most one solution $z \in C^{q+1}(T^p; U)$ satisfying the initial conditions

$$\nabla^k z(x_0) = z^k \qquad (k = 0, 1, \dots, q-1). \tag{2.4.13}$$

For existence of a solution $z \in C^{q+1}(T^p; U)$ to problem (2.4.12)–(2.4.13) *it is necessary that the right-hand side derivative ∇y has the symmetry*

$$\operatorname{sym} y_{i_1 \dots i_p j_1 \dots j_q\, ;\, j_{q+1}} : i_1 \dots i_p (j_1 \dots j_{q+1}), \tag{2.4.14}$$

and the tensors z^k have the symmetries

$$sym\, z^k_{i_1 \dots i_p j_1 \dots j_k} : i_1 \dots i_p (j_1 \dots j_k) \quad (k = 0, 1, \dots, q-1). \tag{2.4.15}$$

If U is simply-connected, then conditions (2.4.14)–(2.4.15) *are sufficient for existence of a solution to problem* (2.4.12)–(2.4.13*). If the right-hand side and initial data have the symmetries*

$$sym\, y_{i_1 \dots i_p j_1 \dots j_q} : (i_1 \dots i_p)(j_1 \dots j_q),$$

$$sym\, z^k_{i_1 \dots i_p j_1 \dots j_k} : (i_1 \dots i_p)(j_1 \dots j_k) \qquad (k = 0, 1, \dots, q-1),$$

then a solution to problem (2.4.12)–(2.4.13) *belongs to $C^{q+1}(S^m; U)$.*

For $p = 0,\ q = 1$ the lemma is equivalent to the claim of necessity and (in the case of a simply-connected U) sufficiency of the conditions $\partial y_i / \partial x^j = \partial y_j / \partial x^i$ for integrability of the Pfaff form $y_i dx^i$. See the proof of this case in [128]. In the same way the claim can be verified for $p \geq 0,\ q = 1$. Now the general case can be easily settled by induction on q.

Now we start proving Theorem 2.2.2.

P r o o f of necessity of conditions (2.2.2), (2.2.4), (2.2.5) for existence a solution to problem (2.2.1), (2.2.3). Let $v \in C^{m+1}(S^{m-1}; U)$ be a solution to problem (2.2.1), (2.2.3). The validity of (2.2.4) follows immediately from (2.2.3). Differentiating (2.2.1) $p-1$ times, we obtain (2.2.5). It remains to prove (2.2.2). To this end, we differentiate (2.2.1) $m-1$ times :

$$\sigma(i_1 \dots i_m) v_{i_1 \dots i_{m-1}\, ;\, i_m j_1 \dots j_{m-1}} = f_{i_1 \dots i_m\, ;\, j_1 \dots j_{m-1}}.$$

By Lemma 2.4.3, it implies that

$$v_{i_1 \dots i_{m-1}\, ;\, i_m j_1 \dots j_{m-1}} = \sigma(i_1 \dots i_{m-1}) \sigma(i_m j_1 \dots j_{m-1}) \sum_{p=1}^{m} (-1)^{p-1} \binom{m}{p} \times$$

$$\times f_{i_1 \dots i_{m-p} i_m j_1 \dots j_{p-1}\, ;\, j_p \dots j_{m-1} i_{m-p+1} \dots i_{m-1}}.$$

Decomposing the operator $\sigma(i_m j_1 \dots j_{m-1})$ on the right-hand side of the last equality with respect to i_m and differentiating the so-obtained relation, we have

$$v_{i_1\dots i_{m-1}\,;\,i_m j_1\dots j_m} = \sigma(i_1 \dots i_{m-1})\sigma(j_1 \dots j_{m-1}) \sum_{p=1}^{m} (-1)^{p-1} \binom{m}{p} \frac{1}{m} \times$$

$$\times \left[p f_{i_1\dots i_{m-p} i_m j_1 \dots j_{p-1}\,;\,j_p \dots j_m i_{m-p+1} \dots i_{m-1}} + (m-p) f_{i_1 \dots i_{m-p} j_1 \dots j_p\,;\,j_{p+1} \dots j_m i_{m-p+1} \dots i_m}\right]. \tag{2.4.16}$$

The tensor $v_{i_1\dots i_{m-1}\,;\,i_m j_1\dots j_m}$ is symmetric with respect to i_m, j_m that can be written as follows:

$$\alpha(i_m j_m) v_{i_1\dots i_{m-1}\,;\,i_m j_1\dots j_m} = 0.$$

Inserting expression (2.4.16) into the last formula, we arrive at the equality

$$\alpha(i_m j_m)\sigma(i_1 \dots i_{m-1})\sigma(j_1 \dots j_{m-1}) g = 0$$

where g denotes the sum on the right-hand side of (2.4.16). Note that all three operators on the left-hand side of the last equality commute; the second summand of the brackets in (2.4.16) is symmetric with respect to i_m, j_m and, consequently, is nullified by the operator $\alpha(i_m j_m)$. Discarding this summand, we arrive at the relation $Vf = 0$ which is equivalent to (2.2.2) by Lemma 2.4.2.

P r o o f of sufficiency of conditions (2.2.2), (2.2.4), (2.2.5) for solvability of problem (2.2.1), (2.2.3). Let U be simply connected and conditions (2.2.2), (2.2.4), (2.2.5) be satisfied. By Lemma 2.4.3, the equation

$$\sigma(i_1 \dots i_m) y_{i_1 \dots i_m j_1 \dots j_{m-1}} = f_{i_1 \dots i_m\,;\,j_1 \dots j_{m-1}} \tag{2.4.17}$$

has a solution given by the formula

$$y_{i_1 \dots i_m j_1 \dots j_{m-1}} = \sigma(i_1 \dots i_{m-1})\sigma(i_m j_1 \dots j_{m-1}) \sum_{p=1}^{m} (-1)^{p-1} \binom{m}{p} \times$$

$$\times f_{i_1 \dots i_{m-p} i_m j_1 \dots j_{p-1}\,;\,j_p \dots j_{m-1} i_{m-p+1} \dots i_{m-1}}. \tag{2.4.18}$$

Consequently, $y \in C^1(T^{2m-1}; U)$. Let us show that the field ∇y has the symmetry

$$\text{sym } y_{i_1 \dots i_m j_1 \dots j_{m-1}\,;\,j_m} : (i_1 \dots i_{m-1})(i_m j_1 \dots j_m). \tag{2.4.19}$$

Indeed, on differentiating (2.4.18) we obtain

$$y_{i_1 \dots i_m j_1 \dots j_{m-1}\,;\,j_m} = \sigma(i_1 \dots i_{m-1})\,\sigma(i_m j_1 \dots j_{m-1}) \sum_{p=1}^{m} (-1)^{p-1} \binom{m}{p} \times$$

$$\times f_{i_1 \dots i_{m-p} i_m j_1 \dots j_{p-1}\,;\,j_p \dots j_m i_{m-p+1} \dots i_{m-1}}. \tag{2.4.20}$$

The derivative ∇y is thus seen to be symmetric with respect to $i_1, \dots, i_{m-1}$ and also with respect to $i_m, j_1, \dots, j_{m-1}$. For this reason, to prove (2.4.19) it suffices to show that

$$\alpha(i_m j_m) y_{i_1 \dots i_m j_1 \dots j_{m-1}\,;\,j_m} = 0. \tag{2.4.21}$$

Inserting the expression (2.4.20) into (2.4.21) and decomposing the operator $\sigma(i_m j_1 \dots j_m)$ in the so-obtained equality with respect to i_m, we see that (2.4.21) is equivalent to the relation $Vf = 0$ and, by Lemma 2.4.2, to relation (2.2.2).

Lemma 2.4.4 and (2.4.19) imply that the problem

$$\nabla^m v = y, \tag{2.4.22}$$

$$\nabla^p v(x_0) = v^p \qquad (p = 0, 1, \dots, m-1) \tag{2.4.23}$$

has a solution $v \in C^{m+1}(S^{m-1}; U)$. Inserting (2.4.22) into (2.4.17), we obtain

$$\sigma(i_1 \dots i_m) v_{i_1 \dots i_{m-1}\,;\, i_m j_1 \dots j_{m-1}} = f_{i_1 \dots i_m\,;\, j_1 \dots j_{m-1}}. \tag{2.4.24}$$

Let us denote

$$z = dv, \tag{2.4.25}$$

and rewrite (2.4.24) as follows:

$$\nabla^{m-1} z = \nabla^{m-1} f. \tag{2.4.26}$$

It follows from (2.4.23) and (2.2.5) that

$$\nabla^p z(x_0) = \nabla^p f(x_0) \qquad (p = 0, 1, \dots, m-2). \tag{2.4.27}$$

Applying lemma 2.4.4, from (2.4.26)–(2.4.27) we obtain $z \equiv f$. By (2.4.25), this is equivalent to (2.2.1). Finally, initial conditions (2.2.3) are satisfied by (2.4.23).

P r o o f of uniqueness of a solution to problem (2.2.1), (2.2.3). Let $v \in C^{m+1}(S^{m-1}; U)$ be a solution of the corresponding homogeneous problem

$$dv = 0, \tag{2.4.28}$$

$$\nabla^p v(x_0) = 0 \qquad (p = 0, 1, \dots, m-1). \tag{2.4.29}$$

Differentiating (2.4.28) $m-1$ times , we have

$$\sigma(i_1 \dots i_m) v_{i_1 \dots i_{m-1}\,;\, i_m j_1 \dots j_{m-1}} = 0.$$

By Lemma 2.4.3, it implies that $\nabla^m v = 0$. Using the last equality, (2.4.29) and Lemma 2.4.4, we obtain $v = 0$.

2.5 The ray transform of a field-distribution

Let $\mathcal{D}(\mathbf{R}^n \times \Omega)$ be the space of smooth compactly-supported functions on $\mathbf{R}^n \times \Omega$ and $\mathcal{D}'(\mathbf{R}^n \times \Omega)$ be the space of distributions on $\mathbf{R}^n \times \Omega$. We assume that $L_{1,\mathrm{loc}}(\mathbf{R}^n \times \Omega)$ is included in $\mathcal{D}'(\mathbf{R}^n \times \Omega)$, identifying a locally integrable function $h(x, \xi)$ with the functional on $\mathcal{D}(\mathbf{R}^n \times \Omega)$ defined by the formula

$$\langle h, \varphi \rangle = \int\limits_{\mathbf{R}^n} \int\limits_{\Omega} h(x, \xi)\, \varphi(x, \xi)\, d\omega(\xi)\, dV^n(x) \qquad (\varphi \in \mathcal{D}(\mathbf{R}^n \times \Omega)), \tag{2.5.1}$$

where $d\omega$ is the angle measure on Ω.

To construct an extension $I : \mathcal{E}'(S^m) \to \mathcal{D}'(\mathbf{R}^n \times \Omega)$ of operator (2.1.10) we note that, for a compactly-supported field $f \in L_{1,\mathrm{loc}}(S^m)$, the field $h = If$ belongs to $L_{1,\mathrm{loc}}(\mathbf{R}^n \times \Omega)$. Consequently, according to (2.5.1) for $\varphi \in \mathcal{D}(\mathbf{R}^n \times \Omega)$,

$$\langle If, \varphi\rangle = \int\limits_{\mathbf{R}^n}\int\limits_{\Omega}\int\limits_{-\infty}^{\infty} \langle f(x + t\xi), \xi^m\rangle\, \varphi(x, \xi)\, dt\; d\omega(\xi)\, dV^n(x).$$

After a simple transformation this expression takes the form:

$$\langle If, \varphi\rangle = \int\limits_{\mathbf{R}^n} f_{i_1\dots i_m}(x) \left[\int\limits_{\Omega} \xi^{i_1}\dots\xi^{i_m} \int\limits_{-\infty}^{\infty} \varphi(x - t\xi, \xi)\, dt\, d\omega(\xi)\right] dV^n(x). \tag{2.5.2}$$

Defining the field $J\varphi \in \mathcal{E}(S^m)$ by the formula

$$(J\varphi)_{i_1\dots i_m} = \int\limits_{\Omega} \xi_{i_1}\dots\xi_{i_m} \left[\int\limits_{-\infty}^{\infty} \varphi(x - t\xi, \xi)\, dt\right] d\omega(\xi), \tag{2.5.3}$$

we can rewrite equality (2.5.2) as follows:

$$\langle If, \varphi\rangle = \int\limits_{\mathbf{R}^n} \langle f(x), \overline{J\varphi(x)}\rangle\, dV^n(x).$$

Comparing this with (2.1.7), we have $\langle If, \varphi\rangle = \langle f, J\varphi\rangle$. The last equality proved for a compactly-supported $f \in L_{1,\mathrm{loc}}(S^m)$, will be taken as grounds for the following definition.

The operator $I : \mathcal{E}'(S^m) \to \mathcal{D}'(\mathbf{R}^n \times \Omega)$ is given by the formula $\langle IF, \varphi\rangle = \langle F, J\varphi\rangle$ where $F \in \mathcal{E}'(S^m)$, $\varphi \in \mathcal{D}(\mathbf{R}^n \times \Omega)$, and $J : \mathcal{D}(\mathbf{R}^n \times \Omega) \to \mathcal{E}(S^m)$ is the continuous operator defined by (2.5.3).

We can now formulate an analog of Theorem 2.2.1 for tensor field-distributions.

Theorem 2.5.1 *For a field $F \in \mathcal{E}'(S^m)$ the next three claims are equivalent:*

(1) $IF = 0$;

(2) *there exists a field $V \in \mathcal{E}'(S^{m-1})$ such that its support is contained in the convex hull of the support of F and $dV = F$;*

(3) $WF = 0$.

For $s \in \mathbf{R}$ we denote by $\mathcal{H}^s(S^m) = \mathcal{H}^s \otimes_{\mathbf{C}} S^m$ the Hilbert space obtained by completing $\mathcal{S}(S^m)$ with respect to the norm $\|f(x)\|_s = \|(1 + |y|)^s \hat f(y)\|_{L_2(S^m)}$ where $f(x) \mapsto \hat f(y)$ is the Fourier transform. Given a compact set $K \subset \mathbf{R}^n$, by $\mathcal{K}^s_K(S^m)$ we mean the subspace of $\mathcal{H}^s(S^m)$ formed by fields whose supports are contained in K. As a differential operator, $d : \mathcal{H}^s(S^m) \to \mathcal{H}^{s-1}(S^{m+1})$ is continuous and so is the Saint Venant operator $W : \mathcal{H}^s(S^m) \to \mathcal{H}^{s-m}(S^m \otimes S^m)$.

We fix a function $\beta \in C_0^\infty(\mathbf{R}^n)$ such that $\beta \ge 0$, $\int_{\mathbf{R}^n} \beta dx = 1$, and $\operatorname{supp}\beta \subset \{x \mid |x| \le 1\}$. For $\varepsilon > 0$, we put $\beta_\varepsilon(x) = \beta(x/\varepsilon)/\varepsilon^n$. For a field-distribution $T \in \mathcal{D}'(S^m)$ the convolution $T * \beta_\varepsilon \in \mathcal{E}(S^m)$ is defined. We need its properties that are listed in the next

Lemma 2.5.2 *The following are valid:*

(1) *if* T *has compact support, then so is* $T*\beta_\varepsilon$; *and if* $K = \mathrm{supp}T$, *then* $\mathrm{supp}(T*\beta_\varepsilon) \subset \{x \mid \rho(x,K) \le \varepsilon\}$ *where* ρ *is the Euclidean distance in* $\mathbf{R}^n$;

(2) *if* $T \in \mathcal{E}'(S^m)$, *then* $T*\beta_\varepsilon$ *converges to* T *in the space* $\mathcal{E}'(S^m)$ *as* $\varepsilon \to 0$;

(3) *if* $T \in \mathcal{H}^s(S^m)$, *then* $T*\beta_\varepsilon$ *converges to* T *in the space* $\mathcal{H}^s(S^m)$ *as* $\varepsilon \to 0$, *for every* $s \in \mathbf{R}$;

(4) *the operator* $T \mapsto T*\beta_\varepsilon$ *commutes with* d *and* W;

(5) *the operator* $T \mapsto T*\beta_\varepsilon$ *commutes with* I *in the following sense: for* $T \in \mathcal{E}'(S^m)$

$$I(T*\beta_\varepsilon) = (IT) \overset{x}{*} \beta_\varepsilon$$

where $\overset{x}{*}$ *denotes the partial convolution in* $\mathbf{R}^n \times \Omega$ *in the space variable* $x \in \mathbf{R}^n$ *and that does not use the angle variable* $\xi \in \Omega$;

(6) *if* $T \in \mathcal{D}'(\mathbf{R}^n \times \Omega)$, *then* $T \overset{x}{*} \beta_\varepsilon$ *converges to* T *in the space* $\mathcal{D}'(\mathbf{R}^n \times \Omega)$ *as* $\varepsilon \to 0$.

P r o o f. The first three claims and the last one are well-known properties of convolution (compare with Proposition 2.10 in [111]). The fourth is a special case of permutability of convolution with any constant coefficient differential operator. To prove the fifth claim, we note that definition (2.5.3) implies the equality $\beta_\varepsilon * J\varphi = J(\beta_\varepsilon \overset{x}{*} \varphi)$ that is valid for $\varphi \in \mathcal{D}(\mathbf{R}^n \times \Omega)$. From the equality, by putting $\tilde\beta_\varepsilon(x) = \beta_\varepsilon(-x)$, for $T \in \mathcal{E}'(S^m)$, we obtain

$$\langle I(T*\beta_\varepsilon), \varphi\rangle = \langle T*\beta_\varepsilon, J\varphi\rangle = \langle T, \tilde\beta_\varepsilon * J\varphi\rangle =$$

$$= \langle T, J(\tilde\beta_\varepsilon \overset{x}{*} \varphi)\rangle = \langle IT, \tilde\beta_\varepsilon \overset{x}{*} \varphi\rangle = \langle (IT) \overset{x}{*} \beta_\varepsilon, \varphi\rangle.$$

Let us begin proving Theorem 2.5.1.

P r o o f of the implication (1) → (2). Let claim (1) be valid for $F \in \mathcal{E}'(S^m)$, and $K' = \mathrm{supp}\, F$. For every $\nu = 1, 2, \dots$, we denote the convex hull of $\{x \mid \rho(x, K') \le 1/\nu\}$ by K_ν and put $K = K_1$.

Having a compact support, the field F has finite order (see [36], Chapter II), and, consequently, $F \in \mathcal{K}^s_K(S^m)$ for some $s \in \mathbf{R}$. By Lemma 2.5.2, the fields $f_\nu = F * \beta_{1/\nu}$ have the next properties:

$$\mathrm{supp} f_\nu \subset K_\nu, \tag{2.5.4}$$

$$f_\nu \to F \quad \text{in} \quad \mathcal{K}^s_K(S^m) \quad \text{as} \quad \nu \to \infty, \tag{2.5.5}$$

$$If_\nu = 0. \tag{2.5.6}$$

For any $\nu = 1, 2, \dots$, applying Theorem 2.2.1 to f_ν, we construct a compactly-supported field $v_\nu \in C^{m+1}(S^{m-1})$ such that

$$dv_\nu = f_\nu, \tag{2.5.7}$$

$$\mathrm{supp}\, v_\nu \subset K_\nu \subset K. \tag{2.5.8}$$

The operator d is elliptic in the sense of the definition given in [111]. Indeed, its symbol $\sigma_1(d)$ is easily seen to be given by the formula $\sigma_1(d)(\xi, u) = i_\xi u$ for $\xi \in \mathbf{R}^n,\ u \in S^m$. By the direct calculation in coordinates, the equality $(m+1)j_\xi i_\xi = |\xi|^2 E + m i_\xi j_\xi$ can be proved (E is the identity operator). Later we shall prove more general formula; see Lemma 3.3.3. Since the operators i_ξ and j_ξ are dual to each other, the last equality implies that $j_\xi i_\xi u \ne 0$ for $\xi \ne 0,\ u \ne 0$. Consequently, by Theorem 3.1 of [111], the estimate $\|v_\nu - v_\mu\|_{s+1} \le C\|d(v_\nu - v_\mu)\|_s$ is valid with a constant C independent of ν, μ. From this estimate with help of (2.5.5) and (2.5.7), we see that the sequence v_ν is a Cauchy sequence in $\mathcal{K}^{s+1}_K(S^{m-1})$. Thus it converges in this space and, consequently, in $\mathcal{E}'(S^{m-1})$ as well. Let $V \in \mathcal{E}'(S^{m-1})$ be the limit of the sequence v_ν. It follows from (2.5.8) that $\operatorname{supp} V$ is contained in $\bigcap_\nu K_\nu$, i.e., in the convex hull of the support of F. Taking the limit in (2.5.7) as $\nu \to \infty$, we obtain $dV = F$.

P r o o f of the implication (2) $\to$ (3). Let the equality $dV = F$ be valid for $V \in \mathcal{E}'(S^{m-1}),\ F \in \mathcal{E}'(S^m)$. We find a sequence $v_\nu \in \mathcal{D}(S^{m-1})\ \ (\nu = 1, 2, \dots)$ converging to V in $\mathcal{E}'(S^{m-1})$. Then the sequence $f_\nu = dv_\nu$ converges to F in $\mathcal{E}'(S^m)$. Applying Theorem 2.2.1 to f_ν, we obtain $Wf_\nu = 0$. Taking the limit in the last equality as $\nu \to \infty$, we get the one desired: $WF = 0$.

P r o o f of the implication (3) $\to$ (1). Let the equality $WF = 0$ is valid for $F \in \mathcal{E}'(S^m)$. As above, we construct a sequence $f_\nu = F * \beta_{1/\nu}$. By Lemma 2.5.2, f_ν converges to F in $\mathcal{E}'(S^m)$ as $\nu \to \infty$, and $Wf_\nu = 0$. Applying Theorem 2.2.1 to f_ν, we verify validity of (2.5.6). Transforming this equality in accordance with the fifth claim of Lemma 2.5.2, we get $IF \overset{x}{*} \beta_{1/\nu} = 0$. Taking the limit in the last equality as $\nu \to \infty$ and using the sixth claim of Lemma 2.5.2, we arrive at $IF = 0$.

2.6 Decomposition of a tensor field into potential and solenoidal parts

A field $f \in C^\infty(S^m)$ is called *potential* if it can be represented in the form $f = dv$ for $v \in C^\infty(S^{m-1})$ such that $v(x) \to 0$ as $|x| \to \infty$. If f is potential, then $If = 0$ as we have seen above. Thus if we can find a subspace $E \subset C^\infty(S^m)$ which does not intersect the space of potential fields and whose sum with the latter equals $C^\infty(S^m)$, then the operator I is uniquely determined by its restriction to E. Since the dual operator of d is $-\delta$, we may naturally take as E the space of solenoidal tensor fields , i.e., the space of fields satisfying the equation $\delta f = 0$.

It is well known that a vector field can be decomposed into the sum of solenoidal and potential fields. If the fields are considered in a bounded domain, then for uniqueness of such decomposition we have to require that the potential vanishes on the boundary of the domain. If the fields are defined on the whole $\mathbf{R}^n$, then we have to require that both summands tend to zero at infinity. It turns out that the last fact is valid for symmetric tensor fields of arbitrary degree. In this section we shall establish it for fields considered on the whole $\mathbf{R}^n$. To prove it we shall use the method that is based on the next observation: after application of the Fourier transform, the relations $f = {}^s\!f + dv,\ \delta\, {}^s\!f = 0$ give the equations $\hat f(y) = \widehat{{}^s\!f}(y) + \sqrt{-1} i_y \hat v(y),\ \ j_y \widehat{{}^s\!f}(y) = 0$ which can be easily solved with respect to $\widehat{{}^s\!f}(y)$ and $\hat v(y)$.

We recall that S^1 is identified with $\mathbf{C}^n$ and, consequently, $\mathbf{R}^n \subset S^1$. For $x \in \mathbf{R}^n$, let $i_x : S^m \to S^{m+1}$ and $j_x : S^m \to S^{m-1}$ denote the operator of symmetric multiplication by x and that of convolution with x.

Lemma 2.6.1 *For every $f \in S^m$ and $0 \neq x \in \mathbf{R}^n$, there exist uniquely determined ${}^t\!f \in S^m$ and $v \in S^{m-1}$ such that*

$$f = {}^t\!f + i_x v, \qquad j_x\, {}^t\!f = 0. \tag{2.6.1}$$

The tensor ${}^tf$ *is expressed through* f *and* x *according to the formula*

$$ {}^tf_{i_1\dots i_m} = \lambda^{j_1\dots j_m}_{i_1\dots i_m}(x) f_{j_1\dots j_m} \qquad (2.6.2)$$

in which

$$\lambda^{j_1\dots j_m}_{i_1\dots i_m}(x) = \left(\delta^{j_1}_{i_1} - \frac{x_{i_1}x^{j_1}}{|x|^2}\right)\dots\left(\delta^{j_m}_{i_m} - \frac{x_{i_m}x^{j_m}}{|x|^2}\right), \qquad (2.6.3)$$

and δ^j_i *is the Kronecker symbol. If* $\xi \in \mathbf{R}^n$ *and* $\langle x, \xi\rangle = 0$, *then*

$$\langle {}^tf, \xi^m\rangle = \langle f, \xi^m\rangle \qquad (\langle x, \xi\rangle = 0). \qquad (2.6.4)$$

P r o o f. The operators i_x and j_x are dual to one other. Consequently, the orthogonal decomposition $S^m = \operatorname{Ker} j_x \oplus \operatorname{Im} i_x$ is valid. This implies existence and uniqueness of ${}^tf$ and v satisfying (2.6.1). The validity of (2.6.2) will be clear after opening the parentheses on the right-hand side of (2.6.3) and inserting the so-obtained expression into (2.6.2). Relation (2.6.4) follows from (2.6.2) and (2.6.3). The lemma is proved.

For a field $f \in C^\infty(S^m; \mathbf{R}^n_0)$, if decomposition (2.6.1) takes place for all $x \in \mathbf{R}^n_0$, i.e.,

$$f(x) = {}^tf(x) + i_x v(x), \qquad j_x\, {}^tf(x) = 0, \qquad (2.6.5)$$

then ${}^tf \in C^\infty(S^m; \mathbf{R}^n_0)$ and $v \in C^\infty(S^{m-1}; \mathbf{R}^n_0)$, as one can see from (2.6.2). We shall call the fields ${}^tf(x)$ and $i_x v(x)$ the *tangential* and *radial* components of the field $f(x)$. The terms are given because, in the case $m = 1$, the vector ${}^tf(x)$ is tangent to the sphere $|x| = \text{const}$, and a vector $i_x v(x)$ is parallel to x.

Theorem 2.6.2 *Let* $n \ge 2$. *For every field* $f \in \mathcal{S}(S^m; \mathbf{R}^n)$ *there exist uniquely-determined fields* ${}^sf \in C^\infty(S^m)$ *and* $v \in C^\infty(S^{m-1})$ *such that*

$$f = {}^sf + dv, \qquad \delta\, {}^sf = 0, \qquad (2.6.6)$$

$${}^sf(x) \to 0, \quad v(x) \to 0 \quad \textit{as} \quad |x| \to \infty. \qquad (2.6.7)$$

These fields satisfy the estimates

$$|{}^sf(x)| \le C(1+|x|)^{1-n}, \quad |v(x)| \le C(1+|x|)^{2-n}, \quad |dv(x)| \le C(1+|x|)^{1-n}. \qquad (2.6.8)$$

The fields ${}^sf$ *and* dv *belong to* $L_2(S^m)$. *The Fourier transform* $\widehat{{}^sf}(y)$ *of the field* ${}^sf(x)$ *belongs to* $C^\infty(S^m; \mathbf{R}^n_0)$, *is bounded on* $\mathbf{R}^n$ *and decreases rapidly as* $|y| \to \infty$.

P r o o f of existence. Let $\hat f(y)$ be the Fourier transform of the field $f(x)$. By Lemma 2.6.1, for every $y \ne 0$, there exist uniquely-determined ${}^t\hat f(y) \in S^m$ and $\hat v(y) \in S^{m-1}$ such that

$$\hat f(y) = {}^t\hat f(y) + i_y \hat v(y), \qquad j_y\, {}^t\hat f(y) = 0. \qquad (2.6.9)$$

Starting with (2.6.2) and (2.6.3), by induction on $|\alpha|$, we verify validity of the representations

$$D^\alpha\, {}^t\hat f_{i_1\dots i_m}(y) = |y|^{-2|\alpha|-2m} \sum_{|\beta|\le|\alpha|} P^{\alpha j_1\dots j_m}_{\beta i_1\dots i_m}(y) D^\beta \hat f_{j_1\dots j_m}(y),$$

$$D^\alpha \hat v_{i_1\dots i_{m-1}}(y) = |y|^{-2|\alpha|-2m} \sum_{|\beta|\le|\alpha|} Q^{\alpha j_1\dots j_m}_{\beta i_1\dots i_{m-1}}(y) D^\beta \hat f_{j_1\dots j_m}(y)$$

in which $P^{\alpha j_1\dots j_m}_{\beta i_1\dots i_m}$ ($Q^{\alpha j_1\dots j_m}_{\beta i_1\dots i_{m-1}}$) are homogeneous polynomials of degree $2m+|\alpha|+|\beta|$ ($2m+|\alpha|+|\beta|-1$). We use the ordinary notation: $D_j=-i\partial/\partial x^j$ and $D^\alpha=D_1^{\alpha_1}\dots D_n^{\alpha_n}$ for a multi-index $\alpha=(\alpha_1\dots\alpha_n)$. These representations imply that the fields ${}^t\hat f(y)$, $\hat v(y)$ are smooth on $\mathbf{R}^n_0$, decrease rapidly as $|y|\to\infty$, and satisfy the estimates

$$|D^\alpha\, {}^t\hat f(y)|\le|y|^{-|\alpha|},\qquad |D^\alpha\hat v(y)|\le|y|^{-|\alpha|-1} \tag{2.6.10}$$

for $|y|\le1$. Consequently, ${}^t\hat f,\ i_y\hat v\in L_2(S^m)$. It follows from (2.6.10) that the field $D^\alpha\,{}^t\hat f$ is summable on $\mathbf{R}^n$ for $|\alpha|\le n-1$, and $D^\alpha\hat v$ is summable on $\mathbf{R}^n$ for $|\alpha|\le n-2$.

Let ${}^sf(x)=\overline{F}[{}^t\hat f(y)]$ and $v(x)=-\sqrt{-1}\,\overline{F}[\hat v(y)]$, where $\overline{F}$ is the inverse Fourier transform. Now ${}^sf\in C^\infty(S^m)$, $v\in C^\infty(S^{m-1})$ since ${}^t\hat f(y)$ and $\hat v(y)$ are rapidly decreasing. Applying $\overline{F}$ to (2.6.9), we obtain (2.6.6). Relations (2.6.7) are satisfied since ${}^t\hat f$ and $\hat v$ are summable on $\mathbf{R}^n$. Estimates (2.6.8) follow from the above established summability of $D^\alpha\,{}^t\hat f$ and $D^\alpha\hat v$. Finally, by the Fourier-Plancherel theorem, the containment ${}^t\hat f,\ i_y\hat v\in L_2(S^m)$ implies that ${}^sf,\ dv\in L_2(S^m)$.

P r o o f of uniqueness. Let ${}^sf\in C^\infty(S^m)$ and $v\in C^\infty(S^{m-1})$ satisfy (2.6.7) and relations ${}^sf+dv=0$, $\delta\,{}^sf=0$. In particular, ${}^sf\in\mathcal{S}'(S^m)$, $v\in\mathcal{S}'(S^{m-1})$ where $\mathcal{S}'$ is the space of tempered distributions on $\mathbf{R}^n$. Thus the Fourier images $\widehat{{}^sf}\in\mathcal{S}'(S^m)$ and $\hat v\in\mathcal{S}'(S^{m-1})$ are defined. Applying the Fourier transform to the equalities ${}^sf+dv=0$ and $\delta\,{}^sf=0$, we obtain $\widehat{{}^sf}(y)+\sqrt{-1}i_y\hat v(y)=0$, $j_y\,\widehat{{}^sf}(y)=0$. From this, by Lemma 2.6.1, we see that $\widehat{{}^sf}|_{\mathbf{R}^n_0}=\hat v|_{\mathbf{R}^n_0}=0$, i.e., the supports of these distributions are concentrated at zero. Consequently, each of their components is a finite linear combination of the derivatives of the δ-function. With the help of (2.6.7), we conclude from this that ${}^sf=v=0$. The theorem is proved.

The fields ${}^sf$ and dv satisfying (2.6.6) and (2.6.7) will be called the *solenoidal* and *potential* parts of the field f.

We shall need one more version of the decomposition theorem. To formulate it we first give two definitions.

Let $K(S^m)$ be the closure, in $L_2(S^m;\mathbf{R}^n)$, of the set of the fields $f\in C^\infty(S^m)\cap L_2(S^m)$ that satisfy the equation $\delta f=0$ and the estimate $|f(x)|\le C(1+|x|)^{1-n}$. Let $D(S^m)$ be the closure, in $L_2(S^m)$, of the set of the fields $f\in L_2(S^m)$ that can be represented in the form $f=dv$ for some $v\in C^\infty(S^{m-1})$ satisfying the condition: $v(x)\to0$ as $|x|\to\infty$.

Theorem 2.6.3 *We have the orthogonal decomposition*

$$L_2(S^m)=K(S^m)\oplus D(S^m). \tag{2.6.11}$$

P r o o f. First of all we show that the summands of decomposition (2.6.11) are orthogonal to one other. Let $f\in K(S^m)$, $g\in D(S^m)$. There are fields $f_k\in C^\infty(S^m)\cap L_2(S^m)$ and $v_k\in C^\infty(S^{m-1})$ ($k=1,2,\dots$) such that $\delta f_k=0$, $dv_k\in L_2(S^m)$,

$$|f_k(x)|\le C_k(1+|x|)^{1-n},\quad v_k(x)\to0\quad\text{as}\quad |x|\to\infty \tag{2.6.12}$$

and $f_k\to f$, $dv_k\to g$ in $L_2(S^m)$ as $k\to\infty$. Inserting $u=v_k$, $v=f_k$ and $G=\{x\mid |x|\le A\}$ into the Green's formula (2.1.8), we obtain

$$\int\limits_{|x|\le A}\langle f_k,dv_k\rangle\,dx=\int\limits_{|x|=A}\langle f_k,i_\nu v_k\rangle\,d\sigma.$$

The right-hand side of this equality tends to zero as $A \to \infty$, which is easy from (2.6.12). Consequently, $(f_k, dv_k) = 0$. Passing to the limit in k in the last equality, we obtain $(f, g) = 0$.

For $f \in L_2(S^m)$, choose a sequence $f_k \in \mathcal{S}(S^m)$ that converges to f in $L_2(S^m)$. By Theorem 2.6.2, there is a representation $f_k = {}^s\!f_k + dv_k$, such that ${}^s\!f_k \in C^\infty(S^m) \cap L_2(S^m)$, $\delta\, {}^s\!f_k = 0$, $v_k \in C^\infty(S^{m-1})$, $dv_k \in L_2(S^m)$ and (2.6.12) is satisfied. For every k and l, $f_k - f_l = ({}^s\!f_k - {}^s\!f_l) + d(v_k - v_l)$. The summands on the right-hand side of this equality are orthogonal to each other and, consequently, $\|f_k - f_l\|^2 = \|{}^s\!f_k - {}^s\!f_l\|^2 + \|dv_k - dv_l\|^2$. Since the sequence f_k is Cauchy, the last equality implies that ${}^s\!f_k$ and dv_k are also Cauchy. Consequently, they converge to some ${}^s\!f,\ h \in L_2(S^m)$. The above listed properties of ${}^s\!f_k$ and v_k imply that ${}^s\!f \in K(S^m),\ h \in D(S^m)$. Passing to the limit in the equality $f_k = {}^s\!f_k + dv_k$ with respect to k, we obtain $f = {}^s\!f + h$. The theorem is proved.

2.7 A theorem on the tangent component

From (2.6.2), one can see that the tangent component of a smooth field has, as a rule, a singularity at the point $x = 0$. In this respect the question arises: assume that a tensor field h is smooth on $\mathbf{R}^n_0$ and satisfies the condition $j_x h(x) = 0$; under which supplementary conditions on h does there exist a field f smooth on the whole $\mathbf{R}^n$ and such that its tangent component is h? The question is answered by

Theorem 2.7.1 *For a field $h \in C^\infty(S^m; \mathbf{R}^n_0)$ $(n \geq 3)$ to be the tangent component of some $f \in C^\infty(S^m; \mathbf{R}^n)$, it is necessary and sufficient that h satisfies the relation*

$$j_x h(x) = 0 \tag{2.7.1}$$

and there exists a function $\varphi \in C^\infty(T\Omega)$ connected with h by the equality

$$\langle h(x), \xi^m \rangle = \varphi(x, \xi) \qquad ((x, \xi) \in T\Omega;\ x \neq 0). \tag{2.7.2}$$

Necessity follows from (2.6.4). Beginning the proof of sufficiency, we extend φ to $\mathbf{R}^n \times \mathbf{R}^n_0$ by the equality

$$\varphi(x, \xi) = |\xi|^m \varphi(P_\xi x, \xi/|\xi|) \qquad ((x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n_0) \tag{2.7.3}$$

where $P_\xi x = x - \langle x, \xi \rangle \xi / |\xi|^2$ is the orthogonal projection of $\mathbf{R}^n$ onto $\xi^\perp$. Then φ is in $C^\infty(\mathbf{R}^n \times \mathbf{R}^n_0)$ and satisfies the relations

$$\varphi(x, t\xi) = t^m \varphi(x, \xi) \qquad ((x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n_0;\ t > 0), \tag{2.7.4}$$

$$\varphi(x + t\xi, \xi) = \varphi(x, \xi) \qquad ((x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n_0;\ t \in \mathbf{R}), \tag{2.7.5}$$

$$\xi^i \frac{\partial \varphi}{\partial x^i} = 0 \qquad ((x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n_0), \tag{2.7.6}$$

$$\langle h(x), \xi^m \rangle = \varphi(x, \xi) \qquad ((x, \xi) \in \mathbf{R}^n_0 \times \mathbf{R}^n_0;\ \langle x, \xi \rangle = 0). \tag{2.7.7}$$

Equality (2.7.6) follows from (2.7.5) by differentiation with respect to t.

Let $x, \xi \in \mathbf{R}^n_0,\ x \neq t\xi\ (t \in \mathbf{R})$; then the representation $\xi = P_x\xi + tx$ is possible with $P_x\xi \neq 0$. By (2.7.1) and (2.7.7),

$$\langle h(x), \xi^m\rangle = \langle h(x), (P_x\xi)^m\rangle = \varphi(x, P_x\xi) \qquad (x, \xi \in \mathbf{R}^n_0;\ x \neq t\xi). \tag{2.7.8}$$

Differentiating this equality m times with respect to ξ and putting then $\langle x, \xi\rangle = 0$, we obtain

$$h_{i_1\dots i_m}(x) = \lambda^{j_1\dots j_m}_{i_1\dots i_m}(x)\frac{\partial^m\varphi}{\partial\xi^{j_1}\dots\partial\xi^{j_m}}(x,\xi) \qquad ((x,\xi) \in \mathbf{R}^n_0 \times \mathbf{R}^n_0; \langle x, \xi\rangle = 0). \tag{2.7.9}$$

Note that (2.6.3) implies the relations

$$x^{i_1}\lambda^{j_1\dots j_m}_{i_1\dots i_m}(x) = x_{j_1}\lambda^{j_1\dots j_m}_{i_1\dots i_m}(x) = 0, \tag{2.7.10}$$

$$y^{i_1}\dots y^{i_m}\lambda^{j_1\dots j_m}_{i_1\dots i_m}(x) = y^{j_1}\dots y^{j_m} \qquad (\langle x, y\rangle = 0). \tag{2.7.11}$$

Lemma 2.7.2 *Under the assumptions of Theorem* 2.7.1, *there exists a sequence of tensors* $c^l = (c^l_{i_1\dots i_m j_1\dots j_l}) \in S^m \otimes S^l$ $(l = 0, 1, \dots)$, *such that the relations*

$$h_{i_1\dots i_m}(x) = \lambda^{p_1\dots p_m}_{i_1\dots i_m}(x)\sum_{s=0}^{l} c^s_{p_1\dots p_m j_1\dots j_s}x^{j_1}\dots x^{j_s} + O(|x|^{l+1}) \qquad (x \to 0), \tag{2.7.12}$$

$$\frac{1}{l!}\frac{\partial^l\varphi}{\partial x^{j_1}\dots\partial x^{j_l}}(0,\xi) = \lambda^{q_1\dots q_l}_{j_1\dots j_l}(\xi)c^l_{i_1\dots i_m q_1\dots q_l}\xi^{i_1}\dots\xi^{i_m} \qquad (\xi \in \Omega), \tag{2.7.13}$$

$$\lambda^{p_1\dots p_m}_{i_1\dots i_m}(\eta)\left[\frac{1}{l!}\frac{\partial^{m+l}\varphi}{\partial\xi^{p_1}\dots\partial\xi^{p_m}\partial x^{j_1}\dots\partial x^{j_l}}(0,\xi) - c^l_{p_1\dots p_m j_1\dots j_l}\right]\eta^{j_1}\dots\eta^{j_l} = 0$$
$$(\xi, \eta \in \Omega;\ \langle\xi, \eta\rangle = 0) \tag{2.7.14}$$

are valid for any $l \geq 0$.

P r o o f. We shall construct tensors c^l by induction on l. To avoid considering the case $l = 0$, we note that the claim of the lemma makes sense for $l = -1$ as well. In the last case the lemma asserts only that $h(x) = O(1)$. From (2.7.2), one can easily see that $h(x)$ is bounded for $|x| \leq 1$. Assuming the validity of the lemma for $l = -1, 0, \dots, k-1 \geq -1$, we expand $\varphi(x, \xi)$ into the Taylor series in x :

$$\varphi(x,\xi) = \sum_{s=0}^{k}\frac{1}{s!}\frac{\partial^s\varphi}{\partial x^{j_1}\dots\partial x^{j_s}}(0,\xi)x^{j_1}\dots x^{j_s} + O(|x|^{k+1}) \qquad (\xi \in \Omega).$$

Using the inductive hypothesis (2.7.13) on the derivatives of order less than k, we transform the last equality to the form

$$\varphi(x,\xi) = \xi^{i_1}\dots\xi^{i_m}\sum_{s=0}^{k-1}\lambda^{q_1\dots q_s}_{j_1\dots j_s}(\xi)c^s_{i_1\dots i_m q_1\dots q_s}x^{j_1}\dots x^{j_s} + b^k_{j_1\dots j_k}(\xi)x^{j_1}\dots x^{j_k} + O(|x|^{k+1})$$
$$(\xi \in \Omega), \tag{2.7.15}$$

where

$$b^k_{j_1\dots j_k}(\xi) = \frac{1}{k!}\frac{\partial^k\varphi}{\partial x^{j_1}\dots\partial x^{j_k}}(0,\xi) \qquad (\xi \in \Omega). \tag{2.7.16}$$

By (2.7.11), if $\langle x,\xi\rangle = 0$, then equality (2.7.15) looks like

$$\varphi(x,\xi) = \xi^{i_1}\dots\xi^{i_m}\sum_{s=0}^{k-1} c^s_{i_1\dots i_m j_1\dots j_s} x^{j_1}\dots x^{j_s} + b^k_{j_1\dots j_k}(\xi)\, x^{j_1}\dots x^{j_k} + O(|x|^{k+1}) \qquad ((x,\xi)\in T\Omega). \tag{2.7.17}$$

For $(x,\xi)\in T\Omega$, we replace the right-hand side of (2.7.8) with (2.7.17) and obtain

$$\left[h_{i_1\dots i_m}(x) - \sum_{s=0}^{k-1} c^s_{i_1\dots i_m j_1\dots j_s} x^{j_1}\dots x^{j_s}\right]\xi^{i_1}\dots\xi^{i_m} = b^k_{j_1\dots j_k}(\xi) x^{j_1}\dots x^{j_k} + O(|x|^{k+1}) \qquad ((x,\xi)\in T\Omega;\ x\neq 0). \tag{2.7.18}$$

Using (2.7.11), the last formula can be rewritten as follows:

$$\left[h_{i_1\dots i_m}(x) - \lambda^{p_1\dots p_m}_{i_1\dots i_m}(x)\sum_{s=0}^{k-1} c^s_{p_1\dots p_m j_1\dots j_s} x^{j_1}\dots x^{j_s}\right]\xi^{i_1}\dots\xi^{i_m} = b^k_{j_1\dots j_k}(\xi) x^{j_1}\dots x^{j_k} + O(|x|^{k+1}) \qquad ((x,\xi)\in T\Omega;\ x\neq 0). \tag{2.7.19}$$

For $\eta\in\Omega$, we will establish existence of the limit

$$a^k_{i_1\dots i_m}(\eta) = \lim_{\substack{x\to 0\\ x/|x|\to\eta}} \frac{1}{|x|^k}\left[h_{i_1\dots i_m}(x) - \lambda^{p_1\dots p_m}_{i_1\dots i_m}(x)\sum_{s=0}^{k-1} c^s_{p_1\dots p_m j_1\dots j_s} x^{j_1}\dots x^{j_s}\right]. \tag{2.7.20}$$

To this end, we expand the function $\partial^m\varphi(x,\xi)/\partial\xi^{p_1}\dots\partial\xi^{p_m}$ into the Taylor series in x:

$$\frac{\partial^m\varphi(x,\xi)}{\partial\xi^{p_1}\dots\partial\xi^{p_m}} = \sum_{s=0}^{k}\frac{1}{s!}\frac{\partial^{m+s}\varphi}{\partial\xi^{p_1}\dots\partial\xi^{p_m}\partial x^{j_1}\dots\partial x^{j_s}}(0,\xi)\, x^{j_1}\dots x^{j_s} + O(|x|^{k+1}).$$

Inserting the last expression into (2.7.9) and using the inductive hypothesis (2.7.14), we obtain

$$h_{i_1\dots i_m}(x) - \lambda^{p_1\dots p_m}_{i_1\dots i_m}(x)\sum_{s=0}^{k-1} c^s_{p_1\dots p_m j_1\dots j_s} x^{j_1}\dots x^{j_s} = \frac{1}{k!}\lambda^{p_1\dots p_m}_{i_1\dots i_m}(x)\frac{\partial^{m+k}\varphi}{\partial\xi^{p_1}\dots\partial\xi^{p_m}\partial x^{j_1}\dots\partial x^{j_k}}(0,\xi)\, x^{j_1}\dots x^{j_k} + O(|x|^{k+1}) \qquad ((x,\xi)\in T\Omega;\ x\neq 0). \tag{2.7.21}$$

It follows from (2.6.3) that $\lim\lambda^{p_1\dots p_m}_{i_1\dots i_m}(x) = \lambda^{p_1\dots p_m}_{i_1\dots i_m}(\eta)$ as $x\to 0$ and $x/|x|\to\eta$. Dividing equality (2.7.21) by $|x|^k$ and passing to the limit, we obtain existence of the limit in (2.7.20) as well as validity of the relation

$$a^k_{i_1\dots i_m}(\eta) = \frac{1}{k!}\lambda^{p_1\dots p_m}_{i_1\dots i_m}(\eta)\frac{\partial^{m+k}\varphi}{\partial\xi^{p_1}\dots\partial\xi^{p_m}\partial x^{j_1}\dots\partial x^{j_k}}(0,\xi)\,\eta^{j_1}\dots\eta^{j_k} \qquad (\langle\xi,\eta\rangle = 0). \tag{2.7.22}$$

Dividing (2.7.19) by $|x|^k$ and passing to the limit as $x\to 0,\ x/|x|\to\eta$, we obtain

$$\langle a^k(\eta),\xi^m\rangle = \langle b^k(\xi),\eta^k\rangle \qquad (\xi,\eta\in\Omega;\ \langle\xi,\eta\rangle = 0). \tag{2.7.23}$$

Let us prove that

$$j_\eta a^k(\eta) = 0 \quad (\eta\in\Omega). \tag{2.7.24}$$

To this end, we note that, by (2.7.10), condition (2.7.1) can be rewritten as

$$x^q\left[h_{qi_2\dots i_m}(x)-\lambda^{p_1\dots p_m}_{qi_2\dots i_m}(x)\sum_{s=0}^{k-1}c^s_{p_1\dots p_m j_1\dots j_s}x^{j_1}\dots x^{j_s}\right]=0.$$

Dividing this equality by $|x|^{k+1}$ and passing to the limit, from (2.7.20) we obtain (2.7.24).

Now we check that

$$j_\xi b^k(\xi)=0\qquad(\xi\in\Omega).\tag{2.7.25}$$

Indeed, if we differentiate equality (2.7.6) $k-1$ times with respect to x and put $x=0$, then the so-obtained relation coincides with (2.7.25), by (2.7.16).

Equalities (2.7.23)–(2.7.25), together with smoothness of $b^k(\xi)$ which follows from (2.7.16), imply that the maps $a^k:\Omega\to S^m$ and $b^k:\Omega\to S^k$ satisfy the conditions of Theorem 2.8.1 formulated below. By this theorem, there exists a tensor $c^k=(c^k_{i_1\dots i_m j_1\dots j_k})\in S^m\otimes S^k$ such that

$$a^k_{i_1\dots i_m}(\eta)=\lambda^{p_1\dots p_m}_{i_1\dots i_m}(\eta)c^k_{p_1\dots p_m j_1\dots j_k}\eta^{j_1}\dots\eta^{j_k}\qquad(\eta\in\Omega),\tag{2.7.26}$$

$$b^k_{j_1\dots j_k}(\xi)=\lambda^{q_1\dots q_k}_{j_1\dots j_k}(\xi)c^k_{i_1\dots i_m q_1\dots q_k}\xi^{i_1}\dots\xi^{i_m}\qquad(\xi\in\Omega).\tag{2.7.27}$$

Replacing the left-hand side of (2.7.22) with (2.7.26), we obtain (2.7.14) for $l=k$. Relations (2.7.16) and (2.7.27) imply (2.7.13) for $l=k$. Writing (2.7.14) in the form

$$\frac{1}{k!}\lambda^{p_1\dots p_m}_{i_1\dots i_m}(x)\frac{\partial^{m+k}\varphi}{\partial\xi^{p_1}\dots\partial\xi^{p_m}\partial x^{j_1}\dots\partial x^{j_k}}(0,\xi)x^{j_1}\dots x^{j_k}=\lambda^{p_1\dots p_m}_{i_1\dots i_m}(x)c^k_{p_1\dots p_m j_1\dots j_k}x^{j_1}\dots x^{j_k}$$
$$((x,\xi)\in T\Omega;\ x\neq 0)$$

and inserting this expression into the right-hand side of (2.7.21), we arrive at (2.7.12) for $l=k$. Thus the inductive step is finished and the lemma is proved.

P r o o f of Theorem 2.7.1. Let $u_{i_1\dots i_m}\in C^\infty(\mathbf{R}^n)$ be a function such that its Taylor series is

$$u_{i_1\dots i_m}(x)\sim\sum_{k=0}^{\infty}c^k_{i_1\dots i_m j_1\dots j_k}x^{j_1}\dots x^{j_k}\tag{2.7.28}$$

where c^k are tensors constructed in Lemma 2.7.2. Then $u=(u_{i_1\dots i_m})\in C^\infty(S^m)$ and, by (2.7.12),

$$h_{i_1\dots i_m}(x)=\lambda^{p_1\dots p_m}_{i_1\dots i_m}(x)u_{p_1\dots p_m}(x)+o(|x|^k),\tag{2.7.29}$$

for every $k\geq 0$. Defining the field $\tilde u\in C^\infty(S^m;\mathbf{R}^n_0)$ by the equality

$$\tilde u_{i_1\dots i_m}(x)=\lambda^{p_1\dots p_m}_{i_1\dots i_m}(x)\,u_{p_1\dots p_m}(x),\tag{2.7.30}$$

we can rewrite (2.7.29) as follows:

$$h(x)=\tilde u(x)+o(|x|^k)\qquad(x\in\mathbf{R}^n_0,\ x\to 0;\ k=0,1,\dots).\tag{2.7.31}$$

From the structure of $\lambda^{p_1\dots p_m}_{i_1\dots i_m}(x)$ defined by (2.6.3), we see that (2.7.30) can be written in the form

$$\tilde u(x)=u(x)-i_x v(x)\tag{2.7.32}$$

with some $v \in C^\infty(S^{m-1}; \mathbf{R}^n_0)$. Using (2.7.10) and (2.7.11), from (2.7.30) we obtain the relations

$$j_x \tilde{u}(x) = 0 \quad (x \in \mathbf{R}^n_0), \tag{2.7.33}$$

$$\langle \tilde{u}(x), \xi^m \rangle = \langle u(x), \xi^m \rangle \qquad ((x,\xi) \in T\Omega;\ x \neq 0). \tag{2.7.34}$$

For $x \in \mathbf{R}^n$, we define $f(x) \in S^m$ by

$$f(x) = \begin{cases} h(x) + i_x v(x), & \text{if} \quad x \neq 0, \\ c^0, & \text{if} \quad x = 0. \end{cases} \tag{2.7.35}$$

This definition implies that h is the tangent component of the field f. Thus the theorem will be proved if the containment $f \in C^\infty(S^m; \mathbf{R}^n)$ will be established. To this end, it is sufficient to prove that $\tilde{h} = f - u \in C^\infty(S^m; \mathbf{R}^n)$. By (2.7.28), $u(0) = c^0 = f(0)$ and, consequently, $\tilde{h}(0) = 0$. It follows from (2.7.31), (2.7.32) and (2.7.35) that

$$\tilde{h}(x) = o(|x|^k) \qquad (x \in \mathbf{R}^n,\ x \to 0;\ k = 1, 2, \dots). \tag{2.7.36}$$

By (2.7.32) and (2.7.35), $\tilde{h}(x) = h(x) - \tilde{u}(x)$ for $x \in \mathbf{R}^n_0$. With the help of (2.7.33), (2.7.34) and conditions (2.7.1), (2.7.2) of the theorem; from this we conclude that

$$j_x \tilde{h}(x) = 0 \qquad (x \in \mathbf{R}^n_0), \tag{2.7.37}$$

$$\langle \tilde{h}(x), \xi^m \rangle = \tilde{\varphi}(x, \xi) \qquad ((x,\xi) \in T\Omega;\ x \neq 0), \tag{2.7.38}$$

where the function $\tilde{\varphi} \in C^\infty(T\Omega)$ is defined by the equality

$$\tilde{\varphi}(x,\xi) = \varphi(x,\xi) - \langle u(x), \xi^m \rangle \qquad ((x,\xi) \in T\Omega). \tag{2.7.39}$$

Relations (2.7.37) and (2.7.38) show that the field $\tilde{h}$ satisfies the assumptions of the theorem as well as h. As compared with h, the field $\tilde{h}$ has the advantage of satisfying (2.7.36).

By definition, $\tilde{h} \in C^\infty(S^m; \mathbf{R}^n_0)$. Thus, to prove the containment $\tilde{h} \in C^\infty(S^m; \mathbf{R}^n)$ it is sufficient to verify that the asymptotic estimate

$$D^\alpha \tilde{h}(x) = o(|x|^k) \quad \text{as} \quad x \to 0 \tag{2.7.40}$$

is valid for every multi-index α and every $k \geq 0$.

Replacing h by $\tilde{h}$, we repeat the discussion which is placed just after the formulation of Theorem 2.7.1. We extend $\tilde{\varphi}$ to $\mathbf{R}^n \times \mathbf{R}^n_0$ by a formula similar to (2.7.3) and verify that

$$\langle \tilde{h}(x), \xi^m \rangle = \tilde{\varphi}(x, P_x \xi) \qquad (x, \xi \in \mathbf{R}^n_0;\ x \neq t\xi). \tag{2.7.41}$$

From (2.7.39) and (2.7.3), one can see that the functions φ and $\tilde{\varphi}$ are connected by the relation $\tilde{\varphi}(x,\xi) = \varphi(x,\xi) - \langle u(P_\xi x), \xi^m \rangle$ for $(x,\xi) \in \mathbf{R}^n \times \mathbf{R}^n_0$. Differentiating this equality with respect to x and then putting $x = 0$, we obtain

$$\frac{\partial^k \tilde{\varphi}}{\partial x^{j_1} \dots \partial x^{j_k}}(0,\xi) = \frac{\partial^k \varphi}{\partial x^{j_1} \dots \partial x^{j_k}}(0,\xi) - \lambda^{q_1 \dots q_k}_{j_1 \dots j_k}(\xi) \frac{\partial^k u_{i_1 \dots i_m}}{\partial x^{q_1} \dots \partial x^{q_k}}(0)\, \xi^{i_1} \dots \xi^{i_m} \qquad (\xi \in \Omega).$$

Inserting the value (2.7.13) of $\frac{\partial^k \varphi}{\partial x^{j_1} \dots \partial x^{j_k}}(0,\xi)$ and the value (2.7.28) of $\frac{\partial^k u_{i_1 \dots i_m}}{\partial x^{q_1} \dots \partial x^{q_k}}(0)$ into the last equality, we see that $D^\alpha_x \tilde{\varphi}(0,\xi) = 0$ for $\xi \in \Omega$ and for every multi-index

α. This implies that $D_x^\alpha\tilde\varphi(0,\xi)=0$ for $\xi\in\mathbf{R}_0^n$, by homogeneity of $\tilde\varphi(x,\xi)$ in its second argument. Differentiating the last relation with respect to ξ, we obtain $D_x^\alpha D_\xi^\beta\tilde\varphi(0,\xi)=0$ for every α,β. From this, one can easily prove existence of a constant $C_{\alpha\beta k}$ such that

$$|D_x^\alpha D_\xi^\beta\tilde\varphi(x,\xi)|\le C_{\alpha\beta k}|x|^k \qquad (x\in\mathbf{R}^n,\ |x|\le 1,\ \xi\in\Omega) \tag{2.7.42}$$

for every α,β and $k\ge 0$.

Differentiating (2.7.41) with respect to ξ, we obtain

$$\tilde h_{i_1\dots i_m}=|x|^{-2m}Q^{0,p_1\dots p_m}_{i_1\dots i_m}(x,\xi)\frac{\partial^m\tilde\varphi}{\partial\xi^{p_1}\dots\partial\xi^{p_m}}(x,P_x\xi) \qquad (x,\xi\in\mathbf{R}_0^n;\ x\ne t\xi),$$

where

$$Q^{0,p_1\dots p_m}_{i_1\dots i_m}(x,\xi)=(|x|^2\delta^{p_1}_{i_1}-x_{i_1}x^{p_1})\dots(|x|^2\delta^{p_m}_{i_m}-x_{i_m}x^{p_m}).$$

From this, by induction on $|\alpha|$, we infer the representation

$$D^\alpha\tilde h_{i_1\dots i_m}=|x|^{-4|\alpha|-2m}\sum_{\beta+\gamma\le\alpha}Q^{\alpha\beta\gamma}_{i_1\dots i_m}(x,\xi)D_x^\beta D_\xi^\gamma\tilde\varphi(x,P_x\xi) \qquad (x,\xi\in\mathbf{R}_0^n;\ x\ne t\xi)$$

with some polynomials $Q^{\alpha\beta\gamma}_{i_1\dots i_m}(x,\xi)$. This representation taken at $(x,\xi)\in T\Omega$, together with inequalities (2.7.42), implies estimate (2.7.40). The theorem is proved.

2.8 A theorem on conjugate tensor fields on the sphere

By Lemma 2.6.1, the decomposition $S^m=\operatorname{Ker} j_x\oplus\operatorname{Im} i_x$ is valid for every $x\in\Omega$. Let P_x be the orthogonal projection of S^m onto the first summand of this decomposition. The relation ${}^tf=P_xf$ is expressed in coordinate form by equality (2.6.2). A mapping $a:\Omega\to S^k$ is called the *tangent tensor field on the sphere* if $j_xa(x)=0$ for $x\in\Omega$. Two fields $a:\Omega\to S^k$ and $b:\Omega\to S^m$ are called *conjugate* if the equality $\langle a(x),y^k\rangle=\langle b(y),x^m\rangle$ is valid for every $x,y\in\Omega$ satisfying the condition $\langle x,y\rangle=0$.

Theorem 2.8.1 *Let $n\ge 3$, $k\ge 0$ and $m\ge 0$ be integers; $\Omega=\Omega^{n-1}$. If $a:\Omega\to S^k$ and $b:\Omega\to S^m$ are conjugate tangent tensor fields on the sphere and one of them is smooth, then there exists a linear operator $A:S^m\to S^k$ such that*

$$a(x)=P_xAx^m,\quad b(x)=P_xA^*x^k \qquad (x\in\Omega),$$

where $A^*:S^k\to S^m$ is the dual of A. In coordinate form this claim is formulated as follows: there exists a tensor $A=(A_{i_1\dots i_kj_1\dots j_m})\in S^k\otimes S^m$ such that

$$a_{i_1\dots i_k}(x)=\lambda^{p_1\dots p_k}_{i_1\dots i_k}(x)A_{p_1\dots p_kj_1\dots j_m}x^{j_1}\dots x^{j_m} \qquad (x\in\Omega),$$

$$b_{j_1\dots j_m}(x)=\lambda^{q_1\dots q_m}_{j_1\dots j_m}(x)A_{i_1\dots i_kq_1\dots q_m}x^{i_1}\dots x^{i_k} \qquad (x\in\Omega).$$

Starting with the proof, first of all we extend a and b to the mappings $a : \mathbf{R}^n_0 \to S^k$ and $b : \mathbf{R}^n_0 \to S^m$ in such a way that a becomes positively homogeneous of degree m nd b of degree k. Then the relations

$$j_x a(x) = 0, \quad j_x b(x) = 0 \qquad (x \in \mathbf{R}^n_0), \tag{2.8.1}$$

$$\langle a(x), y^k \rangle = \langle b(y), x^m \rangle \qquad (x, y \in \mathbf{R}^n_0;\ \langle x, y \rangle = 0), \tag{2.8.2}$$

are valid. They imply the equalities

$$\left.\begin{array}{l} \langle a(x), y^k \rangle = \langle b(P_x y), x^m \rangle \\ \langle b(y), x^m \rangle = \langle a(P_y x), y^k \rangle \end{array}\right\} \qquad (x, y \in \mathbf{R}^n_0;\ x \neq ty). \tag{2.8.3}$$

By assumption, one of the two mappings is smooth; let it be a. We can now prove smoothness of the other mapping. Indeed, the right-hand side of the second equality in (2.8.3) is smooth on $\mathbf{R}^n \times \mathbf{R}^n_0$. Consequently, the left-hand side of this equality is smooth on $\{(x, y) \in \mathbf{R}^n_0 \times \mathbf{R}^n_0 \mid x \neq ty\}$. This implies smoothness of b on $\mathbf{R}^n_0$.

Lemma 2.8.2 *Let U be a domain in $\mathbf{R}^n$ and $f \in C^\infty(U)$. Given an integer k, if $D^\alpha f \equiv 0$ in U for $|\alpha| > k$, then f is a polynomial of degree at most k.*

We omit the proof of this lemma because it is evident.

Lemma 2.8.3 *Let $f \in C^\infty(\mathbf{R}^n_0)$. If f, restricted to every two-dimensional plane passing through the origin of the coordinates, is a polynomial of degree at most k, then f is a polynomial of degree at most k.*

P r o o f. Every straight line can be included into some two-dimensional plane passing through the origin. Consequently, f restricted to every straight line is a polynomial of degree at most k. From this, $\partial^{k+1} f / \partial x_i^{k+1} = 0$ for $x \in \mathbf{R}^n_0$. Thus $D^\alpha f = 0$ for $|\alpha| > kn$. Applying Lemma 2.8.2, we see that f is a polynomial. Let us show that its degree N is at most k. To this end we represent f as $f = P + Q$ where $P \neq 0$ is a homogeneous polynomial of degree N, and the degree of Q is less then N. There exists $a \in \mathbf{R}^n_0$ such that $P(a) \neq 0$. Then the restriction of f to the straight line $x = at$ has degree N. Consequently, $N \leq k$.

Lemma 2.8.4 *Theorem* 2.8.1 *is valid for $m = 0$.*

P r o o f. In this case (2.8.2) looks like: $\langle a(x), y^k \rangle = b(y)$ $(x, y \in \mathbf{R}^n_0;\ \langle x, y \rangle = 0)$. This implies that the restriction of b to every hyperplane passing through the origin is a polynomial of degree k. By Lemma 2.8.3, we conclude that b is a homogeneous polynomial of degree k

$$b(y) = A_{i_1 \dots i_k} y^{i_1} \dots y^{i_k}. \tag{2.8.4}$$

The first of equalities (2.8.3) looks now like: $\langle a(x), y^k \rangle = b(P_x y)$ $(x, y \in \mathbf{R}^n_0;\ x \neq ty)$. Differentiating this relation, we obtain

$$a_{i_1 \dots i_k}(x) = \frac{1}{k!} \frac{\partial^k (b(P_x y))}{\partial y^{i_1} \dots \partial y^{i_k}} = \frac{1}{k!} \frac{\partial^k b}{\partial y^{j_1} \dots \partial y^{j_k}} (P_x y) \lambda^{j_1 \dots j_k}_{i_1 \dots i_k}(x).$$

By (2.8.4), it can be written in the form

$$a_{i_1 \dots i_k}(x) = A_{j_1 \dots j_k} \lambda^{j_1 \dots j_k}_{i_1 \dots i_k}(x). \tag{2.8.5}$$

Relations (2.8.4) and (2.8.5) give the claim of the theorem. The lemma is proved.

Lemma 2.8.5 *Let the assumptions of Theorem* 2.8.1 *be satisfied. Extend a and b to $\mathbf{R}^n_0$ in such a way that a (b) be positively homogeneous of degree m (k) and put*

$$\tilde{a}(x,y) = |x|^{2k}\langle a(x), y^k\rangle; \quad \tilde{b}(x,y) = |y|^{2m}\langle b(y), x^m\rangle \qquad (x, y \in \mathbf{R}^n_0). \tag{2.8.6}$$

Then $\tilde{a}(x,y)$ $(\tilde{b}(x,y))$ is a homogeneous polynomial of degree $2k+m$ (m) in x and a homogeneous polynomial of degree k $(k+2m)$ in y.

P r o o f. Differentiating the second equality in (2.8.3) m times with respect to x and putting $\langle x,y\rangle = 0$, we obtain

$$|y|^{2m} b_{i_1\ldots i_m}(y) = \frac{|y|^{2m}}{m!}\frac{\partial^m a_{j_1\ldots j_k}(x)}{\partial x^{p_1}\ldots\partial x^{p_m}}\lambda^{p_1\ldots p_m}_{i_1\ldots i_m}(y) y^{j_1}\ldots y^{j_k} \qquad (x,y\in\mathbf{R}^n_0; \langle x,y\rangle = 0). \tag{2.8.7}$$

We fix indices $i_1,\ldots,i_m$ and put

$$\tilde{b}(y) = |y|^{2m} b_{i_1\ldots i_m}(y). \tag{2.8.8}$$

The right-hand side of equality (2.8.7) is a homogeneous polynomial of degree $k+2m$ in y, i.e., this equality can be written in the form

$$\tilde{b}(y) = a'_{j_1\ldots j_{k+2m}}(x) y^{j_1}\ldots y^{j_{k+2m}} \qquad (x,y\in\mathbf{R}^n_0;\ \langle x,y\rangle = 0) \tag{2.8.9}$$

with some $a' \in C^\infty(S^{k+2m}; \mathbf{R}^n_0)$. Let $\tilde{a}(x)$ be the tangent component of the field $a'(x)$. It follows from Lemma 2.6.1 and (2.8.9) that $\langle \tilde{a}(x), y^{k+2m}\rangle = \tilde{b}(y)$ for $x,y\in\mathbf{R}^n_0$, $\langle x,y\rangle = 0$. We thus see that the mappings $\tilde{a}:\Omega\to S^{k+2m}$ and $\tilde{b}:\Omega\to S^0 = \mathbf{C}$ satisfy the conditions of Theorem 2.8.1. By Lemma 2.8.4, the assertion of Theorem 2.8.1 is valid for these fields. In particular, $\tilde{b}(y)$ is a homogeneous polynomial of degree $k+2m$. Now (2.8.8) implies the desired statement about $\tilde{b}(x,y)$. The statement on $\tilde{a}(x,y)$ can be proved in the same way. The lemma is proved.

Lemma 2.8.6 *Let $f(x,y)$ be a real polynomial in variables $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$. For $x,y\in\mathbf{R}^n$, if the equality $\langle x,y\rangle = 0$ implies $f(x,y) = 0$, then $f(x,y)$ is divisible by $\langle x,y\rangle = \sum\limits_i x_i y_i$.*

We omit the proof which can be implemented by elementary algebraic methods.

Lemma 2.8.7 *Let $n\ge 3$, $k\ge 0$ and $m\ge 0$ be integers, $f(x,y)$ be a complex polynomial in $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$. If there exist polynomials $g(x,y)$ and $h(x,y)$ such that*

$$|y|^{2k} f(x,y) = |x|^{2m} g(x,y) + \langle x,y\rangle h(x,y), \tag{2.8.10}$$

where $|x|^2 = \sum\limits_i x_i^2$, then there exist polynomials $\alpha(x,y)$ and $\beta(x,y)$ such that

$$f(x,y) = |x|^{2m}\alpha(x,y) + \langle x,y\rangle\beta(x,y). \tag{2.8.11}$$

If $f(x,y)$ has real coefficients, then $\alpha(x,y)$ and $\beta(x,y)$ in (2.8.11) *can be selected with real coefficients. If $f(x,y)$ is bihomogeneous in x,y, then $\alpha(x,y)$ and $\beta(x,y)$ can also be selected to be bihomogeneous.*

The proof of this lemma will be given in the next section.

P r o o f of Theorem 2.8.1. Multiplying equality (2.8.2) by $|x|^{2k}|y|^{2m}$, we obtain $|y|^{2m}\tilde{a}(x,y) = |x|^{2k}\tilde{b}(x,y)$ for $\langle x,y\rangle = 0$. By Lemma 2.8.5, $\tilde{a}(x,y)$ and $\tilde{b}(x,y)$ are bihomogeneous polynomials of degree $(2k+m,k)$ and $(m,k+2m)$ respectively. Applying Lemma 2.8.6, we obtain $|y|^{2m}\tilde{a}(x,y) - |x|^{2k}\tilde{b}(x,y) = \langle x,y\rangle h(x,y)$, where $h(x,y)$ is a polynomial. With the help of Lemma 2.8.7 we conclude that

$$\tilde{a}(x,y) = |x|^{2k}\alpha(x,y) + \langle x,y\rangle\beta(x,y) \tag{2.8.12}$$

with bihomogeneous real polynomials α,β. As one can see from (2.8.12) the degree of α is equal to (m,k), i.e.,

$$\alpha(x,y) = A_{i_1\dots i_k j_1\dots j_m} y^{i_1}\dots y^{i_k}x^{j_1}\dots x^{j_m} \tag{2.8.13}$$

with some $A \in S^k \otimes S^m$. Relation (2.8.12) together with (2.8.6) gives

$$\langle a(x), y^k\rangle = \alpha(x,y) + |x|^{-2k}\langle x,y\rangle\beta(x,y) \qquad (x \in \mathbf{R}^n_0,\ y \in \mathbf{R}^n). \tag{2.8.14}$$

Relations (2.8.3) imply that $\langle a(x), y^k\rangle = \langle a(x), (P_x y)^k\rangle$. Comparing the last equality with (2.8.14), we obtain

$$\langle a(x), y^k\rangle = \alpha(x, P_x y) \qquad (x \in \mathbf{R}^n_0,\ y \in \mathbf{R}^n). \tag{2.8.15}$$

It follows from (2.8.14) and the second equality in (2.8.3) that

$$\langle b(y), x^m\rangle = \alpha(P_y x, y) \qquad (x \in \mathbf{R}^n,\ y \in \mathbf{R}^n_0). \tag{2.8.16}$$

Equalities (2.8.13), (2.8.15) and (2.8.16) obtained above are equivalent to the claim of Theorem 2.8.1. The theorem is proved.

We will demonstrate an interesting example related to the question: to what extent is the assumption of smoothness essential for validity of Theorem 2.8.1? For $k = m = 1$, i.e., in the case of conjugate tangent vector fields on the sphere, the assumption of smoothness can be replaced by the assumption of continuity of a, b. Without this assumption, the theorem becomes wrong. To exhibit a corresponding example, we will recall a definition.

A *derivative of the field* $\mathbf{R}$ is a function $D : \mathbf{R} \to \mathbf{R}$, satisfying two conditions: 1) $D(x+y) = D(x)+D(y)$; 2) $D(xy) = yD(x)+xD(y)$. Each derivative vanishes on the field of rational numbers. A nonzero derivative, from the standpoint of elementary analysis, is rather exotic function: it is discontinuous at any point and unbounded in a neighborhood of any point. Derivatives of the field $\mathbf{R}$ form an infinite-dimensional vector space.

Let $D : \mathbf{R} \to \mathbf{R}$ be a derivative and $e_1,\dots,e_n$ be an orthonomalized basis for $\mathbf{R}^n$. Define $a_D : \Omega \to \mathbf{R}^n$, by putting $a_D(\sum\limits_i x_i e_i) = \sum\limits_i D(x_i)e_i$. Then

$$\langle a_D(x), x\rangle = \sum_i x_i D(x_i) = \frac{1}{2}D\left(\sum_i x_i^2\right) = \frac{1}{2}D(1) = 0,$$

i.e., the field a_D is tangent to Ω. A similar calculation shows that $\langle a_D(x), y\rangle + \langle x, a_D(y)\rangle = 0$ for $\langle x,y\rangle = 0$, i.e., that the fields a_D and $-a_D$ are conjugate.

It can be proved that all pairs, of conjugate vector fields, are reduced to the cases described in Theorem 2.8.1 and the last example. To be more exact, for every pair of conjugate tangent vector fields a and b on Ω^{n-1} (for $n \ge 3$), there exist a linear operator $A : \mathbf{R}^n \to \mathbf{R}^n$ and a derivative D such that $a(x) = P_x Ax + a_D(x),\ b(x) = P_x A^* x - a_D(x)$.

2.9 Primality of the ideal $(|x|^2, \langle x, y\rangle)$

Let $\mathbf{C}[x,y] = \mathbf{C}[x_1,\dots,x_n,y_1,\dots,y_n]$ be the algebra of complex polynomials in variables $x_1,\dots,x_n,\ y_1,\dots,y_n$ and $(|x|^2, \langle x,y\rangle)$ be its ideal generated by the polynomials $|x|^2 = \sum\limits_i x_i^2$ and $\langle x,y\rangle = \sum\limits_i x_i y_i$.

Theorem 2.9.1 *For $n \ge 3$, the algebra $\mathbf{C}[x,y]\,/\,(|x|^2, \langle x,y\rangle)$ has no zero divisor.*

We do not give the proof of this theorem here since the methods of the proof are beyond the scope of the book. The reader interested in the proof is referred to [70].

We show that Lemma 2.8.7 is a consequence of Theorem 2.9.1. First of all we note that the last two claims of Lemma 2.8.7 follow from the first. Indeed, assume that f has real coefficients and equality (2.8.11) is valid for some α and β having complex coefficients. Taking the real parts of this equality, we obtain (2.8.11) with real α,β. Similarly, given a bihomogeneous f, assume (2.8.11) valid for some $\alpha,\ \beta$. Taking the bihomogeneous components of both sides of (2.8.11), we arrive at a similar relation with bihomogeneous α,β. We shall prove the first claim by induction on k. First of all note that for $k=0$ or $m=0$ this claim is trivial. For $m=1$ equality (2.8.10) means that the comparison $|y|^{2k} f \equiv 0\,(\mathrm{mod}\,(|x|^2, \langle x,y\rangle))$ is valid in the algebra $\mathbf{C}[x,y]$. By Theorem 2.9.1, this implies that $f \equiv 0\,(\mathrm{mod}\,(|x|^2, \langle x,y\rangle))$ because $|y|^{2k} \not\equiv 0\,(\mathrm{mod}\,(|x|^2, \langle x,y\rangle))$. Thus (2.8.11) is valid in this case. Let now $k>0,\ m>1$. Rewriting (2.8.10) in the form $|x|^{2m} g = |y|^2 f_1 - \langle x,y\rangle h$ where $f_1 = |y|^{2k-2} f$ and using the already-proved part of Lemma 2.8.7, we can find a polynomials g_1 and h_1 such that $g = |y|^2 g_1 + \langle x,y\rangle h_1$. Inserting this expression into (2.8.10), we obtain $|y|^2(|y|^{2k-2} f - |x|^{2m} g_1) = \langle x,y\rangle(|x|^{2m} h_1 + h)$. Since the polynomial $\langle x,y\rangle$ is irreducible and is not a factor of $|y|^2$, the polynomial $\langle x,y\rangle$ is a factor of $|y|^{2k-2} f - |x|^{2m} g_1$. Thus we have $|y|^{2k-2} f = |x|^{2m} g_1 + \langle x,y\rangle h_2$. It remains to use the inductive hypothesis. The lemma is proved.

2.10 Description of the range of the ray transform

In Section 2.1 we have shown that $I : \mathcal{S}(S^m) \to \mathcal{S}(T\Omega)$. The next theorem describes the range of this operator.

Theorem 2.10.1 *Let $n \ge 3$. A function $\varphi \in \mathcal{S}(T\Omega)$ can be represented as $\varphi = If$ for some field $f \in \mathcal{S}(S^m; \mathbf{R}^n)$ if and only if the next two conditions are satisfied:*

(1) $\varphi(x,-\xi) = (-1)^m \varphi(x,\xi)$;

(2) *the function $\psi(x,\xi)$ defined on $\mathbf{R}^n \times \mathbf{R}^n_0$ by equality* (2.1.13) *satisfies the equations* $(1 \le i_1, j_1, \dots, i_{m+1}, j_{m+1} \le n)$

$$\left(\frac{\partial^2}{\partial x^{i_1}\partial \xi^{j_1}} - \frac{\partial^2}{\partial x^{j_1}\partial \xi^{i_1}}\right)\dots\left(\frac{\partial^2}{\partial x^{i_{m+1}}\partial \xi^{j_{m+1}}} - \frac{\partial^2}{\partial x^{j_{m+1}}\partial \xi^{i_{m+1}}}\right)\psi = 0. \tag{2.10.1}$$

Equations (2.10.1) are called the John conditions for the reason mentioned in the introduction to the current chapter.

P r o o f of necessity. Necessity of the first condition follows from (2.1.11). Let us prove necessity of the second condition. The function ψ defined from φ by formula (2.1.13) satisfies relation (2.1.10). Differentiating this relation, we obtain

$$\frac{\partial^2\psi(x,\xi)}{\partial x^i\partial\xi^j} = \int\limits_{-\infty}^{\infty} t\left\langle \frac{\partial^2 f}{\partial x^i\partial x^j}(x+t\xi),\xi^m\right\rangle dt + m\int\limits_{-\infty}^{\infty}\frac{\partial f_{i_1\dots i_{m-1}j}}{\partial x^i}(x+t\xi)\xi^{i_1}\dots\xi^{i_{m-1}}dt.$$

The first summand on the right-hand side of this equality is symmetric with respect to i,j. If we define the field $h_{ij}\in\mathcal{S}(S^{m-1})$ by putting

$$(h_{ij})_{i_1\dots i_{m-1}} = m\left(\frac{\partial f_{i_1\dots i_{m-1}j}}{\partial x^i} - \frac{\partial f_{i_1\dots i_{m-1}i}}{\partial x^j}\right),$$

then the next relation is satisfied:

$$\left(\frac{\partial^2}{\partial x^i\partial\xi^j} - \frac{\partial^2}{\partial x^j\partial\xi^i}\right)\psi(x,\xi) = \int\limits_{-\infty}^{\infty}\langle h_{ij}(x+t\xi),\xi^{m-1}\rangle\,dt. \qquad (2.10.2)$$

Now the validity of (2.10.1) can be easily proved by induction on m. Indeed, for $m=0$, $h_{ij}=0$ and (2.10.1) follows from (2.10.2). By the induction hypothesis,

$$\left(\frac{\partial^2}{\partial x^{i_1}\partial\xi^{j_1}} - \frac{\partial^2}{\partial x^{j_1}\partial\xi^{i_1}}\right)\dots\left(\frac{\partial^2}{\partial x^{i_m}\partial\xi^{j_m}} - \frac{\partial^2}{\partial x^{j_m}\partial\xi^{i_m}}\right)\psi_{i_{m+1}j_{m+1}} = 0 \qquad (2.10.3)$$

where $\psi_{ij}(x,\xi) = \int_{-\infty}^{\infty}\langle h_{ij}(x+t\xi),\xi^{m-1}\rangle\,dt$. Comparing the last equality with (2.10.2), we see that

$$\left(\frac{\partial^2}{\partial x^{i_{m+1}}\partial\xi^{j_{m+1}}} - \frac{\partial^2}{\partial x^{j_{m+1}}\partial\xi^{i_{m+1}}}\right)\psi = \psi_{i_{m+1}j_{m+1}}. \qquad (2.10.4)$$

(2.10.3) and (2.10.4) imply (2.10.1).

Lemma 2.10.2 *Let a function $\varphi\in\mathcal{S}(T\Omega)$ satisfy conditions* (1) *and* (2) *of Theorem* 2.10.1, *$\hat\varphi\in\mathcal{S}(T\Omega)$ be the Fourier transform of φ. There exists a field $\hat h\in C^\infty(S^m;\mathbf{R}^n_0)$ such that*

(1) *$j_y\hat h(y)=0$ for $y\in\mathbf{R}^n_0$;*
(2) *The functions $\hat h_{i_1\dots i_m}(y)$ and all their derivatives decrease rapidly as $|y|\to\infty$;*
(3) *$\langle\hat h(y),\xi^m\rangle = \hat\varphi(y,\xi)$ for $(y,\xi)\in T\Omega,\ y\neq 0$.*

The proof of this lemma will be given later. Using the lemma, let us now complete the proof of Theorem 2.10.1.

P r o o f of sufficiency in Theorem 2.10.1. Let $\hat h\in C^\infty(S^m;\mathbf{R}^n_0)$ be a field existing by Lemma 2.10.2. Using Theorem 2.7.1, we find a field $\hat f\in\mathcal{S}(S^m)$ such that $\hat h$ is the tangent component of $\hat f$. i.e.,

$$\hat f(y) = \hat h(y) + i_y\hat v(y) \qquad (y\in\mathbf{R}^n_0)$$

where $\hat v\in C^\infty(S^{m-1};\mathbf{R}^n_0)$ and $\hat v(y)=0$ for $|y|\geq 1$. By Lemma 2.6.1,

$$\langle\hat f(y),\xi^m\rangle = \hat\varphi(y,\xi) \qquad ((y,\xi)\in T\Omega). \qquad (2.10.5)$$

Let $f \in \mathcal{S}(S^m)$ be a field such that its Fourier transform is $\hat{f}$. We shall show that $If = (2\pi)^{1/2}\varphi$ and thus complete the proof of Theorem 2.10.1. To this end, we recall that, by (2.1.15), $(2\pi)^{1/2} \times \langle \hat{f}(y), \xi^m \rangle = \widehat{If}(y, \xi)$ for $(y, \xi) \in T\Omega$. Comparing this with (2.10.5), we see that $\widehat{If} = (2\pi)^{1/2}\hat{\varphi}$. Since the Fourier transform is bijective on $\mathcal{S}(T\Omega)$, the last equality implies that $If = (2\pi)^{1/2}\varphi$, and Theorem 2.10.1 is proved.

P r o o f of Lemma 2.10.2. Let a function $\varphi \in \mathcal{S}(T\Omega)$ satisfy the conditions (1), (2) of Theorem 2.10.1 and $\hat{\varphi} \in \mathcal{S}(T\Omega)$ be the Fourier transform of φ. It follows from condition (1) that

$$\hat{\varphi}(y, -\xi) = (-1)^m \hat{\varphi}(y, \xi). \tag{2.10.6}$$

Define the function $\hat{\psi} \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n_0)$ by putting

$$\hat{\psi}(y, \xi) = |\xi|^m \hat{\varphi}\left(y - \frac{\langle y, \xi \rangle}{|\xi|^2}\xi, \frac{\xi}{|\xi|}\right). \tag{2.10.7}$$

The next properties of this function follow from (2.10.6) and (2.10.7):

$$\hat{\psi}(y, t\xi) = t^m \hat{\psi}(y, \xi) \qquad (0 \neq t \in \mathbf{R}), \tag{2.10.8}$$

$$\hat{\psi}(y + t\xi, \xi) = \hat{\psi}(y, \xi) \qquad (t \in \mathbf{R}). \tag{2.10.9}$$

Let $\psi \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n_0)$ be a function defined from φ by equality (2.1.13); it has the properties (2.1.11) and (2.1.12). We now find the relation of ψ to $\hat{\psi}$ and, in particular, express the John conditions (2.10.1) by means of $\hat{\psi}$. From (2.10.7), (2.1.11) and the definition of the Fourier transform on $\mathcal{S}(T\Omega)$, we obtain

$$\hat{\psi}(y, \xi) = (2\pi)^{(1-n)/2}|\xi| \int\limits_{\xi^\perp} \psi(x, \xi)\, e^{-i\langle y, x \rangle} dx.$$

Inverting the Fourier transform on $\xi^\perp$, we have

$$\psi(x, \xi) = (2\pi)^{(1-n)/2}|\xi|^{-1} \int\limits_{\xi^\perp} \hat{\psi}(y, \xi)\, e^{i\langle x, y \rangle} dy.$$

Using the δ-function, we write the right-hand side of the last equality in the form of the n-dimensional integral:

$$\psi(x, \xi) = (2\pi)^{(1-n)/2} \int\limits_{\mathbf{R}^n} \hat{\psi}(y, \xi)\, \delta(\langle y, \xi \rangle)\, e^{i\langle x, y \rangle} dy.$$

Differentiating this equality, we obtain

$$\frac{\partial^2 \psi(x, \xi)}{\partial x^j \partial \xi^k} = i(2\pi)^{(1-n)/2} \int\limits_{\mathbf{R}^n} y_j \frac{\partial \hat{\psi}(y, \xi)}{\partial \xi^k}\, \delta(\langle y, \xi \rangle)\, e^{i\langle y, x \rangle} dy +$$

$$+ i(2\pi)^{(1-n)/2} \int\limits_{\mathbf{R}^n} y_j y_k\, \hat{\psi}(y, \xi)\, \delta'(\langle y, \xi \rangle)\, e^{i\langle y, x \rangle} dy.$$

The second integral is symmetric in j,k and, consequently, vanishes after alternation with respect to these indices. Accomplishing the alternation and returning to integration over $\xi^\perp$, we have

$$\left(\frac{\partial^2}{\partial x^j\partial\xi^k}-\frac{\partial^2}{\partial x^k\partial\xi^j}\right)\psi(x,\xi)=i(2\pi)^{(1-n)/2}|\xi|^{-1}\int\limits_{\xi^\perp}\left(y_j\frac{\partial}{\partial\xi^k}-y_k\frac{\partial}{\partial\xi^j}\right)\hat\psi(y,\xi)\,e^{i\langle y,x\rangle}dy.$$

Repeating this procedure and using bijectivity of the Fourier transform on $\xi^\perp$, we make sure that the John equations (2.10.1) are equivalent to the following:

$$\left(y_{i_1}\frac{\partial}{\partial\xi^{j_1}}-y_{j_1}\frac{\partial}{\partial\xi^{i_1}}\right)\dots\left(y_{i_{m+1}}\frac{\partial}{\partial\xi^{j_{m+1}}}-y_{j_{m+1}}\frac{\partial}{\partial\xi^{i_{m+1}}}\right)\hat\psi(y,\xi)=0$$
$$((y,\xi)\in\mathbf{R}^n\times\mathbf{R}^n_0,\ \langle y,\xi\rangle=0).\tag{2.10.10}$$

Let us fix $y\in\mathbf{R}^n_0$. If considered as vectors, the operators $y_i\dfrac{\partial}{\partial\xi^j}-y_j\dfrac{\partial}{\partial\xi^i}$ belong to the space $y^\perp$ and span it. Consequently, equation (2.10.10) means that the restriction of the function $\hat\psi_y(\xi)=\hat\psi(y,\xi)$ to the space $y^\perp$ is such that all its derivatives of order $m+1$ are equal to zero. Together with (2.10.8), this means that the restriction of $\hat\psi_y$ to $y^\perp$ is a homogeneous polynomial of degree m. Recalling relation (2.10.7) between $\hat\psi$ and $\hat\varphi$, we obtain the representation

$$\hat\varphi(y,\xi)=\hat f_{i_1\dots i_m}(y)\xi^{i_1}\dots\xi^{i_m}\qquad((y,\xi)\in T\Omega;\ y\neq0).$$

Thus, given an arbitrary $y\in\mathbf{R}^n_0$, we have established existence of a tensor $\hat f(y)\in S^m$ such that

$$\langle\hat f(y),\xi^m\rangle=\hat\varphi(y,\xi)\qquad((y,\xi)\in T\Omega;\ y\neq0).$$

Applying Lemma 2.6.1, for every $y\in\mathbf{R}^n_0$, we can find a tensor $\hat h(y)\in S^m$ such that $j_y\hat h(y)=0$ and

$$\langle\hat h(y),\xi^m\rangle=\hat\varphi(y,\xi)\qquad((y,\xi)\in T\Omega;\ y\neq0).$$

To complete the proof of Lemma 2.10.2 it remains to apply the next

Lemma 2.10.3 *Let $\varphi\in\mathcal{S}(T\Omega)$ and, for every $x\in\mathbf{R}^n_0$, there exists a tensor $h(x)\in S^m$ such that*

$$j_xh(x)=0\qquad(x\in\mathbf{R}^n_0),\tag{2.10.11}$$

$$\langle h(x),\xi^m\rangle=\varphi(x,\xi)\qquad((x,\xi)\in T\Omega;\ x\neq0).\tag{2.10.12}$$

Then $h\in C^\infty(S^m;\mathbf{R}^n_0)$ and functions $h_{i_1\dots i_m}(x)$ decrease rapidly, together with all their derivatives, as $|x|\to\infty$. In other words, for every $k\geq0$ and every multi-index α,

$$\sup_{|x|\geq1}\{|x|^k|D^\alpha h_{i_1\dots i_m}(x)|\}<\infty.\tag{2.10.13}$$

P r o o f. If $(x,\xi)\in\mathbf{R}^n_0\times\mathbf{R}^n_0$, $x\neq t\xi$, then $P_x\xi\neq0$ and, consequently, the function $\psi(x,\xi)=|P_x\xi|^m\varphi(x,P_x\xi/|P_x\xi|)$ belongs to $C^\infty(U)$ where $U=\{(x,\xi)\in\mathbf{R}^n_0\times\mathbf{R}^n_0\mid x\neq t\xi\}$. One can easily see that the condition $\varphi\in\mathcal{S}(T\Omega)$ implies that

$$\sup_{(x,\xi)\in T\Omega,\ |x|\geq1}\left\{|x|^k|D^\alpha_xD^\beta_\xi\psi(x,\xi)|\right\}<\infty,\tag{2.10.14}$$

for every $k \geq 0$ and for all multi-indices α, β.

Given $(x, \xi) \in U$, (2.10.11) and (2.10.12) imply that

$$\langle h(x), \xi^m \rangle = \langle h(x), (P_x\xi)^m \rangle = |P_x\xi|^m \varphi(x, P_x\xi/|P_x\xi|) = \psi(x, \xi).$$

Since the right-hand side of these equalities is smooth on U, the same is true for the left-hand side. This implies that $h \in C^\infty(S^m; \mathbf{R}^n_0)$ and the next equalities are valid:

$$h_{i_1\dots i_m}(x, \xi) = \frac{1}{m!} \frac{\partial^m \psi(x, \xi)}{\partial \xi^{i_1} \dots \partial \xi^{i_m}} \qquad ((x, \xi) \in U). \tag{2.10.15}$$

From (2.10.14) and (2.10.15), we obtain (2.10.13). The lemma is proved.

2.11 Integral moments of the function If

We now turn to the problem of inverting the ray transform. We pose this problem as follows: given the function If, determine the solenoidal part of the field f or (and) the value Wf of the Saint Venant operator.

From now to the end of the current chapter, we will use only Cartesian coordinate systems on $\mathbf{R}^n$. In this case there is no difference between covariant and contravariant coordinates, and we shall use only lower indices in the notation of coordinates. The summation rule looks now like: on repeating indices (not necessarily standing at different heights) i a monomial the summation from 1 to n is assumed.

We define the operator $\mu^m : C^\infty(\mathbf{R}^n \times \mathbf{R}^n_0) \to C^\infty(S^m)$ sending a function $\varphi(x, \xi)$ to the set of its integral moments with respect to its second argument:

$$(\mu^m \varphi)_{i_1\dots i_m}(x) = \frac{1}{\omega_n} \int\limits_{\Omega} \xi_{i_1} \dots \xi_{i_m} \varphi(x, \xi)\, d\omega(\xi). \tag{2.11.1}$$

Recall that Ω is the unit sphere in $\mathbf{R}^n$, $\omega_n = 2\pi^{n/2}/\Gamma\left(n/2\right)$ is the volume of Ω and $d\omega$ is the angle measure on Ω.

If a function $\varphi \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n_0)$ satisfies the equation

$$\xi_i \frac{\partial \varphi}{\partial x_i} = 0, \tag{2.11.2}$$

then the field $\mu^m \varphi$ is solenoidal: $\delta(\mu^m \varphi) = 0$. Indeed,

$$(\delta \mu^m \varphi)_{i_1\dots i_{m-1}} = \frac{\partial (\mu^m \varphi)_{i_1\dots i_{m-1}j}}{\partial x_j} = \frac{1}{\omega_n} \int\limits_{\Omega} \xi_{i_1} \dots \xi_{i_{m-1}} \left(\xi_j \frac{\partial \varphi}{\partial x_j} \right) d\omega(\xi) = 0.$$

For $f \in \mathcal{S}(S^m)$, the function $\varphi = If$ satisfies (2.11.2), which insures from (2.1.12). Thus the field $\mu^m If$ is solenoidal for every $f \in \mathcal{S}(S^m)$.

Let us calculate the composition $\mu^m I$ where $I : \mathcal{S}(S^m) \to C^\infty(\mathbf{R}^n \times \mathbf{R}^n_0)$ is the ray transform. Given $f \in \mathcal{S}(S^m)$, it follows from (2.1.10) and (2.11.1) that

$$(\mu^m If)_{i_1\dots i_m}(x) = \frac{2}{\omega_n} \int\limits_{\Omega} \xi_{i_1} \dots \xi_{i_m} \left[\int\limits_0^\infty \langle f(x + t\xi), \xi^m \rangle\, dt \right] d\omega(\xi).$$

Changing variables in the integral according to the equality $x + t\xi = x'$, we have

$$(\mu^m If)_{i_1\dots i_m} = \frac{2}{\omega_n} f_{j_1\dots j_m} * \frac{x_{i_1}\dots x_{i_m} x_{j_1}\dots x_{j_m}}{|x|^{2m+n-1}}. \tag{2.11.3}$$

We like now to apply the Fourier transform to the last equality. This application needs some justification. By condition, the first factor on the right-hand side of (2.11.3) belongs to $\mathcal{S}$. The second factor is a function locally summable over $\mathbf{R}^n$ and bounded for $|x| \geq 1$. Consequently, the second factor can be considered as an element of the space $\mathcal{S}'$. It is well known [138] that, for $u \in \mathcal{S}$ and $v \in \mathcal{S}'$, the convolution $u * v$ is defined and belongs to the space θ_M consisting of smooth functions, on $\mathbf{R}^n$, whose every derivative increases at most as a polynomial. In this case the usual rule is valid: $(u * v)^\wedge = (2\pi)^{n/2}\hat{u}\hat{v}$. Thus (2.11.3) implies

$$F[(\mu^m If)_{i_1\dots i_m}] = (2\pi)^{n/2}\frac{2}{\omega_n}\hat{f}_{j_1\dots j_m} F\left[\frac{x_{i_1}\dots x_{i_m} x_{j_1}\dots x_{j_m}}{|x|^{2m+n-1}}\right]. \tag{2.11.4}$$

The product on the right-hand side is now understood to be a product of a function in $\mathcal{S}$ and a distribution in $\mathcal{S}'$. Soon we shall see that the second factor is a locally summable function and, consequently, this product can be understood in the conventional sense. Calculating the second factor in accordance with the usual rules of treating the Fourier transform [48], we obtain

$$F[(\mu^m If)_{i_1\dots i_m}] = \frac{(-1)^m}{2^{2m-1}} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(-m+\frac{1}{2}\right)}{\Gamma\left(m+\frac{n}{2}-\frac{1}{2}\right)} \hat{f}_{j_1\dots j_m}(y)\, \partial_{i_1\dots i_m j_1\dots j_m}|y|^{2m-1}, \tag{2.11.5}$$

where $\partial_{i_1\dots i_k} = \partial^k/\partial y_{i_1}\dots\partial y_{i_k}$.

Define the functions $e_i \in C^\infty(\mathbf{R}^n_0)$, putting $e_i(y) = y_i/|y|$, and the tensor field $\varepsilon \in C^\infty(S^2;\mathbf{R}^n_0)$, by the equality $\varepsilon_{ij}(y) = \delta_{ij} - e_i(y)e_j(y)$. The next lemma can be easily proved by induction on m with the help of the Leibniz formula; we omit the proof

Lemma 2.11.1 *For an integer $m \geq 0$,*

$$\partial^{2m}|y|^{2m-1} = ((2m-1)!!)^2|y|^{-1}\varepsilon^m(y)$$

or, in a more detailed coordinate form,

$$\partial_{i_1\dots i_{2m}}|y|^{2m-1} = ((2m-1)!!)^2|y|^{-1}\sigma(\varepsilon_{i_1i_2}\dots\varepsilon_{i_{2m-1}i_{2m}}).$$

Here the notation $k!! = k(k-2)(k-4)\dots$ is used and it is supposed that $(-1)!! = 1$.

By Lemma 2.11.1, equality (2.11.5) can be rewritten as

$$(\mu^m If)^\wedge(y) = b(m,n)|y|^{-1}\varepsilon^m(y)/\hat{f}(y), \tag{2.11.6}$$

where

$$b(m,n) = (-1)^m \frac{((2m-1)!!)^2}{2^{2m-1}} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(-m+\frac{1}{2}\right)}{\Gamma\left(m+\frac{n}{2}-\frac{1}{2}\right)}. \tag{2.11.7}$$

Since the field $|y|^{-1}\varepsilon^m(y)$ is locally summable, the product on the right-hand side of (2.11.6) can be understood in the ordinary sense. Consequently, the same is true for the initial equality (2.11.4).

Let us estimate the behavior of the field $\mu^m If(x)$ as $|x| \to \infty$. To this end, we note that (2.11.6) implies the representation

$$(D^\alpha(\mu^m If)^\wedge)_{i_1\dots i_m}(y) = |y|^{-2m-2|\alpha|-1} \sum_{|\beta|\le|\alpha|} P^{\alpha\beta}_{i_1\dots i_m j_1\dots j_m}(y) D^\beta \hat f_{j_1\dots j_m}(y),$$

where $P^{\alpha\beta}_{i_1\dots i_m j_1\dots j_m}$ is a homogeneous polynomial of degree $2m+|\alpha|+|\beta|$. It follows from the representation that $D^\alpha(\mu^m If)^\wedge$ decreases rapidly as $|y| \to \infty$ and satisfies the estimate $|D^\alpha(\mu^m If)^\wedge(y)| \le C|y|^{-|\alpha|-1}$ for $|y| \le 1$. Consequently, the field $D^\alpha(\mu^m If)\hat{}$ is summable over $\mathbf{R}^n$ for $|\alpha| \le n-2$. Thus we have proved

Theorem 2.11.2 *Let $n \ge 2$ and $m \ge 0$ be integers. For $f \in \mathcal{S}(S^m)$, the field $\mu^m If$ belongs to $C^\infty(S^m)$, tends to zero as $|x| \to \infty$, satisfies the estimate $|\mu^m If(x)| \le C(1+|x|)^{2-n}$ and has zero divergence: $\delta\mu^m If = 0$.*

2.12 Inversion formulas for the ray transform

Since we have agreed to use only Cartesian coordinates, the metric tensor g coincides with the Kronecker tensor $\delta = (\delta_{ij})$ (the author hopes that the reader is not confused by coincidence of the notations of this tensor and the divergence). Recall that by i, j we mean the operators of symmetric multiplication by δ and the operator of convolution with δ. Let $f \in \mathcal{S}(S^m)$. By Theorem 2.6.2, we have the decomposition

$$f = {}^s\!f + dv, \qquad \delta\, {}^s\!f = 0 \tag{2.12.1}$$

of the field f to the solenoidal and potential parts. Applying the Fourier transform to these equalities, we have

$$\hat f(y) = \widehat{{}^s\!f}(y) + \sqrt{-1}\, i_y \hat v(y), \qquad j_y\, \widehat{{}^s\!f}(y) = 0. \tag{2.12.2}$$

Inserting (2.12.2) into (2.11.6), we obtain

$$(\mu^m If)^\wedge(y) = b(m,n)|y|^{-1}\varepsilon^m(y) \,/\, \widehat{{}^s\!f}(y) + \sqrt{-1}\, b(m,n)|y|^{-1}\varepsilon^m(y) \,/\, i_y\hat v(y). \tag{2.12.3}$$

By Theorem 2.6.2, the fields $\widehat{{}^s\!f}(y)$ and $i_y\hat v(y)$ are bounded on $\mathbf{R}^n$. Consequently, both summands on the right-hand side of (2.12.3) are locally summable and this equality can be understood in the ordinary sense. Let us show that the second summand is equal to zero. Indeed, the expression

$$(\varepsilon^m / i_y \hat v)_{i_1\dots i_m} = \{\sigma\,[(\delta_{i_1j_1} - e_{i_1}e_{j_1})\dots(\delta_{i_mj_m} - e_{i_m}e_{j_m})]\}\,\{\sigma(y_{j_1}\hat v_{j_2\dots j_m})\}$$

is a sum whose every summand has a factor of the type $(\delta_{ij} - e_ie_j)y_j = 0$. Consequently, (2.12.3) can be rewritten as follows:

$$b(m,n)\varepsilon^m(y) \,/\, \widehat{{}^s\!f}(y) = |y|(\mu^m If)^\wedge(y). \tag{2.12.4}$$

By Theorem 2.11.2, $\delta\mu^m If = 0$. Applying the Fourier transform to the last equality, we obtain $j_y(\mu^m If)^\wedge(y) = 0$. Thus (2.12.4), considered as a system of linear algebraic equations relative to the components of the field $\widehat{{}^s\!f}(y)$, has a unique solution, as the next theorem shows.

Theorem 2.12.1 *Let* $y \in \mathbf{R}^n_0$, $\varepsilon = \varepsilon(y)$, $h \in S^m$ $(m \geq 0)$. *The equation*

$$\varepsilon^m / f = h \tag{2.12.5}$$

has a solution $f \in S^m$ *if and only if, when its right-hand side satisfies the condition*

$$j_y h = 0. \tag{2.12.6}$$

In this case equation (2.12.5) *has a unique solution satisfying the condition*

$$j_y f = 0. \tag{2.12.7}$$

The solution is expressed from the right-hand side according to the formula

$$f = \frac{(2m)!}{2^m m!} \sum_{k=0}^{[m/2]} \frac{(-1)^k}{2^k k!(m-2k)!} \frac{n+2m-1}{(n+2m-1)(n+2m-3)\dots(n+2m-2k-1)} \varepsilon^k (j^k h), \tag{2.12.8}$$

where $[m/2]$ *is the integral part of* $m/2$.

The proof of this theorem will be given in the next section. On using this theorem, we shall now obtain the inversion formulas for the ray transform.

Applying Theorem 2.12.1 to equation (2.12.4), we arrive at the equality

$$\widehat{{}^s f}(y) = |y| \sum_{k=0}^{[m/2]} c_k (i_\varepsilon)^k j^k \hat{h}, \tag{2.12.9}$$

where

$$c_k = (-1)^k \frac{\Gamma\left(\frac{n-1}{2}\right)}{2\sqrt{\pi}(n-3)!!\Gamma\left(\frac{n}{2}\right)} \frac{(n+2m-2k-3)!!}{2^k k!(m-2k)!}. \tag{2.12.10}$$

(recall that $(-1)!! = 1$) and the notation $h = \mu^m I f$ is used for brevity. Since the operator j is purely algebraic (independent of y), it commutes with the Fourier transform F. Consequently, (2.12.9) can be rewritten as follows:

$$F[{}^s f] = |y| \sum_{k=0}^{[m/2]} c_k (i_\varepsilon)^k F[j^k h]. \tag{2.12.11}$$

Let us find the operator whose Fourier-image is i_ε. From the equalities $i_\varepsilon = i - (i_y)^2/|y|^2$, $iF = Fi$ and $i_y F = \sqrt{-1} F d$, we obtain

$$i_\varepsilon F = Fi + |y|^{-2} F d^2. \tag{2.12.12}$$

It is well known that

$$|y|^{-2} F = -F \Delta^{-1}, \tag{2.12.13}$$

where Δ^{-1} is the operator of convolution with the fundamental solution to the Laplace equation: $\Delta^{-1} u = u * E$,

$$E(x) = \begin{cases} -\dfrac{\Gamma\left(\frac{n}{2}-1\right)}{4\pi^2} |x|^{2-n} & \text{for} \quad n \geq 3, \\ (2\pi)^{-1} \ln |x| & \text{for} \quad n = 2. \end{cases}$$

Inserting (2.12.12) and (2.12.13) into (2.12.11), we obtain

$$F[{}^sf] = |y|F\sum_{k=0}^{[m/2]} c_k(i-\Delta^{-1}d^2)^k j^k h. \tag{2.12.14}$$

It is well known that $|y|F = F(-\Delta)^{1/2}$ where $(-\Delta)^{1/2}u = -\pi^{-(n+1)/2}\Gamma\left(\frac{n+1}{2}\right)u * |x|^{-n-1}$. Thus (2.12.14) implies the final formula:

$${}^sf = (-\Delta)^{1/2}\left[\sum_{k=0}^{[m/2]} c_k(i-\Delta^{-1}d^2)^k j^k\right]h.$$

Let us formulate the result we have obtained.

Theorem 2.12.2 *Let $n \ge 2$ and $m \ge 0$ be integers. For every field $f \in \mathcal{S}(S^m;\mathbf{R}^n)$, the solenoidal part ${}^sf$ can be recovered from the ray transform If according to the formula*

$${}^sf = (-\Delta)^{1/2}\left[\sum_{k=0}^{[m/2]} c_k(i-\Delta^{-1}d^2)^k j^k\right]\mu^m If, \tag{2.12.15}$$

where Δ is the Laplacian (acting componentwise); d is the operator of inner differentiation; i and j are the operators defined by the equalities $iu = u\delta$, $ju = u/\delta$ where $\delta = (\delta_{ij})$ is the Kronecker tensor; the coefficients c_k are given by formula (2.12.10); $[m/2]$ *is the integral part of* $m/2$.

Let us consider the question about the domain of definition of the operator $A = (-\Delta)^{1/2}\sum c_k(i-\Delta^{-1}d^2)^k j^k$ on the right-hand side of (2.12.15). By $\mathcal{A}$ we mean the space of the distributions $f \in \mathcal{S}'$ whose Fourier transform $\hat{f}(y)$ is summable in a neighborhood of the point $y = 0$. Let $\mathcal{A}(S^m)$ be the space of tensor fields whose components are in $\mathcal{A}$. The operator A is defined on $\mathcal{A}(S^m)$ and maps this space into itself, as one can see from (2.12.9). It follows from (2.11.6) that, for $f \in \mathcal{S}(S^m)$, the field $\mu^m If$ belongs to $\mathcal{A}(S^m)$.

Let $W : C^\infty(S^m) \to C^\infty(S^m \otimes S^m)$ be the Saint Venant operator. We know that $Wdv = 0$ for every $v \in C^\infty(S^{m-1})$. Consequently, (2.12.1) implies that

$$Wf = W\,{}^sf. \tag{2.12.16}$$

The operators i, d, Δ commute with one other, as one can verify by an easy calculation in coordinates. Thus (2.12.15) can be rewritten in the form:

$${}^sf = (-\Delta)^{1/2}\left(\sum_k c_k i^k j^k\right)\mu^m If + du, \tag{2.12.17}$$

where

$$u = (\Delta^{-1/2}d)\left[\sum_{s=1}^{[m/2]}(\Delta^{-1}d^2)^{s-1}\sum_{k=s}^{[m/2]} a_{ks}i^{k-s}j^k\right]\mu^m If \tag{2.12.18}$$

with some coefficients a_{ks}. Applying the Fourier transform to (2.12.18) and using (2.11.6), we see that the field $\hat{u}(y)$ is summable over $\mathbf{R}^n$ and decreases rapidly as $|y| \to 0$. Consequently, $u \in C^\infty(S^{m-1})$ and therefore $Wdu = 0$. Applying W to equality (2.12.17) and taking (2.12.16) into account, we arrive at the next

Theorem 2.12.3 *Let $n \geq 2$ and $m \geq 0$ be integers. For $f \in \mathcal{S}(S^m; \mathbf{R}^n)$, the next equality is valid*

$$Wf = W(-\Delta)^{1/2}\left(\sum_{k=0}^{[m/2]} c_k i^k j^k\right)\mu^m If. \tag{2.12.19}$$

Here W is the Saint Venant operator, the rest of the notations are explained in Theorem 2.12.2.

Theorems 2.12.2 and 2.12.3 answer to the two versions of the question about inversion of the ray transform which are formulated in the beginning of the previous section.

It is pertinent to compare (2.12.15) with the inversion formula for the Radon transform $\mathcal{R}$ (Theorem 3.1 from [48]):

$$f = c_n(-\Delta)^{(n-1)/2}\mathcal{R}^{\#}\mathcal{R}f.$$

Here $\mathcal{R}^{\#}$, the dual of the Radon transform, is an analog of our operator μ^0. Thus, passing from scalar integral geometry to tensor integral geometry, the inversion formula acquires not only the algebraic operators i and j but also the operator $\Delta^{-1}d^2$ which can be considered as a pseudodifferential operator of zero degree.

2.13 Proof of Theorem 2.12.1

We fix $0 \neq y \in \mathbf{R}^n$ and recall that $e_i = y_i/|y|$, $\varepsilon_{ij} = \delta_{ij} - e_ie_j$. Now we prove necessity of condition (2.12.6) for solvability of equation (2.12.5). Indeed, assume that equation (2.12.5) has a solution $f \in S^m$. Then

$$(j_y h)_{i_1\dots i_{m-1}} = y_{i_m}(\varepsilon^m/f)_{i_1\dots i_m} = y_{i_m}\sigma\left[(\delta_{i_1j_1} - e_{i_1}e_{j_1})\dots(\delta_{i_mj_m} - e_{i_m}e_{j_m})f_{j_1\dots j_m}\right].$$

The right-hand side is a sum each of whose summands has a factor of the type $y_{i_m}(\delta_{i_mj_k} - e_{i_m}e_{j_k}) = 0$.

Similar arguments show that (2.12.6) implies the equalities $j_y(\varepsilon^k(j^kh)) = 0$. Consequently, (2.12.7) follows from (2.12.8).

Thus operators $A : f \mapsto h$ and $B : h \mapsto f$ defined by equalities (2.12.5) and (2.12.8) map the space $S^m_y = \{f \in S^m \mid j_yf = 0\}$ into itself. Consequently, to prove the theorem it is sufficient to show that BA is the identical mapping of the space S^m_y, i.e., to verify validity, on S^m_y, of the identity obtained from (2.12.8) by replacing h by ε^m/f. We shall divide the proof into the several lemmas.

Lemma 2.13.1 *For $f \in S^m_y$, the next equality is valid:*

$$\varepsilon^m/f = \frac{2^m(m!)^3}{(2m)!}\sum_{k=0}^{[m/2]}\frac{1}{2^{2k}(k!)^2(m-2k)!}\varepsilon^k(j^kf). \tag{2.13.1}$$

P r o o f. We choose an orthonormalized basis for $\mathbf{R}^n$ such that $y = (0, \dots, 0, y_n)$. In this basis $e_i = \delta_{in}$; $\varepsilon_{ij} = \delta_{ij}$ for $1 \leq i, j \leq n-1$; and $\varepsilon_{ij} = 0$ if $i = n$ or $j = n$. Condition (2.12.7) now looks like: $f_{ni_2\dots i_m} = 0$. Thus, for all tensors involved in equality (2.13.1), the following is true: if a component of the tensor contains n among the indices, then this component is equal to zero. Consequently, proving equality (2.13.1), we can do in such a way as if the n-th coordinate is absent. According to the last observation, we agree to use Greek letters for designation of indices that vary from 1 to $n-1$. On repeating Greek indices, the summation from 1 to $n-1$ is understood.

Let us prove the equality

$$(\varepsilon^m)_{\alpha_1\ldots\alpha_m\beta_1\ldots\beta_m} = \frac{2^m(m!)^3}{(2m)!}\sigma(\alpha_1\ldots\alpha_m)\sigma(\beta_1\ldots\beta_m)\times$$

$$\times\sum_{k=0}^{[m/2]}\frac{2^{-2k}(k!)^{-2}}{(m-2k)!}\delta_{\alpha_1\alpha_2}\ldots\delta_{\alpha_{2k-1}\alpha_{2k}}\delta_{\beta_1\beta_2}\ldots\delta_{\beta_{2k-1}\beta_{2k}}\delta_{\alpha_{2k+1}\beta_{2k+1}}\ldots\delta_{\alpha_m\beta_m}. \tag{2.13.2}$$

Indeed, denoting $(\gamma_1,\ldots,\gamma_m,\gamma_{m+1},\ldots,\gamma_{2m}) = (\alpha_1,\ldots,\alpha_m,\beta_1,\ldots,\beta_m)$, we have

$$(\varepsilon^m)_{\alpha_1\ldots\alpha_m\beta_1\ldots\beta_m} = \frac{1}{(2m)!}\sum_{\pi\in\Pi_{2m}}\delta_{\gamma_{\pi(1)}\gamma_{\pi(2)}}\ldots\delta_{\gamma_{\pi(2m-1)}\gamma_{\pi(2m)}} \tag{2.13.3}$$

where Π_{2m} is the group of all permutations of the set

$$\{\alpha_1,\ldots,\alpha_m,\beta_1,\ldots,\beta_m\} = \{\gamma_1,\ldots,\gamma_{2m}\}$$

which consists of $2m$ different symbols $\alpha_1,\ldots,\beta_m$ (rather than of their numerical values). We shall write such a permutation as an ordered sequence of pairs:

$$\pi = ((\gamma_{\pi(1)},\gamma_{\pi(2)}),\ldots,(\gamma_{\pi(2m-)},\gamma_{\pi(2m)})).$$

We introduce some equivalence relation on the set Π_{2m} by declaring the permutations π_1 and π_2 equivalent if the first of them can be transformed to the second by transpositions of elements in the set $\alpha_1,\ldots,\alpha_m$, transpositions of elements in the set $\beta_1,\ldots,\beta_m$, transpositions of pairs and transpositions of elements in pairs. This definition implies the next more simple criterion: two permutations are equivalent if and only if they contain the same number of pairs of the type (α_i,α_j). Let $\Pi_{2m} = \bigcup_k B_k$ be the decomposition into classes where B_k is the equivalence class of the permutation

$$((\alpha_1,\alpha_2),\ldots,(\alpha_{2k-1},\alpha_{2k}),(\beta_1,\beta_2),\ldots,(\beta_{2k-1},\beta_{2k}),(\alpha_{2k+1},\beta_{2k+1}),\ldots,(\alpha_m,\beta_m)).$$

In (2.13.3), we group together the summands that correspond to equivalent permutations. Then we have

$$(\varepsilon^m)_{\alpha_1\ldots\alpha_m\beta_1\ldots\beta_m} = \frac{1}{(2m)!}\sigma(\alpha_1\ldots\alpha_m)\sigma(\beta_1\ldots\beta_m)\sum_{k=0}^{[m/2]}c_{km}\times$$

$$\times\delta_{\alpha_1\alpha_2}\ldots\delta_{\alpha_{2k-1}\alpha_{2k}}\delta_{\beta_1\beta_2}\ldots\delta_{\beta_{2k-1}\beta_{2k}}\delta_{\alpha_{2k+1}\beta_{2k+1}}\ldots\delta_{\alpha_m\beta_m}, \tag{2.13.4}$$

where c_{km} is the number of elements in the set B_k. We calculate this number as follows. From m elements $\alpha_1,\ldots,\alpha_m$, we have to choose $2k$ elements that enter in the pairs (α_i,α_j); it can be done in $\binom{m}{2k}$ ways. Similarly, from $\beta_1,\ldots\ldots,\beta_m$, the elements entering in the pairs (β_i,β_j) can be chosen in $\binom{m}{2k}$ ways. k pairs (α_i,α_j) can occupy m positions in $\binom{m}{k}$

ways. The remaining $m-k$ positions can be occupied by k pairs (β_i, β_j) in $\binom{m-k}{k}$ ways. The chosen $2k$ elements α_i can be arranged in the chosen pairs in $(2k)!$ ways. Similarly, the chosen $2k$ elements β_i can be arranged in the chosen pairs in $(2k)!$ ways. The remaining $m-2k$ elements α_i and $m-2k$ elements β_j can be grouped into pairs of the type (α_i, β_j) in $2^{m-2k}((m-2k)!)^2$ ways. As a result, we have

$$c_{km} = \binom{m}{2k}\binom{m}{2k}\binom{m}{k}\binom{m-k}{k}(2k)!(2k)!2^{m-2k}((m-2k)!)^2 = \frac{(m!)^3 2^{m-2k}}{(k!)^2(m-2k)!}.$$

Inserting this expression into (2.13.4), we arrive at (2.13.2).

With the help of (2.13.2), we can easily complete the proof of the lemma:

$$(\varepsilon^m/f)_{\alpha_1\dots\alpha_m} = (\varepsilon^m)_{\alpha_1\dots\alpha_m\beta_1\dots\beta_m} f_{\beta_1\dots\beta_m} =$$

$$= \sigma(\alpha_1\dots\alpha_m)\sum_k c_{km}\delta_{\alpha_1\alpha_2}\dots\delta_{\alpha_{2k-1}\alpha_{2k}}\delta_{\beta_1\beta_2}\dots\delta_{\beta_{2k-1}\beta_{2k}}\delta_{\alpha_{2k+1}\beta_{2k+1}}\dots\delta_{\alpha_m\beta_m} f_{\beta_1\dots\beta_m} =$$

$$= \sigma(\alpha_1\dots\alpha_m)\sum_k c_{km}\delta_{\alpha_1\alpha_2}\dots\delta_{\alpha_{2k-1}\alpha_{2k}} f_{\beta_1\beta_1\dots\beta_k\beta_k\alpha_{2k+1}\dots\alpha_m} = \left[\sum_k c_{km}\varepsilon^k(j^k f)\right]_{\alpha_1\dots\alpha_m}.$$

Lemma 2.13.2 $j\varepsilon^k = \dfrac{n+2k-3}{2k-1}\,\varepsilon^{k-1}$.

P r o o f. We choose a coordinate system as in the proof of Lemma 2.13.1. Then

$$(\varepsilon^k)_{\alpha_1\dots\alpha_{2k-2}\beta\gamma} = \frac{1}{(2k)!}\sigma(\alpha_1\dots\alpha_{2k-2})\times$$

$$\times(c_1\delta_{\alpha_1\alpha_2}\dots\delta_{\alpha_{2k-3}\alpha_{2k-2}}\delta_{\beta\gamma} + c_2\delta_{\alpha_1\beta}\delta_{\alpha_2\gamma}\delta_{\alpha_3\alpha_4}\dots\delta_{\alpha_{2k-3}\alpha_{2k-2}}). \qquad (2.13.5)$$

Here c_1 denotes the number of permutations, of the set $\{\alpha_1,\dots,\alpha_{2k-2},\beta,\gamma\}$, in which β and γ enter in the same pair; and c_2 denotes the number of the other permutations. These numbers are easily computable from the arguments similar to those used in the proof of Lemma 2.13.1: $c_1 = (2k)!/(2k-1)$; $c_2 = (2k-2)(2k)!/(2k-1)$. Insert the last values into (2.13.5). Putting then $\gamma = \beta$ in (2.13.5) and summing over β, we arrive at the claim of the lemma.

Lemma 2.13.3 *For $u \in S^m$ and $v \in S^l$ the equality*

$$j(uv) = \frac{1}{(m+l)(m+l-1)}\left[m(m-1)(ju)v + 2ml\,u\wedge v + l(l-1)u(jv)\right]$$

is valid where $u\wedge v$ denotes the tensor in S^{m+l-2} that is defined by the formula

$$(u\wedge v)_{i_1\dots i_{m+l-2}} = \sigma(u_{i_1\dots i_{m-1}j}v_{i_m\dots i_{m+l-2}j}).$$

We omit the proof that is obtained from straightforward calculation in coordinates.

Lemma 2.13.4 *For $u \in S^m_y$, the next relation is valid: $\varepsilon^k\wedge u = \varepsilon^{k-1}u$.*

The proof is omitted.

Lemma 2.13.5 *For $f \in S_y^m$ and $0 \le l \le [m/2]$ the equality*

$$j^l(\varepsilon^m/f) = \sum_{k=0}^{[m/2]-l} c_{lk}^m \varepsilon^k(j^{l+k}f), \tag{2.13.6}$$

is valid where

$$c_{lk}^m = \frac{2^m(m!)^2}{(2m)!}\frac{2^{-2k-l}(m-2l)!}{k!(k+l)!(m-2k-2l)!}\frac{(n+2m-1)(n+2m-3)\dots(n+2m-2l-1)}{n+2m-1}. \tag{2.13.7}$$

P r o o f proceeds by induction on l. For $l=0$ the claim coincides with Lemma 2.13.1. Let (2.13.6) and (2.13.7) be valid for $l = p < [m/2]$. Applying the operator j to equality (2.13.6), we obtain

$$j^{p+1}(\varepsilon^m/f) = \sum_{k=0}^{[m/2]-p} c_{pk}^m j(\varepsilon^k(j^{p+k}f)). \tag{2.13.8}$$

By Lemmas 2.13.3, 2.13.2 and 2.13.4,

$$j(\varepsilon^k(j^{p+k}f)) = \frac{1}{(m-2p)(m-2p-1)}[2k(n+2m-4p-2k-3)\varepsilon^{k-1}(j^{p+k}f) +$$

$$+ (m-2p-2k)(m-2p-2k-1)\varepsilon^k(j^{p+k+1}f)].$$

Inserting this expression into (2.13.8), we arrive at equality (2.13.6), for $l = p+1$ in which

$$c_{p+1,k}^m = \frac{1}{(m-2p)(m-2p-1)}[2(k+1)(n+2m-4p-2k-5)c_{p,k+1}^m +$$

$$+ (m-2p-2k)(m-2p-2k-1)c_{pk}^m].$$

In the last equality, replacing $c_{p,k+1}^m$ and c_{pk}^m by their expressions from the inductive hypothesis (2.13.7), we obtain validity of (2.13.7) for $l = p+1$. The lemma is proved.

Lemma 2.13.6 *For $f \in S_y^m$ and $0 \le l \le [m/2]$ the equality*

$$j^{[m/2]-l}f = \sum_{k=0}^{l}(-1)^k b_{lk}^m \varepsilon^k \left(j^{[m/2]-l+k}(\varepsilon^m/f)\right) \tag{2.13.9}$$

is valid where

$$b_{lk}^m = \frac{(2m)!}{2^m(m!)^2}\frac{2^{[m/2]}([m/2]-l)!(2l+\kappa_m)!}{2^{l+k}k!(2l-2k+\kappa_m)!}\times$$

$$\times\frac{n+2m-1}{(n+2m-1)(n+2m-3)\dots(n+m+\kappa_m+2l-2k-1)} \tag{2.13.10}$$

and

$$\kappa_m = m - 2[m/2] = \begin{cases} 0, & \text{if } m \text{ is even,} \\ 1, & \text{if } m \text{ is odd.} \end{cases}$$

P r o o f proceeds by induction on l. By putting $l = [m/2]$ in Lemma 2.13.5, we obtain the claim being proved for $l = 0$. Assume that the claim is valid for all l satisfying the inequalities $0 \le l < p \le [m/2]$. Putting $l = [m/2] - p$ in (2.13.6), we have

$$j^{[m/2]-p} f = \left(c^m_{[m/2]-p,0}\right)^{-1} \left(j^{[m/2]-p}(\varepsilon^m/f) - \sum_{k=1}^{p} c^m_{[m/2]-p,k}\varepsilon^k(j^{[m/2]-p+k} f)\right). \qquad (2.13.11)$$

By the inductive hypothesis, for $1 \le k \le p$,

$$j^{[m/2]-p+k} f = \sum_{s=0}^{p-k} (-1)^s b^m_{p-k,s}\varepsilon^s \left(j^{[m/2]-p+k+s}(\varepsilon^m/f)\right).$$

Inserting this expression into (2.13.11), after the easy transformations we arrive at equality (2.13.9), for $l = p$, with the coefficients

$$b^m_{p0} = \left(c^m_{[m/2]-p,0}\right)^{-1}; \quad b^m_{pk} = -b^m_{p0}\sum_{t=1}^{k}(-1)^t b^m_{p-t,k-t} c^m_{[m/2]-p,t} \qquad (1 \le k \le p). \qquad (2.13.12)$$

Replacing the coefficients $c^m_{[m/2]-p,t}$ and $b^m_{p-t,k-t}$ in (2.13.12) by their values given by (2.13.7) and the inductive hypothesis (2.13.10), we obtain (2.13.10) for $l = p$. The lemma is proved.

P r o o f of Theorem 2.12.1. Putting $l = [m/2]$ in (2.13.9) and (2.13.10), we obtain relation (2.12.8) in which h is replaced by ε^m/f. As was noted in the beginning of the current section, this equality proves the theorem.

2.14 Inversion of the ray transform on the space of field-distributions

The inversion formulas (2.12.15) and (2.12.19) for the ray transform, having been proved for $f \in \mathcal{S}(S^m)$, can be easily transferred to broader classes of tensor fields. Here we will briefly describe how the main definitions and results concerning this formulas can be extended to the space $\mathcal{E}'(S^m)$. We omit the proofs that are based on the fact of density of the space $\mathcal{D}(S^m)$ in $\mathcal{S}(S^m)$ as well as in $\mathcal{E}'(S^m)$.

As we have shown in Section 2.5, the ray transform $I : \mathcal{D}(S^m) \to C^\infty(\mathbf{R}^n \times \mathbf{R}^n_0)$ can be uniquely extended to the continuous operator $I : \mathcal{E}'(S^m) \to \mathcal{D}'(\mathbf{R}^n \times \mathbf{R}^n_0)$. Similarly, the operator $\mu^m : C^\infty(\mathbf{R}^n \times \mathbf{R}^n_0) \to C^\infty(S^m)$ defined in Section 2.11 has a unique continuous extension $\mu^m : \mathcal{D}'(\mathbf{R}^n \times \mathbf{R}^n_0) \to \mathcal{D}'(S^m)$.

We say that a field $u \in \mathcal{S}'(S^m)$ tends to zero at infinity if $u(x)$ is continuous outside some compact set and $u(x) \to 0$ as $|x| \to \infty$. For $\mathcal{E}'(S^m)$, the theorem on decomposition of a tensor field into potential and solenoidal parts remains valid in the following formulation:

Theorem 2.14.1 *Let $n \ge 2$. For $f \in \mathcal{E}'(S^m)$, there exist uniquely determined fields ${}^sf \in \mathcal{S}'(S^m)$ and $v \in \mathcal{S}'(S^{m-1})$ tending to zero at infinity and satisfying* (2.6.6). *The fields ${}^sf$ and v are smooth outside* $\operatorname{supp} f$ *and satisfy estimates* (2.6.8) *outside some compact set.*

For $f \in \mathcal{E}'(S^m)$, equality (2.11.6) remains valid which implies that the field $\mu^m If$ belongs to the space $\mathcal{A}(S^m)$ defined after the formulation of Theorem 2.12.2. Consequently, for $f \in \mathcal{E}'(S^m)$, the right-hand sides of equalities (2.12.15) and (2.12.19) are defined and belong to $\mathcal{A}(S^m) \subset \mathcal{S}'(S^m)$. Now Theorems 2.12.2 and 2.12.3 translate word by word to the case when $f \in \mathcal{E}'(S^m)$.

We will finish the current section by a more detailed discussion of formula (2.12.19) in the case of tensor fields of degree 1 and 2 on a plane. We shall need the results of the discussion in Section 2.16 that is devoted to a physical application of the ray transform. According to the physical tradition, in Section 2.16 and in the remainder of the current section we will use slightly-modified notations: indices of tensors will be marked with the same letters x, y, z as coordinates.

First we consider the case of a vector field on a plane ($n = 2, m = 1$). Let D be a domain, on the (x, y)-plane, bounded by a closed C^1-smooth curve γ; and $u = (u_x, u_y)$ be some vector field that is defined and continuous in the closed domain $D \bigcup \gamma$ and has continuous first-order derivatives in D. It is convenient to consider u as the field defined on the whole plane, putting $u = 0$ outside $D \bigcup \gamma$. Thus the field u has discontinuity on the curve γ. Consequently, the derivatives in the Saint Venant operator must be understood in the distribution sense. According to the observation, let us agree about notations: by ∂_x, ∂_y we denote the derivatives in the distribution sense and by $\partial/\partial x, \partial/\partial y$ we denote the classical derivatives.

We introduce coordinates (x, y, α) on $\mathbf{R}^2 \times \Omega^1$, putting $\xi = (\cos\alpha, \sin\alpha)$ for $\xi \in \Omega^1$. The ray transform of a vector field on a plane is given by the formula

$$Iu(x, y, \alpha) = \int\limits_{-\infty}^{\infty} [u_x(x + t\cos\alpha, y + t\sin\alpha)\cos\alpha + u_y(x + t\cos\alpha, y + t\sin\alpha)\sin\alpha]\, dt. \tag{2.14.1}$$

In the case under consideration ($n = 2,\ m = 1$) the Saint Venant operator has one nonzero component. Consequently, Wu can be considered as the scalar function: $Wu = \partial_x u_y - \partial_y u_x$. It is clear that

$$Wu = W_D u + W_\gamma u \tag{2.14.2}$$

where

$$W_D u = \begin{cases} \dfrac{\partial u_y}{\partial x} - \dfrac{\partial u_x}{\partial y}, & \text{if} \quad (x, y) \in D, \\ 0, & \text{if} \quad (x, y) \notin D, \end{cases} \tag{2.14.3}$$

and $W_\gamma u$ is a distribution whose support is a subset of γ. Now we find the distribution.

According to the definition of the derivative of a distribution, for $\varphi \in \mathcal{D}(\mathbf{R}^2)$, we have

$$\langle Wu, \varphi\rangle = \langle \partial_x u_y - \partial_y u_x, \varphi\rangle = \left\langle u_x, \frac{\partial\varphi}{\partial y}\right\rangle - \left\langle u_y, \frac{\partial\varphi}{\partial x}\right\rangle = \iint\limits_D (u_x \frac{\partial\varphi}{\partial y} - u_y \frac{\partial\varphi}{\partial x})\, dxdy.$$

Transforming the last integral with the help of the Green's formula, we obtain

$$\langle Wu, \varphi\rangle = \iint\limits_D \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \varphi\, dxdy + \oint\limits_\gamma (u_x\nu_y - u_y\nu_x)\varphi\, ds \tag{2.14.4}$$

where $\nu = (\nu_x, \nu_y)$ is the unit outer normal vector of γ and s is the arc length of γ. Recall that the δ-function concentrated on γ is the distribution δ_γ defined by the equality $\langle \delta_\gamma, \varphi\rangle = \oint_\gamma \varphi\, ds$. Comparing (2.14.2)–(2.14.4), we see that

$$W_\gamma u = (u_x\nu_y - u_y\nu_x)\delta_\gamma. \tag{2.14.5}$$

The algorithm for determination of $W_D u$ from the ray transform $Iu(x,y,\alpha)$ afforded by the formula (2.12.19) is described as follows. First we define the vector field $\mu = (\mu_x, \mu_y)$ according to the formulas

$$\mu_x(x,y) = \frac{1}{4\pi}\int\limits_0^{2\pi} Iu(x,y,\alpha)\cos\alpha\,d\alpha;$$
$$\mu_y(x,y) = \frac{1}{4\pi}\int\limits_0^{2\pi} Iu(x,y,\alpha)\sin\alpha\,d\alpha. \tag{2.14.6}$$

Then we find the field $\lambda = (\lambda_x, \lambda_y)$, putting

$$\lambda_x = (-\Delta)^{1/2}\mu_x, \quad \lambda_y = (-\Delta)^{1/2}\mu_y. \tag{2.14.7}$$

At last we find

$$W_D u = W_D\lambda = \frac{\partial\lambda_y}{\partial x} - \frac{\partial\lambda_x}{\partial y}. \tag{2.14.8}$$

We can determine the second summand on the right-hand side of (2.14.2) from Iu, starting with the same formula (2.12.19); but it is can easier to proceed as follows. We assume that the domain D is strictly convex. By $\gamma(s) = (\gamma_x(s), \gamma_y(s))$ we mean the parametrization, of the curve γ by the arc length s, which is chosen in a way such that the increase of s corresponds to counterclockwise going along the curve. By $\alpha(s, \Delta s)$ we mean the angle from the Ox-axis to the vector $\xi(s,\Delta s) = (\gamma(s+\Delta s)-\gamma(s))/|(\gamma(s+\Delta s)-\gamma(s)|$. Then

$$Iu(\gamma_x(s), \gamma_y(s), \alpha(s,\Delta s)) =$$
$$= \int\limits_0^{\Delta' s} \left[u_x(\gamma(s) + t\xi(s,\Delta s))\cos\alpha(s,\Delta s) + u_y(\gamma(s) + t\xi(s,\Delta s))\sin\alpha(s,\Delta s)\right]dt$$

where $\Delta' s = |\gamma(s+\Delta s) - \gamma(s)|$. The last equality implies that

$$(u_x\dot\gamma_x + u_y\dot\gamma_y)(s) = \lim_{\Delta s\to 0}\frac{1}{\Delta s} Iu\left(\gamma_x(s), \gamma_y(s), \alpha(s,\Delta s)\right)$$

where the dot denotes differentiation with respect to s. Since $\dot\gamma_x = -\nu_y$ and $\dot\gamma_y = \nu_x$, the previous relation can be rewritten in the form

$$(u_x\nu_y - u_y\nu_x)(s) = -\lim_{\Delta s\to 0}\frac{1}{\Delta s} Iu(\gamma_x(s), \gamma_y(s), \alpha(s,\Delta s)). \tag{2.14.9}$$

Equalities (2.14.5) and (2.14.9) express $W_\gamma u$ through Iu.

Now we consider the case of a tensor field of degree 2 on a plane ($n = m = 2$). Let D be the same domain as above; $v = (v_{xx}, v_{xy}, v_{yy})$ be a symmetric tensor field which is defined and continuous, together with its first-order derivatives, in $D\bigcup\gamma$, and has continuous second-order derivatives in D. We put $v = 0$ outside $D\bigcup\gamma$. In the case under consideration the ray transform is defined by the formula

$$Iv(x,y,\alpha) = \int\limits_{-\infty}^{\infty}\left[v_{xx}(r)\cos^2\alpha + 2v_{xy}(r)\cos\alpha\sin\alpha + v_{yy}(r)\sin^2\alpha\right]_{r=(x+t\cos\alpha, y+t\sin\alpha)} dt. \tag{2.14.10}$$

The Saint Venant operator has one nonzero component:

$$Wv = 2\partial_{xy}v_{xy} - \partial_{yy}v_{xx} - \partial_{xx}v_{yy}.$$

With the help of the arguments similar to those used in obtaining (2.13.5), one can easily show that

$$Wv = W_D v + W_\gamma v, \tag{2.14.11}$$

where

$$W_D v = \begin{cases} 2\dfrac{\partial^2 v_{xy}}{\partial x \partial y} - \dfrac{\partial^2 v_{xx}}{\partial y^2} - \dfrac{\partial^2 v_{yy}}{\partial x^2}, & \text{if} \quad (x,y) \in D, \\ 0, & \text{if} \quad (x,y) \notin D, \end{cases} \tag{2.14.12}$$

and $W_\gamma v$ is a distribution whose support is a subset of γ. The distribution is defined by the equality

$$\langle W_\gamma v, \varphi \rangle = \oint\limits_\gamma \left[\left(\frac{\partial v_{yy}}{\partial x} - \frac{\partial v_{xy}}{\partial y} \right) \nu_x + \left(\frac{\partial v_{xx}}{\partial y} - \frac{\partial v_{xy}}{\partial x} \right) \nu_y \right] \varphi \, ds -$$
$$- \oint\limits_\gamma \left[(v_{yy}\nu_x - v_{xy}\nu_y) \frac{\partial \varphi}{\partial x} + (v_{xx}\nu_y - v_{xy}\nu_x) \frac{\partial \varphi}{\partial y} \right] ds. \tag{2.14.13}$$

The algorithm, for determination of W_D from the ray transform Iv, afforded by the formula (2.12.19) is described as follows. Fist we define the tensor field $\mu = (\mu_{xx}, \mu_{xy}, \mu_{yy})$ by formulas

$$\begin{aligned} \mu_{xx}(x,y) &= \frac{1}{4\pi} \int\limits_0^{2\pi} Iv(x,y,\alpha) \cos^2 \alpha \, d\alpha, \\ \mu_{xy}(x,y) &= \frac{1}{4\pi} \int\limits_0^{2\pi} Iv(x,y,\alpha) \cos \alpha \sin \alpha \, d\alpha, \\ \mu_{yy}(x,y) &= \frac{1}{4\pi} \int\limits_0^{2\pi} Iv(x,y,\alpha) \sin^2 \alpha \, d\alpha. \end{aligned} \tag{2.14.14}$$

Then we define the field $\lambda = (\lambda_{xx}, \lambda_{xy}, \lambda_{yy})$ by the equalities

$$\lambda_{xx} = (-\Delta)^{1/2}(\mu_{xx} - \frac{1}{2}\mu_{yy}), \quad \lambda_{xy} = \frac{3}{2}(-\Delta)^{1/2}\mu_{xy}, \quad \lambda_{yy} = (-\Delta)^{1/2}(\mu_{yy} - \frac{1}{2}\mu_{xx}). \tag{2.14.15}$$

At last we put

$$W_D v = W_D \lambda = 2\frac{\partial^2 \lambda_{xy}}{\partial x \partial y} - \frac{\partial^2 \lambda_{xx}}{\partial y^2} - \frac{\partial^2 \lambda_{yy}}{\partial x^2}. \tag{2.14.16}$$

The second summand on the right-hand side of equality (2.14.11) is expressible, in terms Iv, which possibility is neglected here.

2.15 The Plancherel formula for the ray transform

Let $H(T\Omega)$ be the Hilbert space obtained by completing the space $\mathcal{S}(T\Omega)$ with respect to the norm corresponding to the scalar product

$$(\varphi, \psi)_{H(T\Omega)} = (2\pi)^{-1} \int\limits_\Omega \int\limits_{\xi^\perp} |y| \hat{\varphi}(x,\xi) \overline{\hat{\psi}(y,\xi)} \, dy d\omega(\xi), \tag{2.15.1}$$

where $\varphi \mapsto \hat{\varphi}$ is the Fourier transform on $\mathcal{S}(T\Omega)$ defined in Section 2.1.

Theorem 2.15.1 *The ray transform $I : \mathcal{S}(S^m) \to \mathcal{S}(T\Omega)$ can be extended in a unique way to a continuous operator $I : L_2(S^m) \to H(T\Omega)$. The kernel of the resulting operator coincides with the second summand of decomposition (2.6.11). For $f^1, f^2 \in L_2(S^m)$, the next Plancherel formula is valid:*

$$(If^1, If^2)_{H(T\Omega)} = \sum_{k=0}^{[m/2]} a_k (j^k P f^1, j^k P f^2)_{L_2(S^{m-2k})}, \tag{2.15.2}$$

where $[m/2]$ is the integral part of $m/2$, $P : L_2(S^m) \to K(S^m)$ is the orthogonal projection onto the second summand of decomposition (2.6.11), $j : S^m \to S^{m-2}$ is the operator of convolution with the metric tensor and the coefficients a_k are given by the equalities

$$a_k = a_k(m,n) = \frac{2^{m+1}\pi^{(n-2)/2}\Gamma\left(m+\frac{1}{2}\right)}{(2m)!\Gamma\left(m+\frac{n-1}{2}\right)} \frac{1}{2^{2k}(k!)^2(m-2k)!}. \tag{2.15.3}$$

The proof of the theorem will be divided into several lemmas.

Lemma 2.15.2 *Let $f^\alpha \in \mathcal{S}(S^m)$ $(\alpha = 1,2)$, ${}^s\!f^\alpha$ be the solenoidal part of the field f^α (i.e., ${}^s\!f^\alpha = Pf^\alpha$) and $\widehat{{}^s\!f^\alpha}$ be the Fourier transform of the field ${}^s\!f^\alpha$. The next relation is valid:*

$$(If^1, If^2)_{H(T\Omega)} = \int\limits_{\Omega}\int\limits_{\xi^\perp} |y| \, \langle \widehat{{}^s\!f^1}(y), \xi^m \rangle \, \overline{\langle \widehat{{}^s\!f^2}(y), \xi^m \rangle} \, dy \, d\omega(\xi). \tag{2.15.4}$$

P r o o f. First of all we note that the integral on the right-hand side of this equality converges, by the properties, of the field $\widehat{{}^s\!f^\alpha}$, mentioned in Theorem 2.6.2. By (2.15.1) and (2.1.15),

$$(If^1, If^2)_{H(T\Omega)} = \int\limits_{\Omega}\int\limits_{\xi^\perp} |y| \, \langle \widehat{f^1}(y), \xi^m \rangle \, \overline{\langle \widehat{f^2}(y), \xi^m \rangle} \, dy \, d\omega(\xi). \tag{2.15.5}$$

According to the last claim of Lemma 2.6.1, the fields $\widehat{f^\alpha}$ and $\widehat{{}^s\!f^\alpha}$ are connected by the relation $\langle \widehat{f^\alpha}(y), \xi^m \rangle = \langle \widehat{{}^s\!f^\alpha}(y), \xi^m \rangle$ for $(y,\xi) \in T\Omega$. Inserting this expression into the right-hand side of (2.15.5), we arrive at (2.15.4). The lemma is proved.

Lemma 2.15.3 *Let $\varphi(x,\xi)$ be a continuous function on $T\Omega$ that decreases sufficiently rapidly in x. Then the next equality is valid:*

$$\int\limits_{\Omega}\int\limits_{\xi^\perp} \varphi(x,\xi) \, dx \, d\omega(\xi) = \int\limits_{\mathbf{R}^n} \frac{1}{|x|} \int\limits_{\Omega \cap x^\perp} \varphi(x,\xi) \, d\omega^{(n-2)}(\xi) \, dx, \tag{2.15.6}$$

where $d\omega^{(n-2)}(\xi)$ denotes the $(n-2)$-dimensional angle measure on $\Omega \cap x^\perp$.

P r o o f. Let $\psi(x,\xi)$ be a continuous extension of $\varphi(x,\xi)$ to $\mathbf{R}^n \times \Omega$. With the help of the δ-function, we replace the inner integral on the left-hand side of (2.15.6) by the integral over $\mathbf{R}^n$ and then change the limits of integration. As a result we obtain

$$\int\limits_{\Omega}\int\limits_{\xi^\perp} \varphi(x,\xi) \, dx \, d\omega(\xi) = \int\limits_{\mathbf{R}^n}\int\limits_{\Omega} \psi(x,\xi) \, \delta(\langle x, \xi \rangle) \, d\omega(\xi) \, dx.$$

By Theorem 6.1.5 of [53], $\delta(\langle x,\xi\rangle)d\omega(\xi) = d\omega^{(n-2)}(\xi)/|x|$. Consequently,

$$\int\limits_{\Omega}\int\limits_{\xi^\perp} \varphi(x,\xi)\,dx\,d\omega(\xi) = \int\limits_{\mathbf{R}^n} \frac{1}{|x|} \int\limits_{\Omega\cap x^\perp} \psi(x,\xi)\,d\omega^{(n-2)}(\xi)\,dx =$$

$$= \int\limits_{\mathbf{R}^n} \frac{1}{|x|} \int\limits_{\Omega\cap x^\perp} \varphi(x,\xi)\,d\omega^{(n-2)}(\xi)\,dx.$$

The lemma is proved.

We define the field $\varepsilon \in C^\infty(S^2;\mathbf{R}^n_0)$ by putting $\varepsilon_{ij} = \delta_{ij} - x_i x_j/|x|^2$.

Lemma 2.15.4 *For* $0 \neq x \in \mathbf{R}^n$,

$$\int\limits_{\Omega\cap x^\perp} \xi_{i_1}\dots\xi_{i_{2m}} d\omega^{(n-2)}(\xi) = \frac{2\Gamma\left(m+\frac{1}{2}\right)\pi^{(n-2)/2}}{\Gamma\left(m+\frac{n-1}{2}\right)}\,\varepsilon^m_{i_1\dots i_{2m}}(x). \tag{2.15.7}$$

P r o o f. Both the left- and right-hand sides of equality (2.15.7) do not depend on $|x|$. Consequently, it is sufficient to consider the case of $|x| = 1$. Note that, for coincidence $f = g$ of two symmetric tensors of degree m, it is necessary and sufficient that $f_{i_1\dots i_m}\eta_{i_1}\dots\eta_{i_m} = g_{i_1\dots i_m}\eta_{i_1}\dots\eta_{i_m}$, for every $\eta \in \mathbf{R}^n$. Applying this rule, we see that to prove (2.15.7) it suffices to show that

$$\int\limits_{\Omega\cap x^\perp} \langle\xi,\eta\rangle^{2m}\,d\omega^{(n-2)}(\xi) = \frac{2\Gamma\left(m+\frac{1}{2}\right)\pi^{(n-2)/2}}{\Gamma\left(m+\frac{n-1}{2}\right)}\,\langle\varepsilon^m(x),\eta^{2m}\rangle. \tag{2.15.8}$$

for every $\eta \in \mathbf{R}^n$

From the definition of the tensor ε, one can easily obtain that, for $|x| = 1$,

$$\langle\varepsilon^m(x),\eta^{2m}\rangle = |\eta - \langle\eta,x\rangle x|^{2m}. \tag{2.15.9}$$

On the other hand,

$$\int\limits_{\Omega\cap x^\perp} \langle\xi,\eta\rangle^{2m}\,d\omega^{(n-2)}(\xi) = |\eta - \langle\eta,x\rangle x|^{2m} \int\limits_{\Omega\cap x^\perp} \langle\xi,\eta'\rangle^{2m}\,d\omega^{(n-2)}(\xi) \tag{2.15.10}$$

where the notation $\eta' = \eta'(x,\eta) = (\eta - \langle x,\eta\rangle x)/|\eta - \langle x,\eta\rangle x|$ is used. Note that $\eta' \in \Omega\cap x^\perp$. One can easily see that the integral on the right-hand side of (2.15.10) does not depend of the choice of the vector $\eta' \in \Omega\cap x^\perp$ and is equal to [112]

$$\int\limits_{\Omega\cap x^\perp} \langle\xi,\eta'\rangle^{2m}\,d\omega^{(n-2)}(\xi) = \int\limits_{\Omega^{n-2}} \xi_1^{2m}\,d\omega(\xi) = 2\frac{\Gamma\left(m+\frac{1}{2}\right)\pi^{(n-2)/2}}{\Gamma\left(m+\frac{n-1}{2}\right)}.$$

Inserting this expression into (2.15.10) and substituting then (2.15.9), (2.15.10) into (2.15.8), we obtain the claim of the lemma.

Lemma 2.15.5 *The Plancherel formula* (2.15.2) *is valid for* $f^\alpha \in \mathcal{S}(S^m)$ $(\alpha = 1,2)$.

P r o o f. Let ${}^s\!f^\alpha = Pf^\alpha$. Using Lemma 2.15.3, we change the limits of integration in claim (2.15.4) of Lemma 2.15.2:

$$(If^1, If^2)_{H(T\Omega)} = \int\limits_{\mathbf{R}^n} \widehat{{}^s\!f^1}_{i_1\dots i_m}(y)\, \overline{\widehat{{}^s\!f^2}_{j_1\dots j_m}(y)} \left[\int\limits_{\Omega\cap y^\perp} \xi_{i_1}\dots\xi_{i_m}\xi_{j_1}\dots\xi_{j_m}\,d\omega^{(n-2)}(\xi)\right] dy.$$

Replacing the inner integral by its value indicated in Lemma 2.15.4, we have

$$(If^1, If^2)_{H(T\Omega)} = 2\frac{\Gamma\left(m+\frac{1}{2}\right)\pi^{(n-2)/2}}{\Gamma\left(m+\frac{n-1}{2}\right)} \int\limits_{\mathbf{R}^n} \widehat{{}^s\!f^1}_{i_1\dots i_m}(y)\, \overline{\widehat{{}^s\!f^2}_{j_1\dots j_m}(y)}\varepsilon^m_{i_1\dots i_m j_1\dots j_m}(y)\,dy. \tag{2.15.11}$$

By the condition $\delta\,{}^s\!f^\alpha = 0$, the relations $y_j\widehat{{}^s\!f^\alpha}_{ji_2\dots i_m}(y) = 0$ are valid. Consequently, the tensor ε^m on (2.15.11) can be replaced by δ^m where $\delta = (\delta_{ij})$ is the Kronecker tensor. Repeating the arguments that were used in obtaining relation (2.13.2), we obtain

$$(\delta^m)_{i_1\dots i_m j_1\dots j_m} = \frac{2^m(m!)^3}{(2m)!}\sigma(i_1\dots i_m)\sigma(j_1\dots j_m)\sum_{k=0}^{[m/2]}\frac{1}{2^{2k}(k!)^2(m-2k)!}\times$$
$$\times\delta_{i_1i_2}\dots\delta_{i_{2k-1}i_{2k}}\delta_{j_1j_2}\dots\delta_{j_{2k-1}j_{2k}}\delta_{i_{2k+1}j_{2k+1}}\dots\delta_{i_mj_m}.$$

Substituting this expression for $\varepsilon^m_{i_1\dots j_m}$ into (2.15.11), we have

$$(If^1, If^2)_{H(T\Omega)} =$$
$$= \sum_{k=0}^{[m/2]} a_k \int\limits_{\mathbf{R}^n} \widehat{{}^s\!f^1}_{i_1\dots i_m}(y)\overline{\widehat{{}^s\!f^2}_{j_1\dots j_m}(y)}\delta_{i_1i_2}\dots\delta_{i_{2k-1}i_{2k}}\delta_{j_1j_2}\dots\delta_{j_{2k-1}j_{2k}}\delta_{i_{2k+1}j_{2k+1}}\dots\delta_{i_mj_m}dy =$$
$$= \sum_{k=0}^{[m/2]} a_k \left(j^k(\widehat{{}^s\!f^1}), j^k(\widehat{{}^s\!f^2})\right)_{L_2(S^{m-2k})}$$

with the coefficients a_k defined by formula (2.15.3). Since the operator j is purely algebraic, it commutes with the Fourier transform. Using the Plancherel formula for the Fourier transform $(\hat{u},\hat{v})_{L_2(S^k)} = (u,v)_{L_2(S^k)}$, we finally obtain

$$(If^1, If^2)_{H(T\Omega)} = \sum_{k=0}^{[m/2]} a_k \left(j^k({}^s\!f^1), j^k({}^s\!f^2)\right)_{L_2(S^{m-2k})}.$$

The lemma is proved.

P r o o f of Theorem 2.15.1. Uniqueness of a continuous extension, of the operator $I : \mathcal{S}(S^m) \to H(T\Omega)$, to $L_2(S^m)$ follows from density of $\mathcal{S}(S^m)$ in $L_2(S^m)$. Let us prove existence of such an extension. For $f \in L_2(S^m)$, we choose a sequence $f_k \in \mathcal{S}(S^m)$ $(k = 1, 2, \dots)$ that converges to f in $L_2(S^m)$. By Lemma 2.15.5, If_k is a Cauchy sequence in $H(T\Omega)$. Consequently, there exists $\varphi \in H(T\Omega)$ such that $If_k \to \varphi$. We put $\varphi = If$. It can easily be verified that φ is independent of the choice of the sequence f_k converging to f and that the Plancherel formula (2.15.2) is valid, for the so-defined operator $I : L_2(S^m) \to H(T\Omega)$,. From this formula the continuity of I follows together with the fact that the kernel of I coincides with the second summand of decomposition (2.6.11). The theorem is proved.

2.16 Application of the ray transform to an inverse problem of photoelasticity

A certain methods of polarization optics for investigation of stresses (more briefly: methods of photoelasticity) are based on the Brewster effect. It asserts that most transparent materials obtain optical anisotropy in the presence of mechanical stresses. If a plane-polarized electromagnetic wave falls onto such a body, then the wave acquires elliptic polarization after passing the body. The extent of polarization is determined by the values of the stress tensor along a light ray. In photoelasticity there are known a few methods of using this effect for determining stresses. One of these methods is the so-called method of integrated photoelasticity [1]. It can be briefly described as follows: we illuminate the investigated body by the plane-polarized light from various directions and with various planes of polarization of the incoming light, measuring the polarization of the outcoming light. As compared with the others, this method has the advantage of simplicity and preciseness of optical measurements. On the other hand, this method leads to difficult mathematical problems, most of which are still not solved. One of these problems is considered in the current section.

We start with the main differential equations of photoelasticity. The relationship between the stress tensor σ and the dielectric permeability tensor ε is described by the Maxwell-Neumann law:

$$\varepsilon_{jk} = \varepsilon_0\delta_{jk} + C\sigma_{jk} + C_1\sigma_{pp}\delta_{jk}$$

where C and C_1 are the so-called photoelasticity constants. In this section we shall assume that the medium under consideration is homogeneous, i.e., that C, C_1 and ε_0 are independent of a point. We fix a straight line π in $\mathbf{R}^3$ and choose a Cartesian coordinate system $t\eta\zeta$ such that π coincides with the t-axis. The evolution of the polarization ellipse of electromagnetic wave along the light ray π is described by the next equations (see [1]; later, in Chapters 5 and 6, we shall derive these equations in a more general situation)

$$\begin{aligned} \frac{dE_\eta}{dt} &= -\frac{iC}{2}(\sigma_{\eta\eta} - \sigma_{\zeta\zeta})E_\eta + iC\sigma_{\eta\zeta}E_\zeta, \\ \frac{dE_\zeta}{dt} &= iC\sigma_{\eta\zeta}E_\eta + \frac{iC}{2}(\sigma_{\eta\eta} - \sigma_{\zeta\zeta})E_\zeta, \end{aligned} \tag{2.16.1}$$

where E is the normalized electric vector.

We write down (2.16.1) in matrix form

$$\frac{dE}{dt} = CUE \tag{2.16.2}$$

where

$$U = \frac{i}{2}\begin{pmatrix} \sigma_{\zeta\zeta} - \sigma_{\eta\eta} & 2\sigma_{\eta\zeta} \\ 2\sigma_{\eta\zeta} & \sigma_{\eta\eta} - \sigma_{\zeta\zeta} \end{pmatrix}. \tag{2.16.3}$$

The solution to equation (2.16.2) can be represented by the Neumann series

$$E(t) = A(t)E(t_0),$$

$$A(t) = I + C\int_{t_0}^{t} U(\tau)d\tau + C^2\int_{t_0}^{t} U(\tau)d\tau\int_{t_0}^{\tau} U(\tau_1)d\tau_1 + \ldots \tag{2.16.4}$$

where I is the identity matrix. If the constant C is small, then we can neglect the nonlinear terms in (2.16.4) and write

$$A(t) - I = C \int\limits_{t_0}^{t} U(\tau)d\tau. \qquad (2.16.5)$$

We speak about slight optical anisotropy, if equality (2.16.5) is valid.

From (2.16.3) and (2.16.5), we see that the degree of polarization along the light ray π is defined by two integrals

$$L(\sigma, \pi) = \int\limits_{\pi} \sigma_{\eta\zeta} dt, \quad S(\sigma, \pi) = \int\limits_{\pi} (\sigma_{\eta\eta} - \sigma_{\zeta\zeta}) dt. \qquad (2.16.6)$$

Physically measured quantities are the isocline parameter φ and optical retardation Δ connected with integrals (2.16.6) by the equalities

$$\Delta \cos 2\varphi = CS(\sigma, \pi), \quad \Delta \sin 2\varphi = 2CL(\sigma, \pi). \qquad (2.16.7)$$

In [65] and [2] formulas (2.16.6)–(2.16.7) are obtained from physical arguments. Our derivation of these relations from differential equations (2.16.1) is more simple and straightforward.

The technical conditions of measurement impose some restrictions on the family of straight lines π over which integrals (2.16.6) can be measured. From the standpoint of technical realization, the most appropriate situation is that in which the measurements are done for horizontal straight lines, i.e., for those straight lines that are parallel to the plane $z = 0$ in some (laboratory) coordinate system xyz. According to this remark, in the current section we investigate the mathematical formulation of the question: to what extent is the stress tensor field determined by integrals (2.16.6) measured over all horizontal straight lines in the layer $a < z < b$. In studying we shall assume that the stress tensor satisfies the equilibrium equations only, while not considering strains. Consequently, our results are valid for any model (elastic, thermo-elastic etc.) of solid stressed media. The only essential assumptions are as follows: 1) the dielectric permeability tensor depends linearly on the stress tensor, and 2) the approximation of slight optical anisotropy is valid.

We assume that the stressed medium is contained in a cylindrical domain $G = D \times (a, b) = \{(x, y, z) \in \mathbf{R}^3 \mid (x, y) \in D,\ a < z < b\}$, where D is a two-dimensional domain on the (x, y)-plane and the boundary γ of D is a closed strictly convex C^1-smooth curve. Let $B = \gamma \times (a, b) = \{(x, y, z) \mid (x, y) \in \gamma,\ a < z < b\}$ be the lateral surface of the cylinder G. The stress tensor σ is supposed to have continuous second-order derivatives in G and continuous first-order derivatives up to B :

$$\sigma \in C^2(G) \cap C^1(G \cup B). \qquad (2.16.8)$$

The equilibrium equations are assumed to be satisfied in G (we suppose that the volume forces are absent):

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} &= 0, \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} &= 0, \\ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= 0, \end{aligned} \qquad (2.16.9)$$

and the boundary conditions corresponding to the absence of the external forces are assumed to be satisfied on B :

$$\begin{aligned} \sigma_{xx}\nu_x + \sigma_{xy}\nu_y &= 0, \\ \sigma_{yx}\nu_x + \sigma_{yy}\nu_y &= 0, \\ \sigma_{zx}\nu_x + \sigma_{zy}\nu_y &= 0, \end{aligned} \tag{2.16.10}$$

where $\nu = (\nu_x, \nu_y)$ is the unit vector normal to γ. Some of our results do not depend on the boundary conditions (2.16.10). It is convenient to suppose that the tensor field σ is defined in the whole layer $\{(x, y, z) \in \mathbf{R}^3 \mid a < z < b\}$ by putting $\sigma = 0$ outside $G \bigcup B$.

The family of horizontal straight lines in $\mathbf{R}^3$ depends on three parameters, but it will be convenient for us to use four parameters. We denote by $\pi(x_0, y_0, \alpha, z_0)$ the horizontal straight line defined by the parametric equations $x = x_0 + t\cos\alpha,\ y = y_0 + t\sin\alpha,\ z = z_0$. Note that $\pi(x, y, \alpha, z)$ depends actually on three parameters $(x\sin\alpha - y\cos\alpha, \alpha, z)$. One gets two functions

$$L_\sigma(x, y, \alpha; z) = L(\sigma, \pi(x, y, \alpha, z)); \quad S_\sigma(x, y, \alpha; z) = S(\sigma, \pi(x, y, \alpha, z))$$

by performing optical measurements along all horizontal straights lines in the layer $a < z < b$. Our problem is: to what extent is the tensor field σ, satisfying (2.16.8)–(2.16.10), determined by the functions L_σ and S_σ that are known for $a < z < b$ and for all x, y, α?

In the current section we will obtain the next three main results on this problem.

1. The functions L_σ and S_σ are related. More exactly, if we know $L_\sigma(x, y, \alpha; z)$ and $S_\sigma(x, y, \alpha; z_0)$, then we can find $S_\sigma(x, y, \alpha; z)$; here z_0 is any number satisfying the condition $a < z_0 < b$.

2. The component σ_{zz} is uniquely determined by the functions $L_\sigma(x, y, \alpha; z)$ and $S_\sigma(x, y, \alpha; z)$. An algorithm for determination is given.

3. No information about the tensor field σ, except for σ_{zz}, can be extracted from the functions L_σ and S_σ. More exactly: if two tensor fields σ^1 and σ^2 satisfy conditions (2.16.8)–(2.16.10) and $\sigma^1_{zz} \equiv \sigma^2_{zz}$, then $L_{\sigma^1} \equiv L_{\sigma^2}$ and $S_{\sigma^1} \equiv S_{\sigma^2}$.

The straight line $\pi(x_0, y_0, \alpha, z_0)$ coincides with the t-axis of the coordinate system $t\eta\zeta$ that is related to xyz by the formulas

$$x = t\cos\alpha - \eta\sin\alpha + x_0, \quad y = t\sin\alpha + \eta\cos\alpha + y_0, \quad z = z_0,$$

The components of the tensor σ in these coordinates are related as follows:

$$\begin{aligned} \sigma_{\eta\eta}(t, 0, 0) &= \sigma_{xx}(r_0 + t\xi)\sin^2\alpha - 2\sigma_{xy}(r_0 + t\xi)\cos\alpha\sin\alpha + \\ &\quad + \sigma_{yy}(r_0 + t\xi)\cos^2\alpha, \\ \sigma_{\eta\zeta}(t, 0, 0) &= -\sigma_{xz}(r_0 + t\xi)\sin\alpha + \sigma_{yz}(r_0 + t\xi)\cos\alpha, \\ \sigma_{\zeta\zeta}(t, 0, 0) &= \sigma_{zz}(r_0 + t\xi), \end{aligned}$$

where we put for brevity:

$$r_0 = (x_0, y_0, z_0), \qquad \xi = (\cos\alpha, \sin\alpha, 0).$$

Inserting these expressions into (2.16.6), we obtain ($r = (x, y, z)$)

$$L_\sigma(x, y, \alpha; z) = \int\limits_{-\infty}^{\infty} \left[-\sigma_{xz}(r + t\xi)\sin\alpha + \sigma_{yz}(r + t\xi)\cos\alpha\right] dt, \tag{2.16.11}$$

$$S_\sigma(x, y, \alpha; z) = \int\limits_{-\infty}^{\infty} \Big[(\sigma_{xx} - \sigma_{zz})(r + t\xi)\sin^2\alpha - 2\sigma_{xy}(r + t\xi)\cos\alpha\,\sin\alpha\ + $$
$$+\ (\sigma_{yy} - \sigma_{zz})(r + t\xi)\cos^2\alpha\Big]\,dt. \tag{2.16.12}$$

We fix $z_0 \in (a, b)$ and define on the plane $z = z_0$ the vector field $u = (u_x, u_y)$ and symmetric tensor field $v = (v_{xx}, v_{xy}, v_{yy})$ by putting

$$u_x = \sigma_{yz}, \qquad u_y = -\sigma_{xz}, \tag{2.16.13}$$

$$v_{xx} = \sigma_{yy} - \sigma_{zz}, \quad v_{xy} = -\sigma_{xy}, \quad v_{yy} = \sigma_{xx} - \sigma_{zz}. \tag{2.16.14}$$

Inserting these expressions into (2.16.11), (2.16.12) and comparing the so-obtained equalities with (2.14.1) and (2.14.10), we see that

$$L_\sigma(x, y, \alpha; z_0) = Iu(x, y, \alpha; z_0), \qquad S_\sigma(x, y, \alpha; z_0) = Iv(x, y, \alpha; z_0), \tag{2.16.15}$$

where I is the ray transform and (x, y, α) are considered as the coordinates on $\mathbf{R}^2_{z_0} \times \Omega^1 = \{(x, y, z_0; \cos\alpha, \sin\alpha)\}$.

We know that the ray transform If of a tensor field f of degree m on $\mathbf{R}^n$ and the value Wf of the Saint Venant operator are uniquely determined from each other. Let us denote:

$$l_\sigma(x, y; z) = W_D u(x, y; z), \qquad s_\sigma(x, y; z) = W_D v(x, y; z), \tag{2.16.16}$$

where W_D is the regular part of the Saint Venant operator introduced in Section 2.14. By (2.16.15), we can assert that the functions l_σ and s_σ are uniquely determined from the functions L_σ and S_σ respectively. The explicit algorithms for determination of l_σ and s_σ from L_σ and S_σ are given by formulas (2.14.6)–(2.14.8) and (2.14.14)–(2.14.16).

Inserting the values (2.16.13) and (2.16.14) of the components of u and v into equalities (2.14.3) and (2.14.12), we have

$$l_\sigma(x, y; z) = -\frac{\partial\sigma_{xz}}{\partial x} - \frac{\partial\sigma_{yz}}{\partial y}, \tag{2.16.17}$$

$$s_\sigma(x, y; z) = \frac{\partial^2(\sigma_{zz} - \sigma_{xx})}{\partial x^2} + \frac{\partial^2(\sigma_{zz} - \sigma_{yy})}{\partial y^2} - 2\frac{\partial^2\sigma_{xy}}{\partial x\partial y}. \tag{2.16.18}$$

Assuming l_σ and s_σ to be known, let us now consider (2.16.17) and (2.16.18) as a system of equations for σ; and add the equilibrium equations (2.16.9) and the boundary conditions (2.16.10) to this system. It turns out that the so-obtained boundary problem can be easily investigated.

By the third of the equilibrium equations (2.16.9), relation (2.16.17) is equivalent to the next one:

$$\frac{\partial\sigma_{zz}}{\partial z} = l_\sigma. \tag{2.16.19}$$

We differentiate the first of the equations (2.16.9) with respect to x, the second, with respect to y and add the so-obtained equalities to (2.16.18). We thus come to the equation

$$\frac{\partial^2\sigma_{zz}}{\partial x^2}+\frac{\partial^2\sigma_{zz}}{\partial y^2}+\frac{\partial}{\partial z}\left(\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}\right)=s_\sigma$$

which is, by (2.16.17), equivalent to the next one:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\sigma_{zz}=\frac{\partial l_\sigma}{\partial z}+s_\sigma. \qquad (2.16.20)$$

Comparing (2.16.19) and (2.16.20), we see that l_σ and s_σ are connected by the relation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}\right)l_\sigma=\frac{\partial s_\sigma}{\partial z} \qquad (2.16.21)$$

which implies the first of the above-mentioned results.

Note that equations (2.16.19)–(2.16.21) were obtained without using boundary conditions (2.16.10).

Let us show that the value of σ_{zz} on the lateral surface B of the cylinder G is determined by the function S_σ. To this end, we consider two close points $r=(x,y,z)$ and $r'=(x',y',z)$ belonging to B and denote the horizontal straight line that passes through r and r' by $\pi(r,r')$. By (2.16.12),

$$S(\sigma,\pi(r,r'))=$$

$$=\int\limits_0^{|r'-r|}\Big[(\sigma_{xx}-\sigma_{zz})(r+t\xi)\,\xi_y^2-2\sigma_{xy}(r+t\xi)\,\xi_x\xi_y+(\sigma_{yy}-\sigma_{zz})(r+t\xi)\,\xi_x^2\Big]\,dt \qquad (2.16.22)$$

where $\xi=(\xi_x,\xi_y,0)=(r'-r)/|r'-r|$. If the point r' tends to r, then the vector ξ tends to $(\tau_x,\tau_y,0)$ where $\tau=(\tau_x,\tau_y)$ is the unit tangent vector, of the curve γ, at the point (x,y). Applying the mean value theorem to integral (2.16.22), we obtain

$$\lim_{r'\to r}\frac{S(\sigma,\pi(r,r'))}{|r'-r|}=\sigma_{xx}(r)\tau_y^2-2\sigma_{xy}(r)\tau_x\tau_y+\sigma_{yy}(r)\tau_x^2-\sigma_{zz}(r). \qquad (2.16.23)$$

The tangent vector $\tau=(\tau_x,\tau_y)$ is expressed through the normal vector $\nu=(\nu_x,\nu_y)$ by the formulas $\tau_x=-\nu_y,\ \tau_y=\nu_x$. Inserting these expressions into (2.16.23) and using (2.16.10), we arrive at the equality

$$\sigma_{zz}(r)=-\lim_{r'\to r}\frac{S(\sigma,\pi(r,r'))}{|r'-r|}\qquad (r,r'\in B;\ z=z'). \qquad (2.16.24)$$

Relations (2.16.19), (2.16.20) and (2.16.24) permit us to assert that the component σ_{zz} is uniquely determined by the functions L_σ and S_σ. Many versions of numerical methods for determination of σ_{zz} are possible because of overdeterminedness of system (2.16.19), (2.16.20). We will briefly discuss two of them.

1. We fix z_0 $(a<z_0<b)$ and find $\sigma_{zz}(r)$ for $r=(x,y,z_0)\in B$ by (2.16.24) from the measured values of $S_\sigma(x,y,\alpha;z_0)$. Then we find functions $l_\sigma(x,y,z)$ and $s_\sigma(x,y,z_0)$ from

$L_\sigma(x, y, \alpha; z)$ and $S_\sigma(x, y, \alpha; z_0)$ by procedures, of inversion of the ray transform, given by formulas (2.14.6)–(2.14.8) and (2.14.14)–(2.14.16). Then we solve the Dirichlet problem for equation (2.16.20) at the level $z = z_0$, by using known values $\sigma_{zz}(x, y, z_0)|_{(x,y)\in\gamma}$, and find $\sigma_{zz}(x, y, z_0)$. The last step is integration of equation (2.16.19), using known $l_\sigma(x, y, z)$ and $\sigma_{zz}(x, y, z_0)$.

2. First we find $\sigma_{zz}|_B$ from $S_\sigma(x, y, \alpha; z)$ by (2.16.24). Then we determine $l_\sigma(x, y, z)$ and $s_\sigma(x, y, z)$ by inverting the ray transform. At last we find $\sigma_{zz}(x, y, z)$ by solving the Dirichlet problem for equation (2.16.20) at all levels.

The first method, in contrast to the other one, needs fewer calculations since it requires inverting the ray transform of a tensor field of degree two only once (for $z = z_0$). Thus, in the first method, most calculations are spent for inverting the ray transform of a vector field (determination of l_σ from L_σ), which must be executed for all z. In the author's opinion, the only deficiency of the first method is that the error of determination of $\sigma_{zz}(x, y, z)$ can increase with $|z - z_0|$, while all levels are equal in rights in the second method. Relation (2.16.21) can be used for the control of calculations in the second method. Various combinations of these methods are possible.

The first method has been realized numerically. It has given good results on test data. In the author's opinion, the algorithm is ready for application to processing real measurement.

We will now prove that no information about a tensor field σ, except for σ_{zz}, can be determined from L_σ and S_σ. Let σ^1 and σ^2 be two tensor fields satisfying (2.16.8)–(2.16.10), and $\sigma^1_{zz} \equiv \sigma^2_{zz}$. Then the difference $\sigma = \sigma^1 - \sigma^2$ satisfies (2.16.8)–(2.16.10), and

$$\sigma_{zz} \equiv 0. \tag{2.16.25}$$

We have to prove that

$$L_\sigma \equiv 0, \quad S_\sigma \equiv 0. \tag{2.16.26}$$

Let us fix z_0 and define the tensor fields u and v on the plane $z = z_0$ by formulas (2.16.13), (2.16.14). Then equalities (2.16.15) are valid. Thus, to prove (2.16.26) it is sufficient to show that $Iu = 0, Iv = 0$. By Theorem 2.5.1, the last equalities are equivalent to

$$Wu = 0, \tag{2.16.27}$$

$$Wv = 0. \tag{2.16.28}$$

Inserting expressions (2.16.13) into (2.14.2), (2.14.3) and (2.14.5), we obtain

$$Wu = -\left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y}\right) + (\sigma_{xz}\nu_x + \sigma_{yz}\nu_y)\delta_\gamma.$$

Using the third of the equilibrium equations (2.16.9), the last equality can be rewritten as

$$Wu = \frac{\partial \sigma_{zz}}{\partial z} + (\sigma_{zx}\nu_x + \sigma_{zy}\nu_y)\delta_\gamma.$$

Comparing this relation with (2.16.25) and (2.16.10), we arrive at (2.16.27).

Inserting expression (2.16.14) into equalities (2.14.11)–(2.14.13), we have

$$W_D v = -\left(\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y}\right), \tag{2.16.29}$$

$$\langle W_\gamma v, \varphi\rangle = \oint\limits_\gamma \left[\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y}\right)\nu_x + \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}\right)\nu_y\right]\varphi\, ds -$$
$$- \oint\limits_\gamma \left[(\sigma_{xx}\nu_x + \sigma_{xy}\nu_y)\frac{\partial \varphi}{\partial x} + (\sigma_{xy}\nu_x + \sigma_{yy}\nu_y)\frac{\partial \varphi}{\partial y}\right]\varphi\, ds. \tag{2.16.30}$$

Differentiating the first of the equations (2.16.9) with respect to x, the second, with respect to y and summing the so-obtained equalities, we get

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = -\frac{\partial}{\partial z}\left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y}\right).$$

According to the third of the equations (2.16.9), the last relation can be rewritten as follows:

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = \frac{\partial^2 \sigma_{zz}}{\partial z^2}.$$

Comparing this equality with (2.16.25) and (2.16.29), we see that

$$W_D v = 0. \tag{2.16.31}$$

By (2.16.10), the integrand of the second integral on the right-hand side of (2.16.30) is identical zero. By (2.16.9), the integrand of the first integral can be transformed to as follows:

$$\langle W_\gamma v, \varphi\rangle = -\oint\limits_\gamma \frac{\partial}{\partial z}(\sigma_{xz}\nu_x + \sigma_{yz}\nu_y)\,\varphi\, ds.$$

Comparing the last equality with the third of the boundary conditions (2.16.10), we see that $W_\gamma v = 0$. Together with (2.16.31), the last equality gives $Wv = 0$ which completes the proof.

2.17 Further results

The presentation of the current chapter mainly follows the papers [113, 115, 117, 119]. Here we will deliver a survey of some results close to the main content of the chapter.

In Section 2.10, describing the range of the operator I, we left aside the ray transform on a plane ($n = 2$). It was not accidental. Indeed, let us observe that, for $n > 2$, the John conditions (2.10.1), that are necessary and sufficient for a function to be in the range of the operator I, are of differential nature. In the case $n = 2, m = 0$ the ray transform coincides with the Radon transform, for which the corresponding conditions are of integral nature [35]. It turns out that the situation is similar in the case of arbitrary m. Thus the two-dimensional case radically differs from the case $n > 2$. The precise formulation of the result, due to E. Yu. Pantjukhina [100], is as follows:

Theorem 2.17.1 *If a function $\varphi \in \mathcal{S}(T\Omega^{n-1})$ can be represented as $\varphi = If$ for some field $f \in \mathcal{S}(S^m; \mathbf{R}^n)$, then (1) the identity $\varphi(x, -\xi) = (-1)^m \varphi(x, \xi)$ is valid and (2) for every integral $k \geq 0$, on $\mathbf{R}^n$ there exists a homogeneous polynomial $P^k_{i_1 \dots i_m}(x)$ of degree k such that the equality*

$$\int\limits_{\xi^\perp} \varphi(x', \xi)\langle x, x'\rangle^k\, dx' = P^k_{i_1 \dots i_m}(x)\xi^{i_1} \dots \xi^{i_m} \tag{2.17.1}$$

is valid for all $(x,\xi)\in T\Omega$. *Here* dx' *is the* $(n-1)$*-dimensional Lebesgue measure on* $\xi^\perp$. *In the case* $n=2$ *conditions* (1) *and* (2) *are sufficient for existence a field* $f\in\mathcal{S}(S^m)$ *such that* $\varphi = If$.

In the case $m=0$ conditions (2.17.1) are formulated in [48].

In [115] the author considered the next generalization of the ray transform

$$I^q f = \int\limits_{-\infty}^{\infty} t^q\langle f(x+t\xi),\xi^m\rangle\,dt.$$

The subsequent claim is proved in full analogy with Theorem 2.5.1.

Theorem 2.17.2 *Let* $0\le p\le m$ *and* $n\ge 2$ *be integers. For a field* $F\in\mathcal{E}'(S^m;\mathbf{R}^n)$ *the next three assertions are equivalent:*

(1) $I^qF=0$ *for* $q=0,1,\dots,p$;

(2) *there exists a field* $V\in\mathcal{E}'(S^{m-p-1})$ *such that its support is contained in the convex hull of the support of* F *and* $d^{p+1}V=F$;

(3) $W^pF=0$ *where* $W^p: C^\infty(S^m)\to C^\infty(S^{m-p}\otimes S^m)$ *is the differential operator of order* p *defined by the formula*

$$(W^pf)_{i_1\dots i_{m-p}j_1\dots j_m} = \sigma(i_1\dots i_{m-p})\sigma(j_1\dots j_m)\sum_{l=0}^{m-p}(-1)^l\binom{m-p}{l}\times$$
$$\times f_{i_1\dots i_{m-p-l}j_1\dots j_{p+l}\,;\,j_{p+l+1}\dots j_m i_{m-p-l+1}\dots i_{m-p}}.$$

In particular, a compactly-supported tensor field F of degree m is uniquely determined by the first $m+1$ integral moments $I^0F,\dots,I^mF$ along all straight lines. It is possible that explicit inversion formulas similar to (2.12.15) exist in this case; encountered algebraic difficulties in the way of deriving such formulas are not overcome yet.

In conclusion, we present results, due to L. B. Vertgeim [136], on the complex analog of the ray transform.

We fix the Hermitian form $\langle z,w\rangle = z^i\overline{w}^i$ on $\mathbf{C}^n$. Given integers $p,q\ge 0$, by T_p^q we mean the space of tensors of bidegree (p,q) on $\mathbf{C}^n$, i.e., the space of functions $\mathbf{C}^n\times\ldots\times\mathbf{C}^n\to\mathbf{C}$ (there are $p+q$ factors to the left of the arrow) which are $\mathbf{C}$-linear in each of the first p arguments and are $\mathbf{C}$-antilinear in each of the last q arguments. Let $S_p^q=S_p^q(\mathbf{C}^n)$ be the subspace of T_p^q that consists of tensors symmetric in the first p and last q arguments; let $\sigma: T_p^q\to S_p^q$ be the operator of symmetrization with respect to these groups of arguments. Any tensor $u\in T_p^q$ is uniquely represented as

$$u = u^{j_1\dots j_q}_{i_1\dots i_p}\,dz^{i_1}\otimes\ldots\otimes dz^{i_p}\otimes d\overline{z}^{j_1}\otimes\ldots\otimes d\overline{z}^{j_q}.$$

The scalar product on T_p^q is defined by the equality

$$\langle u,v\rangle = u^{j_1\dots j_q}_{i_1\dots i_p}\,\overline{v}^{j_1\dots j_q}_{i_1\dots i_p}.$$

For $u\in S_p^q$ and $v\in S_m^l$ the symmetric product $uv=\sigma(u\otimes v)\in S_{p+m}^{q+l}$ is defined. As above, by i_u we mean the operator of symmetric multiplication by u, and by j_u we mean the dual of i_u.

Two operators of inner differentiation and two divergences are defined on $C^\infty(S^q_p)$:

$$(df)^{j_1\dots j_q}_{i_1\dots i_{p+1}} = \sigma\left(\frac{\partial}{\partial z^{i_{p+1}}} f^{j_1\dots j_q}_{i_1\dots i_p}\right), \qquad (\overline{d}f)^{j_1\dots j_{q+1}}_{i_1\dots i_p} = \sigma\left(\frac{\partial}{\partial \overline{z}^{j_{q+1}}} f^{j_1\dots j_q}_{i_1\dots i_p}\right),$$

$$(\overline{\delta} f)^{j_1\dots j_q}_{i_1\dots i_{p-1}} = \sum_{i=1}^{n} \frac{\partial}{\partial \overline{z}^i} f^{j_1\dots j_q}_{i_1\dots i_{p-1} i}, \qquad (\delta f)^{j_1\dots j_{q-1}}_{i_1\dots i_p} = \sum_{i=1}^{n} \frac{\partial}{\partial z^j} f^{j_1\dots j_{q-1} j}_{i_1\dots i_p},$$

where the usual notation is used:

$$\frac{\partial}{\partial z^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial y^k}\right), \quad \frac{\partial}{\partial \overline{z}^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\right).$$

Decomposition into potential and solenoidal parts is realizable as follows:

Theorem 2.17.3 *Given a field $f \in \mathcal{S}(S^q_p)$, there exists a unique field ${}^s\!f \in C^\infty(S^q_p)$ such that, for some $v \in C^\infty(S^q_{p-1})$ and $w \in C^\infty(S^{q-1}_p)$ the next relations are valid:*

$$f = {}^s\!f + dv + \overline{d}w, \quad \overline{\delta}\, {}^s\!f = 0, \quad \delta\, {}^s\!f = 0,$$

$${}^s\!f(z) \to 0, \quad v(z) \to 0, \quad w(z) \to 0 \quad as \quad |z| \to \infty.$$

The field ${}^s\!f$ satisfies the estimate $|{}^s\!f(z)| \le C(1+|z|)^{1-2n}$; the fields v and w can be chosen such that

$$|v(z)| \le C(1+|z|)^{2-2n}, \qquad |w(z)| \le C(1+|z|)^{2-2n}.$$

Let $\mathbf{C}^n_0 = \mathbf{C}^n \setminus \{0\}$. By the *ray transform* of a field $f \in C^\infty(S^q_p)$ we mean the function defined on $\mathbf{C}^n \times \mathbf{C}^n_0$ by the formula

$$If(z,\xi) = \int\limits_{\mathbf{C}} f^{j_1\dots j_q}_{i_1\dots i_p}(z+t\xi)\, \xi^{i_1} \dots \xi^{i_p} \overline{\xi}^{j_1} \dots \overline{\xi}^{j_q} ds(t)$$

under the condition that the integral converges. Here $ds(t)$ is the area form on $\mathbf{C}$.

To give the inversion formula for the ray transform we have, at first, to introduce an analog of operator (2.11.1). For $f \in C^\infty(S^q_p)$, we define the tensor field $\mu If \in C^\infty(S^q_p)$ by the equality

$$(\mu If)^{j_1\dots j_q}_{i_1\dots i_p}(z) = \int\limits_{\mathbf{CP}^{n-1}} \frac{\overline{\xi}^{i_1} \dots \overline{\xi}^{i_p} \xi^{j_1} \dots \xi^{j_q}}{|\xi|^{2p+2q-2}} If(z,\xi)\, dV_{\mathbf{P}}(\xi), \tag{2.17.2}$$

where $dV_{\mathbf{P}}(\xi)$ is the volume form, on the complex projective space $\mathbf{CP}^{n-1}$, generated by the Fubini-Study metric

$$ds^2 = \frac{\langle \xi,\xi\rangle\langle d\xi, d\xi\rangle - \langle \xi, d\xi\rangle\langle d\xi,\xi\rangle}{\langle \xi,\xi\rangle^2}.$$

In the last two formulas ξ is considered as the vector of homogeneous coordinates on $\mathbf{CP}^{n-1}$. Note that the ray transform has the next homogeneity:

$$If(z,\tau\xi) = \frac{\tau^p \overline{\tau}^q}{|\tau|^2} If(z,\xi).$$

Consequently, the integrand in (2.17.2) has complex homogeneity of zero degree in ξ, i.e., it is defined on $\mathbf{CP}^{n-1}$.

Theorem 2.17.4 *The solenoidal part ${}^s\!f$ of a field $f \in \mathcal{S}(S^q_p)$ can be recovered from the ray transform If by the formula*

$$ {}^s\!f = -\frac{1}{4\pi^n}\Delta \sum_{l=0}^{\min(p,q)} \frac{(-1)^l(n+p+q-l-2)!}{l!(p-l)!(q-l)!}(i-\Delta^{-1}d\bar{d})^l j^l(\mu If), $$

where $i = i_\delta,\ j = j_\delta,\ \delta = (\delta^i_j) \in S^1_1$ is the Kronecker tensor.

Note that in the case $p = 0$ or $q = 0$ this formula has the local character, i.e., a value ${}^s\!f(z)$ is determined by integrals over complex straight lines that intersect any neibourhood of the point z. In other cases the formula is nonlocal.

To describe the range of the ray transform we define the manifold $M = \{(z,\xi) \mid \langle z,\xi\rangle = 0, \xi \neq 0\}$. By $\mathcal{S}(M)$ we mean the space of functions that are smooth on M and decrease, together with all derivatives, rapidly in z and uniformly in ξ belonging to any compact subset of $\mathbf{C}^n_0$.

Theorem 2.17.5 *Let $n \geq 3$. A function $\psi \in C^\infty(\mathbf{C}^n \times \mathbf{C}^n_0)$ is the ray transform of some field $f \in \mathcal{S}(S^q_p)$ if and only if the next four conditions are satisfied:*

(1) $\psi|_M \in \mathcal{S}(M)$;

(2) $\psi(z+\tau\xi,\xi) = \psi(z,\xi)$ *for* $\tau \in \mathbf{C}$;

(3) $\psi(z,\tau\xi) = (\tau^p\bar{\tau}^q/|\tau|^2)\psi(z,\xi)$;

(4) *For all indices,*

$$ \left(\frac{\partial^2}{\partial z^{i_1}\partial\xi^{j_1}} - \frac{\partial^2}{\partial z^{j_1}\partial\xi^{i_1}}\right)\dots\left(\frac{\partial^2}{\partial z^{i_{p+1}}\partial\xi^{j_{p+1}}} - \frac{\partial^2}{\partial z^{j_{p+1}}\partial\xi^{i_{p+1}}}\right)\psi = 0, $$

$$ \left(\frac{\partial^2}{\partial \bar{z}^{i_1}\partial\bar{\xi}^{j_1}} - \frac{\partial^2}{\partial \bar{z}^{j_1}\partial\bar{\xi}^{i_1}}\right)\dots\left(\frac{\partial^2}{\partial \bar{z}^{i_{q+1}}\partial\bar{\xi}^{j_{q+1}}} - \frac{\partial^2}{\partial \bar{z}^{j_{q+1}}\partial\bar{\xi}^{i_{q+1}}}\right)\psi = 0. $$

Chapter 3

Some questions of tensor analysis

Here we will expose some notions and facts of tensor analysis that are used in the next chapters for investigating integral geometry of tensor fields on Riemannian manifolds.

The first two sections contain a survey of tensor algebra and the theory of connections on manifolds including the definition of a Riemannian connection. This survey is not a systematic introduction to the subject; here the author's only purpose is to represent main notions and formulas in the form convenient for use in the book. Our presentation of the connection theory is nearest to that of the book [61]; although there are many other excellent textbooks on the subject [41, 81, 25].

In Section 3.3 the operators of inner differentiation and divergence are introduced on the bundle of symmetric tensors of a Riemannian manifold; their duality is established. Then we prove the theorem on decomposition of a tensor field, on a compact manifold, into the sum of solenoidal and potential fields.

Sections 3.4–3.6 are devoted to exposing the main tools that are applied in the next chapters to studying the kinetic equation on Riemannian manifolds. The so-called semibasic tensors are defined on the space of the tangent bundle; two differential operators, vertical and horizontal covariant derivatives, are introduced on the bundle of semibasic tensors. For these operators, formulas of Gauss-Ostrogradskiĭ type are established. The term "semibasic tensor field" is adopted from the book [39] in which a corresponding notion is considered for exterior differential forms.

In Sections 3.3–3.6 our presentation mainly follows the paper [102].

3.1 Tensor fields

We shall use the terminology and notation of vector bundle theory. The reader is however assumed to be acquainted only with the first notions of the theory. They can be found at the first pages of the book [54].

By a *manifold* we mean a smooth Hausdorff paracompact manifold, possibly with a boundary. The term "smooth" is used as a synonym of "infinitely differentiable."

Let $\mathbf{F}$ be either the field $\mathbf{C}$ of complex numbers or the field $\mathbf{R}$ of real numbers. For a manifold M, by $C^\infty(M, \mathbf{F})$ we denote the ring of smooth $\mathbf{F}$-valued functions on M. We will write more briefly $C^\infty(M)$ instead of $C^\infty(M, \mathbf{C})$. Elements of the ring $C^\infty(M)$ are called *smooth functions* while elements of $C^\infty(M, \mathbf{R})$ are called *smooth real functions.*

Given a smooth vector $\mathbf{F}$-bundle $\alpha = (E, p, M)$ over a manifold M and open $U \subset M$, we mean by $C^\infty(\alpha; U)$ the $C^\infty(U, \mathbf{F})$-module of *smooth sections* of the bundle α over U, and by $C_0^\infty(\alpha; U)$, the submodule of $C^\infty(\alpha; U)$ that consists of sections whose supports are compact and contained in U. The notation $C^\infty(\alpha; M)$ will usually be abbreviated to $C^\infty(\alpha)$. For $\alpha = (E, p, M)$, let α' be the dual bundle, E_x be the fiber of α over a point x. If N is a manifold and $f : N \to M$ is a smooth map, then by $f^*\alpha$ we mean the induced bundle over N. Given two $\mathbf{F}$-bundles $\alpha = (E, p, M)$ and $\beta = (F, q, M)$, let $L(\alpha, \beta)$ be the $\mathbf{F}$-bundle whose fiber over x is the space of all $\mathbf{F}$-linear mappings $E_x \to F_x$. The elements of $C^\infty(M, \mathbf{F})$-module $C^\infty(L(\alpha, \beta))$ are called *homomorphisms* (over M) of the bundle α into the bundle β. Thus a homomorphism of the bundle α into the bundle β is a smooth mapping $E \to F$ linear on fibers and identical on the base. We shall use only finite-dimensional vector bundles except the case in which we encounter graded vector bundles of the type $\alpha^* = \bigoplus_{m=0}^{\infty} \alpha^m$, where each α^m is of finite dimension; such an object can be viewed as a sequence of finite-dimensional bundles.

Given a manifold M, by $\tau_M = (TM, p, M)$ we denote the *tangent bundle* of M. Its dual $\tau'_M = (T'M, p', M)$ is called the *cotangent bundle* of the manifold M. Note that these bundles are real. Their fibers T_xM and T'_xM are the *spaces tangent* and *cotangent* to M at the point x. Their sections are called *real vector* and *covector fields* on M. A real vector field $v \in C^\infty(\tau_M)$ can be considered as a *derivative* of the ring $C^\infty(M, \mathbf{R})$, i.e., as an $\mathbf{R}$-linear mapping $C^\infty(M, \mathbf{R}) \to C^\infty(M, \mathbf{R})$ such that

$$v(\varphi \cdot \psi) = \psi \cdot v\varphi + \varphi \cdot v\psi. \tag{3.1.1}$$

The number $v\varphi(x)$ is called the *derivative of the function φ at the point x in the direction* $v(x)$. For two such derivatives v and w, their commutator $[v, w] = vw - wv$ is also a derivative, i.e., a real vector field. It is called the *Lie commutator* of the fields v and w.

For nonnegative integers r and s, let ${}_{\mathbf{R}}\tau_s^r M = ({}_{\mathbf{R}}T_s^r M, p_s^r, M)$ be the real vector bundle defined by the equality

$${}_{\mathbf{R}}\tau_s^r M = \tau_M \otimes \ldots \otimes \tau_M \otimes \tau'_M \otimes \ldots \otimes \tau'_M, \tag{3.1.2}$$

where the factor τ_M is repeated r times and τ'_M is repeated s times; the tensor products are taken over $\mathbf{R}$. It is convenient to assume that ${}_{\mathbf{R}}\tau_s^r M = 0$ for $r < 0$ or $s < 0$. Let $\tau_s^r M = (T_s^r M, p_s^r, M)$ be the complexification of the bundle ${}_{\mathbf{R}}\tau_s^r M$, i.e.,

$$\tau_s^r M = \mathbf{C} \otimes_{\mathbf{R}} ({}_{\mathbf{R}}\tau_s^r M) \tag{3.1.3}$$

where $\mathbf{C}$ is the trivial one-dimensional complex bundle. $\tau_s^r M$ is called the *bundle of tensors of degree* (r, s) on M, and its sections are called *tensor fields of degree* (r, s). A tensor $u \in T_s^r M$ is said to be r times contravariant and s times covariant. The fiber of $\tau_s^r M$ over $x \in M$ is denoted by $T_{s,x}^r M$.

By definition (3.1.3), in the complex vector bundle $\tau_s^r M$ the $\mathbf{R}$-subbundle ${}_{\mathbf{R}}\tau_s^r M$ of real tensors is distinguished. Consequently, any tensor $u \in T_s^r M$ can uniquely be represented as $u = v + iw$ with real v and w.

Tensor fields of degrees $(0, 0), (1, 0)$ and $(0, 1)$ are just smooth (complex) functions, vector and covector fields respectively. A vector field $v \in C^\infty(\tau_0^1 M)$ can be considered as a derivative of the ring $C^\infty(M)$, i.e., as a $\mathbf{C}$-linear mapping $v : C^\infty(M) \to C^\infty(M)$ satisfying (3.1.1).

The bundles $\tau_s^r M$ and $\tau_r^s M$ are dual to one other and, consequently, $C^\infty(\tau_s^r M)$ and $C^\infty(\tau_r^s M)$ are the mutually dual $C^\infty(M)$-modules. This implies, in particular, that a covariant tensor field $u \in C^\infty(\tau^0_s M)$ can be considered as $C^\infty(M)$-multilinear mapping $u : C^\infty(\tau^1_0 M)\times\ldots\times C^\infty(\tau^1_0 M) \to C^\infty(M)$. The last observation allows us to define the **C**-linear map $d : C^\infty(M) \to C^\infty(\tau^0_1 M)$ that sends a function $\varphi \in C^\infty(M)$ to its *differential* $d\varphi \in C^\infty(\tau^0_1 M)$ by the formula $d\varphi(v) = v\varphi$. Similarly, a field $u \in C^\infty(\tau^1_s M)$ can be considered as a $C^\infty(M)$-multilinear mapping $u : C^\infty(\tau^1_0 M)\times\ldots\times C^\infty(\tau^1_0 M) \to C^\infty(\tau^1_0 M)$.

We will now list some algebraic operations which are defined on tensors and tensor fields.

Since $C^\infty(\tau^r_s M)$ is a $C^\infty(M)$-module, tensor fields of the same degree can be summed and multiplied by smooth functions.

Every permutation π of the set $\{1,\ldots,r\}$ (of the set $\{1,\ldots,s\}$) determines the automorphism ρ^π (automorphism ρ_π) of the bundle $\tau^r_s M$ by the corresponding permutation of first r (of last s) factors on the right-hand side of equality (3.1.2) and, consequently, it determines the automorphism ρ^π (automorphism ρ_π) of the $C^\infty(M)$-module $C^\infty(\tau^r_s M)$. The automorphism $\rho^\pi(\rho_\pi)$ is called the *operator of transposition* of upper (lower) indices.

For $1 \le k \le r$ and $1 \le l \le s$ the canonical pairing of the k-th factor in the first group of (3.1.2) and the l-th factor in the second group defines the homomorphism $C^k_l : \tau^r_s M \to \tau^{r-1}_{s-1} M$ which is called the *convolution* with respect to k-th upper and l-th lower indices.

By the permutation of the factors $T_x M$ and $T'_x M$, the isomorphism

$$T^r_{s,x}M \otimes_{\mathbf{C}} T^{r'}_{s',x}M = (\mathbf{C} \otimes \underbrace{T_xM \otimes\ldots\otimes T_xM}_{r} \otimes \underbrace{T'_xM \otimes\ldots\otimes T'_xM}_{s})\otimes_{\mathbf{C}}$$

$$\otimes_{\mathbf{C}}(\mathbf{C} \otimes \underbrace{T_xM \otimes\ldots\otimes T_xM}_{r'} \otimes \underbrace{T'_xM \otimes\ldots\otimes T'_xM}_{s'}) \cong$$

$$\cong \mathbf{C} \otimes \underbrace{T_xM \otimes\ldots\otimes T_xM}_{r+r'} \otimes \underbrace{T'_xM \otimes\ldots\otimes T'_xM}_{s+s'} = T^{r+r'}_{s+s',x}M$$

is defined. The composition of the last isomorphism with the canonical projection $T^r_{s,x}M \times T^{r'}_{s',x}M \to T^r_{s,x}M \otimes T^{r'}_{s',x}M$ allows us, given tensors $u \in T^r_{s,x}M$ and $v \in T^{r'}_{s',x}M$, to define the *tensor product* $u \otimes v \in T^{r+r'}_{s+s',x}M$. We thus turn $\tau^*_* M = \bigoplus_{r,s=0}^\infty \tau^r_s M$ into a bundle of bigraded **C**-algebras, and $C^\infty(\tau^*_* M) = \bigoplus_{r,s=0}^\infty C^\infty(\tau^r_s M)$ into a bigraded $\mathbf{C}^\infty(M)$-algebra.

We shall often use coordinate representation of tensor fields. If $(x^1,\ldots,x^n)$ is a local coordinate system defined in a domain $U \subset M$, then the *coordinate vector fields* $\partial_i = \partial/\partial x^i \in C^\infty(\tau_M; U)$ and the *coordinate covector fields* $dx^i \in C^\infty(\tau'_M; U)$ are defined. Any tensor field $u \in C^\infty(\tau^r_s M; U)$ can be uniquely represented as

$$u = u^{i_1\ldots i_r}_{j_1\ldots j_s}\frac{\partial}{\partial x^{i_1}} \otimes\ldots\otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes\ldots\otimes dx^{j_s}, \tag{3.1.4}$$

where $u = u^{i_1\ldots i_r}_{j_1\ldots j_s} \in C^\infty(U)$ are called the *coordinates* (or the *components*) of the field u in the given coordinate system. Assuming that the choice of coordinates is clear from the context, we will usually abbreviate equality (3.1.4) as follows:

$$u = (u^{i_1\ldots i_r}_{j_1\ldots j_s}). \tag{3.1.5}$$

For a point $x \in U$ and a tensor $u \in T^r_{s,x}M$ equalities (3.1.4) and (3.1.5) also make sense, but in this case the coordinates are complex numbers. If $(x'^1,\ldots,x'^n)$ is a second coordinate system defined in a domain U', then in $U \cap U'$ the components of a tensor field with respect to two coordinate systems are connected by the transformation formulas:

$$u'^{i_1\ldots i_r}_{j_1\ldots j_s} = \frac{\partial x'^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial x'^{i_r}}{\partial x^{k_r}} \frac{\partial x^{l_1}}{\partial x'^{j_1}} \cdots \frac{\partial x^{l_s}}{\partial x'^{j_s}} u^{k_1\ldots k_r}_{l_1\ldots l_s}. \tag{3.1.6}$$

The above-listed operations on tensor fields are expressed in coordinate form by the formulas

$$(\varphi u + \psi v)^{i_1\ldots i_r}_{j_1\ldots j_s} = \varphi u^{i_1\ldots i_r}_{j_1\ldots j_s} + \psi v^{i_1\ldots i_r}_{j_1\ldots j_s} \quad (\varphi, \psi \in C^\infty(M)),$$

$$(\rho^\pi u)^{i_1\ldots i_r}_{j_1\ldots j_s} = u^{i_{\pi(1)}\ldots i_{\pi(r)}}_{j_1\ldots j_s}, \quad (\rho_\pi u)^{i_1\ldots i_r}_{j_1\ldots j_s} = u^{i_1\ldots i_r}_{j_{\pi(1)}\ldots j_{\pi(s)}}, \tag{3.1.7}$$

$$(C^k_l u)^{i_1\ldots i_{r-1}}_{j_1\ldots j_{s-1}} = u^{i_1\ldots i_{k-1} p i_k \ldots i_{r-1}}_{j_1\ldots j_{l-1} p j_l \ldots j_{s-1}}, \tag{3.1.8}$$

$$(u \otimes v)^{i_1\ldots i_{r+r'}}_{j_1\ldots j_{s+s'}} = u^{i_1\ldots i_r}_{j_1\ldots j_s} v^{i_{r+1}\ldots i_{r+r'}}_{j_{s+1}\ldots j_{s+s'}}. \tag{3.1.9}$$

Note that the tensor fields $\partial/\partial x^i$ and dx^j commute with respect to tensor product, i.e., $\partial/\partial x^i \otimes dx^j = dx^j \otimes \partial/\partial x^i$, while dx^i and dx^j (and also $\partial/\partial x^i$ and $\partial/\partial x^j$) do not commute. Moreover, if U is diffeomorphic to $\mathbf{R}^n$, then the $C^\infty(U)$-algebra $C^\infty(\tau^*_* M; U)$ is obtained from the free $C^\infty(U)$-algebra with generators $\partial/\partial x^i$ and dx^i by the defining relations $\partial/\partial x^i \otimes dx^j = dx^j \otimes \partial/\partial x^i$.

3.2 Covariant differentiation

A *connection* on a manifold M is a mapping $\nabla : C^\infty(\tau_M) \times C^\infty(\tau_M) \to C^\infty(\tau_M)$ sending a pair of real vector fields u, v into the third real vector field $\nabla_u v$ that is $\mathbf{R}$-linear in the second argument, and $C^\infty(M, \mathbf{R})$-linear in the first argument, while satisfying the relation:

$$\nabla_u(\varphi v) = \varphi \nabla_u v + (u\varphi)v, \tag{3.2.1}$$

for $\varphi \in C^\infty(M, \mathbf{R})$.

By one of remarks in the previous section, $C^\infty(_{\mathbf{R}}\tau^1_1 M)$ is canonically identified with the set of $C^\infty(M, \mathbf{R})$-linear mappings $C^\infty(\tau_M) \to C^\infty(\tau_M)$. Consequently, a given connection defines the $\mathbf{R}$-linear mapping (which is denoted by the same letter)

$$\nabla : C^\infty(\tau_M) \to C^\infty(_{\mathbf{R}}\tau^1_1 M) \tag{3.2.2}$$

by the formula $(\nabla v)(u) = \nabla_u v$. Relation (3.2.1) is rewritten as:

$$\nabla(\varphi v) = \varphi \cdot \nabla v + v \otimes d\varphi. \tag{3.2.3}$$

Passing to complexifications in (3.2.2), we obtain the $\mathbf{C}$-linear mapping

$$\nabla : C^\infty(\tau^1_0 M) \to C^\infty(\tau^1_1 M), \tag{3.2.4}$$

that satisfies (3.2.3) for $\varphi \in C^\infty(M)$. Conversely, any $\mathbf{C}$-linear mapping (3.2.4), satisfying condition (3.2.3) and sending real vector fields into real tensor fields, defines a connection. The tensor field ∇v is called the *covariant derivative* of the vector field v (with respect to the given connection).

The covariant differentiation, having been defined on vector fields, can be transferred to tensor fields of arbitrary degree, as the next theorem shows.

Theorem 3.2.1 *Given a connection, there exist uniquely determined* **C**-*linear mappings*

$$\nabla : C^\infty(\tau^r_s M) \to C^\infty(\tau^r_{s+1} M), \tag{3.2.5}$$

for all integers r *and* s, *such that*

(1) $\nabla\varphi = d\varphi$ *for* $\varphi \in C^\infty(M) = C^\infty(\tau^0_0 M)$;

(2) *For* $r = 1$ *and* $s = 0$, *mapping* (3.2.5) *coincides with the above-defined mapping* (3.2.4);

(3) *For* $1 \le k \le r$ *and* $1 \le l \le s$, *operator* (3.2.5) *commutes with the convolution operator* C^k_l;

(4) *the operator* ∇ *is a derivative of the algebra* $C^\infty(\tau^*_* M)$ *in the following sense: for* $u \in C^\infty(\tau^r_s M)$ *and* $v \in C^\infty(\tau^{r'}_{s'} M)$,

$$\nabla(u \otimes v) = \rho_{s+1}(\nabla u \otimes v) + u \otimes \nabla v, \tag{3.2.6}$$

where ρ_{s+1} *is the transposition operator for lower indices corresponding to the permutation* $\{1, \dots, s, s+2, \dots, s+s'+1, s+1\}$.

We omit the proof of the theorem which can be accomplished in a rather elementary way based on the local representation (3.1.4).

For a connection ∇, the mappings $R : C^\infty(\tau_M) \times C^\infty(\tau_M) \times C^\infty(\tau_M) \to C^\infty(\tau_M)$ and $T : C^\infty(\tau_M) \times C^\infty(\tau_M) \to C^\infty(\tau_M)$ defined by the formulas

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w,$$

$$T(u, v) = \nabla_u v - \nabla_v u - [u, v],$$

are $C^\infty(M, \mathbf{R})$-linear in all arguments and, consequently, they are real tensor fields of degrees (1, 3) and (1, 2) respectively. They are called the *curvature tensor* and the *torsion tensor* of the connection ∇. A connection with the vanishing torsion tensor is called *symmetric*.

Let us present the coordinate form of the covariant derivative. If $(x^1, \dots, x^n)$ is a local coordinate system defined in a domain $U \subset M$, then the *Christoffel symbols* of the connection ∇ are defined by the equalities

$$\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k, \tag{3.2.7}$$

where $\partial_i = \partial/\partial x^i$ are the coordinate vector fields. We emphasize that the functions $\Gamma^k_{ij} \in C^\infty(U, \mathbf{R})$ are not components of any tensor field; under a change of the coordinates, they are transformed by the formulas

$$\Gamma'^k_{ij} = \frac{\partial x'^k}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^i} \frac{\partial x^\gamma}{\partial x'^j} \Gamma^\alpha_{\beta\gamma} + \frac{\partial x'^k}{\partial x^\alpha} \frac{\partial^2 x^\alpha}{\partial x'^i \partial x'^j}. \tag{3.2.8}$$

The torsion tensor and curvature tensor are expressed through the Christoffel symbols by the equalities

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji},$$

$$R^k{}_{lij} = \frac{\partial}{\partial x^i} \Gamma^k_{jl} - \frac{\partial}{\partial x^j} \Gamma^k_{il} + \Gamma^k_{ip} \Gamma^p_{jl} - \Gamma^k_{jp} \Gamma^p_{il}. \tag{3.2.9}$$

For a field

$$u = u^{i_1\dots i_r}_{j_1\dots j_s}\frac{\partial}{\partial x^{i_1}}\otimes\dots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_s},$$

the components of the field ∇u are denoted by $u^{i_1\dots i_r}_{j_1\dots j_s\,;\,k}$ or by $\nabla_k u^{i_1\dots i_r}_{j_1\dots j_s}$, i.e.,

$$\nabla u = u^{i_1\dots i_r}_{j_1\dots j_s\,;\,k}\frac{\partial}{\partial x^{i_1}}\otimes\dots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_s}\otimes dx^k.$$

We emphasize that the factor dx^k, corresponding to the number of the coordinate with respect to which "the differentiation is taken", is situated in the final position. Of course, this rule is not obligatory, but some choice must be done. Our choice stipulates the appearance the operator ρ_{s+1} in equality (3.2.6). According to our choice, the notation $u^{i_1\dots i_r}_{j_1\dots j_s\,;\,k}$ is preferable to $\nabla_k u^{i_1\dots i_r}_{j_1\dots j_s}$, since it has the index k in the final position. Nevertheless, we will also use the second notation because it is convenient to interpret ∇_k as "the covariant partial derivative". The components of the field ∇u are expressed through the components of u by the formulas

$$\nabla_k u^{i_1\dots i_r}_{j_1\dots j_s} = u^{i_1\dots i_r}_{j_1\dots j_s\,;\,k} = \frac{\partial}{\partial x^k}u^{i_1\dots i_r}_{j_1\dots j_s} +$$

$$+\sum_{m=1}^{r}\Gamma^{i_m}_{kp}u^{i_1\dots i_{m-1}pi_{m+1}\dots i_r}_{j_1\dots j_s} - \sum_{m=1}^{s}\Gamma^{p}_{kj_m}u^{i_1\dots i_r}_{j_1\dots j_{m-1}pj_{m+1}\dots j_s}. \qquad (3.2.10)$$

The second-order covariant derivatives satisfy the commutation relations:

$$(\nabla_k\nabla_l - \nabla_l\nabla_k)u^{i_1\dots i_r}_{j_1\dots j_s} =$$

$$= \sum_{m=1}^{r}R^{i_m}_{pkl}u^{i_1\dots i_{m-1}pi_{m+1}\dots i_r}_{j_1\dots j_s} - \sum_{m=1}^{s}R^{p}_{\;j_mkl}u^{i_1\dots i_r}_{j_1\dots j_{m-1}pj_{m+1}\dots j_s}. \qquad (3.2.11)$$

We recall that a *Riemannian metric* on a manifold M is a real tensor field $g = (g_{ij}) \in C^\infty(\tau^0_2 M)$ such that the matrix $(g_{ij}(x))$ is symmetric and positive-definite for every point $x \in M$. A manifold M together with a fixed Riemannian metric is called *Riemannian manifold*. We denote a Riemannian manifold by (M, g) or simply by M if it is clear what metric is assumed. Given $\xi, \eta \in T_x M$, by $\langle\xi,\eta\rangle = g_{ij}(x)\xi^i\eta^j$ we mean the scalar product. A Riemannian metric defines the canonical isomorphism of the bundles τ_M and τ'_M by the equality $\xi(\eta) = \langle\xi,\eta\rangle$ and, consequently, defines isomorphism of $\tau^r_s M$ and $\tau^s_r M$. By this reason we will not distinguish co- and contravariant tensors on a Riemannian manifold and will speak about co- and contravariant coordinates of the same tensor. In coordinate form this fact is expressed by the well-known rules of raising and lowering indices of a tensor:

$$u_{i_1\dots i_m} = g_{i_1j_1}\dots g_{i_mj_m}u^{j_1\dots j_m}; \quad u^{i_1\dots i_m} = g^{i_mj_m}\dots g^{i_1j_1}u_{j_1\dots j_m},$$

where (g^{ij}) is the matrix inverse to (g_{ij}).

The scalar product is extendible to $\tau^0_r M$ by equality (2.1.3). Declaring $\tau^0_r M$ and $\tau^0_s M$ orthogonal to one other in the case $r \neq s$, we turn $\tau_* M = \bigoplus_{r=0}^{\infty}\tau^0_r M$ into a Hermitian vector bundle. Consequently, we can define the scalar product in $C^\infty_0(\tau_* M)$ by formula (2.1.6) in which

$$dV^n(x) = [\det(g_{ij})]^{1/2}dx^1\wedge\dots\wedge dx^n \qquad (3.2.12)$$

is the Riemannian volume and integration is performed over M.

A connection ∇ on a Riemannian manifold is called *compatible with the metric*, if $v\langle\xi,\eta\rangle = \langle\nabla_v\xi,\eta\rangle + \langle\xi,\nabla_v\eta\rangle$ for every vector fields $v,\xi,\eta \in C^\infty(\tau_M)$. It is known that on a Riemannian manifold there is a unique symmetric connection compatible with the metric; it is called the *Riemannian connection.* Its Christoffel symbols are expressed through the components of the metric tensor by the formulas

$$\Gamma^k_{ij} = \frac{1}{2}g^{kp}\left(\frac{\partial g_{jp}}{\partial x^i} + \frac{\partial g_{ip}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^p}\right). \tag{3.2.13}$$

From now on we will use only this connection on a Riemannian manifold, unless we state otherwise.

A smooth mapping $\gamma : (a,b) \to M$ is called a (*parametrized*) *curve* in the manifold M, and $C^\infty(\gamma^*\tau^r_s M)$ is called the *space of tensor fields along the curve* γ. In particular, the vector field $\dot\gamma \in C^\infty(\gamma^*\tau_M)$ along γ defined by the equality $\dot\gamma = d\gamma(d/dt)$ is called the *tangent vector field of the curve* γ. A connection ∇ on M induces the *operator of total differentiation* $D/dt : C^\infty(\gamma^*\tau^r_s M) \to C^\infty(\gamma^*\tau^r_s M)$ along γ which is defined in coordinate form by the equality $D/dt = \dot\gamma^i\nabla_i$. A field u is called *parallel along* γ if $Du/dt = 0$. If u is parallel along γ, we say that the tensor $u(\gamma(t))$ is obtained from $u(\gamma(0))$ by parallel displacement along γ.

A curve γ in a Riemannian manifold is called a *geodesic* if its tangent field $\dot\gamma$ is parallel along γ. In coordinate form the equations of geodesics coincide with (1.2.5). Given a Riemannian manifold M without boundary, a geodesic $\gamma : (a,b) \to M$ $(-\infty \le a < b \le \infty)$ is called *maximal* if it is not extendible to a geodesic $\gamma' : (a-\varepsilon_1, b+\varepsilon_2) \to M$ where $\varepsilon_1 \ge 0$, $\varepsilon_2 \ge 0$ and $\varepsilon_1 + \varepsilon_2 > 0$. It is known that there is a unique geodesic issuing from any point in any direction. More exactly, for every $x \in M$ and $\xi \in T_xM$, there exists a unique maximal geodesic $\gamma_{x,\xi} : (a,b) \to M$ $(-\infty \le a < 0 < b \le \infty)$ such that the initial conditions $\gamma_{x,\xi}(0) = x$ and $\dot\gamma_{x,\xi}(0) = \xi$ are satisfied. In geometry the notation $\exp_x(t\xi)$ is widely used instead of $\gamma_{x,\xi}(t)$, but the notation $\gamma_{x,\xi}(t)$ is more convenient for our purposes and it will be always used in the book.

Let R_{ijkl} be the curvature tensor of the Riemannian manifold M. For a point $x \in M$ and a two-dimensional subspaces $\sigma \subset T_xM$, the number

$$K(x,\sigma) = R_{ijkl}\xi^i\xi^k\eta^j\eta^k / \left(|\xi|^2|\eta|^2 - \langle\xi,\eta\rangle^2\right) \tag{3.2.14}$$

is independent of the choice of the basis ξ,η for σ. It is called the *sectional curvature* of the manifold M at the point x and in the two-dimensional direction σ. This notion is very popular in differential geometry [41].

3.3 Symmetric tensor fields

By $S^m\tau'_M = (S^mT'M, p^m, M)$ we denote the subbundle of $\tau^0_m M$ which consists of tensors that are invariant with respect to all transpositions of the indices. Another equivalent definition explaining the notation is possible: $S^m\tau'_M$ is the complexification of the m-th symmetric degree of the bundle τ'_M. We call $S^*\tau'_M = \bigoplus_{m=0}^\infty S^m\tau'_M$ the *bundle of* (*covariant*) *symmetric tensors*, and its sections are called the *symmetric tensor fields.* Let $\sigma : \tau^0_m M \to S^m\tau'_M$ be the *canonical projection* (*symmetrization*) defined by the equality $\sigma = \frac{1}{m!}\sum\limits_{\pi\in\Pi_m}\rho_\pi$ where Π_m is the group of all purmutations of degree m. The *symmetric* *product* $uv = \sigma(u \otimes v)$ turns $S^*\tau'_M$ into the bundle of the commutative graded algebras. For a Riemannian manifold, $S^*\tau'_M$ is a Hermitian vector bundle.

From now on we assume in this section that M is a Riemannian manifold. Let $S^mT'_xM$ be the fiber of the bundle $S^m\tau'_M$ over the point x, and $S^*T'_xM = \bigoplus_{m=0}^{\infty} S^mT'_xM$. Given $u \in S^*T'_xM$, by $i_u : S^*T'_xM \to S^*T'_xM$ we denote the operator of symmetric multiplication by u, and by j_u we denote the dual of i_u. In coordinate form these operators are expressed by formulas (2.1.5). For $u \in C^\infty(S^*\tau'_M)$, the homomorphisms $i_u, j_u : S^*\tau'_M \to S^*\tau'_M$ are introduced by the relations $i_u(x) = i_{u(x)},\ j_u(x) = j_{u(x)}$.

The *operator of inner differentiation* $d : C^\infty(S^m\tau'_M) \to C^\infty(S^{m+1}\tau'_M)$ is defined by the equality $d = \sigma\nabla$. The *divergence operator* $\delta : C^\infty(S^{m+1}\tau'_M) \to C^\infty(S^m\tau'_M)$ is defined in coordinate form by the formula $(\delta u)_{i_1\dots i_m} = u_{i_1\dots i_m j\,;\,k}g^{jk}$.

Theorem 3.3.1 *The operators d and $-\delta$ are formally dual to one other. Moreover, for a compact domain $D \subset M$ bounded by piecewise smooth hypersurface ∂D and for every fields $u, v \in C^\infty(S^*\tau'_M)$, the next Green's formula is valid:*

$$\int\limits_D [\langle du, v\rangle + \langle u, \delta v\rangle]\, dV^n = \int\limits_{\partial D} \langle i_\nu u, v\rangle\, dV^{n-1}, \tag{3.3.1}$$

where dV^n and dV^{n-1} are the Riemannian volumes on M and ∂D respectively, ν is the exterior unit normal vector to ∂D.

P r o o f. It is known [55] that, for a vector field $\xi \in C^\infty(\tau_M)$, the next Gauss-Ostrogradskiĭ formula is valid:

$$\int\limits_D (\delta\xi)\, dV^n = \int\limits_D \xi^i{}_{;\,i}\, dV^n = \int\limits_{\partial D} \langle \xi, \nu\rangle\, dV^{n-1}. \tag{3.3.2}$$

Given $u \in C^\infty(S^m\tau'_M)$ and $v \in C^\infty(S^{m+1}\tau'_M)$, we write

$$\langle du, v\rangle + \langle u, \delta v\rangle = u_{i_1\dots i_m\,;\,i_{m+1}}\overline{v}^{i_1\dots i_{m+1}} + u_{i_1\dots i_m}\overline{v}^{i_1\dots i_{m+1}}{}_{;\,i_{m+1}} = \left(u_{i_1\dots i_m}\overline{v}^{i_1\dots i_{m+1}}\right)_{;\,i_{m+1}}.$$

Introducing the vector field ξ by the equality $\xi^j = u_{i_1\dots i_m}\overline{v}^{i_1\dots i_m j}$ and applying (3.3.2) to ξ, we arrive at (3.3.1).

We recall [97] that, for a smooth complex vector bundle α over a compact manifold M and an integer $k \geq 0$, the topological Hilbert space $H^k(\alpha)$ is defined of sections whose components are locally square-integrable together with all derivatives up to the order k in every local coordinate system. Let M be a compact Riemannian manifold with boundary, $S^m\tau'_M|_{\partial M}$ be the restriction of the bundle $S^m\tau'_M$ to ∂M. By fixing a finite atlas on M and a partition of unity subordinate to the atlas, we provide $H^k(S^m\tau'_M)$ and $H^k(S^m\tau'_M|_{\partial M})$ with the structures of Hilbert spaces. The norms of these spaces will be denoted by $\|\cdot\|_k$. We recall that, for $k \geq 1$, the trace operator $H^k(S^m\tau'_M) \to H^{k-1}(S^m\tau'_M|_{\partial M}),\ u \mapsto u|_{\partial M}$ is bounded.

The next theorem generalizes the well-known fact about decomposition of a vector field ($m = 1$) into potential and solenoidal parts to symmetric tensor fields of arbitrary degree.

Theorem 3.3.2 *Let M be a compact Riemannian manifold with boundary; let $k \geq 1$ and $m \geq 0$ be integers. For every field $f \in H^k(S^m\tau'_M)$, there exist uniquely determined ${}^s\!f \in H^k(S^m\tau'_M)$ and $v \in H^{k+1}(S^{m-1}\tau'_M)$ such that*

$$f = {}^s\!f + dv, \quad \delta\, {}^s\!f = 0, \quad v|_{\partial M} = 0. \tag{3.3.3}$$

The estimates

$$\|{}^s\!f\|_k \leq C\|f\|_k, \quad \|v\|_{k+1} \leq C\|\delta f\|_{k-1} \tag{3.3.4}$$

are valid where a constant C is independent of f. In particular, ${}^s\!f$ and v are smooth if f is smooth.

We call the fields ${}^s\!f$ and dv the *solenoidal* and *potential* parts of the field f.

P r o o f. Assuming existence of ${}^s\!f$ and v which satisfy (3.3.3) and applying the operator δ to the first of these equalities, we see that v is a solution to the boundary problem $\delta dv = \delta f$, $v|_{\partial M} = 0$. Conversely, if we establish that, for any $u \in H^{k-1}(S^m\tau'_M)$, the boundary problem

$$\delta dv = u, \quad v|_{\partial M} = 0 \tag{3.3.5}$$

has a unique solution $v \in H^{k+1}(S^m\tau'_M)$ satisfying the estimate

$$\|v\|_{k+1} \leq C\|u\|_{k-1}, \tag{3.3.6}$$

then we shall arrive at the claim of the theorem by putting $u = \delta f$ and ${}^s\!f = f - dv$.

We will show that problem (3.3.5) is elliptic with zero kernel and zero cokernel. After this, applying the theorem on normal solvability [139], we shall obtain existence and uniqueness of the solution to problem (3.3.5) as well as estimate (3.3.6).

To check ellipticity of problem (3.3.5) we have to show that the symbol $\sigma_2(\delta d)$ of the operator δd is elliptic and to verify the Lopatinskiĭ condition for the problem.

We use the definition and notation, for symbols of differential operators on vector bundles, that are given in [97]. It is straightforward from the definition that the symbols of operators d and δ are expressed by the formulas

$$\sigma_1 d(x,\xi) = i_\xi, \quad \sigma_1\delta(x,\xi) = j_\xi \quad (\xi \in T'_xM),$$

where i_ξ and j_ξ are the operators defined in the previous section. Thus, $\sigma_2(\delta d)(\xi, u) = j_\xi i_\xi u$. Now we use the next

Lemma 3.3.3 *Let M be a Riemannian manifold, $x \in M$ and $0 \neq \xi \in T_xM$. For an integer $m \geq 0$, the equality*

$$j_\xi i_\xi = \frac{1}{(m+1)}|\xi|^2 E + \frac{m}{(m+1)} i_\xi j_\xi \tag{3.3.7}$$

holds on $S^mT'_xM$, where E is the identity operator.

The lemma will be proved at the end of the section, and now we continue the proof of the theorem. The operator $i_\xi j_\xi$ is nonnegative, as a product of two mutually dual operators. Consequently, formula (3.3.7) implies positiveness of $j_\xi i_\xi$ for $\xi \neq 0$. Thus ellipticity of the symbol $\sigma_2(\delta d)$ is proved.

It will be convenient for us to verify the Lopatinskiĭ condition in the form presented in [139] (condition III of this paper; we note simultaneously that condition II of regular ellipticity is satisfied since equation (3.3.5) has real coefficients). We choose a local coordinates $x^1, \dots, x^{n-1}, x^n = t \ge 0$ in a neighbourhood of a point $x_0 \in \partial M$ in such a way that the boundary ∂M is determined by the equation $t = 0$ $g_{ij}(x_0) = \delta_{ij}$. For brevity we denote $d_0(D) = \sigma_1 d(x_0, D)$ and $\delta_0(D) = \sigma_1 \delta(x_0, D)$ where $D = (D_j),\ D_j = -i\partial/\partial x^j$. Then

$$(d_0(D)v)_{j_1\dots j_{m+1}} = i\sigma(j_1 \dots j_{m+1})(D_{j_1} v_{j_2\dots j_{m+1}}), \tag{3.3.8}$$

$$(\delta_0(D)v)_{j_1\dots j_{m-1}} = i\sum_{k=1}^{n} D_k v_{kj_1\dots j_{m-1}}. \tag{3.3.9}$$

To verify the Lopatinskiĭ condition for problem (3.3.5) we have to consider the next boundary problem for a system of ordinary differential equations:

$$\delta_0(\xi', D_t)\, d_0(\xi', D_t)\, v(t) = 0, \tag{3.3.10}$$

$$v(0) = v_0, \tag{3.3.11}$$

where $D_t = -id/dt$; and to prove that this problem has a unique solution in $\mathcal{N}_+$ for every $0 \ne \xi' \in \mathbf{R}^{n-1}$ and every tensor $v_0 \in S^m(\mathbf{R}^n)$. Here $\mathcal{N}_+$ is the space of solutions, to system (3.3.10), which tend to zero as $t \to \infty$.

Since the equation $\det(\delta_0(\xi', \lambda) d_0(\xi', \lambda)) = 0$ has real coefficients and has not a real root for $\xi' \ne 0$ as we have seen above, the space $\mathcal{N}$ of all solutions to system (3.3.10) can be represented as the direct sum: $\mathcal{N} = \mathcal{N}_+ \bigoplus \mathcal{N}_-$ where $\mathcal{N}_-$ is the space of solutions tending to zero as $t \to -\infty$. Moreover, $\dim \mathcal{N}_+ = \dim \mathcal{N}_- = \dim S^m(\mathbf{R}^n)$. Consequently, to verify the Lopatinskiĭ condition it is sufficient to show that the homogeneous problem

$$\delta_0(\xi', D_t)\, d_0(\xi', D_t)\, v(t) = 0, \quad v(0) = 0 \tag{3.3.12}$$

has only zero solution in the space $\mathcal{N}_+$. Before proving this, we will establish a Green's formula.

Let $u(t)$ and $v(t)$ be symmetric tensors, on $\mathbf{R}^n$ of degree $m+1$ and m respectively, which depend smoothly on $t \in [0, \infty)$ and decrease rapidly together with all their derivatives as $t \to 0$. If $v(0) = 0$ then

$$\int_0^\infty \langle \delta_0(\xi', D_t)u, v\rangle\, dt = -\int_0^\infty \langle u, d_0(\xi', D_t)v\rangle\, dt. \tag{3.3.13}$$

The scalar product is understood here according to definition (2.1.3) for $g_{ij} = \delta_{ij}$. Indeed,

$$\int_0^\infty \langle \delta_0(\xi', D_t)u, v\rangle\, dt = i\int_0^\infty \left(D_t u_{nj_1\dots j_m} + \sum_{k=1}^{n-1} \xi'_k u_{kj_1\dots j_m}\right) \overline{v}^{j_1\dots j_m} dt =$$

$$= i\int_0^\infty \left[u^{nj_1\dots j_m}\overline{(D_t v_{j_1\dots j_m})} + \sum_{k=1}^{n-1} \xi'_k u_{kj_1\dots j_m} \overline{v}^{j_1\dots j_m}\right] dt.$$

Putting $\xi = (\xi'_1, \dots, \xi'_{n-1}, D_t)$, we can rewrite this equality as:

$$\int_0^\infty \langle \delta_0(\xi', D_t)u, v\rangle\, dt = i\int_0^\infty u^{j_1\dots j_{m+1}} \overline{\xi_{j_1} v_{j_2\dots j_{m+1}}}\, dt. \tag{3.3.14}$$

By (3.3.8), we have $(d_0(\xi', D_t)v)_{j_1\dots j_{m+1}} = \sigma(j_1\dots j_{m+1})(\xi_{j_1}v_{j_2\dots j_{m+1}})$. Consequently, $\langle u, d_0(\xi', D_t)v\rangle = -iu^{j_1\dots j_{m+1}}\overline{\xi_{j_1}v_{j_2\dots j_{m+1}}}$. Comparing the last relation with (3.3.14), we arrive at (3.3.13).

Let $v(t) \in \mathcal{N}_+$ be a solution to problem (3.3.12). Putting $u(t) = d_0(\xi', D_t)v(t)$ in (3.3.13), we obtain

$$d_0(\xi', D_t)v(t) = 0. \tag{3.3.15}$$

Let us now prove that (3.3.15) and the initial condition $v(0) = 0$ imply that $v(t) \equiv 0$. Definition (3.3.8) for the operator $d_0(\xi)$ can be rewritten as

$$(d_0(\xi)v)_{j_1\dots j_{m+1}} = \frac{i}{m+1}\sum_{k=1}^{m+1}\xi_{j_k}v_{j_1\dots\widehat{j_k}\dots j_{m+1}},$$

where the symbol $\wedge$ posed over j_k designates that this index is omitted. Putting $\xi = (\xi', D_t)$, $j_{m+1} = n$ in the last equality and taking (3.3.15) into account, we obtain

$$(d_0(\xi', D_t)v)_{nj_1\dots j_m} = \frac{i}{m+1}\left[(l+1)D_tv_{j_1\dots j_m} + \sum_{j_k\neq n}\xi_{j_k}v_{nj_1\dots\widehat{j_k}\dots j_m}\right] = 0. \tag{3.3.16}$$

Here $l = l(j_1,\dots,j_m)$ is the number of occurrences of the index n in $(j_1,\dots,j_m)$. Thus the field $v(t)$ satisfies the homogeneous system (3.3.16) which is resolved with respect to derivatives. The last claim, together with the initial condition $v(0) = 0$, implies that $v(t) \equiv 0$. Ellipticity of problem (3.3.5) is proved.

For a field $u \in C^\infty(S^m\tau'_M)$ and a geodesic $\gamma : (a,b) \to M$, the next equality is valid:

$$\frac{d}{dt}\left[u_{i_1\dots i_m}(\gamma(t))\dot\gamma^{i_1}(t)\dots\dot\gamma^{i_m}(t)\right] = (du)_{i_1\dots i_{m+1}}(\gamma(t))\dot\gamma^{i_1}(t)\dots\dot\gamma^{i_{m+1}}(t). \tag{3.3.17}$$

It can be easily proved with the help of the operator $D/dt = \dot\gamma^i\nabla_i$ of total differentiation along γ. Indeed, using the equality $D\dot\gamma/dt = 0$, we obtain

$$\frac{d}{dt}\left(u_{i_1\dots i_m}\dot\gamma^{i_1}\dots\dot\gamma^{i_m}\right) = \frac{D}{dt}\left(u_{i_1\dots i_m}\dot\gamma^{i_1}\dots\dot\gamma^{i_m}\right) = \left(\frac{Du}{dt}\right)_{i_1\dots i_m}\dot\gamma^{i_1}\dots\dot\gamma^{i_m} =$$

$$= u_{i_1\dots i_m\,;\,j}\dot\gamma^j\dot\gamma^{i_1}\dots\dot\gamma^{i_m} = (du)_{i_1\dots i_{m+1}}\dot\gamma^{i_1}\dots\dot\gamma^{i_{m+1}}.$$

Let us prove that problem (3.3.5) has the trivial kernel; i.e., that the homogeneous problem $\delta du = 0,\ u|_{\partial M} = 0$ has only zero solution. By ellipticity, we can assume the field u to be smooth. Putting $v = du$ and $D = M$ in the Green's formula (3.3.1), we obtain $du = 0$. Let $x_0 \in M \setminus \partial M$, and x_1 be a point in the boundary ∂M which is nearest to x_0. There exists a geodesic $\gamma : [-1, 0] \to M$ such that $\gamma(-1) = x_1$ and $\gamma(0) = x_0$. For a vector $\xi \in T_{x_0}M$, let γ_ξ be the geodesic defined by the initial conditions $\gamma_\xi(0) = x_0,\ \dot\gamma_\xi(0) = \xi$. If ξ is sufficiently close to $\dot\gamma(0)$, then γ_ξ intersects ∂M for some $t_0 = t_0(\xi) < 0$. Using (3.3.17), we obtain

$$u_{i_1\dots i_m}(x_0)\xi^{i_1}\dots\xi^{i_m} = u_{i_1\dots i_m}(\gamma_\xi(t_0))\dot\gamma_\xi^{i_1}(t_0)\dots\dot\gamma_\xi^{i_m}(t_0) +$$

$$+\int\limits_{t_0}^{0}(du)_{i_1\dots i_{m+1}}(\gamma_\xi(t))\dot\gamma_\xi^{i_1}(t)\dots\dot\gamma_\xi^{i_{m+1}}(t)\,dt = 0.$$

Since the last equality is valid for all ξ in a neighbourhood of the vector $\dot\gamma(0)$ in $T_{x_0}M$, it implies that $u(x_0)=0$. This means that $u\equiv 0$ because x_0 is arbitrary.

Let us prove that problem (3.3.5) has the trivial cokernel. Let a field $f\in C^\infty(S^m\tau'_M)$ be orthogonal to the image of the operator of the boundary problem:

$$\int\limits_M \langle f,\delta du\rangle\,dV^n=0 \tag{3.3.18}$$

for every field $u\in C^\infty(S^m\tau'_M)$ satisfying the boundary condition

$$u|_{\partial M}=0. \tag{3.3.19}$$

We have to show that $f=0$. We first take u such that $\operatorname{supp}u\subset M\setminus\partial M$. From (3.3.18) with the help of the Green's formula, we obtain

$$0=\int\limits_M\langle f,\delta du\rangle\,dV^n=-\int\limits_M\langle df,du\rangle\,dV^n=\int\limits_M\langle\delta d\,f,u\rangle\,dV^n.$$

Since $u\in C_0^\infty(S^m\tau'_M)$ is arbitrary, the last equality implies that

$$\delta d\,f=0. \tag{3.3.20}$$

Now let $v\in C^\infty(S^m\tau'_M|_{\partial M})$ be arbitrary. One can easily see that there exists $u\in C^\infty(S^m\tau'_M)$ such that

$$u|_{\partial M}=0,\qquad j_\nu du|_{\partial M}=v. \tag{3.3.21}$$

From (3.3.18), (3.3.20) and (3.3.21) with the help of the Green's formula, we obtain

$$0=\int\limits_M\langle f,\delta du\rangle\,dV^n=-\int\limits_M\langle d\,f,du\rangle\,dV^n+\int\limits_{\partial M}\langle f,j_\nu du\rangle\,dV^{n-1}=$$

$$=\int\limits_M\langle\delta d\,f,u\rangle\,dV^n+\int\limits_{\partial M}\langle f,v\rangle\,dV^{n-1}=\int\limits_{\partial M}\langle f,v\rangle\,dV^{n-1}.$$

Thus $\int_{\partial M}\langle f,v\rangle\,dV^{n-1}=0$ for every $v\in C^\infty(S^m\tau'_M|_{\partial M})$ and, consequently, $f|_{\partial M}=0$. As we know, the last equality and (3.3.20) imply that $f=0$. The theorem is proved.

P r o o f of Lemma 3.3.3. For a symmetric tensor u of degree m, we obtain

$$(j_\xi i_\xi u)_{i_1\dots i_m}=\xi^{i_{m+1}}\sigma(i_1\dots i_{m+1})(u_{i_1\dots i_m}\xi_{i_{m+1}})$$

where $\sigma(i_1\dots i_{m+1})$ is the symmetrization in the indices $i_1\dots i_{m+1}$. Using the decomposition of the symmetrization given by Lemma 2.4.1, we transform the right-hand side of the last equality as follows:

$$(j_\xi i_\xi u)_{i_1\dots i_m}=\frac{1}{m+1}\xi^{i_{m+1}}\sigma(i_1...i_m)(u_{i_1\dots i_m}\xi_{i_{m+1}}+mu_{i_2\dots i_{m+1}}\xi_{i_1})=$$

$$=\frac{1}{m+1}\sigma(i_1...i_m)(u_{i_1\dots i_m}\xi_{i_{m+1}}\xi^{i_{m+1}}+m\xi_{i_1}u_{i_2\dots i_{m+1}}\xi^{i_{m+1}})=$$

$$=\Big(\frac{1}{m+1}|\xi|^2u+\frac{m}{m+1}i_\xi j_\xi u\Big)_{i_1\dots i_m}.$$

The lemma is proved.

3.4 Semibasic tensor fields

The modern mathematical style presumes that invariant (independent of the choice of coordinates) notions are introduced by invariant definitions. Risking to look old-fashioned, in the current and next sections the author consciously chooses the opposite approach. The notions under consideration will first be introduced with the help of local coordinates. We will pay particular attention to the rule of transformation of the quantities under definition with respect to a change of coordinates. Only at the end of the sections we will briefly discuss the possibility of an invariant definition.

Let M be a manifold of dimension n and $\tau_M = (TM, p, M)$ be its tangent bundle. Points of the manifold TM are designated by the pairs (x, ξ) where $x \in M$, $\xi \in T_xM$. If $(x^1, \dots, x^n)$ is a local coordinate system defined in a domain $U \subset M$, then by $\partial_i = \partial/\partial x^i \in C^\infty(\tau_M; U)$ we mean the coordinate vector fields and by $dx^i \in C^\infty(\tau'_M; U)$ we mean the coordinate covector fields. We recall that the coordinates of a vector $\xi \in T_xM$ are the coefficients of the expansion $\xi = \xi^i\partial/\partial x^i$. On the domain $p^{-1}(U) \subset TM$, the family of the functions $(x^1, \dots, x^n, \xi^1, \dots, \xi^n)$ is a local coordinate system (strictly speaking, we have to write $x^i \circ p$; nevertheless we will use a more brief notation x^i, hoping that it will not lead to misunderstanding) which is called *associated* with the system $(x^1, \dots, x^n)$. A local coordinate system on TM will be called a *natural coordinate system* if it is associated with some local coordinate system on M. From now on we will use only such coordinate systems on TM. If $(x'^1, \dots, x'^n)$ is another coordinate system defined in a domain $U' \subset M$, then in $p^{-1}(U \cap U')$ the associated coordinates are related by the transformation formulas

$$x'^i = x'^i(x^1, \dots, x^n); \quad \xi'^i = \frac{\partial x'^i}{\partial x^j}\xi^j. \tag{3.4.1}$$

Unlike the case of general coordinates, these formulas have the next peculiarity: the first n transformation functions are independent of ξ^i while the last n functions depend linearly on these variables. This peculiarity is the base of all further constructions in the current section.

The algebra of tensor fields of the manifold TM is generated locally by the coordinate fields $\partial/\partial x^i$, $\partial/\partial \xi^i$, dx^i, $d\xi^i$. Differentiating (3.4.1), we obtain the next rules for transforming the fields with respect to change of natural coordinates:

$$\frac{\partial}{\partial \xi^i} = \frac{\partial x'^j}{\partial x^i}\frac{\partial}{\partial \xi'^j}, \qquad dx'^i = \frac{\partial x'^i}{\partial x^j}dx^j, \tag{3.4.2}$$

$$\frac{\partial}{\partial x^i} = \frac{\partial x'^j}{\partial x^i}\frac{\partial}{\partial x'^j} + \frac{\partial^2 x'^j}{\partial x^i \partial x^k}\xi^k\frac{\partial}{\partial \xi'^j}, \quad d\xi'^i = \frac{\partial^2 x'^i}{\partial x^j \partial x^k}\xi^k dx^j + \frac{\partial x'^i}{\partial x^j}d\xi^j. \tag{3.4.3}$$

We note that formulas (3.4.2) contain only the first-order derivatives of the transformation functions and take the observation as the basis for the next definition.

A tensor $u \in T^r_{s,(x,\xi)}(TM)$ of degree (r, s) at a point (x, ξ) of the manifold TM is called *semibasic* if in some (and, consequently, in any) natural coordinate system it can be represented as:

$$u = u^{i_1 \dots i_r}_{j_1 \dots j_s}\frac{\partial}{\partial \xi^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s} \tag{3.4.4}$$

with complex coefficients $u^{i_1\dots i_r}_{j_1\dots j_s}$ that are called the *coordinates* (or *components*) of the tensor u. Assuming the choice of the natural coordinate system to be clear from the context (or arbitrary), we will abbreviate equality (3.4.4) to the next one:

$$u = (u^{i_1\dots i_r}_{j_1\dots j_s}). \tag{3.4.5}$$

It follows from (3.4.2) that, under change of a natural coordinate system, the components of a semibasic tensor are transformed by the formulas

$$u'^{i_1\dots i_r}_{j_1\dots j_s} = \frac{\partial x'^{i_1}}{\partial x^{k_1}} \dots \frac{\partial x'^{i_r}}{\partial x^{k_r}} \frac{\partial x^{l_1}}{\partial x'^{j_1}} \dots \frac{\partial x^{l_s}}{\partial x'^{j_s}} u^{k_1\dots k_r}_{l_1\dots l_s} \tag{3.4.6}$$

which are identical in form with formulas (3.1.6) for transforming components of an ordinary tensor on M. The set of all semibasic tensors of degree (r,s) constitutes the subbundle in $\tau^r_s(TM)$. We shall denote the subbundle by $\beta^r_s M = (B^r_s M, p^r_s, TM)$. Sections of this bundle are called *semibasic tensor fields of degree* (r,s). For such a field $u \in C^\infty(\beta^r_s M)$, equalities (3.4.4) and (3.4.5) are valid in the domain $p^{-1}(U)$ in which a natural coordinate system acts; here $u^{i_1\dots i_r}_{j_1\dots j_s} \in C^\infty(p^{-1}(U))$. Note that $C^\infty(\beta^0_0 M) = C^\infty(TM)$, i.e., semibasic tensor fields of degree $(0,0)$ are just smooth functions on TM. The elements of $C^\infty(\beta^1_0 M)$ are called the *semibasic vector fields*, and the elements of $C^\infty(\beta^0_1 M)$ are called *semibasic covector fields*.

Formula (3.4.6) establishes a formal analogy between ordinary tensors and semibasic tensors. Using the analogy, we introduce some algebraic and differential operations on semibasic tensor fields.

The set $C^\infty(\beta^r_s M)$ is a $C^\infty(TM)$-module, i.e., the semibasic tensor fields of the same degree can be summed and multiplied by functions $\varphi(x,\xi)$ depending smoothly on $(x,\xi) \in TM$.

For $u \in C^\infty(\beta^r_s M)$ and $v \in C^\infty(\beta^{r'}_{s'} M)$ the *tensor product* $u \otimes v \in C^\infty(\beta^{r+r'}_{s+s'} M)$ is defined in coordinate form by formula (3.1.9). With the help of (3.4.6) by standard arguments, one proves correctness of this definition, i.e., that the field $u \otimes v$ is independent of the choice of a natural coordinate system participating in the definition. The so-obtained operation turns $C^\infty(\beta^*_* M) = \bigoplus_{r,s=0}^{\infty} C^\infty(\beta^r_s M)$ into a bigraded $C^\infty(TM)$-algebra. This algebra is generated locally by the coordinate semibasic fields $\partial/\partial\xi^i$ and dx^i.

The *operations of transposition of upper and lower indices* are defined by formulas (3.1.7), and the *convolution operators* $C^k_l : \beta^r_s M \to \beta^{r-1}_{s-1} M$ are defined by (3.1.8). With the help of (3.4.6), one verifies correctness of these definitions.

Tensor fields on M can be identified with the semibasic tensor fields on TM whose components are independent of the second argument ξ. Let us call such the fields *basic fields*. Formula (3.4.6) implies that this property is independent of choice of natural coordinates. Thus we obtain the canonical imbedding

$$\kappa : C^\infty(\tau^r_s M) \subset C^\infty(\beta^r_s M) \tag{3.4.7}$$

which is compatible with all algebraic operations introduced above. Note that $\kappa(\partial/\partial x^i) = \partial/\partial\xi^i$ and $\kappa(dx^i) = dx^i$.

Given $u \in C^\infty(\beta^r_s M)$, it follows from (3.4.6) that the set of the functions

$$\overset{v}{\nabla}_k u^{i_1\dots i_r}_{j_1\dots j_s} = \frac{\partial}{\partial \xi^k} u^{i_1\dots i_r}_{j_1\dots j_s} \tag{3.4.8}$$

is the set of components of a semibasic field of degree $(r, s+1)$. The equality

$$\overset{v}{\nabla} u = \overset{v}{\nabla}_k u^{i_1 \dots i_r}_{j_1 \dots j_s} \frac{\partial}{\partial \xi^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s} \otimes dx^k \tag{3.4.9}$$

defines correctly the differential operator $\overset{v}{\nabla}$: $C^\infty(\beta^r_s M) \rightarrow C^\infty(\beta^r_{s+1} M)$ which will be called the *vertical covariant derivative*. One can verify directly that $\overset{v}{\nabla}$ commutes with the convolution operators and is related to the tensor product by the equality

$$\overset{v}{\nabla}(u \otimes v) = \rho_{s+1}(\overset{v}{\nabla} u \otimes v) + u \otimes \overset{v}{\nabla} v \tag{3.4.10}$$

for $u \in C^\infty(\beta^r_s M)$ where ρ_{s+1} is the same as in (3.2.6).

In conclusion of the section we will formulate the invariant definitions of the above-introduced notions.

Let $p^*\tau^r_s M$ be the bundle, over TM, induced by projection p of the tangent bundle from the bundle $\tau^r_s M$. By the definition of induced bundle, the fiber of $p^*\tau^r_s M$ over a point $(x, \xi) \in TM$ can be identified with $T^r_{s,x} M$. With the help of this identification, the operation on $\tau^*_* M$ introduced above (tensor product, convolution operators and transpositions of indices) are transferred to $p^*\tau^*_* M = \bigoplus_{r,s=0}^{\infty} p^*\tau^r_s M$.

We could formulate the simplest invariant definition of the bundle of semibasic tensors by declaring $\beta^r_s M$ coincident with $p^*\tau^r_s M$. However a semibasic tensor field defined in this manner would not be a tensor field on TM. Therefore we will proceed otherwise by constructing a monomorphism $0 \rightarrow p^*\tau^r_s M \overset{\lambda}{\rightarrow} \tau^r_s(TM)$ and defining $\beta^r_s M$ as the image of the monomorphism.

For $x \in M$ we consider the next sequence of manifolds and smooth mappings:

$$T_x M \overset{i}{\subset} TM \overset{p}{\rightarrow} M, \tag{3.4.11}$$

where i is the imbedding and p is the projection of the tangent bundle. The leftmost manifold in the sequence is provided by the structure of vector space and, consequently, its tangent space can be identified with $T_x M$ itself. The differentials of mappings (3.4.11) form the exact sequence

$$0 \rightarrow T_x M \overset{i_*}{\longrightarrow} T_{(x,\xi)}(TM) \overset{p_*}{\longrightarrow} T_x M \rightarrow 0 \tag{3.4.12}$$

of vector spaces and linear mappings (recall that a sequence of vector spaces and linear mappings is called exact if the image of every mapping coincides with the kernel of the subsequent mapping). Passing to the dual spaces and mappings, we obtain the second exact sequence

$$0 \leftarrow T'_x M \overset{i^*}{\longleftarrow} T'_{(x,\xi)}(TM) \overset{p^*}{\longleftarrow} T'_x M \leftarrow 0. \tag{3.4.13}$$

We define the monomorphism

$$0 \rightarrow T^r_{s,x} M \overset{\lambda_{(x,\xi)}}{\longrightarrow} T^r_{s,(x,\xi)}(TM) \tag{3.4.14}$$

as the complexification of the mapping $i_* \otimes \dots \otimes i_* \otimes p^* \otimes \dots \otimes p^*$. Then $\lambda_{(x,\xi)}$ depends smoothly on $(x, \xi) \in TM$ and defines the monomorphism of the bundles

$$0 \rightarrow p^*\tau^r_s M \overset{\lambda}{\longrightarrow} \tau^r_s(TM). \tag{3.4.15}$$

We define the bundle $\beta^r_s M$ of semibasic tensors as the image of this monomorphism. By definition,

$$\beta^r_s M \subset \tau^r_s(TM) \tag{3.4.16}$$

and there is the isomorphism (denoted by the same letter)

$$0 \to p^*\tau^r_s M \xrightarrow{\lambda} \beta^r_s M \to 0, \tag{3.4.17}$$

The tensor product, convolutions and transpositions of indices defined above on $p^*\tau^r_s M$ are transferred to $\beta^r_s M$ with the help of isomorphism (3.4.17).

For a natural coordinate system, the monomorphisms i_* and p^* in (3.4.12) and (3.4.13) satisfy the relations $i_*(\partial/\partial x^i) = \partial/\partial\xi^i$ and $p^*(dx^i) = dx^i$. Consequently, $C^\infty(\beta^*_* M)$ is a $C^\infty(TM)$-subalgebra, of the algebra $C^\infty(\tau^*_*(TM))$, generated by the fields $\partial/\partial\xi^i, dx^i$. Therefore the given invariant definition of semibasic tensor field is equivalent to the local definition (3.4.4).

Note that the operators of tensor multiplication and transposition of indices defined on the bundles $\beta^*_* M$ and $\tau^*_*(TM)$ are compatible with imbedding (3.4.16), whereas the convolution operators are not. Indeed, in the sense of $\tau^*_*(TM)$ each convolution of every semibasic tensor is equal to zero. Therefore, speaking about semibasic tensors, we will always use the convolution in the sense of $\beta^*_* M$.

The invariant definition of imbedding (3.4.7) is given by the equality $(\kappa u)(x,\xi) = \lambda_{(x,\xi)}u(x)$, for $u \in C^\infty(\tau^r_s M)$, where $\lambda_{(x,\xi)}$ is monomorphism (3.4.14).

To formulate the invariant definition of the vertical covariant derivative we note preliminary that, for a smooth mapping $f : A \to B$ of finite-dimensional vector spaces, the differential of f can be considered as the smooth mapping $d\,f : A \to B \otimes A'$.

Let $u \in C^\infty(\beta^r_s M)$. Given $x \in M$, we define the mapping $u_x : T_x M \to T^r_{s,x}M$ by the equality $u_x(\xi) = \lambda^{-1}u(x,\xi)$ where λ is isomorphism (3.4.17) and the fiber of $p^*\tau^r_s M$ over (x,ξ) is identified with $T^r_{s,x}M$ as above. Let

$$du_x : T_x M \to T^r_{s,x}M \otimes T'_x M = T^r_{s+1,x}M$$

be the differential of the mapping u_x. We define $\overset{v}{\nabla} u(x,\xi) = \lambda_{(x,\xi)}du_x(\xi)$, where $\lambda_{(x,\xi)}$ is monomorphism (3.4.14).

3.5 The horizontal covariant derivative

In this section M is a Riemannian manifold with metric tensor g.

First of all we will give some heuristic argument which is to be considered as hint leading to the formal definition introduced in the next paragraph. The reader not acquainted with Chapter 1 can miss this paragraph. In Section 1.2 we saw that, for a function $u \in C^\infty(TM)$, there exists a function $Hu \in C^\infty(TM)$ such that in a natural coordinate system it is given by the formula

$$Hu = \left(\xi^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk}\xi^j\xi^k\frac{\partial}{\partial \xi^i}\right)u = \xi^i\left(\frac{\partial u}{\partial x^i} - \Gamma^p_{iq}\xi^q\frac{\partial u}{\partial \xi^p}\right). \tag{3.5.1}$$

The first factor on the right-hand side of this equality is the component of the semibasic vector field $\xi = (\xi^i)$. Invariance of the function Hu suggests that the second factor on the right-hand side of (3.5.1) is also the component of some semibasic covector field. This observation we use as a basis for the next definition.

The *horizontal covariant derivative* of a function $u \in C^\infty(TM) = C^\infty(\beta^0_0 M)$ is the semibasic covector field $\overset{h}{\nabla} u \in C^\infty(\beta^0_1 M)$ given in a natural coordinate system by the equalities

$$\overset{h}{\nabla} u = (\overset{h}{\nabla}_k u) dx^k, \qquad \overset{h}{\nabla}_k u = \frac{\partial u}{\partial x^k} - \Gamma^p_{kq} \xi^q \frac{\partial u}{\partial \xi^p}. \tag{3.5.2}$$

To show correctness of the definition we have to prove that, under a change of the natural coordinate system, the functions (3.5.2) are transformed by formulas (3.4.6) for $r = 0$ and $s = 1$. Using (3.2.8), (3.4.2) and (3.4.3), we obtain

$$\overset{h}{\nabla}{}'_k u = \left(\frac{\partial}{\partial x'^k} - \Gamma'^p_{kq} \xi'^q \frac{\partial}{\partial \xi'^p} \right) u =$$

$$= \left[\frac{\partial x^\alpha}{\partial x'^k} \frac{\partial}{\partial x^\alpha} + \frac{\partial^2 x^\alpha}{\partial x'^k \partial x'^i} \xi'^i \frac{\partial}{\partial \xi^\alpha} - \left(\frac{\partial x'^p}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^k} \frac{\partial x^\gamma}{\partial x'^q} \Gamma^\alpha_{\beta\gamma} + \frac{\partial x'^p}{\partial x^\alpha} \frac{\partial^2 x^\alpha}{\partial x'^k \partial x'^q} \right) \xi'^q \frac{\partial x^\varepsilon}{\partial x'^p} \frac{\partial}{\partial \xi^\varepsilon} \right] u.$$

Changing the notation of summation indices, we rewrite this equality as follows:

$$\overset{h}{\nabla}{}'_k u = \frac{\partial x^\alpha}{\partial x'^k} \left[\frac{\partial}{\partial x^\alpha} - \left(\frac{\partial x'^p}{\partial x^\beta} \frac{\partial x^\varepsilon}{\partial x'^p} \right) \frac{\partial x^\gamma}{\partial x'^q} \xi'^q \Gamma^\beta_{\alpha\gamma} \frac{\partial}{\partial \xi^\varepsilon} \right] u +$$

$$+ \frac{\partial^2 x^\alpha}{\partial x'^k \partial x'^q} \xi'^q \left[\frac{\partial}{\partial \xi^\alpha} - \left(\frac{\partial x'^p}{\partial x^\alpha} \frac{\partial x^\varepsilon}{\partial x'^p} \right) \frac{\partial}{\partial \xi^\varepsilon} \right] u.$$

Using (3.4.1) and taking it into account that the matrices $\partial x'/\partial x$ and $\partial x/\partial x'$ are inverse to one other, we finally obtain

$$\overset{h}{\nabla}{}'_k u = \frac{\partial x^\alpha}{\partial x'^k} \left(\frac{\partial u}{\partial x^\alpha} - \Gamma^\beta_{\alpha\gamma} \xi^\gamma \frac{\partial u}{\partial \xi^\beta} \right) = \frac{\partial x^\alpha}{\partial x'^k} \overset{h}{\nabla}_\alpha u.$$

Thus correctness of the definition of the operator $\overset{h}{\nabla}\colon C^\infty(\beta^0_0 M) \to C^\infty(\beta^0_1 M)$ is proved.

By analogy with Theorem 3.2.1 we formulate the next

Theorem 3.5.1 *Let M be a Riemannian manifold. For all integers r and s there exist uniquely determined* **C***-linear operators*

$$\overset{h}{\nabla}\colon C^\infty(\beta^r_s M) \to C^\infty(\beta^r_{s+1} M) \tag{3.5.3}$$

such that

(1) *on the basis tensor fields, $\overset{h}{\nabla}$ coincides with the operator ∇ of covariant differentiation with respect to the Riemannian connection, i.e., $\overset{h}{\nabla}(\kappa u) = \kappa(\nabla u)$, for $u \in C^\infty(\tau^r_s M)$, where κ is imbedding* (3.4.7);

(2) *on $C^\infty(\beta^0_0 M)$, $\overset{h}{\nabla}$ coincides with operator* (3.5.2);

(3) *$\overset{h}{\nabla}$ commutes with the convolution operators C^k_l for $1 \le k \le r,\ 1 \le l \le s$;*

(4) *$\overset{h}{\nabla}$ is related to the tensor product by the equality*

$$\overset{h}{\nabla}(u \otimes v) = \rho_{s+1}(\overset{h}{\nabla} u \otimes v) + u \otimes \overset{h}{\nabla} v \tag{3.5.4}$$

for $u \in C^\infty(\beta^r_s M)$, *where* ρ_{s+1} *is the same as in* (3.2.6).

In a natural coordinate system, for $u \in C^\infty(\beta^r_s M)$, *the next local representation is valid:*

$$\overset{h}{\nabla} u = \overset{h}{\nabla}_k u^{i_1 \dots i_r}_{j_1 \dots j_s} \frac{\partial}{\partial \xi^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s} \otimes dx^k, \tag{3.5.5}$$

where

$$\overset{h}{\nabla}_k u^{i_1 \dots i_r}_{j_1 \dots j_s} = \frac{\partial}{\partial x^k} u^{i_1 \dots i_r}_{j_1 \dots j_s} - \Gamma^p_{kq} \xi^q \frac{\partial}{\partial \xi^p} u^{i_1 \dots i_r}_{j_1 \dots j_s} +$$

$$+ \sum_{m=1}^{r} \Gamma^{i_m}_{kp} u^{i_1 \dots i_{m-1} p i_{m+1} \dots i_r}_{j_1 \dots j_s} - \sum_{m=1}^{s} \Gamma^p_{k j_m} u^{i_1 \dots i_r}_{j_1 \dots j_{m-1} p j_{m+1} \dots j_s}. \tag{3.5.6}$$

Pay attention to a formal analogy between the formulas (3.2.10) and (3.5.6): comparing with (3.2.10), the right-hand side of (3.5.6) contains one additional summand related to dependence of components of the field u on the coordinates ξ^i.

P r o o f. Let operators (3.5.3) satisfy conditions (1)–(4) of the theorem; let us prove the validity of the local representation (3.5.5)–(3.5.6).

The tensor fields

$$\frac{\partial}{\partial \xi^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s} = \kappa \left(\frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s} \right) \tag{3.5.7}$$

are basic. By the first condition of the theorem,

$$\overset{h}{\nabla} \left(\frac{\partial}{\partial \xi^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s} \right) =$$

$$= \sum_{m=1}^{r} \frac{\partial}{\partial \xi^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial \xi^{i_{m-1}}} \otimes \Gamma^p_{k i_m} \frac{\partial}{\partial \xi^p} \otimes \frac{\partial}{\partial \xi^{i_{m+1}}} \otimes \dots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s} \otimes dx^k -$$

$$- \sum_{m=1}^{s} \frac{\partial}{\partial \xi^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_{m-1}} \otimes \Gamma^{j_m}_{kp} dx^p \otimes dx^{j_{m+1}} \otimes \dots \otimes dx^{j_s} \otimes dx^k. \tag{3.5.8}$$

Given $u \in C^\infty(\beta^r_s M)$, we apply the fourth condition of the theorem and obtain

$$\overset{h}{\nabla} u = \overset{h}{\nabla} \left(u^{i_1 \dots i_r}_{j_1 \dots j_s} \frac{\partial}{\partial \xi^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s} \right) =$$

$$= \rho_1 \left[\left(\overset{h}{\nabla} u^{i_1 \dots i_r}_{j_1 \dots j_s} \right) \otimes \frac{\partial}{\partial \xi^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s} \right] +$$

$$+ u^{i_1 \dots i_r}_{j_1 \dots j_s} \overset{h}{\nabla} \left(\frac{\partial}{\partial \xi^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s} \right), \tag{3.5.9}$$

where the expression $\overset{h}{\nabla} u^{i_1 \dots i_r}_{j_1 \dots j_s}$ denotes the result of applying $\overset{h}{\nabla}$ to the scalar function $u^{i_1 \dots i_r}_{j_1 \dots j_s} \in C^\infty(\beta^0_0 M)$. By the second condition of the theorem, this expression can be found by formula (3.5.2). Along the same lines by using (3.5.8), we transform equality (3.5.9) as follows:

$$\overset{h}{\nabla} u = \left(\frac{\partial}{\partial x^k} u^{i_1 \dots i_r}_{j_1 \dots j_s} - \Gamma^p_{kq} \xi^q \frac{\partial}{\partial \xi^p} u^{i_1 \dots i_r}_{j_1 \dots j_s} \right) \frac{\partial}{\partial \xi^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s} \otimes dx^k +$$

$$+\sum_{m=1}^{r}\Gamma^{p}_{ki_m}u^{i_1\dots i_r}_{j_1\dots j_s}\frac{\partial}{\partial\xi^{i_1}}\otimes\dots\otimes\frac{\partial}{\partial\xi^{i_{m-1}}}\otimes\frac{\partial}{\partial\xi^{p}}\otimes\frac{\partial}{\partial\xi^{i_{m+1}}}\otimes\dots\otimes\frac{\partial}{\partial\xi^{i_r}}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_s}\otimes dx^k-$$

$$-\sum_{m=1}^{s}\Gamma^{j_m}_{kp}u^{i_1\dots i_r}_{j_1\dots j_s}\frac{\partial}{\partial\xi^{i_1}}\otimes\dots\otimes\frac{\partial}{\partial\xi^{i_r}}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_{m-1}}\otimes dx^{p}\otimes dx^{j_{m+1}}\otimes\dots\otimes dx^{j_s}\otimes dx^k.$$

Changing the limits of summation over the indices i_m and p in the first sum of the right-hand side and changing the limits of summation over j_m and p in the second sum, we arrive at (3.5.5) and (3.5.6).

Conversely, let us define the operators $\overset{h}{\nabla}$ by formulas (3.5.5)–(3.5.6) in a natural coordinate system, With the help of arguments similar to those we have used just after definition (3.5.2), one can prove correctness of this definition. Thereafter validity of claims (1)–(4) of the theorem can easily be proved by a straightforward calculation in coordinate form. The theorem is proved.

Theorem 3.5.2 *The vertical and horizontal derivatives satisfy the next commutation relations:*

$$(\overset{v}{\nabla}_k\overset{v}{\nabla}_l-\overset{v}{\nabla}_l\overset{v}{\nabla}_k)u^{i_1\dots i_r}_{j_1\dots j_s}=0, \tag{3.5.10}$$

$$(\overset{v}{\nabla}_k\overset{h}{\nabla}_l-\overset{h}{\nabla}_l\overset{v}{\nabla}_k)u^{i_1\dots i_r}_{j_1\dots j_s}=0, \tag{3.5.11}$$

$$(\overset{h}{\nabla}_k\overset{h}{\nabla}_l-\overset{h}{\nabla}_l\overset{h}{\nabla}_k)u^{i_1\dots i_r}_{j_1\dots j_s}=-R^{p}_{\ qkl}\xi^q\overset{v}{\nabla}_p u^{i_1\dots i_r}_{j_1\dots j_s}+$$
$$+\sum_{m=1}^{r}R^{i_m}_{\ pkl}u^{i_1\dots i_{m-1}pi_{m+1}\dots i_r}_{j_1\dots j_s}-\sum_{m=1}^{s}R^{p}_{\ j_mkl}u^{i_1\dots i_r}_{j_1\dots j_{m-1}pj_{m+1}\dots j_s}. \tag{3.5.12}$$

We again pay attention to a formal analogy between the formulas (3.2.11) and (3.5.12).

P r o o f. Equality (3.5.10) is evident, since $\overset{v}{\nabla}_k=\partial/\partial\xi^k$. To prove (3.5.11) we differentiate equality (3.5.6) with respect to ξ^l :

$$\overset{v}{\nabla}_l\overset{h}{\nabla}_k u^{i_1\dots i_r}_{j_1\dots j_s}=\frac{\partial}{\partial x^k}\overset{v}{\nabla}_l u^{i_1\dots i_r}_{j_1\dots j_s}-\Gamma^p_{kq}\xi^q\frac{\partial}{\partial\xi^p}\overset{v}{\nabla}_l u^{i_1\dots i_r}_{j_1\dots j_s}-\Gamma^p_{kl}\overset{v}{\nabla}_p u^{i_1\dots i_r}_{j_1\dots j_s}+$$
$$+\sum_{m=1}^{r}\Gamma^{i_m}_{kp}\overset{v}{\nabla}_l u^{i_1\dots i_{m-1}pi_{m+1}\dots i_r}_{j_1\dots j_s}-\sum_{m=1}^{s}\Gamma^{p}_{kj_m}\overset{v}{\nabla}_l u^{i_1\dots i_r}_{j_1\dots j_{m-1}pj_{m+1}\dots j_s}.$$

Including the third summand on the right-hand side into the last sum, we arrive at (3.5.11).

We will prove (3.5.12) only for $r=s=0$. In other cases this formula is proved by similar but more cumbersome calculations. For $u\in C^\infty(\beta^0_0M)$, we obtain

$$\overset{h}{\nabla}_k\overset{h}{\nabla}_l u=\left(\frac{\partial}{\partial x^k}-\Gamma^p_{kq}\xi^q\frac{\partial}{\partial\xi^p}\right)\overset{h}{\nabla}_l u-\Gamma^p_{kl}\overset{h}{\nabla}_p u=$$
$$=\left(\frac{\partial}{\partial x^k}-\Gamma^p_{kq}\xi^q\frac{\partial}{\partial\xi^p}\right)\left(\frac{\partial u}{\partial x^l}-\Gamma^j_{lr}\xi^r\frac{\partial u}{\partial\xi^j}\right)-\Gamma^p_{kl}\left(\frac{\partial u}{\partial x^p}-\Gamma^j_{pq}\xi^q\frac{\partial u}{\partial\xi^j}\right).$$

After opening the parenthesis and changing notation in summation indices, this equality takes the form

$$\overset{h}{\nabla}_k\overset{h}{\nabla}_l u=\frac{\partial^2u}{\partial x^k\partial x^l}-\Gamma^p_{lq}\xi^q\frac{\partial^2u}{\partial x^k\partial\xi^p}-\Gamma^p_{kq}\xi^q\frac{\partial^2u}{\partial x^l\partial\xi^p}+\Gamma^p_{kq}\Gamma^j_{lr}\xi^q\xi^r\frac{\partial^2u}{\partial\xi^p\partial\xi^j}-$$

$$-\Gamma^p_{kl}\frac{\partial u}{\partial x^p}-\left(\frac{\partial \Gamma^p_{lq}}{\partial x^k}-\Gamma^j_{kq}\Gamma^p_{lj}-\Gamma^j_{kl}\Gamma^p_{jq}\right)\xi^q\overset{v}{\nabla}_p u. \tag{3.5.13}$$

Alternating (3.5.13) with respect to k and l, we come to

$$(\overset{h}{\nabla}_k\overset{h}{\nabla}_l-\overset{h}{\nabla}_l\overset{h}{\nabla}_k)u=-\left(\frac{\partial \Gamma^p_{lq}}{\partial x^k}-\frac{\partial \Gamma^p_{kq}}{\partial x^l}+\Gamma^j_{lq}\Gamma^p_{kj}-\Gamma^j_{kq}\Gamma^p_{lj}\right)\xi^q\overset{v}{\nabla}_p u.$$

By (3.2.9), the last equality coincides with (3.5.12) for $r=s=0$. The theorem is proved.

Note that the next relations are valid:

$$\overset{v}{\nabla}_k g_{ij}=\overset{h}{\nabla}_k g_{ij}=0,\quad \overset{v}{\nabla}_k\delta^i_j=\overset{h}{\nabla}_k\delta^i_j=0,\quad \overset{h}{\nabla}_k\xi^i=0,\quad \overset{v}{\nabla}_k\xi^i=\delta^i_k.$$

In what follows we will also use the notation: $\overset{v}{\nabla}{}^i=g^{ij}\overset{v}{\nabla}_j$, $\overset{h}{\nabla}{}^i=g^{ij}\overset{h}{\nabla}_j$.

In conclusion of the section we will formulate an invariant definition of horizontal covariant derivative. To this end we recall that every point x_0 of a Riemannian manifold M has some so-called *normal neighbourhood* $U\subset M$ that is characterized by the following property: for every $x\in U$ there exists a unique geodesic γ_{x_0x} such that it is in U and its endpoints are x_0 and x. For $(x_0,\xi_0)\in TM$ and $x\in U$ we denote by $\eta_{(x_0,\xi_0)}(x)\in T_xM$ the vector obtained from ξ_0 by parallel displacement to the point x along γ_{x_0x}. Thus we have constructed the vector field $\eta_{(x_0,\xi_0)}\in C^\infty(\tau_M;U)$.

Let $u\in C^\infty(\beta^r_sM)$, $(x_0,\xi_0)\in TM$ and U be a normal neighbourhood of the point x_0. We define the section $\tilde u\in C^\infty(\tau^r_sM;U)$ by the equality $\tilde u(x)=\lambda^{-1}_{(x,\eta_0(x))}u(x,\eta_0(x))$ where $\eta_0=\eta_{(x_0,\xi_0)}$ is the vector field constructed in the previous paragraph and $\lambda_{(x,\xi)}$ is isomorphism (3.4.14). We give an invariant definition of horizontal covariant derivative by putting

$$\overset{h}{\nabla}u(x_0,\xi_0)=\lambda_{(x_0,\xi_0)}\nabla\tilde u(x_0). \tag{3.5.14}$$

To show that definitions (3.5.5)–(3.5.6) and (3.5.14) are equivalent it is sufficient to prove this fact for $u\in C^\infty(\beta^0_0M)$, since it is evident that the operator defined by formula (3.5.14) satisfies conditions (1), (3) and (4) of Theorem 5.1.

Let $u\in C^\infty(\beta^0_0M)$ and $(x_0,\xi_0)\in TM$. We choose a local coordinate system in a neighbourhood of the point x_0. The above-constructed vector field $\eta_0=\eta_{(x_0,\xi_0)}$ satisfies the relations

$$\eta_0(x_0)=\xi_0,\quad \frac{\partial\eta^i_0}{\partial x^k}(x_0)=-\Gamma^i_{kq}(x_0)\xi^q_0. \tag{3.5.15}$$

The function $\tilde u\in C^\infty(U)$ participating in the definition (3.5.14) is expressed through u :

$$\tilde u(x)=u(x,\eta_0(x)).$$

Differentiating this equality, we obtain

$$\nabla\tilde u(x)=\frac{\partial\tilde u}{\partial x^k}dx^k=\left[\frac{\partial u}{\partial x^k}(x,\eta_0(x))+\frac{\partial u}{\partial\xi^p}(x,\eta_0(x))\frac{\partial\eta^p_0}{\partial x^k}(x)\right]dx^k.$$

Putting $x=x_0$ here and using (3.5.15), we arrive at

$$\nabla\tilde u(x_0)=\left[\frac{\partial u}{\partial x^k}(x_0,\xi_0)-\Gamma^p_{kq}(x_0)\xi^q_0\frac{\partial u}{\partial\xi^p}(x_0,\xi_0)\right]dx^k.$$

Inserting the last expression into the right-hand side of equality (3.5.14), we obtain (3.5.2).

3.6 Formulas of Gauss-Ostrogradskiĭ type for vertical and horizontal derivatives

We recall that, on the space $T'M$ of the cotangent bundle of a manifold M, there is the symplectic structure defined by the 2-form $\omega = d(\xi_i dx^i) = d\xi_i \wedge dx^i$ (from now on in this section d is the operator of exterior differentiation). The *symplectic volume form* is the $2n$-form $dV^{2n} = (-1)^{n(n-1)/2}\omega^n$.

Let now (M, g) be a Riemannian manifold. As mentioned in Section 3.2, in this case we have the canonical isomorphism between the tangent and cotangent bundles of M. The isomorphism transfers dV^{2n} to some $2n$-form on TM which will again be called the symplectic volume form and be designated by the same symbol dV^{2n}. In a natural coordinate system this form is given by the equality

$$dV^{2n} = \det(g_{ij})\, d\xi \wedge dx = \det(g_{ij}) d\xi^1 \wedge \ldots \wedge d\xi^n \wedge dx^1 \wedge \ldots \wedge dx^n. \tag{3.6.1}$$

In the domain of the natural coordinate system on the space TM of the tangent bundle of a Riemannian manifold we introduce $(2n-1)$-forms:

$$\overset{v}{\omega}_i = (-1)^{i-1} g\, d\xi^1 \wedge \ldots \wedge \widehat{d\xi^i} \wedge \ldots \wedge d\xi^n \wedge dx, \tag{3.6.2}$$

$$\overset{h}{\omega}_i = g\Big[(-1)^{n+i-1} d\xi \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n + \\ + \sum_{j=1}^{n} (-1)^j \Gamma^j_{ip} \xi^p d\xi^1 \wedge \ldots \wedge \widehat{d\xi^j} \wedge \ldots \wedge d\xi^n \wedge dx\Big]. \tag{3.6.3}$$

We recall that the symbol $\wedge$ over a factor designates that the factor is omitted. From now on in this section $g = \det(g_{jk})$.

Lemma 3.6.1 *The forms $\overset{v}{\omega}_i$ and $\overset{h}{\omega}_i$ have the following properties:*

(1) *under a change of a natural coordinate system, each of the families $(\overset{v}{\omega}_i)$ and $(\overset{h}{\omega}_i)$ transforms according to the same rule as components of a semibasic covector field; consequently, for every semibasic vector field $u = (u^i)$, the forms $u^i\overset{v}{\omega}_i$ and $u^i\overset{h}{\omega}_i$ are independent of the choice of a natural coordinate system and are defined globally on TM;*

(2) *For a semibasic vector field $u = (u^i)$ the next equalities are valid:*

$$d(u^i\overset{v}{\omega}_i) = \overset{v}{\nabla}_i u^i\, dV^{2n}, \qquad d(u^i\overset{h}{\omega}_i) = \overset{h}{\nabla}_i u^i\, dV^{2n}. \tag{3.6.4}$$

P r o o f. First of all we note that for coincidence of two $(d-1)$-forms α and β defined on a d-dimensional manifold X it is sufficient that the equality $\alpha \wedge dx^i = \beta \wedge dx^i$ $(i = 1, \ldots, d)$ is valid for any local coordinate system $(x^1, \ldots, x^d)$ on X.

The first claim of the lemma say that, in the intersection of domains of two natural coordinate systems, the next relations are valid:

$$\overset{v}{\omega}{}'_i = \frac{\partial x^j}{\partial x'^i}\overset{v}{\omega}_j, \qquad \overset{h}{\omega}{}'_i = \frac{\partial x^j}{\partial x'^i}\overset{h}{\omega}_j.$$

By the above observation, to prove these equalities it is sufficient to show that, for every $k = 1, \ldots, n$,

$$\overset{v}{\omega}{}'_i \wedge dx'^k = \frac{\partial x^j}{\partial x'^i}\overset{v}{\omega}_j \wedge dx'^k, \quad \overset{v}{\omega}{}'_i \wedge d\xi'^k = \frac{\partial x^j}{\partial x'^i}\overset{v}{\omega}_j \wedge d\xi'^k, \tag{3.6.5}$$

$$\overset{h}{\omega}{}'_i \wedge dx'^k = \frac{\partial x^j}{\partial x'^i}\,\overset{h}{\omega}_j \wedge dx'^k, \quad \overset{h}{\omega}{}'_i \wedge d\xi'^k = \frac{\partial x^j}{\partial x'^i}\,\overset{h}{\omega}_j \wedge d\xi'^k. \tag{3.6.6}$$

Validity of the first of equalities (3.6.5) is evident, since its sides are both equal to zero, as follows from (3.6.2) and (3.4.2). By (3.6.1) and (3.6.2), the left-hand side of the second of the equalities (3.6.5) is equal to $-\delta^k_i dV^{2n}$. We calculate the right-hand side of the second of equalities (3.6.5) with the help of (3.4.3):

$$\frac{\partial x^j}{\partial x'^i}\,\overset{v}{\omega}_j \wedge d\xi'^k = (-1)^{j-1}\frac{\partial x^j}{\partial x'^i} g\, d\xi^1 \wedge \ldots \wedge \widehat{d\xi^j} \wedge \ldots \wedge d\xi^n \wedge dx \wedge \left(\frac{\partial^2 x'^k}{\partial x^p \partial x^q}\xi^p dx^q + \frac{\partial x'^k}{\partial x^l} d\xi^l\right) =$$

$$= -\frac{\partial x^j}{\partial x'^i}\frac{\partial x'^k}{\partial x^l}\,\delta^l_j\, dV^{2n} = -\delta^k_i dV^{2n}.$$

The last equality of this chain is written because the matrices $\left(\partial x^j/\partial x'^i\right)$ and $\left(\partial x'^i/\partial x^j\right)$ are inverse to one other. Thus relations (3.6.5) are proved.

By (3.6.3) and (3.6.1), the left-hand side of the first of the equalities (3.6.6) is equal to $-\delta^k_i dV^{2n}$. The right-hand side of this equality we find with the help of (3.4.2):

$$\frac{\partial x^j}{\partial x'^i}\,\overset{h}{\omega}_j \wedge dx'^k = (-1)^{n+j-1}\frac{\partial x^j}{\partial x'^i}\frac{\partial x'^k}{\partial x^l}\, g\, d\xi \wedge dx^1 \wedge \ldots \wedge \widehat{dx^j} \wedge \ldots \wedge dx^n \wedge dx^l =$$

$$= -\frac{\partial x^j}{\partial x'^i}\frac{\partial x'^k}{\partial x^l}\,\delta^l_j\, dV^{2n} = -\delta^k_i dV^{2n}.$$

By (3.6.3), the left-hand side of the second of the formulas (3.6.6) is equal to

$$\overset{h}{\omega}{}'_i \wedge d\xi'^k = \Gamma'^k_{iq}\xi'^q\, dV^{2n} = \frac{\partial x'^q}{\partial x^p}\Gamma'^k_{iq}\xi^p\, dV^{2n}. \tag{3.6.7}$$

We calculate the right-hand side of this equality with the help of (3.4.3):

$$\frac{\partial x^j}{\partial x'^i}\,\overset{h}{\omega}_j \wedge d\xi'^k = g\frac{\partial x^j}{\partial x'^i}\Big[(-1)^{n+j-1} d\xi \wedge dx^1 \wedge \ldots \wedge \widehat{dx^j} \wedge \ldots \wedge dx^n \; +$$

$$+ \sum_{l=1}^{n}(-1)^l\,\Gamma^l_{jp}\xi^p\, d\xi^1 \wedge \ldots \wedge \widehat{d\xi^l} \wedge \ldots \wedge d\xi^n \wedge dx\Big] \wedge \left(\frac{\partial^2 x'^k}{\partial x^q \partial x^r}\xi^q\, dx^r + \frac{\partial x'^k}{\partial x^q} d\xi^q\right) =$$

$$= g\frac{\partial x^j}{\partial x'^i}\left[(-1)^{n+j-1}\frac{\partial^2 x'^k}{\partial x^q \partial x^r}\xi^q\, d\xi \wedge dx^1 \wedge \ldots \wedge \widehat{dx^j} \wedge \ldots \wedge dx^n \wedge dx^r \; + \right.$$

$$\left. + \sum_{l=1}^{n}(-1)^l\frac{\partial x'^k}{\partial x^q}\Gamma^l_{jp}\,\xi^p\, d\xi^1 \wedge \ldots \wedge \widehat{d\xi^l} \wedge \ldots \wedge d\xi^n \wedge dx \wedge d\xi^q\right] =$$

$$= \frac{\partial x^j}{\partial x'^i}\left[-\frac{\partial^2 x'^k}{\partial x^q \partial x^r}\xi^q\delta^r_j + \sum_{l=1}^{n}\frac{\partial x'^k}{\partial x^q}\Gamma^l_{jp}\xi^p\delta^q_l\right] dV^{2n}.$$

After summing over r and l, we obtain

$$\frac{\partial x^j}{\partial x'^i}\,\overset{h}{\omega}_j \wedge dx'^k = \frac{\partial x^j}{\partial x'^i}\left(-\frac{\partial^2 x'^k}{\partial x^j \partial x^p} + \frac{\partial x'^k}{\partial x^l}\Gamma^l_{jp}\right)\xi^p\, dV^{2n}.$$

Comparing the last relation with (3.6.7), we see that to prove the second of the equalities (3.6.6) it is sufficient to show that

$$\frac{\partial x'^q}{\partial x^p}\Gamma'^k_{iq} = \frac{\partial x^j}{\partial x'^i}\left(-\frac{\partial^2 x'^k}{\partial x^j \partial x^p} + \frac{\partial x'^k}{\partial x^l}\Gamma^l_{jp}\right). \tag{3.6.8}$$

These relations are equivalent to formulas (3.2.8) of transformation of the Christoffel symbols, as one can verify by multiplying (3.6.8) by $\partial x'^i/\partial x^r$ and summing over i. Thus the first claim of the lemma is proved.

The form $\overset{v}{\omega}_i$ is closed, as one can see directly from (3.6.2). We find the differential of the form $\overset{h}{\omega}_i$. From (3.6.3), we obtain

$$d\overset{h}{\omega}_i = (-1)^{n+i-1}\frac{\partial g}{\partial x^k}\, dx^k \wedge d\xi \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n +$$

$$+ g\sum_{j=1}^{n}(-1)^j \Gamma^j_{ip}\, d\xi^p \wedge d\xi^1 \wedge \ldots \wedge \widehat{d\xi^j} \wedge \ldots \wedge d\xi^n \wedge dx = \left(\frac{\partial g}{\partial x^i} - g\Gamma^j_{ij}\right) d\xi \wedge dx.$$

From this, using the relation

$$\Gamma^j_{ij} = \frac{1}{2}\frac{\partial}{\partial x^i}(\ln g), \tag{3.6.9}$$

which follows from (3.2.13), we conclude that

$$d\overset{h}{\omega}_i = \Gamma^j_{ij}\, dV^{2n}. \tag{3.6.10}$$

Let us now prove the second claim of the lemma. Let $u = (u^i)$ be a semibasic vector field. Taking it into account that the form $\overset{v}{\omega}_i$ is close, we obtain

$$d(u^i\overset{v}{\omega}_i) = du^i \wedge \overset{v}{\omega}_i = (-1)^{i-1} g\frac{\partial u^i}{\partial \xi^k}\, d\xi^k \wedge d\xi^1 \wedge \ldots \wedge \widehat{d\xi^i} \wedge \ldots \wedge d\xi^n \wedge dx =$$

$$= \frac{\partial u^i}{\partial \xi^i} dV^{2n} = \overset{v}{\nabla}_i u^i\, dV^{2n}.$$

Similarly, with the help of (3.6.10), we derive

$$d(u^i\overset{h}{\omega}_i) = du^i \wedge \overset{h}{\omega}_i + u^i d\overset{h}{\omega}_i =$$

$$= g\sum_{i=1}^{n}\left(\frac{\partial u^i}{\partial x^k}dx^k + \frac{\partial u^i}{\partial \xi^k}d\xi^k\right) \wedge \Big[(-1)^{n+i-1} d\xi \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n +$$

$$+ \sum_{j=1}^{n}(-1)^j \Gamma^j_{ip}\, \xi^p \wedge d\xi^1 \wedge \ldots \wedge \widehat{d\xi^j} \wedge \ldots \wedge d\xi^n \wedge dx\Big] + u^i\Gamma^j_{ij}\, dV^{2n} =$$

$$= g\Big[\sum_{i=1}^{n}(-1)^{n+i-1}\frac{\partial u^i}{\partial x^k}\, dx^k \wedge d\xi \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n +$$

$$+ \sum_{j=1}^{n}(-1)^j \Gamma^j_{ip}\xi^p \frac{\partial u^i}{\partial \xi^k}\, d\xi^k \wedge d\xi^1 \wedge \ldots \wedge \widehat{d\xi^j} \wedge \ldots \wedge d\xi^n \wedge dx\Big] + u^i\Gamma^j_{ij}\, dV^{2n} =$$

$$= \left(\frac{\partial u^i}{\partial x^i} + \Gamma^j_{ij} u^i - \Gamma^k_{ip} \xi^p \frac{\partial u^i}{\partial \xi^k}\right) dV^{2n} = \overset{h}{\nabla}_i u^i \, dV^{2n}.$$

The lemma is proved.

Applying the Stokes theorem, from (3.6.4), we obtain the next two Gauss-Ostrogradskiĭ formulas for the vertical and horizontal divergences:

$$\int\limits_D \overset{v}{\nabla}_i u^i \, dV^{2n} = \int\limits_{\partial D} u^i \overset{v}{\omega}_i, \qquad \int\limits_D \overset{h}{\nabla}_i u^i \, dV^{2n} = \int\limits_{\partial D} u^i \overset{h}{\omega}_i, \tag{3.6.11}$$

which are valid for a semibasic vector field $u = (u^i)$ and a compact domain $D \subset TM$ with piecewise smooth boundary ∂D.

We will need the next simple assertion whose proof is omitted due to its clarity.

Lemma 3.6.2 *Let α be a $(d-1)$-form on a d-dimensional manifold X, and $Y \subset X$ be a submanifold of codimension one which is determined by an equation $f(x) = 0$ such that $d\,f(x) \neq 0$ for $x \in Y$. The restriction of the form α to the submanifold Y equals zero if and only if $(\alpha \wedge d\,f)(x) = 0$ for all $x \in Y$.*

Formulas (3.6.11) can be simplified essentially for some particular type of a domain D which is of import for us. Let G be a compact domain in M with piecewise smooth boundary ∂G. For $0 < \rho_0 < \rho_1$, by $T_{\rho_0,\rho_1}G$ we denote the domain in TM that is defined by the equality

$$T_{\rho_0,\rho_1}G = \{(x,\xi) \in TM \mid x \in G,\ \rho_0^2 \le |\xi|^2 = g_{ij}\xi^i\xi^j \le \rho_1^2\}.$$

The boundary of the domain is the union of three piecewise smooth manifolds:

$$\partial(T_{\rho_0,\rho_1}G) = \Omega_{\rho_1}G - \Omega_{\rho_0}G + T_{\rho_0,\rho_1}(\partial G), \tag{3.6.12}$$

where

$$\Omega_\rho G = \{(x,\xi) \in TM \mid x \in G,\ |\xi| = \rho\},$$
$$T_{\rho_0,\rho_1}(\partial G) = \{(x,\xi) \in TM \mid x \in \partial G,\ \rho_0 \le |\xi| \le \rho_1\}.$$

We have the canonical diffeomorphism

$$\mu : \Omega_{\rho_0}G \to \Omega_{\rho_1}G, \quad \mu(x,\xi) = (x, \frac{\rho_1}{\rho_0}\xi). \tag{3.6.13}$$

The second summand on the right-hand side of (3.6.12) is furnished with the minus sign to emphasize that it enters into $\partial(T_{\rho_0,\rho_1}G)$ with the orientation opposite to that induced by the diffeomorphism μ.

If the boundary ∂G is smooth near a point $x_0 \in \partial G$, then a coordinate system can be chosen in a neighbourhood of x_0 in such a way that $dx^n = 0$ on $T_{\rho_0,\rho_1}(\partial G)$. Therefore (3.6.2) implies that the restriction to $T_{\rho_0,\rho_1}(\partial G)$ of each of the forms $\overset{v}{\omega}_i$ is equal to zero.

Let us show that the restriction to $\Omega_\rho G$ of each of the forms $\overset{h}{\omega}_i$ is equal to zero. By Lemma 3.6.2, to this end it is sufficient to verify the equality $\overset{h}{\omega}_i \wedge d|\xi|^2 = 0$, since $\Omega_\rho G$ is determined by the equation $|\xi|^2 = \rho^2 = \text{const}$. From (3.6.3), we obtain

$$\overset{h}{\omega}_i \wedge d|\xi|^2 = \overset{h}{\omega}_i \wedge d(g_{kl}\xi^k\xi^l) = g\left[(-1)^{n+i-1} d\xi \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n + \right.$$

$$+\sum_{j=1}^{n}(-1)^j\Gamma^j_{ip}\,\xi^p d\xi^1\wedge\ldots\wedge\widehat{d\xi^j}\wedge\ldots\wedge d\xi^n\wedge dx\Big]\wedge\left(\frac{\partial g_{kl}}{\partial x^r}\xi^k\xi^l dx^r+2g_{kl}\xi^k d\xi^l\right)=$$

$$=g\Big[(-1)^{n+i-1}\frac{\partial g_{kl}}{\partial x^r}\xi^k\xi^l d\xi\wedge dx^1\wedge\ldots\wedge\widehat{dx^i}\wedge\ldots\wedge dx^n\wedge dx^r+$$

$$+2\sum_{j=1}^{n}(-1)^j g_{kl}\Gamma^j_{ip}\,\xi^p\xi^k d\xi^1\wedge\ldots\wedge\widehat{d\xi^j}\wedge\ldots\wedge d\xi^n\wedge dx\wedge d\xi^l\Big]=$$

$$=\left(-\frac{\partial g_{kl}}{\partial x^i}\xi^k\xi^l+2g_{kl}\Gamma^l_{ip}\xi^p\xi^k\right)dV^{2n}.$$

After an evident transformation, the obtained result can be rewritten as:

$$\overset{h}{\omega}_i\wedge d|\xi|^2=\left(g_{ip}\Gamma^p_{kl}+g_{kp}\Gamma^p_{il}-\frac{\partial g_{kl}}{\partial x^i}\right)\xi^k\xi^l dV^{2n}.$$

The expression in parentheses on the right-hand side of this equality is equal to zero, as follows from (3.2.13).

Thus, for $D=T_{\rho_0,\rho_1}G$, formulas (3.6.11) assume the form

$$\int\limits_{T_{\rho_0,\rho_1}G}\overset{v}{\nabla}_i u^i\,dV^{2n}=\int\limits_{\Omega_{\rho_1}G}u^i\overset{v}{\omega}_i-\int\limits_{\Omega_{\rho_0}G}u^i\overset{v}{\omega}_i, \tag{3.6.14}$$

$$\int\limits_{T_{\rho_0,\rho_1}G}\overset{h}{\nabla}_i u^i\,dV^{2n}=\int\limits_{T_{\rho_0,\rho_1}(\partial G)}u^i\overset{h}{\omega}_i. \tag{3.6.15}$$

We will do some further transformation of the obtained formulas. To this end, we consider the $(2n-1)$-form

$$d\Sigma^{2n-1}=\frac{1}{|\xi|}\xi^i\overset{v}{\omega}_i \tag{3.6.16}$$

which is defined on TM for $\xi\neq 0$. It is natural to call its restriction to $\Omega_\rho G$ the *volume form of the manifold* $\Omega_\rho G$, since $d|\xi|\wedge d\Sigma^{2n-1}=dV^{2n}$. For a semibasic vector field $u=(u^i)$, the equality

$$u^i\overset{v}{\omega}_i=\frac{1}{\rho}u^i\xi_i\,d\Sigma^{2n-1} \tag{3.6.17}$$

is valid on $\Omega_\rho G$. Indeed, by Lemma 3.6.2, to prove (3.6.17) it is sufficient to show that

$$u^i\overset{v}{\omega}_i\wedge d|\xi|^2=\frac{1}{|\xi|}u^i\xi_i\,d\Sigma^{2n-1}\wedge d|\xi|^2.$$

Validity of the last equality can be obtained from definitions (3.6.2) and (3.6.16) by calculations similar to those we have already done several times in this section. With the help of (3.6.17), formula (3.6.14) obtains the form

$$\int\limits_{T_{\rho_0,\rho_1}G}\overset{v}{\nabla}_i u^i\,dV^{2n}=\frac{1}{\rho_1}\int\limits_{\Omega_{\rho_1}G}\langle u,\xi\rangle\,d\Sigma^{2n-1}-\frac{1}{\rho_0}\int\limits_{\Omega_{\rho_0}G}\langle u,\xi\rangle\,d\Sigma^{2n-1}. \tag{3.6.18}$$

Let ∂G is smooth near a point $x_0\in\partial G$. We can choose a coordinate system $(x^1,\ldots,x^n)$ in a neighbourhood of the point x_0 in such a way that $g_{in}=\delta_{in}$, ∂G is determined by the equation $x^n = 0$ and $x^n > 0$ outside G (it is one of the so-called *semigeodesic* coordinate systems of the hypersurface ∂G). In these coordinates $\overset{h}{\omega}_\alpha = 0$ $(1 \le \alpha \le n-1)$ on $T_{\rho_0,\rho_1}(\partial G)$, as follows from (3.6.3). One can easily see that the form

$$dV^{2n-1} = \overset{h}{\omega}_n = -g\, d\xi \wedge dx^1 \wedge \ldots \wedge dx^{n-1}, \tag{3.6.19}$$

is independent of the arbitrariness in the choice of the indicated coordinate system and, consequently, is defined globally on $T_{\rho_0,\rho_1}(\partial G)$. It is natural to call this form the *volume form of the manifold* $T_{\rho_0,\rho_1}(\partial G)$, since $dx^n \wedge dV^{2n-1} = dV^{2n}$. Written in the above coordinate system, the integrand of the right-hand side of equality (3.6.15) takes the form $u^i\overset{h}{\omega}_i = u^n\overset{h}{\omega}_n = \langle u, \nu\rangle\, dV^{2n-1}$, where ν is the unit vector of the outer normal to the boundary. Thus formula (3.6.15) can be written as:

$$\int\limits_{T_{\rho_0,\rho_1}G} \overset{h}{\nabla}_i u^i\, dV^{2n} = \int\limits_{T_{\rho_0,\rho_1}(\partial G)} \langle u, \nu\rangle\, dV^{2n-1}. \tag{3.6.20}$$

We will carry out further simplification of formulas (3.6.18) and (3.6.20) under the assumption that the semibasic vector field $u = u(x,\xi)$ is positively homogeneous in its second argument

$$u(x, t\xi) = t^\lambda u(x,\xi) \qquad (t > 0). \tag{3.6.21}$$

In this case the integrands of all integrals in (3.6.18) and (3.6.20) are homogeneous in ξ, and we will make use of this fact.

First we transform the right-hand side of equality (3.6.18), putting $1 = \rho_0 < \rho_1 = \rho$ and denoting $\Omega G = \Omega_1 G$. The integrand of the first integral on the right-hand side of formula (3.6.18) is a positively homogeneous function of degree $\lambda+n$ in ρ, as follows from (3.6.21), (3.6.16) and (3.6.2). In other words, diffeomorphism (3.6.13) (for $\rho_0 = 1,\ \rho_1 = \rho$) satisfies the equality

$$\mu^*(\langle u, \xi\rangle\, d\Sigma^{2n-1}) = \rho^{\lambda+n}\langle u, \xi\rangle\, d\Sigma^{2n-1}.$$

Henceforth in this section, given a smooth mapping $f : X \to Y$ and a differential form α on Y, by $f^*\alpha$ we mean the pushback of α. With the help of the last equality, we obtain

$$\int\limits_{\Omega_\rho G} \langle u, \xi\rangle\, d\Sigma^{2n-1} = \int\limits_{\Omega G} \mu^*(\langle u, \xi\rangle\, d\Sigma^{2n-1}) = \rho^{\lambda+n} \int\limits_{\Omega G} \langle u, \xi\rangle\, d\Sigma^{2n-1}. \tag{3.6.22}$$

Let us now transform the left-hand side of formula (3.6.18). To this end we define the diffeomorphism

$$\chi : [1,\rho] \times \Omega G \to T_{1,\rho}G, \qquad \chi(t; x, \xi) = (x, t\xi). \tag{3.6.23}$$

It satisfies the equality

$$\left[\chi^*(\overset{v}{\nabla}_i u^i\, dV^{2n})\right](t; x, \xi) = t^{\lambda+n-2}(\overset{v}{\nabla}_i u^i)(x,\xi)\, dt \wedge d\Sigma^{2n-1}(x,\xi). \tag{3.6.24}$$

Indeed, (3.6.23) implies that $\chi^*(dx^i) = dx^i,\ \chi^*(d\xi^i) = t\, d\xi^i + \xi^i dt$. Thus,

$$\chi^*(\overset{v}{\nabla}_i u^i\ dV^{2n}) = \chi^*[g\overset{v}{\nabla}_i u^i\, d\xi \wedge dx] =$$

$$= g(\overset{v}{\nabla}_i u^i)\,(x, t\xi)\,(t\, d\xi^1 + \xi^1 dt) \wedge \ldots \wedge (t\, d\xi^n + \xi^n dt) \wedge dx.$$

By (3.6.21), $(\overset{v}{\nabla}_i u^i)(x, t\xi) = t^{\lambda-1}(\overset{v}{\nabla}_i u^i)(x, \xi)$, and the previous formula takes the form

$$\chi^*(\overset{v}{\nabla}_i u^i\, dV^{2n}) = t^{\lambda-1} g(\overset{v}{\nabla}_i u^i)(x, \xi)\times$$

$$\times \left[t^n d\xi + t^{n-1} dt \wedge \sum_{i=1}^{n-1} (-1)^{i-1} \xi^i d\xi^1 \wedge \ldots \wedge \widehat{d\xi^i} \wedge \ldots \wedge d\xi^n \right] \wedge dx.$$

By the equality $|\xi|^2 = 1$, the relation $\xi_i d\xi^i = 0$ is valid on ΩG, and, consequently, $d\xi = 0$. Taking to account the last equality and (3.6.2), we rewrite the previous formula as:

$$\chi^*(\overset{v}{\nabla}_i u^i dV^{2n}) = t^{\lambda+n-2}(\overset{v}{\nabla}_i u^i)(x, \xi)\, dt \wedge \xi^i \overset{v}{\omega}_i.$$

By (3.6.16), the last equality coincides with (3.6.24).

With the help of (3.6.24), the left-hand side of formula (3.6.18) is transformed as follows:

$$\int\limits_{T_{1,\rho}G} \overset{v}{\nabla}_i u^i\, dV^{2n} = \int\limits_{[1,\rho]\times\Omega G} \chi^* \left(\overset{v}{\nabla}_i u^i\, dV^{2n} \right) = \int\limits_1^\rho t^{\lambda+n-2} dt \int\limits_{\Omega G} \overset{v}{\nabla}_i u^i\, d\Sigma^{2n-1} =$$

$$= \frac{\rho^{\lambda+n-1} - 1}{\lambda + n - 1} \int\limits_{\Omega G} \overset{v}{\nabla}_i u^i\, d\Sigma^{2n-1}. \tag{3.6.25}$$

Putting $\rho_0 = 1,\ \rho_1 = \rho$ in (3.6.18) and inserting expressions (3.6.22) and (3.6.25) into this formula, we obtain the final version of the Gauss-Ostrogradskiĭ formula for the vertical divergence:

$$\int\limits_{\Omega G} \overset{v}{\nabla}_i u^i\, d\Sigma^{2n-1} = (\lambda + n - 1) \int\limits_{\Omega G} \langle u, \xi \rangle\, d\Sigma^{2n-1}. \tag{3.6.26}$$

Repeating word by word the arguments that were used in the proof of (3.6.25), we transform the left-hand side of equality (3.6.20):

$$\int\limits_{T_{1,\rho}G} \overset{h}{\nabla}_i u^i\, dV^{2n} = \frac{\rho^{\lambda+n} - 1}{\lambda + n} \int\limits_{\Omega G} \overset{h}{\nabla}_i u^i\, d\Sigma^{2n-1}. \tag{3.6.27}$$

The distinction between the coefficients on the right-hand sides of equalities (3.6.25) and (3.6.27) is due to the fact that the homogeneity degree of the function $\overset{h}{\nabla}_i u^i$ is greater than that of $\overset{v}{\nabla}_i u^i$ by one.

To fulfil a similar transformation of the right-hand side of equality (3.6.20) we introduce the manifold $\partial\Omega G = \{(x, \xi) \in TM \mid x \in \partial G,\ |\xi| = 1\}$ and consider the diffeomorphism

$$\chi : [1, \rho] \times \partial\Omega G \to T_{1,\rho}(\partial G); \qquad \chi(t; x, \xi) = (x, t\xi),$$

which is the restriction of the diffeomorphism (3.6.23) to $[1, \rho] \times \partial\Omega G$. Let $(x^1, \ldots, x^n)$ be the semigeodesic coordinate system used in definition (3.6.19) of the form dV^{2n-1}. In full analogy with the proof of equality (3.6.24), the next relation is verified:

$$[\chi^*(\langle u, \nu \rangle dV^{2n-1})](t; x, \xi) = t^{\lambda+n-1} \langle u, \nu \rangle (x, \xi)\, dt \wedge d\Sigma^{2n-2}(x, \xi), \tag{3.6.28}$$

where the form $d\Sigma^{2n-2}$ is defined in the indicated coordinate system by the equality

$$d\Sigma^{2n-2} = -g\sum_{i=1}^{n}(-1)^{i-1}\xi^i d\xi^1 \wedge \ldots \wedge \widehat{d\xi^i} \wedge \ldots \wedge d\xi^n \wedge dx^1 \wedge \ldots \wedge dx^{n-1}. \tag{3.6.29}$$

One can easily see that this form is independent of the arbitrariness in the choice of our coordinate system and, consequently, is defined globally on $\partial\Omega G$. It is natural to call this form the *volume form of the manifold* $\partial\Omega G$, since $d|\xi| \wedge d\Sigma^{2n-2} = dV^{2n-1}$, as follows from (3.6.19) and (3.6.29). With the help of (3.6.28), the right-hand side of (3.6.20) takes the form:

$$\int\limits_{T_{1,\rho}(\partial G)} \langle u, \nu\rangle \, dV^{2n-1} = \frac{\rho^{\lambda+n}-1}{\lambda+n}\int\limits_{\partial\Omega G} \langle u, \nu\rangle \, d\Sigma^{2n-2}. \tag{3.6.30}$$

Inserting (3.6.27) and (3.6.30) into (3.6.20), we arrive at the final version of the Gauss-Ostrogradskiĭ formula for the horizontal divergence:

$$\int\limits_{\Omega G} \overset{h}{\nabla}_i u^i \, d\Sigma^{2n-1} = \int\limits_{\partial\Omega G} \langle u, \nu\rangle \, d\Sigma^{2n-2}. \tag{3.6.31}$$

The above-presented proof of formula (3.6.26) was fulfilled under the assumption that $\lambda + n - 1 \neq 0$, and the proof of formula (3.6.31) was fulfilled under the assumption that $\lambda + n \neq 0$. Nevertheless, these formulas are valid for an arbitrary λ. Indeed, for $\lambda + n - 1 = 0$, the factor $(\rho^{\lambda+n-1} - 1)/(\lambda + n - 1)$ in formula (3.6.25) is replaced by $\ln\rho - 1$; the remainder of the proof does not change. Similarly, for $\lambda + n = 0$, the factor $(\rho^{\lambda+n} - 1)/(\lambda + n)$ in equalities (3.6.27) and (3.6.30) is replaced by $\ln\rho - 1$; the remainder of the proof goes through without change.

The forms $d\Sigma^{2n-1}$ and $d\Sigma^{2n-2}$ participating in relations (3.6.26) and (3.6.31) have a simple geometrical sense. To clarify it we note that, for every point $x \in M$, the tangent space T_xM is provided by the structure of a Euclidean vector space which is induced by the Riemannian metric. By $dV_x^n(\xi)$ we denote the Euclidean volume form on T_xM. In a local coordinate system it is expressed by the formula

$$dV_x^n(\xi) = g^{1/2}d\xi^1 \wedge \ldots \wedge d\xi^n = g^{1/2}d\xi. \tag{3.6.32}$$

By $d\omega_x(\xi)$ we denote the angle measure, on the unit sphere $\Omega_xM = \{\xi \in T_xM \mid |\xi|^2 = g_{ij}(x)\xi^i\xi^j = 1\}$ of the space T_xM, induced by the Euclidean structure of the space. In coordinates this form is expressed as follows:

$$d\omega_x(\xi) = g^{1/2}\sum_{i=1}^{n}(-1)^{i-1}\xi^i d\xi^1 \wedge \ldots \wedge \widehat{d\xi^i} \wedge \ldots \wedge d\xi^n. \tag{3.6.33}$$

This equality can be verified with the help of Lemma 3.6.2. Indeed, it follows from (3.6.32) and (3.6.33) that, for $|\xi| = 1$, the relation $d|\xi| \wedge d\omega_x(\xi) = dV_x^n(\xi)$ is valid. The last equality is just the definition of the angle measure on Ω_xM.

Comparing definitions (3.6.16) and (3.6.29) of the forms $d\Sigma^{2n-1}$ and $d\Sigma^{2n-2}$ with equality (3.6.33), we see that

$$d\Sigma^{2n-1} = d\omega_x(\xi) \wedge dV^n(x), \quad d\Sigma^{2n-2} = (-1)^n d\omega_x(\xi) \wedge dV^{n-1}(x) \tag{3.6.34}$$

where $dV^n(x) = g^{1/2}dx$ is the Riemannian volume form on M and $dV^{n-1}(x)$ is the Riemannian volume form on ∂G. In the semigeodesic coordinate system have been used in definition (3.6.29), the last form is given by the formula $dV^{n-1}(x) = g^{1/2}dx^1 \wedge \ldots \wedge dx^{n-1}$.

Let us formulate the obtained results.

Theorem 3.6.3 *Let M be a Riemannian manifold of dimension n and $u = u(x,\xi)$ be a semibasic vector field on TM positively homogeneous of degree λ in ξ. For every compact domain $G \subset M$ with piecewise smooth boundary ∂G the next Gauss-Ostrogradskiĭ formulas are valid:*

$$\int\limits_G \int\limits_{\Omega_x M} \overset{v}{\nabla}_i u^i \, d\omega_x(\xi)\, dV^n(x) = (\lambda + n - 1) \int\limits_G \int\limits_{\Omega_x M} \langle u, \xi \rangle \, d\omega_x(\xi)\, dV^n(x), \qquad (3.6.35)$$

$$\int\limits_G \int\limits_{\Omega_x M} \overset{h}{\nabla}_i u^i \, d\omega_x(\xi)\, dV^n(x) = (-1)^n \int\limits_{\partial G} \int\limits_{\Omega_x M} \langle u, \nu \rangle \, d\omega_x(\xi)\, dV^{n-1}(x). \qquad (3.6.36)$$

Here $dV^n(x)$ and $dV^{n-1}(x)$ are the Riemannian volumes on M and ∂G respectively; $d\omega_x$ is the angle measure, on the unit sphere $\Omega_x M = \{\xi \in T_x M \mid |\xi| = 1\}$, induced by the Riemannian metric; ν is the unit vector of the outer normal to ∂G.

Chapter 4

The ray transform on a Riemannian manifold

In Section 1.1, linearizing the problem of determining a metric from its hodograph, we stated the definition of the ray transform of a symmetric tensor field on a simple Riemannian manifold.

Unlike the Euclidean version considered in Chapter 2, the ray transform theory for general Riemannian metrics is not abundant in results. Even an answer to the next main question is not found yet: in what cases is the solenoidal part of a tensor field f of degree m uniquely determined by the ray transform If? A positive answer to this question is obtained only for $m = 0$ and $m = 1$ in the case of a simple metric.

In Section 4.1 we introduce a class of so-called dissipative Riemannian metrics. The ray transform can be defined in a natural way for dissipative metrics. This class essentially extends the class of simple metrics.

In Section 4.2 we introduce the ray transform on a compact dissipative Riemannian manifold and prove that it is bounded with respect to the Sobolev norms.

In Section 4.3 we formulate a main result of the current chapter, Theorem 4.3.3, which gives a positive answer to the above-mentioned question and a stability estimate under some restrictions on the sectional curvature. The restrictions are of integral nature and mean, roughly speaking, that positive values of the sectional curvature must not accumulate along geodesics. We emphasize that no restrictions are imposed on the negative values of the sectional curvature.

Sections 4.4–4.5 contain three auxiliary claims which are used in the proof of Theorem 4.3.3. Two of them, the Pestov identity and the Poincaré inequality for semibasic tensor fields have certain significance besides the proof of Theorem 4.3.3; use of them will be made in the next chapters.

In Sections 4.6–4.7 we expose the proof of Theorem 4.3.3. In Section 4.6 the theorem is reduced to an inverse problem for the kinetic equation with the right-hand side depending polynomially on the direction; some quadratic integral identity is proved for the equation. In Section 4.7, for the summands of the last identity, some estimates are obtained which lead to the claim of Theorem 4.3.3.

In Section 4.8 we prove two corollaries, Theorems 4.8.1 and 4.8.2, which relate to the nonlinear problem of determining a metric from its hodograph. Together with these theorems, a reduction of the problem to an inverse problem for a system of kinetic equations is presented here which is of some interest by itself.

4.1 Compact dissipative Riemannian manifolds

Let M be a Riemannian manifold with boundary ∂M. For a point $x \in \partial M$, the *second quadratic form of the boundary*

$$\mathrm{II}\,(\xi, \xi) = \langle \nabla_\xi \nu, \xi \rangle \qquad (\xi \in T_x(\partial M))$$

is defined on the space $T_x(\partial M)$ where $\nu = \nu(x)$ is the unit outer normal vector to the boundary and ∇ is the Riemannian connection. We say that the boundary is *strictly convex* if the form is positive-definite for all $x \in \partial M$.

A compact Riemannian manifold M with boundary is called a *compact dissipative Riemannian manifold* (CDRM briefly), if it satisfies two conditions: 1) the boundary ∂M is strictly convex; 2) for every point $x \in M$ and every vector $0 \neq \xi \in T_x M$, the maximal geodesic $\gamma_{x,\xi}(t)$ satisfying the initial conditions $\gamma_{x,\xi}(0) = x$ and $\dot\gamma_{x,\xi}(0) = \xi$ is defined on a finite segment $[\tau_-(x,\xi), \tau_+(x,\xi)]$. We recall simultaneously that a geodesic $\gamma : [a,b] \to M$ is called maximal if it cannot be extended to a segment $[a - \varepsilon_1, b + \varepsilon_2]$, where $\varepsilon_i \geq 0$ and $\varepsilon_1 + \varepsilon_2 > 0$.

The second of the conditions participating in the definition of CDRM is equivalent to the absence of a geodesic of infinite length in M.

Recall that by $TM = \{(x,\xi) \mid x \in M,\ \xi \in T_x M\}$ we denote the space of the tangent bundle of the manifold M, and by $\Omega M = \{(x,\xi) \in TM \mid |\xi|^2 = g_{ij}(x)\xi^i\xi^j = 1\}$ we denote its submanifold that consists of unit vectors. We introduce the next submanifolds of TM :

$$T^0 M = \{(x,\xi) \in TM \mid \xi \neq 0\},$$

$$\partial_\pm \Omega M = \{(x,\xi) \in \Omega M \mid x \in \partial M;\ \pm\langle \xi,\ \nu(x)\rangle \geq 0\}.$$

Note that $\partial_+\Omega M$ and $\partial_-\Omega M$ are compact manifolds with the common boundary $\partial_0 \Omega M = \Omega M \bigcap T(\partial M)$, and $\partial \Omega M = \partial_+\Omega M \bigcup \partial_-\Omega M$.

While defining a CDRM, we have determined two functions $\tau_\pm : T^0 M \to \mathbf{R}$. It is evident that they have the next properties:

$$\gamma_{x,\xi}(\tau_\pm(x,\xi)) \in \partial M; \tag{4.1.1}$$

$$\begin{gathered} \tau_+(x,\xi) \geq 0, \quad \tau_-(x,\xi) \leq 0, \quad \tau_+(x,\xi)) = -\tau_-(x,-\xi); \\ \tau_\pm(x, t\xi) = t^{-1}\tau_\pm(x,\xi) \quad (t > 0); \\ \tau_+|_{\partial_+\Omega M} = \tau_-|_{\partial_-\Omega M} = 0. \end{gathered} \tag{4.1.2}$$

We now consider the smoothness properties of the functions $\tau_\pm$. With the help of the implicit function theorem, one can easily see that $\tau_\pm(x,\xi)$ is smooth near a point (x,ξ) such that the geodesic $\gamma_{x,\xi}(t)$ intersects ∂M transversely for $t = \tau_\pm(x,\xi)$. By strict convexity of ∂M, the last claim is valid for all $(x,\xi) \in T^0 M$ except for the points of the set $\partial_0 T^0 M = T^0 M \bigcap T(\partial M)$. Thus we conclude that $\tau_\pm$ are smooth on $T^0 M \setminus \partial_0 T^0 M$. All points of the set $\partial_0 T^0 M$ are singular points for $\tau_\pm$, since one can easily see that some derivatives of these functions are unbounded in a neighbourhood of such a point. Nevertheless, the next claim is valid:

Lemma 4.1.1 *Let (M, g) be a CDRM. The function $\tau : \partial\Omega M \to \mathbf{R}$ defined by the equality*

$$\tau(x,\xi) = \begin{cases} \tau_+(x,\xi), & \textit{if} \quad (x,\xi) \in \partial_-\Omega M, \\ \tau_-(x,\xi), & \textit{if} \quad (x,\xi) \in \partial_+\Omega M \end{cases} \tag{4.1.3}$$

is smooth. In particular, $\tau_- : \partial_+\Omega M \to \mathbf{R}$ is a smooth function.

P r o o f. In some neighbourhood U of a point $x_0 \in \partial M$, a semigeodesic coordinate system $(x^1, \dots, x^n) = (y^1, \dots, y^{n-1}, r)$ can be introduced such that the function r coincides with the distance (in the metric g) from the point (y, r) to ∂M and $g_{in} = \delta_{in}$. In this coordinate system, the Christoffel symbols satisfy the relations $\Gamma^n_{in} = \Gamma^i_{nn} = 0$, $\Gamma^\alpha_{\beta n} = -g^{\alpha\gamma}\Gamma^n_{\beta\gamma}$ (in this and subsequent formulas, Greek indices vary from 1 to $n-1$; on repeating Greek indices, the summation from 1 to $n-1$ is assumed), the unit vector of the outer normal has the coordinates $(0, \dots, 0, -1)$. Putting $j = n$ in (3.2.7), we see that the Christoffel symbols $\Gamma^n_{\alpha\beta}$ coincide with the coefficients of the second quadratic form. Consequently, the condition of strict convexity of the boundary means that

$$\Gamma^n_{\alpha\beta}(y, 0)\eta^\alpha\eta^\beta \ge a|\eta|^2 = a\sum_{\alpha=1}^{n-1}(\eta^\alpha)^2 \qquad (a > 0). \tag{4.1.4}$$

Let $(y^1, \dots, y^{n-1}, r, \eta^1, \dots, \eta^{n-1}, \rho)$ be the coordinate system on TM associated with $(y^1, \dots, y^{n-1}, r)$. As we have seen before the formulation of the lemma, the function $\tau(y, 0, \eta, \rho)$ is smooth for $\rho \ne 0$. Consequently, to prove the lemma it is sufficient to verify that this function is smooth for $|\eta| \ge 1/2$ and $|\rho| < \varepsilon$ with some $\varepsilon > 0$.

Let $\gamma_{(y,\eta,\rho)}(t) = (\gamma^1_{(y,\eta,\rho)}(t), \dots, \gamma^n_{(y,\eta,\rho)}(t))$ be the geodesic defined by the initial conditions $\gamma_{(y,\eta,\rho)}(0) = (y, 0)$, $\dot\gamma_{(y,\eta,\rho)}(0) = (\eta, \rho)$. Expanding the function $r(t, y, \eta, \rho) = \gamma^n_{(y,\eta,\rho)}(t)$ into the Taylor series in t and using equations (1.2.5) for geodesics, we obtain the representation

$$r(t, y, \eta, \rho) = \rho t - \Gamma^n_{\alpha\beta}(y, 0)\eta^\alpha\eta^\beta t^2 + \varphi(t, y, \eta, \rho)t^3 \tag{4.1.5}$$

with some smooth function $\varphi(t, y, \eta, \rho)$. For small ρ the equation $r(t, y, \eta, \rho) = 0$ has the solutions $t = 0$ and $t = \tau(y, 0, \eta, \rho)$. Consequently, (4.1.5) implies that $\tau = \tau(y, 0, \eta, \rho)$ is a solution to the equation

$$F(\tau, y, \eta, \rho) \equiv \rho - \Gamma^n_{\alpha\beta}(y, 0)\eta^\alpha\eta^\beta\tau + \varphi(\tau, y, \eta, \rho)\tau^2 = 0.$$

It follows from (4.1.4) that $\frac{\partial}{\partial\tau}|_{\tau=0}F(\tau, y, \eta, \rho) \ne 0$. Applying the implicit function theorem, we see that $\tau(y, 0, \eta, \rho)$ is a smooth function. The lemma is proved.

Lemma 4.1.2 *Let M be a CDRM. The function $\tau_+(x,\xi)/(-\langle\xi, \nu(x)\rangle)$ is bounded on the set $\partial_-\Omega M \setminus \partial_0\Omega M$.*

P r o o f. It suffices to prove that the function is bounded on the subset $W_\varepsilon = \{(x,\xi) \mid 0 < -\langle\xi, \nu(x)\rangle < \varepsilon,\ 1/2 \le |\xi| \le 3/2\}$ of the manifold $\partial(TM)$ for some $\varepsilon > 0$. Decreasing ε, one can easily see that it suffices to verify boundedness of the function for $(x,\xi) \in W_\varepsilon$ such that the geodesic $\gamma_{x,\xi} : [0, \tau_+(x,\xi)] \to M$ is wholly in the domain U of the semigeodesic coordinate system introduced in the proof of Lemma 4.1.1. In these coordinates, $(x,\xi) = (y,0,\eta,\rho)$, $0 < -\langle\xi,\nu(x)\rangle = \rho < \varepsilon$, $1/2 \le |\eta| \le 3/2$. The left-hand side of equality (4.1.5) vanishes for $t = \tau_+(x,\xi)$:

$$\left[\Gamma^n_{\alpha\beta}(y,0)\eta^\alpha\eta^\beta - \varphi(\tau_+(x,\xi),y,\eta,\rho)\tau_+(x,\xi)\right]\frac{\tau_+(x,\xi)}{\rho} = 1. \tag{4.1.6}$$

By decreasing ε, we can achieve that $\tau_+(x,\xi) < \delta$ for $(x,\xi) \in W_\varepsilon$ with any $\delta > 0$. Thus the second summand in the brackets of (4.1.6) can be made arbitrarily small. Together with (4.1.4), this implies that the expression in the brackets is bounded from below by some positive constant. Consequently, $0 < -\tau_+(x,\xi)/\langle\xi,\nu(x)\rangle = \tau_+(x,\xi)/\rho \le C$. The lemma is proved.

We need the next claim in Section 4.6.

Lemma 4.1.3 *Let (M,g) be a CDRM and $x_0 \in \partial M$. Let a semigeodesic coordinate system $(x^1,\dots,x^n)$ be chosen in a neighbourhood U of the point x_0 in such a way that x^n coincides with the distance in the metric g from x to ∂M, and let $(x^1,\dots,x^n,\xi^1,\dots,\xi^n)$ be the associated coordinate system on TM. There exists a neighbourhood $U' \subset U$ of the point x_0 such that the derivatives*

$$\frac{\partial\tau_-(x,\xi)}{\partial x^\alpha}\quad(\alpha = 1,\dots,n-1);\qquad \frac{\partial\tau_-(x,\xi)}{\partial\xi^i}\quad(i = 1,\dots,n) \tag{4.1.7}$$

are bounded on the set $\Omega M \bigcap p^{-1}(U'\setminus\partial M)$, where $p : TM \to M$ is the projection of the tangent bundle.

P r o o f. It suffices to prove boundedness of derivatives (4.1.7) only for $(x,\xi) \in \Omega M \bigcap p^{-1}(U'\setminus\partial M)$ such that the geodesic $\gamma_{x,\xi} : [\tau_-(x,\xi),0] \to M$ is wholly in U. By $\gamma^i(t,x,\xi)$ we denote the coordinates of the point $\gamma_{x,\xi}(t)$. The point $\gamma_{x,\xi}(\tau_-(x,\xi))$ is in ∂M. This means that $\gamma^n(\tau_-(x,\xi),x,\xi) = 0$. Differentiating the last equality, we obtain

$$\begin{aligned}\frac{\partial\tau_-(x,\xi)}{\partial x^i} &= -\frac{\partial\gamma^n}{\partial x^i}(\tau_-,x,\xi)/\dot\gamma^n(\tau_-,x,\xi),\\ \frac{\partial\tau_-(x,\xi)}{\partial\xi^i} &= -\frac{\partial\gamma^n}{\partial\xi^i}(\tau_-,x,\xi)/\dot\gamma^n(\tau_-,x,\xi).\end{aligned} \tag{4.1.8}$$

Note that $(\partial\gamma^n/\partial x^\alpha)(0,x,\xi) = 0$ for $1 \le \alpha \le n-1$. Consequently, a representation $(\partial\gamma^n/\partial x^\alpha)(\tau_-,x,\xi) = \varphi_\alpha(\tau_-,x,\xi)\tau_-(x,\xi)$ is possible with some functions $\varphi_\alpha(t',x,\xi)$ smooth on the set

$$W = \{(t',x,\xi) \in \mathbf{R}\times T^0M \mid \tau_-(x,\xi) \le t' \le 0,\ \gamma_{x,\xi}(t) \in U \text{ for } \tau_-(x,\xi) \le t \le 0\}.$$

By the equality $(\partial\gamma^n/\partial\xi^i)(0,x,\xi) = 0$ $(1 \le i \le n)$, a representation $(\partial\gamma^n/\partial\xi^i)(\tau_-,x,\xi) = \psi_i(\tau_-,x,\xi)\tau_-(x,\xi)$ is possible with some functions $\psi_i(t',x,\xi)$ smooth on W. Consequently, (4.1.8) is rewritten as

$$\frac{\partial\tau_-(x,\xi)}{\partial x^\alpha} = \varphi_\alpha(\tau_-,x,\xi)\frac{-\tau_-(x,\xi)}{\dot\gamma^n(\tau_-,x,\xi)};\qquad \frac{\partial\tau_-(x,\xi)}{\partial\xi^i} = \psi_i(\tau_-,x,\xi)\frac{-\tau_-(x,\xi)}{\dot\gamma^n(\tau_-,x,\xi)}. \tag{4.1.9}$$

Since the functions φ_α and ψ_i are smooth on W, they are bounded on any compact subset of W. Consequently, (4.1.9) implies that the proof will be finished if we verify boundedness of the ratio $-\tau_-(x,\xi)/\dot\gamma^n(\tau_-(x,\xi),x,\xi)$ on $\Omega M \bigcap p^{-1}(U\setminus\partial M)$.

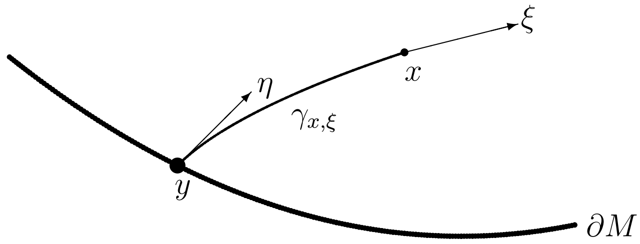

Fig. 1

We denote $y = y(x,\xi) = \gamma_{x,\xi}(\tau_-(x,\xi))$, $\eta = \eta(x,\xi) = \dot\gamma_{x,\xi}(\tau_-(x,\xi))$; then $(y,\eta) \in \partial_-\Omega M \setminus \partial_0\Omega M$, $0 \le -\tau_-(x,\xi) \le \tau_+(y,\eta)$ and $\dot\gamma^n(\tau_-(x,\xi),x,\xi) = -\langle \eta, \nu(y)\rangle$. Consequently,

$$0 \le \frac{-\tau_-(x,\xi)}{\dot\gamma^n(\tau_-(x,\xi),x,\xi)} \le \frac{\tau_+(y,\eta)}{-\langle \eta, \nu(y)\rangle}.$$

The last ratio is bounded on $\partial_-\Omega M \setminus \partial_0\Omega M$, by Lemma 4.1.2. The lemma is proved.

4.2 The ray transform on a CDRM

Let $C^\infty(\partial_+\Omega M)$ be the space of smooth functions on the manifold $\partial_+\Omega M$.

The *ray transform* on a CDRM M is the linear operator

$$I : C^\infty(S^m\tau'_M) \to C^\infty(\partial_+\Omega M) \tag{4.2.1}$$

defined by the equality

$$If(x,\xi) = \int\limits_{\tau_-(x,\xi)}^{0} \langle f(\gamma_{x,\xi}(t)), \dot\gamma^m_{x,\xi}(t)\rangle\, dt = \int\limits_{\tau_-(x,\xi)}^{0} f_{i_1\dots i_m}(\gamma_{x,\xi}(t))\dot\gamma^{i_1}_{x,\xi}(t)\dots\dot\gamma^{i_m}_{x,\xi}(t)\, dt, \tag{4.2.2}$$

where $\gamma_{x,\xi} : [\tau_-(x,\xi), 0] \to M$ is a maximal geodesic satisfying the initial conditions $\gamma_{x,\xi}(0) = x$ and $\dot\gamma_{x,\xi}(0) = \xi$. By Lemma 4.1.1, the right-hand side of equality (4.2.2) is a smooth function on $\partial_+\Omega M$.

Recall that in Section 3.3 the Hilbert space $H^k(S^m\tau'_M)$ was introduced. In a similar way the Hilbert space $H^k(\partial_+\Omega M)$ of functions on $\partial_+\Omega M$ is defined.

Theorem 4.2.1 *The ray transform on a CDRM is extendible to the bounded operator*

$$I : H^k(S^m\tau'_M) \to H^k(\partial_+\Omega M) \tag{4.2.3}$$

for every integer $k \ge 0$.

To prove the theorem we need the next

Lemma 4.2.2 *Let (M,g) be a CDRM, $d\Sigma$ and $d\sigma$ be smooth volume forms (differential forms of the most degree that do not vanish at every point) on ΩM and $\partial_+\Omega M$ respectively. Let D be the closed domain in $\partial_+\Omega M \times \mathbf{R}$ defined by the equality*

$$D = \{(x,\xi;t) \,|\, \tau_-(x,\xi) \le t \le 0\},$$

and the mapping $G : D \to \Omega M$ be defined by $G(x,\xi;t) = G^t(x,\xi)$, where G^t is the geodesic flow. Then the equality

$$(G^* d\Sigma)(x,\xi;t) = a(x,\xi;t)\langle \xi, \nu(x)\rangle d\sigma(x,\xi) \wedge dt \tag{4.2.4}$$

holds on D with some function $a \in C^\infty(D)$ not vanishing at every point.

P r o o f. Only the coefficient a changes in (4.2.4) under the change of the volume form $d\Sigma$ or $d\sigma$. Therefore it suffices to prove the claim for some forms $d\Sigma$ and $d\sigma$.

We define the function $r : M \to \mathbf{R}$ by putting $r(x) = -\rho(x, \partial M)$, where ρ is the distance in the metric g. The function r is smooth in some neighbourhood of ∂M, and $\nabla r(x) = \nu(x)$ for $x \in \partial M$. We extend the form $d\sigma$ to some neighbourhood of $\partial_+\Omega M$ in the manifold ΩM. According to the remark in the preceding paragraph, we can assume that $d\Sigma = d\sigma \wedge dr$ near $\partial\Omega M$.

We first prove validity of representation (4.2.4) for $t = 0$. The differential of the mapping G at a point $(x,\xi;0)$ is identical on $T_{(x,\xi)}(\partial_+\Omega M)$ and maps the vector $\partial/\partial t$ into H. Consequently,

$$(G^* d\Sigma)(x,\xi;0) = G^*(d\sigma \wedge dr)(x,\xi;0) = Hr\, d\sigma(x,\xi) \wedge dt. \tag{4.2.5}$$

By (3.5.1), $Hr = \xi^i \dfrac{\partial r}{\partial x^i} = \langle \xi, \nu(x)\rangle$. Inserting this expression into (4.2.5), we obtain

$$(G^* d\Sigma)(x,\xi;0) = \langle \xi, \nu(x)\rangle\, d\sigma(x,\xi) \wedge dt. \tag{4.2.6}$$

The mapping G satisfies the relation $G(x,\xi;t+s) = G^t(G(x,\xi;s))$. This implies the equality

$$(G^* d\Sigma)(x,\xi;t) = a(x,\xi;t)(G^* d\Sigma)(x,\xi;0) \tag{4.2.7}$$

with some positive function $a \in C^\infty(D)$. Formulas (4.2.6) and (4.2.7) imply (4.2.4). The lemma is proved.

P r o o f of Theorem 4.2.1. Let us agree to denote various constants independent of f by the same letter C.

First we will prove the estimate

$$\|If\|_k \le C\|f\|_k \tag{4.2.8}$$

for $f \in C^\infty(S^m\tau'_M)$. To this end, we define a function $F \in C^\infty(\Omega M)$ by putting $F(x,\xi) = f_{i_1\dots i_m}(x)\xi^{i_1}\dots\xi^{i_m}$. The inequality

$$\|F\|_k \le C\|f\|_k \tag{4.2.9}$$

is evident. With the help of F, equality (4.2.2) is rewritten as:

$$If(x,\xi) = IF(x,\xi) \equiv \int\limits_{\tau_-(x,\xi)}^{0} F(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))\, dt. \tag{4.2.10}$$

By (4.2.10), to prove (4.2.8) it suffices to establish the estimate

$$\|IF\|_k \le C\|F\|_k. \tag{4.2.11}$$

Since operator (4.2.10) is linear, it suffices to prove (4.2.11) for a function $F \in C^\infty(\Omega M)$ such that its support is contained in a domain $V \subset \Omega M$ of some local coordinate system $(z^1, \dots, z^{2n-1})$ on the manifold ΩM.

Let $(y^1, \dots, y^{2n-2})$ be a local coordinate system on $\partial_+\Omega M$ defined in a domain $U \subset \partial_+\Omega M$, and φ be a smooth function whose support is contained in U. To prove (4.2.11) it suffices to establish the estimate

$$\|\varphi \cdot IF\|_{H^k(U)} \le C\|F\|_{H^k(V)}. \tag{4.2.12}$$

Differentiating (4.2.10), we obtain

$$D_y^\alpha[\varphi(x,\xi)IF(x,\xi)] = \sum_{\beta+\gamma=\alpha} (D_y^\gamma\varphi)(x,\xi) \int\limits_{\tau_-(x,\xi)}^{0} D_y^\beta[F(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))]\,dt +$$

$$+ \sum_{\substack{\beta+\gamma+\delta=\alpha\\ \delta<\alpha}} C^\alpha_{\beta\gamma\delta}(D_y^\beta\varphi)(x,\xi)\cdot(D_y^\gamma\tau_-)(x,\xi)\cdot D_y^\delta[F(\gamma_{x,\xi}(\tau_-(x,\xi)), \dot\gamma_{x,\xi}(\tau_-(x,\xi)))]. \tag{4.2.13}$$

We will prove that, for $|\alpha| \le k$, the L_2-norm of each of the summands on the right-hand side of (4.2.13) can be estimated by $C\|F\|_{H^k(V)}$.

By Lemma 4.1.1, the functions $D_y^\gamma\tau_-$ are locally bounded, and the mapping

$$\partial_+\Omega M \to \partial_-\Omega M, \qquad (x,\xi) \mapsto (\gamma_{x,\xi}(\tau_-(x,\xi)), \dot\gamma_{x,\xi}(\tau_-(x,\xi)))$$

is a diffeomorphism. Therefore the L_2-norm of the second sum on the right-hand side of (4.2.13) is not more than $C\|F|_{\partial_-\Omega M}\|_{k-1}$. Using the boundedness of the trace operator $H^k(\Omega M) \to H^{k-1}(\partial_-\Omega M)$, $F \mapsto F|_{\partial_-\Omega M}$, we conclude that the L_2-norm of the second sum on the right-hand side of (4.2.13) is majorized by $C\|F\|_{H^k(V)}$.

We now estimate the L_2-norm of the integral on the right-hand side of (4.2.13). With the help of the Cauchy-Bunyakovskiĭ inequality, we obtain

$$\Big|\int\limits_{\tau_-(x,\xi)}^{0} D_y^\beta[F(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))]\,dt\Big|^2 \le -\tau_-(x,\xi) \int\limits_{\tau_-(x,\xi)}^{0} \Big|D_y^\beta[F(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))]\Big|^2\,dt =$$

$$= -\tau_-(x,\xi) \int\limits_{\tau_-(x,\xi)}^{0} \sum_{\gamma\le\beta} C_\gamma^\beta(x,\xi)\,|(D_z^\gamma F)(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))]|^2\,dt,$$

where $C_\gamma^\beta(x,\xi)$ are smooth functions. Integrating the last inequality, we obtain

$$\Big\|\int\limits_{\tau_-(x,\xi)}^{0} D_y^\beta[F(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))]\,dt\Big\|^2_{L_2(U)} \le$$

$$\le \sum_{\gamma\le\beta} C_\gamma^\beta \int\limits_U \int\limits_{\tau_-(x,\xi)}^{0} |\tau_-(x,\xi)|\,|(D_z^\gamma F)(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))|^2\,dt\,dy. \tag{4.2.14}$$

We change the integration variable in the integral on the right-hand side of (4.2.14) by the formula $z = G(x, \xi; t)$, where G is the mapping constructed in Lemma 4.2.2. By (4.2.4), after the change inequality (4.2.14) takes the form

$$\Big\| \int\limits_{\tau_-(x,\xi)}^{0} D_y^\beta [F(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))]\, dt \Big\|^2_{L_2(U)} \le \sum_{\gamma\le\beta} C_\gamma^\beta \int\limits_V \left| \frac{\tau_-(x,\xi)}{\langle \xi, \nu(x)\rangle} \right| |(D_z^\gamma F)(z)|^2\, dz. \qquad (4.2.15)$$

By Lemma 4.1.1, the ratio $\tau_-(x,\xi)/\langle\xi,\nu(x)\rangle$ is bounded. Therefore (4.2.15) implies the desired estimate

$$\Big\| \int\limits_{\tau_-(x,\xi)}^{0} D_y^\beta [F(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))]\, dt \Big\|_{L_2(U)} \le C\|F\|_{H^k(V)}.$$

Thus, the estimate (4.2.8) is proved for $f \in C^\infty(S^m\tau'_M)$.

Let now $f \in H^k(S^m\tau'_M)$. We define F as above, estimate (4.2.9) remaining valid. From the Fubini theorem we see that the integral on the right-hand side of equality (4.2.10) is finite for almost all $(x,\xi) \in \partial_+\Omega M$ and the function If, which is defined by this equality, belongs to $H^0(\partial_+\Omega M)$. We choose a sequence $f_\nu \in C^\infty(S^m\tau'_M)$ $(\nu = 1, 2, \dots)$ that converges to f in $H^k(S^m\tau'_M)$. The sequence If_ν converges to If in $H^0(\partial_+\Omega M)$. Applying estimate (4.2.8) for $f_\nu - f_\mu$, we see that If_ν is a Cauchy sequence in $H^k(\partial_+\Omega M)$. Consequently, $If \in H^k(\partial_+\Omega M)$ and estimate (4.2.8) is valid. The theorem is proved.

4.3 The problem of inverting the ray transform

Let M be a CDRM. Given a field $v \in C^\infty(S^{m-1}\tau'_M)$ satisfying the boundary condition $v|_{\partial M} = 0$, equality (3.3.17) and definition (4.2.2) of the ray transform imply immediately that $I(dv) = 0$. From this, using Theorem 4.2.1 and boundedness of the trace operator $H^{k+1}(S^m\tau'_M) \to H^k(S^m\tau'_M|_{\partial M}),\ v \mapsto v|_{\partial M}$, we obtain the next

Lemma 4.3.1 *Let M be a CDRM, $k \ge 0$ and $m \ge 0$ be integers. If a field $v \in H^{k+1}(S^m\tau'_M)$ satisfies the boundary condition $v|_{\partial M} = 0$, then $Idv = 0$.*

By Theorem 3.3.2, a field $f \in H^k(S^m\tau'_M)$ $(k \ge 1)$ can be uniquely decomposed into solenoidal and potential parts:

$$f = {}^s\!f + dv, \quad \delta\, {}^s\!f = 0, \quad v|_{\partial M} = 0, \qquad (4.3.1)$$

where ${}^s\!f \in H^k(S^m\tau'_M)$ and $v \in H^{k+1}(S^{m-1}\tau'_M)$. By Lemma 4.3.1, the ray transform pays no heed to the potential part of (4.3.1): $Idv = 0$. Consequently, given the ray transform If, we can hope to recover only the solenoidal part of the field f. Thus we come to the next

Problem 4.3.2 (problem of inverting the ray transform) *For which CDRM can the solenoidal part of any field $f \in H^k(S^m\tau'_M)$ be recovered from the ray transform If?*

The main result of the current chapter, Theorem 4.3.3 stated below, gives an answer for $k = 1$ under some assumption on the curvature of the manifold in question. Let us now formulate the assumption.

Let M be a Riemannian manifold. Recall that, for a point $x \in M$ and a two-dimensional subspace $\sigma \subset T_xM$, by $K(x,\sigma)$ we denote the sectional curvature at the point x and in the two-dimensional direction σ, which is defined by (3.2.14). For $(x,\xi) \in T^0M$ we put

$$K(x,\xi) = \sup_{\sigma\xi} K(x,\sigma), \qquad K^+(x,\xi) = \max\{0, K(x,\xi)\}. \tag{4.3.2}$$

For a CDRM (M,g), we introduce the next characteristic:

$$k^+(M,g) = \sup_{(x,\xi)\in\partial_-\Omega M} \int\limits_0^{\tau_+(x,\xi)} tK^+(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))\, dt. \tag{4.3.3}$$

We recall that here $\gamma_{x,\xi} : [0, \tau_+(x,\xi)] \to M$ is a maximal geodesic satisfying the initial conditions $\gamma_{x,\xi}(0) = x$ and $\dot\gamma_{x,\xi}(0) = \xi$. Note that $k^+(M,g)$ is a dimensionless quantity, i.e., it does not vary under multiplication of the metric g by a positive number.

Recall finally that, for $x \in \partial M$, we denote by $\nu(x)$ the unit vector of the outer normal to the boundary and by $j_\nu : S^mT'_xM \to S^{m-1}T'_xM$, the operator of convolution with the vector ν.

We can now formulate the main result of the current chapter.

Theorem 4.3.3 *Let $m \ge 0$ be an integer. For every compact dissipative Riemannian manifold (M,g) satisfying the condition*

$$k^+(M,g) < 1/(m+1) \tag{4.3.4}$$

and every tensor field $f \in H^1(S^m\tau'_M)$, the solenoidal part ${}^sf$ is uniquely determined by the ray transform If and the next conditional stability estimate is valid:

$$\|{}^sf\|_0^2 \le C\left(m\|j_\nu\, {}^sf|_{\partial M}\|_0 \cdot \|If\|_0 + \|If\|_1^2\right) \le C_1\left(m\|f\|_1 \cdot \|If\|_0 + \|If\|_1^2\right) \tag{4.3.5}$$

where constants C and C_1 are independent of f.

We will make a few remarks on the theorem.

The first summand on the right-hand side of estimate (4.3.5) shows that the problem of recovering ${}^sf$ from If is perhaps of conditionally-correct nature: for stably determining ${}^sf$, we are to have an a priori estimate for $\|f\|_1$. Note that this summand has appeared due to the method applied in our proof; the author knows nothing about any example demonstrating that the problem is conditionally-correct as a matter of fact. The factor m before the first summand is distinguished so as to emphasize that in the case $m = 0$ the problem is correct.

In order to avoid complicated formulations and proofs, in the current and previous chapters we use the spaces H^k only for integral $k \ge 0$. If the reader is familiar with the definition of these spaces for fractional k, he or she can verify, by examining the proof below, that it is possible to replace the factor $\|f\|_1$ in (4.3.5) by $\|f\|_{1/2}$.

We emphasize that (4.3.4) is a restriction only on the positive values of the sectional curvature, which is of an integral nature, moreover.

The right-hand side of equality (4.3.4) takes its maximal value for $m = 0$. If a CDRM (M, g) satisfies the condition

$$k^+(M, g) < 1, \tag{4.3.6}$$

then the next claims are valid: 1) M is diffeomorphic to the ball, and 2) the metric g is simple in the sense of the definition given in Chapter 1. We will not give here the proof, of the claims, which is beyond the scope of our book (and will not use these claims). We will only discuss briefly a possible way of the proof. First of all, condition (4.3.6) implies absence of conjugate points. This fact can be proved as follows: first, by arguments similar to those used in the proof of the theorem on comparing indices [41], we reduce the question to the two-dimensional case; then applying the Hartman-Wintner theorem (Theorem 5.1 of [46]). With the absence of conjugate points available, our claims can be established by arguments similar to those used in the proof of the Hadamard-Cartan theorem [41].

Let us show that Theorem 4.3.3 follows from the next special case of it.

Lemma 4.3.4 *Let a CDRM (M, g) satisfies (4.3.4). For a real field $f \in C^\infty(S^m\tau'_M)$ satisfying the condition*

$$\delta f = 0, \tag{4.3.7}$$

the estimate

$$\|f\|_0^2 \le C\left(m\|j_\nu f|_{\partial M}\|_0 \cdot \|If\|_0 + \|If\|_1^2\right) \tag{4.3.8}$$

holds with a constant C independent of f.

Indeed, we first note that it suffices to consider a real field f, since the general case can be reduced to this one by writing down estimates (4.3.5) for both the real and imaginary parts of f.

Given a real field $f \in H^1(S^m\tau'_M)$, let

$$f = {}^s\!f + dv, \quad \delta\, {}^s\!f = 0, \quad v|_{\partial M} = 0 \tag{4.3.9}$$

be the decomposition into the solenoidal and potential parts, where ${}^s\!f \in H^1(S^m\tau'_M)$ and $v \in H^2(S^{m-1}\tau'_M)$. By Theorem 3.3.2, the estimate

$$\|{}^s\!f\|_1 \le C_1\|f\|_1 \tag{4.3.10}$$

holds. We choose a sequence, of real fields $f_k \in C^\infty(S^m\tau'_M)$ $(k = 1, 2, \dots)$, which converges to f in $H^1(S^m\tau'_M)$. Applying Theorem 3.3.2 to f_k, we obtain the decomposition

$$f_k = {}^s\!f_k + dv_k, \quad \delta\, {}^s\!f_k = 0, \quad v_k|_{\partial M} = 0 \tag{4.3.11}$$

with real ${}^s\!f_k \in C^\infty(S^m\tau'_M)$, $v_k \in C^\infty(S^{m-1}\tau'_M)$. Since ${}^s\!f$ in (4.3.9) depends continuously on f, as have been shown in Theorem 3.3.2,

$${}^s\!f_k \to {}^s\!f \quad \text{in} \quad H^1(S^m\tau'_M) \quad \text{as} \quad k \to \infty. \tag{4.3.12}$$

In the view of boundedness of the trace operator $H^1(S^m\tau'_M) \to H^0(S^m\tau'_M|_{\partial M})$, (4.3.12) implies that

$${}^s\!f_k|_{\partial M} \to {}^s\!f|_{\partial M} \quad \text{in} \quad H^0(S^m\tau'_M|_{\partial M}) \quad \text{as} \quad k \to \infty. \tag{4.3.13}$$

By Lemma 4.3.1, the equalities $v_k|_{\partial M} = 0$ and $v|_{\partial M} = 0$ imply that $I(dv_k) = I(dv) = 0$. Therefore, from (4.3.10) and (4.3.12), we obtain

$$If = I\,{}^s\!f, \quad If_k = I\,{}^s\!f_k. \tag{4.3.14}$$

Applying Lemma 4.3.4 to ${}^s\!f_k$, we have

$$\|{}^s\!f_k\|_0^2 \le C\left(m\|j_\nu {}^s\!f_k|_{\partial M}\|_0 \cdot \|I\,{}^s\!f_k\|_0 + \|I\,{}^s\!f_k\|_1^2\right).$$

By (4.3.14), the last inequality can be rewritten as:

$$\|{}^s\!f_k\|_0^2 \le C\left(m\|j_\nu {}^s\!f_k|_{\partial M}\|_0 \cdot \|If_k\|_0 + \|If_k\|_1^2\right).$$

We pass to the limit in this inequality as $k \to \infty$; and make use of (4.3.12), (4.3.13) and continuity of I proved in Theorem 4.2.1. In such a way we arrive at the estimate

$$\|{}^s\!f\|_0^2 \le C\left(m\|j_\nu {}^s\!f|_{\partial M}\|_0 \cdot \|If\|_0 + \|If\|_1^2\right). \tag{4.3.15}$$

Using (4.3.10) and continuity of the trace operator $H^1(S^m\tau'_M) \to H^0(S^m\tau'_M|_{\partial M})$, we obtain

$$\|j_\nu {}^s\!f|_{\partial M}\|_0 \le C_2\|{}^s\!f|_{\partial M}\|_0 \le C_3\|{}^s\!f\|_1 \le C_4\|f\|_1. \tag{4.3.16}$$

Inequalities (4.3.15) and (4.3.16) give the claim of Theorem 4.3.3.

4.4 Pestov's differential identity

Recall that in Chapter 3 we introduced the bundle $\beta^r_s M = (B^r_s M, p^r_s, TM)$ of semibasic tensors of degree (r, s) over the space TM of the tangent bundle of a Riemannian manifold (M, g) and defined the operators $\overset{v}{\nabla}, \overset{h}{\nabla}\colon C^\infty(\beta^r_s M) \to C^\infty(\beta^r_{s+1} M)$ of vertical and horizontal differentiation. The metric g establishes the canonical isomorphism of the bundles $\beta^r_s M \cong \beta^{r+s}_0 M \cong \beta^0_{r+s} M$; in coordinate form this fact is expressed by the known operations of raising and lowering indices of a tensor; we will use them everywhere. Similar notation will be used for the derivative operators: $\overset{v}{\nabla}{}^i = g^{ij}\overset{v}{\nabla}_j$, $\overset{h}{\nabla}{}^i = g^{ij}\overset{h}{\nabla}_j$.

From now on in the current chapter we restrict ourselves to considering only real tensors and tensor fields. The metric g allows us to introduce the scalar product on the bundle $\beta^r_s M$. Consequently, for $u, v \in C^\infty(\beta^0_m M)$ the scalar product $\langle u, v\rangle$ is a function on TM expressible in coordinate form as

$$\langle u(x,\xi), v(x,\xi)\rangle = u_{i_1\dots i_m}(x,\xi)v^{i_1\dots i_m}(x,\xi). \tag{4.4.1}$$

We also denote $|u(x,\xi)|^2 = \langle u(x,\xi), u(x,\xi)\rangle$. The notations $\langle u(x,\xi), v(x,\xi)\rangle$ and $|u(x,\xi)|^2$ can be considered as convenient abbreviations of the functions on the right-hand side of (4.4.1), and we will make wide use of them.

The operator $H : C^\infty(\beta^0_m M) \to C^\infty(\beta^0_m M)$ defined by the equality $H = \xi^i \overset{h}{\nabla}_i$ is called the *differentiation along geodesics* and the equation

$$Hu(x,\xi) = f(x,\xi), \tag{4.4.2}$$

where $f \in C^\infty(\beta^0_m M)$ is a given tensor field and $u \in C^\infty(\beta^0_m M)$ is an unknown one, is called the *kinetic equation* of the metric g. This operator has a simple interpretation for $m = 0$, since $C^\infty(\beta^0_0 M) = C^\infty(TM)$. In the last case the operator H is expressed in coordinate form as

$$Hu(x,\xi) = \left(\xi^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk}\xi^j\xi^k \frac{\partial}{\partial \xi^i}\right) u(x,\xi). \tag{4.4.3}$$

Like any homogeneous first-order differential operator, H can be considered as a vector field on TM, which is called the *geodesic vector field*. The one-parameter group of diffeomorphisms G^t of the manifold TM generated by H is called the *geodesic flow* (or *geodesic pulverization*) of the metric g. It has a clear geometrical meaning: $G^t(x,\xi) = (\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))$, for $(x,\xi) \in TM$, where $\gamma_{x,\xi}$ is the geodesic defined by the initial conditions $\gamma_{x,\xi}(0) = x$ and $\dot\gamma_{x,\xi}(0) = \xi$. This means, in particular, that the geodesic flow preserves length of tangent vectors. In other words, at a point of the submanifold ΩM the vector field H is tangent to ΩM. Thus H can be considered as differential operator $H : C^\infty(\Omega M) \to C^\infty(\Omega M)$. Besides, the geodesic flow preserves the symplectic volume (3.6.1), by the well-known Liouville theorem [13].

Recall also that a physical interpretation of the kinetic equation (4.4.2), for $m = 0$, was discussed in Section 1.2.

Lemma 4.4.1 *Let M be a Riemannian manifold. For a real semibasic tensor field $u \in C^\infty(\beta^0_m M)$, the next identity is valid on TM:*

$$2\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = |\overset{h}{\nabla} u|^2 + \overset{h}{\nabla}_i v^i + \overset{v}{\nabla}_i w^i -$$

$$- R_{ijkl}\xi^i\xi^k \overset{v}{\nabla}{}^j u^{i_1\dots i_m} \cdot \overset{v}{\nabla}{}^l u_{i_1\dots i_m} - \sum_{k=1}^m R^{i_k}_{pqj}\xi^q u^{i_1\dots i_{k-1} p i_{k+1}\dots i_m} \overset{v}{\nabla}{}^j u_{i_1\dots i_m}, \tag{4.4.4}$$

where the semibasic vector fields v and w are defined by the equalities

$$v^i = \xi^i \overset{h}{\nabla}{}^j u^{i_1\dots i_m} \cdot \overset{v}{\nabla}_j u_{i_1\dots i_m} - \xi^j \overset{v}{\nabla}{}^i u^{i_1\dots i_m} \cdot \overset{h}{\nabla}_j u_{i_1\dots i_m}, \tag{4.4.5}$$

$$w^i = \xi^j \overset{h}{\nabla}{}^i u^{i_1\dots i_m} \cdot \overset{h}{\nabla}_j u_{i_1\dots i_m}. \tag{4.4.6}$$

P r o o f. From the definition of the operator H, we have

$$2\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = 2\overset{h}{\nabla}{}^i u^{i_1\dots i_m} \cdot \overset{v}{\nabla}_i\left(\xi^j \overset{h}{\nabla}_j u_{i_1\dots i_m}\right).$$

Using the relation $\overset{v}{\nabla}_i \xi^j = \delta^j_i$, we obtain

$$2\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = 2\overset{h}{\nabla}{}^i u^{i_1\dots i_m} \cdot \overset{h}{\nabla}_i u_{i_1\dots i_m} + 2\xi^j \overset{h}{\nabla}{}^i u^{i_1\dots i_m} \cdot \overset{v}{\nabla}_i \overset{h}{\nabla}_j u_{i_1\dots i_m}. \tag{4.4.7}$$

We transform the second summand on the right-hand side of the last relation. To this end we define a function φ by the equality

$$2\xi^j \overset{h}{\nabla}{}^i u^{i_1\dots i_m} \cdot \overset{v}{\nabla}_i \overset{h}{\nabla}_j u_{i_1\dots i_m} = \overset{v}{\nabla}_i\left(\xi^j \overset{h}{\nabla}{}^i u^{i_1\dots i_m} \cdot \overset{h}{\nabla}_j u_{i_1\dots i_m}\right) + \overset{h}{\nabla}_j\left(\xi^j \overset{h}{\nabla}{}^i u^{i_1\dots i_m} \cdot \overset{v}{\nabla}_i u_{i_1\dots i_m}\right) -$$

$$-\overset{h}{\nabla}{}^i\left(\xi^j\overset{v}{\nabla}_i u^{i_1\dots i_m}\cdot\overset{h}{\nabla}_j u_{i_1\dots i_m}\right)-\varphi. \tag{4.4.8}$$

Let us show that φ is independent of second-order derivatives of the field u. Indeed, expressing the derivatives of the products on the right-hand side of (4.4.8) through the derivatives of the factors, we obtain

$$\varphi=-2\xi^j\overset{h}{\nabla}{}^i u^{i_1\dots i_m}\cdot\overset{v}{\nabla}_i\overset{h}{\nabla}_j u_{i_1\dots i_m}+\overset{h}{\nabla}{}^i u^{i_1\dots i_m}\cdot\overset{h}{\nabla}_i u_{i_1\dots i_m}+$$

$$+\xi^j\overset{v}{\nabla}_i\overset{h}{\nabla}{}^i u^{i_1\dots i_m}\cdot\overset{h}{\nabla}_j u_{i_1\dots i_m}+\xi^j\overset{h}{\nabla}{}^i u^{i_1\dots i_m}\cdot\overset{v}{\nabla}_i\overset{h}{\nabla}_j u_{i_1\dots i_m}+$$

$$+\xi^j\overset{h}{\nabla}_j\overset{h}{\nabla}{}^i u^{i_1\dots i_m}\cdot\overset{v}{\nabla}_i u_{i_1\dots i_m}+\xi^j\overset{h}{\nabla}{}^i u^{i_1\dots i_m}\cdot\overset{h}{\nabla}_j\overset{v}{\nabla}_i u_{i_1\dots i_m}-$$

$$-\xi^j\overset{h}{\nabla}{}^i\overset{v}{\nabla}_i u^{i_1\dots i_m}\cdot\overset{h}{\nabla}_j u_{i_1\dots i_m}-\xi^j\overset{v}{\nabla}_i u^{i_1\dots i_m}\cdot\overset{h}{\nabla}{}^i\overset{h}{\nabla}_j u_{i_1\dots i_m}.$$

After evident transformations, this equality takes the form

$$\varphi=\overset{h}{\nabla}{}^i u^{i_1\dots i_m}\cdot\overset{h}{\nabla}_i u_{i_1\dots i_m}+\xi^j\overset{h}{\nabla}{}^i u^{i_1\dots i_m}\cdot\left(\overset{h}{\nabla}_j\overset{v}{\nabla}_i-\overset{v}{\nabla}_i\overset{h}{\nabla}_j\right)u_{i_1\dots i_m}+$$

$$+\xi^j\overset{h}{\nabla}_j u^{i_1\dots i_m}\cdot\left(\overset{v}{\nabla}_i\overset{h}{\nabla}{}^i-\overset{h}{\nabla}{}^i\overset{v}{\nabla}_i\right)u_{i_1\dots i_m}+\xi^j\overset{v}{\nabla}{}^i u^{i_1\dots i_m}\cdot\left(\overset{h}{\nabla}_j\overset{h}{\nabla}_i-\overset{h}{\nabla}_i\overset{h}{\nabla}_j\right)u_{i_1\dots i_m}.$$

Using commutation formulas for the operators $\overset{v}{\nabla}$ and $\overset{h}{\nabla}$ which are presented in Theorem 3.5.2, we obtain

$$\varphi=\overset{h}{\nabla}{}^i u^{i_1\dots i_m}\cdot\overset{h}{\nabla}_i u_{i_1\dots i_m}+\xi^j\overset{v}{\nabla}{}^i u^{i_1\dots i_m}\cdot\left(-R^p{}_{qji}\xi^q\overset{v}{\nabla}_p u_{i_1\dots i_m}-\sum_{k=1}^m R^p{}_{i_kji}u_{i_1\dots i_{k-1}pi_{k+1}\dots i_m}\right).$$

Inserting this expression for the function φ into (4.4.8), we have

$$2\xi^j\overset{h}{\nabla}{}^i u^{i_1\dots i_m}\cdot\overset{v}{\nabla}_i\overset{h}{\nabla}_j u_{i_1\dots i_m}=-\overset{h}{\nabla}{}^i u^{i_1\dots i_m}\cdot\overset{h}{\nabla}_i u_{i_1\dots i_m}+\overset{h}{\nabla}_i v^i+\overset{v}{\nabla}_i w^i-$$

$$-R_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^j u^{i_1\dots i_m}\cdot\overset{v}{\nabla}{}^l u_{i_1\dots i_m}-\sum_{k=1}^m R^{i_k}{}_{pqj}\xi^q u^{i_1\dots i_{k-1}pi_{k+1}\dots i_m}\overset{v}{\nabla}{}^j u_{i_1\dots i_m}.$$

Finally, replacing the second summand on the right-hand side of (4.4.7) by the last value, we arrive at (4.4.4). The lemma is proved.

Let us separately write down the claim of Lemma 4.4.1 for $m=0$, i.e., for $u\in C^\infty(TM)$. In this case equalities (4.4.4)–(4.4.6) take the form

$$2\langle\overset{h}{\nabla}u,\overset{v}{\nabla}(Hu)\rangle=|\overset{h}{\nabla}u|^2+\overset{h}{\nabla}_i v^i+\overset{v}{\nabla}_i w^i-R_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^j u\cdot\overset{v}{\nabla}{}^l u, \tag{4.4.9}$$

$$v^i=\xi^i\overset{h}{\nabla}{}^j u\cdot\overset{v}{\nabla}_j u-\xi^j\overset{v}{\nabla}{}^i u\cdot\overset{h}{\nabla}_j u, \tag{4.4.10}$$

$$w^i=\xi^j\overset{h}{\nabla}{}^i u\cdot\overset{h}{\nabla}_j u. \tag{4.4.11}$$

4.5 Poincaré's inequality for semibasic tensor fields

Lemma 4.5.1 *Let M be a CDRM and λ be a continuous nonnegative function on ΩM. For a semibasic tensor field $f \in C^\infty(\beta^0_m M)$ satisfying the boundary condition*

$$f|_{\partial_-\Omega M} = 0, \tag{4.5.1}$$

the next inequality is valid:

$$\int\limits_{\Omega M} \lambda(x,\xi)|f(x,\xi)|^2\,d\Sigma \le \lambda_0 \int\limits_{\Omega M} |Hf(x,\xi)|^2\,d\Sigma, \tag{4.5.2}$$

where

$$\lambda_0 = \sup_{(x,\xi)\in\partial_-\Omega M} \int\limits_0^{\tau_+(x,\xi)} t\lambda(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))\,dt, \tag{4.5.3}$$

$\gamma_{x,\xi} : [0, \tau_+(x,\xi)] \to M$ is a maximal geodesic defined by the initial conditions $\gamma_{x,\xi}(0) = x$ and $\dot\gamma_{x,\xi}(0) = \xi$, $d\Sigma = d\Sigma^{2n-1}$ is the volume form on ΩM defined by formula (3.6.16).

By the Liouville theorem [13], the geodesic flow preserves the volume form $d\Sigma$. Therefore, in the scalar case $f = \varphi \in C^\infty(\beta^0_0 M)$, the lemma coincides, in fact, with the well-known Poincaré inequality [82]. The case of an arbitrary semibasic field $f \in C^\infty(\beta^0_m M)$ is reduced to the scalar one by introducing the function $\varphi(x,\xi) = |f(x,\xi)|$. Unfortunately, in such a way an additional obstacle arises that relates to the singularities of the function φ at zeros of the field f. For this reason we should reproduce the proof of the Poincaré inequality, while taking the nature of the mentioned singularities into account. First of all we will reduce Lemma 4.5.1 to the next claim:

Lemma 4.5.2 *Let M, λ and λ_0 be the same as in Lemma* 4.5.1; *a function $\varphi \in C(\Omega M)$ be smooth on $\Omega_\varphi = \{(x,\xi) \in \Omega M \mid \varphi(x,\xi) \ne 0\}$. Suppose that*

$$\sup_{(x,\xi)\in\Omega_\varphi} |H\varphi(x,\xi)| < \infty. \tag{4.5.4}$$

If φ satisfies the boundary condition

$$\varphi|_{\partial_-\Omega M} = 0, \tag{4.5.5}$$

then the next estimate is valid:

$$\int\limits_{\Omega M} \lambda(x,\xi)\,|\varphi(x,\xi)|^2\,d\Sigma \le \lambda_0 \int\limits_{\Omega_\varphi} |H\varphi(x,\xi)|^2\,d\Sigma. \tag{4.5.6}$$

P r o o f of Lemma 4.5.1. Let a semibasic field f satisfy the conditions of Lemma 4.5.1. We verify that the function $\varphi = |f|$ satisfies the conditions of Lemma 4.5.2. The only nontrivial condition is (4.5.4). The equality $H\varphi = \langle f, Hf\rangle/|f|$ holds on Ω_φ. It implies that $|H\varphi| \le |Hf|$ and, consequently, (4.5.4) holds.

Assuming validity of Lemma 4.5.2, we have inequality (4.5.6). From this inequality we obtain

$$\int\limits_{\Omega M} \lambda|f|^2\,d\Sigma = \int\limits_{\Omega M} \lambda|\varphi|^2\,d\Sigma \le \lambda_0 \int\limits_{\Omega_\varphi} |H\varphi|^2\,d\Sigma =$$

$$= \lambda_0 \int\limits_{\Omega_\varphi} \frac{|\langle f, Hf\rangle|^2}{|f|^2}\, d\Sigma \le \lambda_0 \int\limits_{\Omega_\varphi} |Hf|^2\, d\Sigma \le \lambda_0 \int\limits_{\Omega M} |Hf|^2\, d\Sigma.$$

The lemma is proved.

P r o o f of Lemma 4.5.2. We consider the domain $D = \{(x,\xi;t) \mid 0 \le t \le \tau_+(x,\xi)\}$ in the manifold $\partial_-\Omega M \times \mathbf{R}$ and define a smooth mapping $\mu : D \to \Omega M$ by putting $\mu(x,\xi;t) = (\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))$. It maps the interior of D diffeomorphically onto $\Omega M \backslash T(\partial M)$. Consequently,

$$\int\limits_{\Omega M} \lambda|\varphi|^2 d\Sigma = \int\limits_{D} (\lambda \circ \mu)|\varphi \circ \mu|^2 \mu^*(d\Sigma). \tag{4.5.7}$$

Differentiating the relation $(h \circ \mu)(x,\xi;t) = h(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))$ with respect to t, we obtain $\partial(h \circ \mu)/\partial t = (Hh) \circ \mu$. Valid for every $h \in C^\infty(\Omega M)$, the last equality means that the vector fields $\partial/\partial t$ and H are μ-connected (we recall that, given a diffeomorphism $f : X \to Y$ of two manifolds, vector fields $u \in C^\infty(\tau_X)$ and $v \in C^\infty(\tau_Y)$ are called *f-connected* if the differential of f transforms u into v; compare [41]). Since the form $d\Sigma$ is preserved by the geodesic flow, $\mu^*(d\Sigma)$ is preserved by the flow of the field $\partial/\partial t$. This implies, as is easily seen, that $\mu^*(d\Sigma) = d\sigma \times dt$ for some volume form $d\sigma$ on $\partial_-\Omega M$. Thus (4.5.7) can be rewritten as:

$$\int\limits_{\Omega M} \lambda|\varphi|^2\, d\Sigma = \int\limits_{\partial_-\Omega M} d\sigma(x,\xi) \int\limits_0^{\tau_+(x,\xi)} \rho(x,\xi;t)|\psi(x,\xi;t)|^2 dt, \tag{4.5.8}$$

where $\rho = \lambda \circ \mu$ and $\psi = \varphi \circ \mu$. In a similar way the integral on the right-hand side of inequality (4.5.6) is transformed as follows

$$\int\limits_{\Omega_\varphi} |H\varphi|^2 d\Sigma = \int\limits_{D_\psi} |\partial\psi(x,\xi;t)/\partial t|^2 dt\, d\sigma(x,\xi), \tag{4.5.9}$$

where $D_\psi = \{(x,\xi;t) \mid \psi(x,\xi;t) \ne 0\}$. The function ψ is continuous on D, smooth on D_ψ and, by (4.5.4) and (4.5.5), satisfies the conditions

$$\sup_{D_\psi} |\partial\psi(x,\xi;t)/\partial t| < \infty, \tag{4.5.10}$$

$$\psi(x,\xi;0) = 0. \tag{4.5.11}$$

The constant λ_0 of Lemma 4.5.1 is expressed through ρ:

$$\lambda_0 = \sup_{(x,\xi)\in\partial_-\Omega M} \int\limits_0^{\tau_+(x,\xi)} t\rho(x,\xi;t)\, dt. \tag{4.5.12}$$

We define the function $\dot\psi : D \to \mathbf{R}$ by putting

$$\dot\psi(x,\xi;t) = \begin{cases} \partial\psi(x,\xi;t)/\partial t, & \text{for} \quad (x,\xi;t) \in D_\psi, \\ 0, & \text{for} \quad (x,\xi;t) \notin D_\psi. \end{cases} \tag{4.5.13}$$

By (4.5.8) and (4.5.9), inequality (4.5.6) under proof is equivalent to the next one

$$\int\limits_{\partial_-\Omega M} d\sigma(x,\xi) \int\limits_0^{\tau_+(x,\xi)} \rho(x,\xi;t)|\psi(x,\xi;t)|^2 dt \le \lambda_0 \int\limits_{\partial_-\Omega M} d\sigma(x,\xi) \int\limits_0^{\tau_+(x,\xi)} |\dot\psi(x,\xi;t)|^2 dt. \tag{4.5.14}$$

Let us consider the function $\psi_y(t) = \psi(x,\xi;t)$, for a fixed $y = (x,\xi)$, as a function in the variable $t \in I_y = (0, \tau_+(y))$. We shall prove that it is absolutely continuous on I_y. Indeed, let $J_y = \{t \in I_y \mid \psi_y(t) \ne 0\}$. As an open subset of I_y, the set J_y is a union of pairwise disjoint intervals $J_y = \bigcup_{i=1}^\infty (a_i, b_i)$. The function ψ_y is smooth on each of these intervals and, by (4.5.10), its derivative is bounded:

$$|d\psi_y(t)/dt| \le C \qquad (t \in (a_i, b_i)), \tag{4.5.15}$$

where a constant C is the same for all i. The function ψ_y vanishes on $I_y \setminus J_y$ and is continuous on I_y. The listed properties imply that

$$|\psi_y(t_1) - \psi_y(t_2)| \le C|t_1 - t_2| \tag{4.5.16}$$

for all $t_1, t_2 \in I_y$. In particular, (4.5.16) implies absolute continuity of ψ_y. Consequently, this function is differentiable almost everywhere on I_y and can be recovered from its derivative:

$$\psi_y(t) = \int\limits_0^t \frac{d\psi_y(\tau)}{d\tau}\, d\tau. \tag{4.5.17}$$

While writing down the last equality, we took (4.5.11) into account. The derivative $d\psi_y(t)/dt$ is bounded. From (4.5.17) with the help of the Cauchy-Bunyakovskiĭ inequality, we obtain

$$|\psi_y(t)|^2 \le t \int\limits_0^t \left|\frac{d\psi_y(\tau)}{d\tau}\right|^2 d\tau \le t \int\limits_0^{\tau_+(y)} \left|\frac{d\psi_y(t)}{dt}\right|^2 dt. \tag{4.5.18}$$

Let us show that, almost everywhere on I_y, $d\psi_y(t)/dt$ coincides with the function $\dot\psi(y;t)$ defined by formula (4.5.13). Indeed, by (4.5.13), $d\psi_y(t)/dt = \dot\psi(y;t)$ if $t \in J_y$. If $t \in I_y \setminus J_y$ does not coincide with any of the endpoints of the intervals (a_i, b_i), then t is a limit point of the set $I_y \setminus J_y$. Since $\psi_y|_{I_y \setminus J_y} = 0$, existence of the derivative $d\psi_y(t)/dt$ implies that it is equal to zero. The function $\dot\psi(y;t)$ vanishes on $I_y \setminus J_y$, by definition (4.5.13). Thus the relation $d\psi_y(t)/dt = \dot\psi(y;t)$ is proved for all $t \in I_y$ such that the derivative $d\psi_y(t)/dt$ exists, with the possible exception of the endpoints of the intervals (a_i, b_i).

We can now rewrite (4.5.18) as:

$$|\psi(x,\xi;t)|^2 \le t \int\limits_0^{\tau_+(x,\xi)} |\dot\psi(x,\xi;t)|^2 dt.$$

We multiply this inequality by $\rho(x,\xi;t)$ and integrate it with respect to t

$$\int\limits_0^{\tau_+(x,\xi)} \rho(x,\xi;t)|\psi(x,\xi;t)|^2 dt \le \int\limits_0^{\tau_+(x,\xi)} t\rho(x,\xi;t)\, dt \int\limits_0^{\tau_+(x,\xi)} |\dot\psi(x,\xi;t)|^2 dt.$$

By (4.5.12), the first integral on the right-hand side of the last formula can be replaced by λ_0. Multiplying the so-obtained inequality by $d\sigma(x,\xi)$ and integrating it over $\partial_-\Omega M$, we arrive at (4.5.14). The lemma is proved.

To prove Lemma 4.3.4 we will need the next claim. It is of a purely algebraic nature, although formulated in terms of analysis.

Lemma 4.5.3 *Let M be a compact n-dimensional Riemannian manifold, $f \in C^\infty(S^m\tau'_M)$, $m \geq 1$. Define the function $\varphi \in C^\infty(TM) = C^\infty(\beta_0^0 M)$ and semibasic covector field $F \in C^\infty(\beta_1^0 M)$ by the equalities*

$$\varphi(x,\xi) = f_{i_1\dots i_m}(x)\xi^{i_1}\dots\xi^{i_m}; \qquad F_i(x,\xi) = f_{ii_2\dots i_m}(x)\xi^{i_2}\dots\xi^{i_m}. \tag{4.5.19}$$

Then the next inequality is valid:

$$\int\limits_{\Omega M} |F|^2 d\Sigma \leq \frac{n+2m-2}{m} \int\limits_{\Omega M} |\varphi|^2 d\Sigma. \tag{4.5.20}$$

P r o o f. We shall show that this claim is reduced to a known property of eigenvalues of the Laplacian on sphere.

It follows from (3.6.34) that inequality (4.5.20) is equivalent to the next one:

$$\int\limits_M \left[\int\limits_{\Omega_x M} |F(x,\xi)|^2 d\omega_x(\xi)\right] dV^n(x) \leq \frac{n+2m-2}{m} \int\limits_M \left[\int\limits_{\Omega_x M} |\varphi(x,\xi)|^2 d\omega_x(\xi)\right] dV^n(x).$$

Consequently, to prove the lemma it suffices to show that

$$\int\limits_{\Omega_x M} |F(x,\xi)|^2 d\omega_x(\xi) \leq \frac{n+2m-2}{m} \int\limits_{\Omega_x M} |\varphi(x,\xi)|^2 d\omega_x(\xi). \tag{4.5.21}$$

for every $x \in M$.

Fixing a point x, we introduce coordinates in some of its neighbourhoods so that $g_{ij}(x) = \delta_{ij}$. With the help of these coordinates we identify T_xM and $\mathbf{R}^n$, the latter furnished with the standard Euclidean metric. Then $\Omega_x M$ is identified with the unit sphere Ω of the space $\mathbf{R}^n$; the measure $d\omega_x$, with the standard angle measure $d\omega$; the function $\varphi(x,\xi)$, by (4.5.19), with a homogeneous polynomial ψ of degree m on $\mathbf{R}^n$; the field F, with $\nabla\psi/m$. Thus, to prove (4.5.21) it suffices to verify the inequality

$$\int\limits_\Omega |\nabla\psi|^2 d\omega \leq m(n+2m-2) \int\limits_\Omega |\psi|^2 d\omega \tag{4.5.22}$$

on the space $P_m(\mathbf{R}^n)$ of homogeneous polynomials of degree m. Applying the Green's formula

$$\int\limits_\Omega |\nabla\psi|^2 d\omega = \int\limits_\Omega \psi(m^2 - \Delta_\omega)\,\psi\,d\omega \qquad (\psi \in P_m(\mathbf{R}^n)),$$

where Δ_ω is the spherical Laplacian [131], we see that (4.5.22) is equivalent to the claim: all eigenvalues μ_k of the operator $m^2 - \Delta_\omega$ on the space $P_m(\mathbf{R}^n)$ do not exceed $m(n+2m-2)$. It is known that the eigenvalues of the Laplacian Δ_ω are precisely the numbers $\lambda_k = -k(n+k-2)$ and the spherical harmonics of order k are just the eigenfunctions belonging to λ_k. Therefore the eigenvalues of the operator $m^2 - \Delta_\omega$ on $P_m(\mathbf{R}^n)$ are those of $\mu_k = m^2 - \lambda_k$ for which $k \leq m$. The maximal of them is $m(n+2m-2)$. The lemma is proved.

4.6 Reduction of Theorem 4.3.3 to an inverse problem for the kinetic equation

Let a field $f \in C^\infty(S^m\tau'_M)$ on a CDRM M satisfy the conditions of Lemma 4.3.4. We define the function

$$u(x,\xi) = \int\limits_{\tau_-(x,\xi)}^{0} \langle f(\gamma_{x,\xi}(t)), \dot\gamma^m_{x,\xi}(t)\rangle\, dt \qquad ((x,\xi) \in T^0M) \tag{4.6.1}$$

on T^0M, using the same notation as used in definition (4.2.2) of the ray transform. The difference between equalities (4.2.2) and (4.6.1) is the fact that the first of them is considered only for $(x,\xi) \in \partial_+\Omega M$ while the second one, for all $(x,\xi) \in T^0M$. In particular, we have the boundary condition

$$u|_{\partial_+\Omega M} = If. \tag{4.6.2}$$

Since $\tau_-(x,\xi) = 0$ for $(x,\xi) \in \partial_-\Omega M$, we have the second boundary condition

$$u|_{\partial_-\Omega M} = 0. \tag{4.6.3}$$

The function $u(x,\xi)$ is smooth at the same points at which $\tau_-(x,\xi)$ is smooth. The last is true, as we know, at all points of the open set $T^0M \setminus T(\partial M)$ of the manifold T^0M.

Let us show that the function u satisfies the kinetic equation

$$Hu = f_{i_1\dots i_m}(x)\xi^{i_1}\dots\xi^{i_m} \tag{4.6.4}$$

on $T^0M \setminus T(\partial M)$.

Indeed, let $(x,\xi) \in T^0M \setminus T(\partial M)$ and $\gamma = \gamma_{x,\xi} : [\tau_-(x,\xi), \tau_+(x,\xi)] \to M$ be the geodesic defined by the initial conditions $\gamma(0) = x$ and $\dot\gamma(0) = \xi$. For sufficiently small $s \in \mathbf{R}$, we put $x_s = \gamma(s)$ and $\xi_s = \dot\gamma(s)$. Then $\gamma_{x_s,\xi_s}(t) = \gamma(t+s)$ and $\tau_-(x_s,\xi_s) = \tau_-(x,\xi) - s$. Consequently,

$$u(\gamma(s),\dot\gamma(s)) = u(x_s,\xi_s) = \int\limits_{\tau_-(x_s,\xi_s)}^{0} \langle f(\gamma_{x_s,\xi_s}(t)), \dot\gamma^m_{x_s,\xi_s}(t)\rangle\, dt =$$

$$= \int\limits_{\tau_-(x,\xi)}^{s} \langle f(\gamma_{x,\xi}(t)), \dot\gamma^m_{x,\xi}(t)\rangle\, dt.$$

Differentiating this equality with respect to s and putting $s = 0$ in the so-obtained relation, we come to

$$\dot\gamma^i(0)\frac{\partial u}{\partial x^i} + \ddot\gamma^i(0)\frac{\partial u}{\partial \xi^i} = f_{i_1\dots i_m}(\gamma(0))\dot\gamma^{i_1}(0)\dots\dot\gamma^{i_m}(0). \tag{4.6.5}$$

Inserting $\gamma(0) = x,\ \dot\gamma(0) = \xi$ and the value $\ddot\gamma^i(0) = -\Gamma^i_{jk}(x)\xi^j\xi^k$ from equation (1.2.5) of geodesics into the last relation and taking (4.4.3) into account, we arrive at (4.6.4).

The function $u(x,\xi)$ is homogeneous in its second argument:

$$u(x,\lambda\xi) = \lambda^{m-1}u(x,\xi) \qquad (\lambda > 0). \tag{4.6.6}$$

Indeed, since $\gamma_{x,\lambda\xi}(t) = \gamma_{x,\xi}(\lambda t)$, if follows from (4.6.1) that

$$u(x,\lambda\xi) = \int\limits_{\tau_-(x,\lambda\xi)}^{0} \langle f(\gamma_{x,\lambda\xi}(t)), \dot\gamma^m_{x,\lambda\xi}(t))\rangle\, dt =$$

$$= \int\limits_{\lambda^{-1}\tau_-(x,\xi)}^{0} \langle f(\gamma_{x,\xi}(\lambda t)), \lambda^m \dot\gamma^m_{x,\xi}(\lambda t))\rangle\, dt =$$

$$= \lambda^{m-1} \int\limits_{\tau_-(x,\xi)}^{0} \langle f(\gamma_{x,\xi}(t)), \dot\gamma^m_{x,\xi}(t))\rangle\, dt = \lambda^{m-1} u(x,\xi).$$

Thus, the function $u(x,\xi)$ is a solution to the boundary problem (4.6.2)–(4.6.4) and satisfies the homogeneity condition (4.6.6). Besides, we recall that condition (4.3.7) is imposed upon the field f in the right-hand side of equation (4.6.4). Lemma 4.3.4 thereby reduces to the next problem: one has to estimate the right-hand side of the kinetic equation (4.6.4) by the right-hand side of the boundary condition (4.6.2).

By Lemma 4.4.1, identity (4.4.9) is valid for the function $u(x,\xi)$ on $T^0M \setminus T(\partial M)$. Using (4.6.4), we transform the left-hand side of the identity

$$2\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = 2\overset{h}{\nabla}{}^i u \cdot \overset{v}{\nabla}_i(Hu) = 2\overset{h}{\nabla}{}^i u \cdot \frac{\partial}{\partial \xi^i}\left(f_{i_1\dots i_m}\xi^{i_1}\dots\xi^{i_m}\right) =$$

$$= 2m\overset{h}{\nabla}{}^i u \cdot f_{i\,i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m} = \overset{h}{\nabla}{}^i(2mu f_{i\,i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}) - 2mu(\overset{h}{\nabla}{}^i f_{ii_2\dots i_m})\xi^{i_2}\dots\xi^{i_m} =$$

$$= \overset{h}{\nabla}_i \tilde v^i - 2mu(\delta f)_{i_1\dots i_{m-1}}\xi^{i_1}\dots\xi^{i_{m-1}}, \tag{4.6.7}$$

where

$$\tilde v^i = 2mu\, g^{ip} f_{pi_1\dots i_{m-1}}\xi^{i_1}\dots\xi^{i_{m-1}}. \tag{4.6.8}$$

The second summand on the right-hand side of (4.6.7) vanishes by (4.3.7). Thus, the application of Lemma 4.4.1 to function (4.6.1) leads to the next identity on $T^0M\setminus T(\partial M)$:

$$|\overset{h}{\nabla} u|^2 - R_{ijkl}\xi^i\xi^k \overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u = \overset{h}{\nabla}_i(\tilde v^i - v^i) - \overset{v}{\nabla}_i w^i, \tag{4.6.9}$$

where the semibasic vector fields v, w and $\tilde v$ are defined by formulas (4.4.10), (4.4.11) and (4.6.8).

We are going to integrate equality (4.6.9) over ΩM. In course of integration, some precautions are needed against singularities of the function u on the set $T(\partial M)$. For this reason we will proceed as follows. Let $r : M \to \mathbf{R}$ be the distance to ∂M in the metric g. In some neighbourhood of ∂M this function is smooth, and the boundary of the manifold $M_\rho = \{x \in M \mid r(x) \ge \rho\}$ is strictly convex for sufficiently small $\rho > 0$. The function u is smooth on ΩM_ρ, since $\Omega M_\rho \subset T^0M \setminus T(\partial M)$. We multiply (4.6.9) by the volume form $d\Sigma = d\Sigma^{2n-1}$ and integrate it over ΩM_ρ. Transforming then the right-hand side of the so-obtained equality by the Gauss-Ostrogradskiĭ formulas (3.6.35) and (3.6.36), we obtain

$$\int\limits_{\Omega M_\rho} (|\overset{h}{\nabla} u|^2 - R_{ijkl}\xi^i\xi^k \overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u)\, d\Sigma =$$

$$= \int\limits_{\partial\Omega M_\rho} \langle \tilde{v} - v, \nu \rangle \, d\Sigma^{2n-2} - (n + 2m - 2) \int\limits_{\Omega M_\rho} \langle w, \xi \rangle \, d\Sigma. \tag{4.6.10}$$

The factor $n + 2m - 2$ is written before the last integral because the field $w(x, \xi)$ is homogeneous of degree $2m - 1$ in its second argument, as one can see from (4.4.11) and (4.6.6). Besides, (4.4.11) implies that $\langle w, \xi \rangle = (Hu)^2$ and, consequently, equality (4.6.10) takes the form

$$\int\limits_{\Omega M_\rho} \Big[|\overset{h}{\nabla} u|^2 - R_{ijkl} \xi^i \xi^k \overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u + (n + 2m - 2)(Hu)^2 \Big] \, d\Sigma = \int\limits_{\partial\Omega M_\rho} \langle \tilde{v} - v, \nu \rangle \, d\Sigma^{2n-2}. \tag{4.6.11}$$

Here $\nu = \nu_\rho(x)$ is the unit vector of the outer normal to the boundary of the manifold M_ρ.

We wish now to pass to the limit in (4.6.11) as $\rho \to 0$. To this end, we first note that both the sides of the last equality can be represented as integrals over domains independent of ρ. Indeed, since $\Omega M_\rho \subset \Omega M$, the domain of integration ΩM_ρ for the leftmost integral on (4.6.11) can be replaced with ΩM by multiplying simultaneously the integrand by the characteristic function $\chi_\rho(x)$ of the set M_ρ. The right-hand side of (4.6.11) can be transformed to an integral over $\partial\Omega M$ with the help of the diffeomorphism $\mu : \partial\Omega M \to \partial\Omega M_\rho$ defined by the equality $\mu(x, \xi) = (x', \xi')$, where a point x' is such that the geodesic $\gamma_{xx'}$, whose endpoints are x and x', has length ρ and intersects ∂M orthogonally at the x, and the vector ξ' is obtained by the parallel translation of the vector ξ along $\gamma_{xx'}$.

The integrands of (4.6.11) are smooth on $\Omega M \setminus \partial_0 \Omega M$ and, consequently, converge to their values almost everywhere on $\partial\Omega M$ as $\rho \to 0$. We also note that the first and third summands in the integrand on the left-hand side of (4.6.11) are nonnegative. Therefore, to apply the Lebesgue dominated convergence theorem, it remains to show that: 1) the second summand in the integrand on the left-hand side of (4.6.11) is summable over ΩM and 2) the absolute value of the integrand on the right-hand side of (4.6.11) is majorized by a function independent of ρ and summable over $\partial\Omega M$. We shall demonstrate more, namely, that the absolute values of the integrand on the right-hand side and of the second summand in the integrand on the left-hand side of (4.6.11) are bounded by some constant independent of ρ. Indeed, since these expressions are invariant, i.e., independent of the choice of coordinates, to prove our claim it suffices to show that these functions are bounded in the domain of some local coordinate system.

In a neighbourhood of a point $x_0 \in \partial M$ we introduce a semigeodesic coordinate system similarly as in Lemma 4.1.3. Then $g_{in} = \delta_{in}$, $\nu^i = -\delta^i_n$. It follows from (4.4.10) and (4.6.8) that

$$\langle \tilde{v} - v, \nu \rangle = \xi^n \overset{h}{\nabla}{}^\alpha u \cdot \overset{v}{\nabla}_\alpha u - \xi^\alpha \overset{h}{\nabla}_\alpha u \cdot \overset{v}{\nabla}{}^n u - 2mu \, f_{n i_1 \dots i_{m-1}} \xi^{i_1} \dots \xi^{i_{m-1}}, \tag{4.6.12}$$

In this formula (and in formula (4.6.14) below) the summation from 1 to $n - 1$ over the index α is assumed. It is important that the right-hand side of (4.6.12) does not contain $\overset{h}{\nabla}_n u$. It follows from Lemma 4.1.3 and equality (4.6.1) that the derivatives $\overset{h}{\nabla}_\alpha u$ $(1 \le \alpha \le n - 1)$ and $\overset{v}{\nabla}_i u$ $(1 \le i \le n)$ are locally bounded.

Thus we have shown that passage to the limit is possible in (4.6.11) as $\rho \to 0$. Accomplishing it, we obtain the equality

$$\int\limits_{\Omega M} \Big[|\overset{h}{\nabla} u|^2 - R_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u + (n+2m-2)(Hu)^2\Big]\, d\Sigma =$$

$$= \int\limits_{\partial\Omega M} (Lu - 2mu\, \langle j_\nu f, \xi^{m-1}\rangle)\, d\Sigma^{2n-2}, \tag{4.6.13}$$

where L is the differential operator given in a semigeodesic coordinate system by the formula

$$Lu = \xi^n \overset{h}{\nabla}{}^\alpha u \cdot \overset{v}{\nabla}_\alpha u - \xi^\alpha \overset{h}{\nabla}_\alpha u \cdot \overset{v}{\nabla}{}^n u. \tag{4.6.14}$$

Note that until now we did not use the restriction on sectional curvatures and boundary conditions (4.6.2)–(4.6.3); i.e., integral identity (4.6.13) is valid, for every solenoidal field f on an arbitrary CDRM, in which the function u is defined by formula (4.6.1).

4.7 Proof of Theorem 4.3.3

In view of the boundary conditions (4.6.2)–(4.6.3), equality (4.6.13) can be written as:

$$\int\limits_{\Omega M} \Big[|\overset{h}{\nabla} u|^2 - R_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u + (n+2m-2)(Hu)^2\Big]\, d\Sigma =$$

$$= \int\limits_{\partial_+\Omega M} (L(If) - 2m(If)\, \langle j_\nu f, \xi^{m-1}\rangle)\, d\Sigma^{2n-2}. \tag{4.7.1}$$

If $(y^1, \dots, y^{2n-2})$ is a local coordinate system on $\partial_+\Omega M$, then we see from (4.6.14) that Lu is a quadratic form in variables u, $\partial u/\partial y^i$ and $\partial u/\partial|\xi|$. According to homogeneity (4.6.6), $\partial u/\partial|\xi| = (m-1)u$ and, consequently, L is a quadratic first-order differential operator on the manifold $\partial_+\Omega M$. So the absolute value of the right-hand side of relation (4.7.1) is not greater than $C(m\|If\|_0 \cdot \|j_\nu f|_{\partial M}\|_0 + \|If\|_1^2)$ with some constant C independent of f. Consequently, (4.7.1) implies the inequality

$$\int\limits_{\Omega M} |\overset{h}{\nabla} u|^2 d\Sigma + (n+2m-2)\int\limits_{\Omega M} (Hu)^2\, d\Sigma \le$$

$$\le \int\limits_{\Omega M} R_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u\, d\Sigma + C(m\|If\|_0 \cdot \|j_\nu f|_{\partial M}\|_0 + \|If\|_1^2). \tag{4.7.2}$$

It turns out that the integral on the right-hand side of (4.7.2) can be estimated from above by the left-hand side of this inequality. To prove this fact, we first note that, for $(x, \xi) \in \Omega M$, the integrand on the right-hand side of (4.7.2) can be estimated as:

$$R_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u \le K^+(x,\xi)|\overset{v}{\nabla} u(x,\xi)|^2, \tag{4.7.3}$$

where $K^+(x, \xi)$ is defined by formula (4.3.2).

In view of the boundary condition (4.6.2), the field $\overset{v}{\nabla}u$ satisfies the conditions of Lemma 4.5.1. Applying this lemma, we obtain the estimate

$$\int\limits_{\Omega M} K^+(x,\xi)|\overset{v}{\nabla}u(x,\xi)|^2 d\Sigma \le k^+ \int\limits_{\Omega M} |H\overset{v}{\nabla}u|^2 d\Sigma, \tag{4.7.4}$$

where $k^+ = k^+(M,g)$ is given by equality (4.3.3).

Applying the operator $\overset{v}{\nabla}$ to equation (4.6.4), since $\overset{v}{\nabla}$ and $\overset{h}{\nabla}$ commute, we obtain

$$H\overset{v}{\nabla}u = mF - \overset{h}{\nabla}u, \tag{4.7.5}$$

where F is the semibasic field defined by formula (4.5.19). It follows from (4.7.5) that

$$|H\overset{v}{\nabla}u|^2 \le m^2|F|^2 + 2m|F||\overset{h}{\nabla}u| + |\overset{h}{\nabla}u|^2 \le m(m+1)|F|^2 + (m+1)\,|\overset{h}{\nabla}u|^2, \tag{4.7.6}$$

The arguments of the next paragraph are slightly different in the cases $m=0$ and $m>0$.

First we assume that $m \ge 1$. By Lemma 4.5.3, inequality (4.5.20) holds in which φ is expressed by relation (4.5.19). Comparing (4.5.19) and (4.6.4), we see that $\varphi = Hu$. Consequently, (4.5.20) can be rewritten as:

$$\int\limits_{\Omega M} |F|^2 d\Sigma \le \frac{n+2m-2}{m} \int\limits_{\Omega M} (Hu)^2 d\Sigma. \tag{4.7.7}$$

From (4.7.6) and (4.7.7) we obtain the inequality

$$\int\limits_{\Omega M} |H\overset{v}{\nabla}u|^2 d\Sigma \le (m+1)(n+2m-2) \int\limits_{\Omega M} (Hu)^2 d\Sigma + (m+1) \int\limits_{\Omega M} |\overset{h}{\nabla}u|^2 d\Sigma.$$

This inequality is obtained for $m \ge 1$. But it holds for $m=0$ too, as follows from (4.7.5).

Together with (4.7.3) and (4.7.4), the last inequality gives

$$\int\limits_{\Omega M} R_{ijkl}\xi^i\xi^k \overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u\, d\Sigma \le k^+(m+1)(n+2m-2) \int\limits_{\Omega M} (Hu)^2 d\Sigma + {}$$
$$+ k^+(m+1) \int\limits_{\Omega M} |\overset{h}{\nabla}u|^2 d\Sigma. \tag{4.7.8}$$

Estimating the integral on the right-hand side of (4.7.2) with the help of (4.7.8), we arrive at the inequality

$$[1 - k^+(m+1)] \left(\int\limits_{\Omega M} |\overset{h}{\nabla}u|^2 d\Sigma + (n+2m-2) \int\limits_{\Omega M} (Hu)^2 d\Sigma \right) \le$$
$$\le C\left(m\|If\|_0 \cdot \|j_\nu f|_{\partial M}\|_0 + \|If\|_1^2 \right).$$

Under assumption (4.3.4) of Theorem 4.3.3, the quantity in the brackets is positive and we obtain the estimate

$$\int\limits_{\Omega M} |\overset{h}{\nabla}u|^2 d\Sigma \le C_1(m\|If\|_0 \cdot \|j_\nu f|_{\partial M}\|_0 + \|If\|_1^2). \tag{4.7.9}$$

It remains to note that equation (4.6.4) implies the estimate

$$\|f\|_0^2 \le C_2 \int\limits_{\Omega M} |Hu|^2 d\Sigma \le C_2 \int\limits_{\Omega M} |\overset{h}{\nabla} u|^2 d\Sigma.$$

From (4.7.9) with the help of the last estimate we obtain (4.3.8). Lemma 4.3.4 is proved as well as Theorem 4.3.3.

4.8 Consequences for the nonlinear problem of determining a metric from its hodograph

We start with introducing the next definition that slightly differs from that of Section 1.1: a Riemannian metric on a compact manifold M with boundary is called *simple* if the boundary is strictly convex and every two points $p, q \in M$ are joint by a unique geodesic depending smoothly on p, q. According to the definition, a simple metric is dissipative.

Recall that in Section 1.1 we introduced the hodograph of a simple Riemannian metric and formulated Problem 1.1.1, that of determining metric from its hodograph. In the current section we shall prove the next two claims related to this problem.

Theorem 4.8.1 *Let g^τ $(0 \le \tau \le 1)$ be a one-parameter family, of simple Riemannian metrics on a compact manifold M, which depends smoothly on τ. If the hodograph Γ_{g^τ} is independent of τ and the metric g^0 satisfies the inequality $k^+(M, g^0) < 1/3$, then there exists a diffeomorphism $\varphi : M \to M$ such that $\varphi|_{\partial M} = \mathrm{Id}$ and $\varphi^* g^0 = g^1$.*

Theorem 4.8.2 *Let g^0, g^1 be two simple Riemannian metrics on a compact manifold M. If their hodographs coincide, g^0 satisfies the condition $k^+(M, g^0) < 1/3$ and g^1 is flat (i.e. with zero curvature tensor), then there exists a diffeomorphism $\varphi : M \to M$ such that $\varphi|_{\partial M} = \mathrm{Id}$ and $\varphi^* g^0 = g^1$.*

We say that a CDRM (M, g) satisfies Conjecture I_m if the conclusion of Theorem 4.3.3 is valid for tensor fields of degree m on this manifold, i.e., if the solenoidal part of every field $f \in C^\infty(S^m\tau'_M)$ is uniquely determined by the ray transform If. Let us show that Theorem 4.8.1 follows from the next claim.

Lemma 4.8.3 *Let g^τ $(0 \le \tau \le 1)$ be a one-parameter family of simple metrics on a compact manifold M. If every g^τ satisfies Conjecture I_2 and the hodograph Γ_{g^τ} is independent of τ, then there exists a diffeomorphism $\varphi : M \to M$ such that $\varphi|_{\partial M} = \mathrm{Id}$ and $\varphi^* g^0 = g^1$.*

Indeed, by Theorem 4.3.3, a metric g satisfies Conjecture I_2 if the condition $k^+(M, g) < 1/3$ holds. So, under the conditions of Theorem 4.8.1, there exists $\tau_0 > 0$ such that g^τ satisfies Conjecture I_2 for $0 \le \tau \le \tau_0$. Applying Lemma 4.8.3, we obtain a diffeomorphism $\varphi : M \to M$ such that $\varphi|_{\partial M} = \mathrm{Id}$ and $\varphi^* g^0 = g^{\tau_0}$. Since φ is an isometry of the Riemannian manifold (M, g^0) onto (M, g^{τ_0}), we have $k^+(M, g^{\tau_0}) = k^+(M, g^0) < 1/3$. Now we can repeat the same reasoning for $\tau \in [\tau_0, 1]$, and the proof of Theorem 4.8.1 is accomplished evidently.

Similarly, Theorem 4.8.2 follows from the next claim:

Lemma 4.8.4 *Let g^0, g^1 be two simple metrics on a compact manifold M. Suppose that one of them is flat and the other satisfies Conjectures I_1 and I_2. If their hodographs coincide, then there exists a diffeomorphism $\varphi : M \to M$ such $\varphi|_{\partial M} = \mathrm{Id}$ and $\varphi^* g^0 = g^1$.*

P r o o f of Lemma 4.8.3. Repeating the arguments after the formulation of Problem 1.1.1, we make sure that $I^\tau(\partial g^\tau/\partial\tau) = 0$ for all τ, where I^τ is the ray transform corresponding to the metric g^τ. According to Conjecture I_2, this implies, for every τ, existence of a field $v^\tau \in H^2(\tau'_M)$ such that

$$-2d^\tau v^\tau = \partial g^\tau/\partial\tau, \qquad v^\tau|_{\partial M} = 0. \tag{4.8.1}$$

It follows from (4.8.1) that $v^\tau(x)$ is smooth in $(x, \tau) \in M \times [0, 1]$. Indeed, as was noted in Section 2.5, the operator of inner differentiation d^τ is elliptic (it does not matter that in Section 2.5 operator d is considered for the Euclidean metric and here, for a Riemannian one; since its symbol is the same in the two cases). It is known that an elliptic operator is hypoelliptic [111], i.e., smoothness of the right-hand side of equation (4.8.1) implies smoothness of a solution v^τ.

We consider the system of ordinary differential equations $dy^i/d\tau = (g^\tau)^{ij}(y)v^\tau_j(y, \tau)$ on M. The system has the solution $\varphi^i(x, \tau)$ defined for $0 \le \tau \le 1$ and satisfying the initial condition $\varphi(x, 0) = x$. By boundary condition (4.8.1), the diffeomorphism $\varphi^\tau : x \mapsto \varphi(x, \tau)$ is identical on ∂M. Converting the argument that led us to (4.8.1), we see that $g^0 = (\varphi^\tau)^* g^\tau$. The lemma is proved.

In the remainder of the section we shall show that Problem 1.1.1 is equivalent to some nonlinear inverse problem for a system of kinetic equations. If one of the two metrics is flat, then the above-mentioned problem is linear and we obtain the claim of Lemma 4.8.4. Besides the proof of Lemma 4.8.4, this problem is of some independent interest, to the author's opinion.

Lemma 4.8.5 *Let a CDRM M satisfies Conjecture I_m. If a function $w(x, \xi)$ continuous on ΩM and smooth on $\Omega M \setminus \partial\Omega M$ is a solution of the boundary problem*

$$Hw = f_{i_1\dots i_m}(x)\xi^{i_1}\dots\xi^{i_m}, \quad w|_{x\in\partial M} = v_{i_1\dots i_{m-1}}(x)\xi^{i_1}\dots\xi^{i_{m-1}},$$

then $w(x, \xi)$ is a homogeneous polynomial of degree $m - 1$ in ξ.

P r o o f. We extend v to a field continuous on M and smooth on $M \setminus \partial M$ and put $\tilde{w} = w - v_{i_1\dots i_{m-1}}(x)\xi^{i_1}\dots\xi^{i_{m-1}}$. Then $\tilde{w}$ is a solution to the boundary problem

$$H\tilde{w} = \tilde{f}_{i_1\dots i_m}(x)\xi^{i_1}\dots\xi^{i_m}, \quad \tilde{w}|_{\partial\Omega M} = 0. \tag{4.8.2}$$

Converting the argument that led us to (4.6.2)–(4.6.4), we see that (4.8.2) implies the equality $I\tilde{f} = 0$. By Conjecture I_m, the last equality implies existence of a field $\tilde{v}$ such that $d\tilde{v} = \tilde{f}$ and $\tilde{v}|_{\partial M} = 0$. The function $\tilde{v}_{i_1\dots i_{m-1}}(x)\xi^{i_1}\dots\xi^{i_{m-1}}$ is a solution to the boundary problem (4.8.2) and, consequently, coincides with $\tilde{w}$. The lemma is proved.

The hodograph $\Gamma_g(x, y)$ of a simple metric is a smooth function for $x \ne y$. For $x, y \in \partial M$, let $a = \Gamma_g(x, y); \gamma : [0, a] \to M$ be a geodesic such that $\gamma(0) = x, \gamma(a) = y$. By the formula for the first variation of the length of a geodesic [41], the next equalities hold:

$$\begin{aligned} \langle\dot\gamma(0), \xi\rangle &= -\partial\Gamma_g(x, y)/\partial\xi \qquad \text{for} \quad \xi \in T_x(\partial M), \\ \langle\dot\gamma(a), \xi\rangle &= \partial\Gamma_g(x, y)/\partial\xi \qquad \text{for} \quad \xi \in T_y(\partial M), \end{aligned} \tag{4.8.3}$$

which mean that the angles, at which γ intersects ∂M, are uniquely determined by the hodograph.

Let g^0 and $\tilde g^1$ be two simple metrics whose hodographs coincide. These metrics induce the same Riemannian metric on ∂M, i.e., $g^0_{ij}(x)\xi^i\xi^j = \tilde g^1_{ij}(x)\xi^i\xi^j$ for $(x,\xi)\in T(\partial M)$. By ν^0 and $\tilde\nu^1$ we denote the unit outer normal vectors to ∂M with respect to these metrics. Let us find a diffeomorphism $\psi : M \to M$ such that $\psi|_{\partial M} = \mathrm{Id}$ and $(d\psi)\nu^0 = \tilde\nu^1$, and put $g^1 = \psi^*\tilde g^1$. Then $\Gamma_{g^0} = \Gamma_{g^1}$ and the metrics g^0, g^1 coincide on ∂M :

$$g^0_{ij}(x)\xi^i\xi^j = g^1_{ij}(x)\xi^i\xi^j \quad \text{for} \quad x\in\partial M,\ \xi\in T_xM. \tag{4.8.4}$$

If Problem 1.1.1 has a positive answer for g^0, g^1, then it has also a positive answer for the initial pair $g^0, \tilde g^1$. Thus in studying the problem it suffices to consider pairs of metrics meeting condition (4.8.4).

Lemma 4.8.6 *Let g^0, g^1 be two simple metrics such that their hodographs coincide and condition* (4.8.4) *is satisfied. For $x\in\partial M$ and $0\neq\xi\in T_xM$, if $\gamma^\alpha : [0,a^\alpha]\to M$ is the maximal geodesic, of the metric g^α $(\alpha = 0,1)$, defined by the initial conditions $\gamma^\alpha(0) = x$ and $\dot\gamma^\alpha(0) = \xi$; then the equalities $a^0 = a^1$, $\gamma^0(a^0) = \gamma^1(a^1)$ and $\dot\gamma^0(a^0) = \dot\gamma^1(a^1)$ are valid.*

P r o o f. We put $|\xi| = (g^\alpha_{ij}\xi^i\xi^j)^{1/2}$, $y = \gamma^0(a^0)$. Then $|\xi|a^0 = \Gamma_{g^\alpha}(x,y)$. Since g^1 is simple, there exists a geodesic $\tilde\gamma^1 : [0,a^0]\to M$ of this metric such that $\tilde\gamma^1(0) = x$ and $\tilde\gamma^1(a^0) = y$. The length of the geodesic $\tilde\gamma^1$ is equal to $|\xi|a^0$ and, consequently,

$$|\dot{\tilde\gamma}^1(0)| = |\dot{\tilde\gamma}^1(a^0)| = |\xi| = |\dot\gamma^0(0)| = |\dot\gamma^0(a^0)|. \tag{4.8.5}$$

By (4.8.3), the angles between any vector $\eta\in T_x(\partial M)$ $(\eta\in T_y(\partial M))$ and the vectors $\dot\gamma^0(0)$, $\dot{\tilde\gamma}^1(0)$ (the vectors $\dot\gamma^0(a^0)$, $\dot{\tilde\gamma}^1(a^0)$) are equal. Together with (4.8.5), this gives

$$\xi = \dot\gamma^0(0) = \dot{\tilde\gamma}^1(0), \quad \dot\gamma^0(a^0) = \dot{\tilde\gamma}^1(a^0). \tag{4.8.6}$$

Both the mappings $\gamma^1 : [0,a^1]\to M$ and $\tilde\gamma^1 : [0,a^0]\to M$ are the maximal geodesics of the metric g^1 and satisfy the same initial conditions $\gamma^1(0) = \tilde\gamma^1(0) = x$ and $\dot\gamma^1(0) = \dot{\tilde\gamma}^1(0) = \xi$. Consequently, they coincide, i.e., $a^0 = a^1$ and $\gamma^1(t)\equiv\tilde\gamma^1(t)$. Now (4.8.6) implies the claim of the lemma.

Remark. The function $\tau_+ : \partial_-\Omega M\to\mathbf{R}$ introduced in Section 4.1 can be called the *angle hodograph* of a dissipative metric g. Lemma 4.8.6 means that, if the hodographs of two simple metrics coincide, then their angle hodographs coincide too. The author does not know whether the converse assertion holds. It is just for this reason, that, investigating Problem 1.1.1, we restrict ourselves to considering only simple metrics and not formulating this problem for the class of dissipative metrics.

Let g^0 and g^1 be two simple metrics, on a compact manifold M with boundary, such that their hodographs coincide and condition (4.8.4) is satisfied. Let us construct a diffeomorphism $\Phi : T^0M\to T^0M$ in the following way. Given $(x,\xi)\in T^0M$, let $\gamma^0_{x,\xi} : [a,b]\to M$ be a maximal geodesic of the metric g^0 satisfying the initial conditions $\gamma^0_{x,\xi}(0) = x$ and $\dot\gamma^0_{x,\xi}(0) = \xi$. We put $y = \gamma^0_{x,\xi}(a)$ and $\eta = \dot\gamma^0_{x,\xi}(a)$. Then $y\in\partial M$ and $\langle\eta,\nu(y)\rangle\le 0$. Let $\gamma^1_{y,\eta}(t)$ be the maximal geodesic, of the metric g^1, satisfying the conditions $\gamma^1_{y,\eta}(a) = y$ and $\dot\gamma^1_{y,\eta}(a) = \eta$. By Lemma 4.8.6, such a geodesic is defined on $[a,b]$. We put $\Phi(x,\xi) = (\gamma^1_{y,\eta}(0), \dot\gamma^1_{y,\eta}(0))$. Changing the roles of g^0 and g^1, we see that Φ is a diffeomorphism.

By Lemma 4.8.6, the so-constructed diffeomorphism Φ satisfies the boundary condition

$$\Phi(x,\xi) = (x,\xi) \quad \text{for} \quad x \in \partial M. \tag{4.8.7}$$

It follows from the definition of Φ that it transfers the geodesic flow of the metric g^0 into the geodesic flow of the metric g^1. Consequently, the geodesic vector fields H^0 and H^1 of these metrics are Φ-connected.

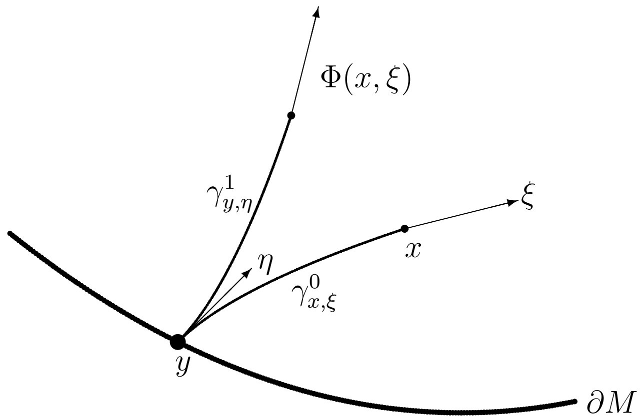

Fig. 2

We will assume for simplicity that some global coordinate system can be introduced over M. Let $\Phi(x,\xi) = (y(x,\xi), \eta(x,\xi))$ be the coordinate form of Φ. Writing down the condition of Φ-connectedness of H^0 and H^1 in coordinate form, we arrive at the system of equations

$$H^0 y^i = \eta^i, \qquad H^0 \eta^i = -\overset{1}{\Gamma}{}^i_{jk}(y)\eta^j\eta^k. \tag{4.8.8}$$

By (4.8.7), the boundary conditions

$$y^i(x,\xi) = x^i, \quad \eta^i(x,\xi) = \xi^i \quad \text{for} \quad x \in \partial M \tag{4.8.9}$$

hold. If Problem 1.1.1 has a positive answer for the given pair of metrics, then $y(x,\xi)$ is independent of ξ and the mapping $x \mapsto y(x)$ coincides with the diffeomorphism φ participating in the formulation of the problem. Thus we arrive at the next question equivalent to Problem 1.1.1: is it true that, for any solution $y(x,\xi), \eta(x,\xi)$ of boundary problem (4.8.8)–(4.8.9), the functions y^i are independent of ξ?

P r o o f of Lemma 4.8.4. If the metric is flat, then in some neighbourhood of every point a coordinate system can be introduced such that its Christoffel symbols vanish everywhere. Given the supplementary condition of simplicity of the metric, one can easily see that such a coordinate system exists on the whole of M globally. So for flat g^1, system (4.8.8) simplifies as follows:

$$H^0 y^i = \eta^i, \qquad H^0 \eta^i = 0. \tag{4.8.10}$$

If g^0 satisfies Conjectures I_1 and I_2, then (4.8.9) and (4.8.10) imply with the help of Lemma 4.8.5 that the functions $\eta^i(x,\xi)$ are linear in ξ and $y^i(x,\xi)$ are independent of ξ. Thus we obtain the diffeomorphism $\varphi : M \to M,\ \varphi : x \mapsto y(x)$ satisfying the boundary condition $\varphi|_{\partial M} = \mathrm{Id}$. The first of the equations (4.8.10) now takes the form $\eta = d_x\varphi(\xi)$ and, consequently, the above-constructed diffeomorphism Φ looks like: $\Phi(x,\xi) = (\varphi(x), d_x\varphi(\xi))$. As we have seen, Φ transforms the geodesic flow of the metric g^0 into the same of g^1. Since a geodesic flow preserves the length of a tangent vector and these metrics coincide on ∂M, we obtain $\varphi^* g^0 = g^1$. The lemma is proved.

4.9 Bibliographical remarks

The general scheme of the method used in the proof of Theorem 4.3.3 is known in mathematical physics for a long time under the name of the method of energy estimates or the method of quadratic integrals. At first, for classical equations, the main relations of the method had a physical sense of energy integrals. While being extended later to wider classes of equations, the method was treated in a more formal manner [30]. Roughly speaking, the principal idea of the method can be explained as follows: given a differential operator D, we try to find another differential operator L such that the product $LuDu$ can be decomposed into the sum of two summands in such a way that the first summand is presented in divergence form and the second one is a positive-definite quadratic form in the higher derivatives of the function u. The Pestov identity (4.4.9) is an example of such decomposition. The first summand on the right-hand side of (4.4.9) is a positive-definite quadratic form in derivatives $\partial u/\partial x^i$, the second and third summands are of divergence form while the last summand is considered, from the viewpoint of the method, as an undesirable term.

In integral geometry the method was at first applied by R. G. Mukhometov [83, 84, 85] to a two-dimensional problem. Thereafter this approach to integral geometry problems was developed by R. G. Mukhometov himself [86, 87, 88, 89] as well as others [7, 10, 12, 90, 108]. For multidimensional problems, the method have obtained a rather complicated form, so almost every of the mentioned articles is not easy for reading. In this series, some papers due to A. Kh. Amirov [3, 4, 5, 6] can be distinguished where some new ideas have arisen.

The first and foremost difference between Mukhometov's approach and Amirov's one, which determines other distinctions, is the choice of the coordinates on the manifold ΩM. A. Kh. Amirov uses the same coordinates $(x,\xi) = (x^1,\dots,x^n,\xi^1,\dots,\xi^n)$ as we have used in this chapter, while R. G. Mukhometov uses the coordinates $(x^1,\dots,x^n,z^1,\dots,z^{n-1})$, where $z \in \partial M$ is a point at which the geodesic $\gamma_{x,\xi}$ meets the boundary. Each of these coordinate systems has its own merits and demerits. For instance, being written in the coordinates (x,z), the kinetic equation is of a more simple structure (dose not contain the derivatives with respect to z). But at the same time, if the right-hand side of the equation depends on ξ in some way (for instance, in this chapter we are interested in the polynomial dependence on ξ), then in the coordinates (x,z) the dependence obtains very complicated character. On the other hand, using the coordinates (x,ξ), there is no problem with the right-hand side. But at the same time, since the equation contains the derivatives with respect to ξ, to apply the method successfully, we have to impose some assumptions, on the coefficients of the equation, which require positive definiteness for a quadratic form.

In [102] L. N. Pestov and V. A. Sharafutdinov implemented the method in covariant terms and demonstrated that, in this approach, the mentioned assumptions on the coefficients of the equation turn into the requirement of nonpositivity for the sectional curvature. Of course, the last requirement is more sensible from a geometrical standpoint.

In [116] the author used the coordinates (x, η) that differ from (x, ξ) as well as from (x, z). Namely η is the vector tangent to the geodesic $\gamma_{x,\xi}$ at the point z. By means of these coordinates, the claim of Theorem 4.3.3 was obtained for a domain $M \subset \mathbf{R}^n$ and for metrics C^2-close to that of Euclidean space. The system (x, η) has the same advantage as (x, z), i.e., the kinetic equation does not contain the derivatives with respect to η. At the same time, if a metric is close to that of Euclidean space, the coordinates (x, η) and (x, ξ) are close. The last circumstance makes application of the method a full success.

Finally, in [121] the author noticed that the Poincaré inequality allows one to obtain estimate (4.7.8) that leads to Theorem 4.3.3 which unites and essentially strengthens the results of [102] and [116]. Note that this observation is of a rather general nature, i.e., it can be applied to other kinds of the kinetic equation, as we will see in the next chapters. On the other hand, this observation allows one to extend essentially the scope of the method, since making it possible to replace the assumptions of positive definiteness of a quadratic form by conditions of the type "of a slightly perturbed system". The last conditions often turns out to be more acceptable for a physical interpretation.

Theorem 4.8.1 generalizes essentially a theorem due to R. Michel [80]. He proved the same claim under the assumption that there exists a function $K_0(x) > 0$ such that the sectional curvature satisfies the inequalities $-1/n < K(x, \sigma)/K_0(x) + 1 < 1/n$ where $n = \dim M$.

Theorem 4.8.2 is a rather special case of a result due to M. Gromov [42]. He obtained the same claim without the condition $k^+(M, g) < 1/3$. A simple proof of the claim is presented in the paper [22] by C. B. Croke.

Chapter 5

The transverse ray transform

It is known that anisotropy of each of the medium characteristics, the dielectric and magnetic permeabilities, and conductivity, is followed by polarization effects for an electromagnetic wave propagating in the medium. A similar assertion is valid for elastic waves in a solid body. A possibility arises of detection and quantitative estimation, of the anisotropy of a medium contained in a bounded domain, by comparing the degrees of polarization of the incoming and outcoming waves. In the framework of geometrical optics, evolution of the polarization ellipse is usually described by a system of ordinary differential equations which connects the values, on a light ray, of the electromagnetic field and of the sought medium characteristic (for definiteness, we say here on electromagnetic waves, although similar consideration is possible for elastic waves too). Consequently, for every ray, the degree of polarization of the outcoming wave depends only on the values of the medium characteristic on the ray, i.e., our problem is of a tomographic nature. Thus, to settle the problem it is natural to use tomographic methods of measurements, i.e., to measure polarization for each ray in a sufficiently large family. By polarization tomography the author means the combination of methods for measurements and data processing which arises in such a way. The domain of potential applications of polarization tomography is rather spacious: lightconductor optics, plasma physics, the earthquake prediction problem and many others. There relate photoelasticity problems one of which is considered in Section 2.16.

The mathematical nature of polarization tomography problems differs radically from that of classical tomography. The difference is responsible for two next circumstances. First, the polarization effects are due to the vector nature of waves. Consequently, it is unnatural and mostly impossible to reduce the polarization tomography problem to that of finding a scalar field, i.e., our problems are concerned to vector tomography as a matter of fact. The second feature is related to the transverse character of electromagnetic waves and is described as follows. The result of each measurement is "an integral along a ray" with the integrand depending only on the component, of the desired medium characteristic, which is orthogonal to the ray.

The first section of a physical nature has the only purpose of demonstrating that the main notion of the current chapter, the transverse ray transform, relates to the above-mentioned feature of polarization tomography rather than presents a product of idle speculation. In Section 5.2 we define the transverse ray transform on a compact dissipative Riemannian manifold and formulate the main result of the chapter, Theorem 5.2.2, which asserts that the transverse ray transform is invertible in sufficiently simple cases and admits a stable estimate. The remainder of the chapter is devoted to the proof of this theorem.

5.1 Electromagnetic waves in quasi-isotropic media

In this section we derive the equations, firstly obtained by Yu. A. Kravtsov [62, 63], describing evolution of the polarization ellipse along a light ray under the assumption that optical anisotropy of the media is comparable with the wavelength. Yu. A. Kravtsov proposed to call such media quasi-isotropic (the term "slightly anisotropic media" is also used). Our exposition as compared with that of [63] differs in systematical use of curvilinear coordinates and covariant differentiation. We shall first obtain the equations in Riemannian form and then show that their Euclidean form coincides with Kravtsov's equations.

At the end of the section we will briefly discuss the nonlinear inverse problem of determining the anisotropic part of the dielectric permeability tensor and show that the linearization of this problem produces a problem of inverting the transverse ray transform for a tensor field of degree 2.

5.1.1 The Maxwell equations

We consider the homogeneous Maxwell system (i.e., on assuming the absence of charges and currents)

$$\begin{aligned} &\operatorname{rot}\mathcal{H} - \frac{1}{c}\frac{\partial}{\partial t}\mathcal{D} = 0, \qquad \operatorname{div}\mathcal{D} = 0,\\ &\operatorname{rot}\mathcal{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathcal{B} = 0, \qquad \operatorname{div}\mathcal{B} = 0, \end{aligned} \tag{5.1.1}$$

complemented by the material equations

$$\mathcal{D}^j = \varepsilon^j_l \mathcal{E}^l, \qquad \mathcal{B} = \mathcal{H}. \tag{5.1.2}$$

The magnetic permeability of the medium under study is assumed to be equal to unity. Thus the only medium characteristic is the dielectric permeability tensor $(\varepsilon^j_l(x))$ that is assumed to depend smoothly on a point $x \in \mathbf{R}^3$.

We restrict ourselves to considering fields that depend harmonically on the time:

$$\mathcal{E}(x,t) = E(x)\, e^{-i\omega t}, \quad \mathcal{D}(x,t) = D(x)\, e^{-i\omega t}, \quad \mathcal{H}(x,t) = H(x)\, e^{-i\omega t}.$$

In this case system (5.1.1)–(5.1.2) amounts to the following

$$\operatorname{rot} H + ikD = 0, \qquad \operatorname{rot} E - ikH = 0, \tag{5.1.3}$$

$$D^j = \varepsilon^j_l E^l. \tag{5.1.4}$$

where $k = \omega/c$ is the wave number,

We will assume that the tensor $\varepsilon^j_l(x)$ is representable as

$$\varepsilon^j_l = \varepsilon\delta^j_l + \frac{1}{k}\chi^j_l, \tag{5.1.5}$$

where $\varepsilon = \varepsilon(x)$ is a smooth positive function. Representation (5.1.5) means that the anisotropy of the medium is comparable in value with the wavelength. Such a medium is called *quasi-isotropic*. In particular, in the case $\chi \equiv 0$ we have an isotropic medium. The function $n = \sqrt{\varepsilon}$ is called the *refraction coefficient*.

The operator rot presumes that the space $\mathbf{R}^3$ is oriented. Therefore in the current section we will use only coordinate systems that are compatible with the orientation. Let x^1, x^2, x^3 be a curviliner coordinate system, and the Euclidean metric is expressed by the quadratic form

$$ds^2 = g_{jk} dx^j dx^k \tag{5.1.6}$$

in these coordinates. As usual, we denote $g = \det(g_{jk})$. The operator rot is written in coordinate form as:

$$(\mathrm{rot}\, a)^j = e^{jkl} a_{k\,;\,l}, \tag{5.1.7}$$

where (e^{jkl}) is the *discriminant tensor* skew-symmetric in each pair of its indices and such that $e^{123} = g^{-1/2}$. One can easily see that these properties actually determine a tensor, i.e., that its components are transformed by formulas (3.1.6) if coordinates are changed in an orientation preserving way.

With the help of the discriminant tensor, system (5.1.3)–(5.1.5) is written as:

$$e^{jkl} H_{k\,;\,l} + ikD^j = 0, \tag{5.1.8}$$

$$e^{jkl} E_{k\,;\,l} - ikH^j = 0, \tag{5.1.9}$$

$$D_j = \varepsilon E_j + \frac{1}{k} \chi_{jk} E^k. \tag{5.1.10}$$

Recall that we use the rule of raising and lowering indices: $E_j = g_{jk} E^k$, $E^j = g^{jk} E_k$; The covariant derivatives participating in (5.1.8)–(5.1.9) are taken with respect to the Euclidean metric (5.1.6).

5.1.2 The eikonal equation

Some version of the method of geometrical optics (or the ray method) is based on the expansion of each of the fields $A = E, H, D$ into the asymptotic series of the type

$$A(x) = e^{ik\tau(x)} \sum_{m=0}^{\infty} \frac{\overset{m}{A}(x)}{(ik)^m}, \tag{5.1.11}$$

which are assumed to be valid in the topology of the space $\mathcal{E}(\mathbf{R}^3)$. We insert series (5.1.11) into equations (5.1.8)–(5.1.10), implement differentiations and equate the coefficients of the same powers of the wave number k on the left- and right-hand sides of the so-obtained equalities. In such a way we arrive at the infinite system of equations

$$\left.\begin{array}{l} e^{jkl}\left(\overset{m}{H}_k \tau_{;\,l} + \overset{m-1}{H}_{k\,;\,l}\right) + \varepsilon \overset{m}{E}{}^j + i\chi^{jk} \overset{m-1}{E}_k = 0 \\ e^{jkl}\left(\overset{m}{E}_k \tau_{;\,l} + \overset{m-1}{E}_{k\,;\,l}\right) - \overset{m}{H}{}^j = 0 \end{array}\right\} \quad (m = 0, 1, \ldots), \tag{5.1.12}$$

where it is assumed that $\overset{-1}{E} = \overset{-1}{H} = 0$.

Consider the initial terms of this chain. Putting $m = 0$ in (5.1.12), we have

$$e^{jkl}\overset{0}{H}_k\tau_{;\,l} + \varepsilon\overset{0}{E}{}^j = 0, \tag{5.1.13}$$

$$e^{jkl}\overset{0}{E}_k\tau_{;\,l} - \overset{0}{H}{}^j = 0. \tag{5.1.14}$$

We see that χ does not participate in (5.1.13)–(5.1.14). So the forthcoming consequences of these equations are identical for isotropic and quasi-isotropic media. Inserting expression (5.1.14) for the vector $\overset{0}{H}$ into (5.1.13), we have

$$e^{jkl}e_{kpq}\overset{0}{E}{}^p\tau_{;\,l}\tau_{;}{}^{q} + \varepsilon\overset{0}{E}{}^j = 0.$$

One can easily verify the relation

$$e^{jkl}e_{kpq} = \delta^j_q\delta^l_p - \delta^p_j\delta^l_q$$

which allows us to rewrite the previous equation as:

$$\overset{0}{E}{}^l\tau_{;\,l}\tau_{;}{}^{j} - \overset{0}{E}{}^j\tau_{;}{}^{l}\tau_{;\,l} + \varepsilon\overset{0}{E}{}^j = 0,$$

or, in invariant form, as:

$$\langle\overset{0}{E}, \nabla\tau\rangle\nabla\tau + \left(\varepsilon - |\nabla\tau|^2\right)\overset{0}{E} = 0$$

where $\nabla\tau$ is the gradient of τ. Taking the scalar product of the last equality and $\nabla\tau$, we obtain

$$\langle\overset{0}{E}, \nabla\tau\rangle = 0. \tag{5.1.15}$$

The previous equality now takes the form $(\varepsilon - |\nabla\tau|^2)\,\overset{0}{E} = 0$. Since we are interested in the case when system (5.1.13)–(5.1.14) has a nontrivial solution, the function τ must satisfy the *eikonal equation*:

$$|\nabla\tau|^2 = n^2 = \varepsilon. \tag{5.1.16}$$

The eikonal equation characteristics (rays) are geodesics of the Riemannian metric

$$d\ell^{\,2} = n^2\,ds^2 = h_{jk}\,dx^j\,dx^k, \qquad h_{jk} = n^2 g_{jk}, \tag{5.1.17}$$

Note also that (5.1.14) implies the relations

$$\langle\overset{0}{H}, \nabla\tau\rangle = \langle\overset{0}{E}, \overset{0}{H}\rangle = 0. \tag{5.1.18}$$

Equalities (5.1.15) and (5.1.18) express the well-known fact of the transverse nature of an electromagnetic wave within the scope of the zero approximation of geometrical optics.

5.1.3 The amplitude of an electromagnetic wave

We fix a solution τ to the eikonal equation (5.1.16). In a neighbourhood of each point one can introduce so-called *ray coordinates*, i.e., curvilinear coordinates x^1, x^2, x^3 such that $x^3 = \tau$ and the coordinate surfaces $x^3 = x_0^3$ are orthogonal to the coordinate lines $x^1 = x_0^1$, $x^2 = x_0^2$ that are the geodesics of metric (5.1.17). In these coordinates the Euclidean metric (5.1.6) has the form

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + n^{-2}d\tau^2. \tag{5.1.19}$$

From now on in the current section Greek indices range over the values 1,2; on repeating Greek indices, summation from 1 to 2 is understood. The gradient of τ has the coordinates

$$\tau_{;\,\alpha} = \tau_{;}{}^{\alpha} = 0, \quad \tau_{;\,3} = 1, \quad \tau_{;}{}^{3} = n^2. \tag{5.1.20}$$

By (3.2.13), the Christoffel symbols of the Euclidean metric in coordinates (5.1.19) look as:

$$\begin{aligned} &\Gamma^3_{\alpha\beta} = -\frac{1}{2}n^2\frac{\partial g_{\alpha\beta}}{\partial\tau}, \qquad \Gamma^\beta_{\alpha 3} = \frac{1}{2}g^{\beta\gamma}\frac{\partial g_{\alpha\gamma}}{\partial\tau}, \qquad \Gamma^3_{\alpha 3} = -n^{-1}\frac{\partial n}{\partial x^\alpha}, \\ &\Gamma^\alpha_{33} = n^{-3}g^{\alpha\beta}\frac{\partial n}{\partial x^\beta}, \qquad \Gamma^3_{33} = -n^{-1}\frac{\partial n}{\partial\tau}. \end{aligned} \tag{5.1.21}$$

We introduce the notations

$$J^2 = \det(g_{\alpha\beta}), \quad g = \det(g_{jk}) = n^{-2}J^2. \tag{5.1.22}$$

The quantity J is called the *geometrical spreading* . It is invariant in the following sense: if the ray coordinates under studying are transformed so that they remain ray coordinates, then J is multiplied by some factor constant on every ray $x^\alpha = x_0^\alpha$. By (5.1.21) and (3.6.9),

$$g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial\tau} = 2\Gamma^\alpha_{\alpha 3} = 2\Gamma^j_{j3} - 2\Gamma^3_{33} = \frac{1}{g}\frac{\partial g}{\partial\tau} + 2n^{-1}\frac{\partial n}{\partial\tau} = \frac{2}{J}\frac{\partial J}{\partial\tau}. \tag{5.1.23}$$

We also need the equalities

$$g^{11} = J^{-2}g_{22}, \quad g^{22} = J^{-2}g_{11}, \quad g^{12} = -J^{-2}g_{12}. \tag{5.1.24}$$

In ray coordinates equations (5.1.13)–(5.1.14) of the zero approximation look as:

$$e^{\alpha\beta 3}\overset{0}{H}_\beta + \varepsilon\overset{0}{E}{}^\alpha = 0, \tag{5.1.25}$$

$$e^{\alpha\beta 3}\overset{0}{E}_\beta - \overset{0}{H}{}^\alpha = 0. \tag{5.1.26}$$

It follows from (5.1.18), (5.1.19) and (5.1.26) that, in ray coordinates, the components of the vectors $\overset{0}{E}$ and $\overset{0}{H}$ satisfy the relations

$$\overset{0}{E}{}^3 = \overset{0}{E}_3 = \overset{0}{H}{}^3 = \overset{0}{H}_3 = 0, \tag{5.1.27}$$

$$\overset{0}{H}{}^1 = g^{-1/2}\overset{0}{E}_2, \qquad \overset{0}{H}{}^2 = -g^{-1/2}\overset{0}{E}_1. \tag{5.1.28}$$

Conversely, if $\overset{0}{E}_1$, and $\overset{0}{E}_2$ are arbitrary functions, then we obtain a solution to system (5.1.25)–(5.1.26), defining $\overset{0}{H}_1$ and $\overset{0}{H}_2$ by equalities (5.1.28). Thus the equations of the zero approximation determine the vector fields $\overset{0}{E}$ and $\overset{0}{H}$ up to the functions $\overset{0}{E}_1$ and $\overset{0}{E}_2$ that remain arbitrary. Differential equations for these functions can be obtained from the equations of the first approximation, that is equations (5.1.12) for $m = 1$, taking the next form in ray coordinates:

$$\varepsilon\overset{1}{E}{}^3 = -e^{3\alpha\beta}\overset{0}{H}_{\alpha\,;\,\beta} - i\chi^{3\alpha}\overset{0}{E}_\alpha, \qquad \overset{1}{H}{}^3 = e^{3\alpha\beta}\overset{0}{E}_{\alpha\,;\,\beta}. \tag{5.1.29}$$

$$e^{\alpha\beta 3}\overset{1}{H}_\beta + \varepsilon\overset{1}{E}{}^\alpha = -e^{\alpha kl}\overset{0}{H}_{k\,;\,l} - i\chi^{\alpha\beta}\overset{0}{E}_\beta, \tag{5.1.30}$$

$$e^{\alpha\beta 3}\overset{1}{E}_\beta - \overset{1}{H}{}^\alpha = -e^{\alpha kl}\overset{0}{E}_{k\,;\,l} \tag{5.1.31}$$

Indeed, equations (5.1.30)–(5.1.31) can be considered as a linear algebraic system of 4 equations with respect to 4 unknown variables $\overset{1}{E}_1, \overset{1}{E}_2, \overset{1}{H}_1, \overset{1}{H}_2$. Note that the matrix of this system coincides with the matrix of the system (5.1.25)–(5.1.26). The eikonal equation (5.1.16) can be considered as the condition that the determinant of the system (5.1.25)–(5.1.26) vanishes. It is essential that the rank of the system (5.1.25)–(5.1.26) is equal to 2 provided (5.1.16) holds. So, for solvability of system (5.1.30)–(5.1.31), its right-hand side must satisfy some pair of linear conditions. The last conditions will give us both desired differential equations on the functions $\overset{0}{E}_1$ and $\overset{0}{E}_2$.

We start implementing the intended program with writing down equations (5.1.30)–(5.1.31) in a more detailed form. Putting $\alpha = 1$ and then $\alpha = 2$ in (5.1.30), we obtain the system

$$\begin{aligned} g^{-1/2}\overset{1}{H}_2 + \varepsilon\overset{1}{E}{}^1 &= g^{-1/2}\left(\overset{0}{H}_{3\,;\,2} - \overset{0}{H}_{2\,;\,3}\right) - i\chi^{1\alpha}\overset{0}{E}_\alpha \\ -g^{-1/2}\overset{1}{H}_1 + \varepsilon\overset{1}{E}{}^2 &= g^{-1/2}\left(\overset{0}{H}_{1\,;\,3} - \overset{0}{H}_{3\,;\,1}\right) - i\chi^{2\alpha}\overset{0}{E}_\alpha. \end{aligned} \tag{5.1.32}$$

Similarly, equation (5.1.31) gives

$$\begin{aligned} \overset{1}{H}{}^1 &= g^{-1/2}\overset{1}{E}_2 + g^{-1/2}\left(\overset{0}{E}_{2\,;\,3} - \overset{0}{E}_{3\,;\,2}\right), \\ \overset{1}{H}{}^2 &= -g^{-1/2}\overset{1}{E}_1 + g^{-1/2}\left(\overset{0}{E}_{3\,;\,1} - \overset{0}{E}_{1\,;\,3}\right). \end{aligned} \tag{5.1.33}$$

Inserting (5.1.33) into the equalities $\overset{1}{H}_\alpha = g_{\alpha\beta}\overset{1}{H}{}^\beta$ and using (5.1.24), we obtain

$$\begin{aligned} \overset{1}{H}_1 &= g^{1/2}n^2\left(\overset{1}{E}{}^2 + \overset{0}{E}{}^2{}_{;\,3} - \overset{0}{E}_{3\,;}{}^2\right), \\ \overset{1}{H}_2 &= g^{1/2}n^2\left(-\overset{1}{E}{}^1 + \overset{0}{E}_{3\,;}{}^1 - \overset{0}{E}{}^1{}_{;\,3}\right). \end{aligned} \tag{5.1.34}$$

Finally, inserting expressions (5.1.34) for the functions $\overset{1}{H}_1$ and $\overset{1}{H}_2$ into (5.1.32), we arrive at the desired differential equations for the vector fields $\overset{0}{E}$ and $\overset{0}{H}$:

$$\begin{aligned} n^2\left(\overset{0}{E}{}^1{}_{;\,3} - \overset{0}{E}_{3\,;}{}^1\right) + g^{-1/2}\left(\overset{0}{H}_{3\,;\,2} - \overset{0}{H}_{2\,;\,3}\right) &= i\chi^{1\alpha}\overset{0}{E}_\alpha, \\ n^2\left(\overset{0}{E}{}^2{}_{;\,3} - \overset{0}{E}_{3\,;}{}^2\right) - g^{-1/2}\left(\overset{0}{H}_{3\,;\,1} - \overset{0}{H}_{1\,;\,3}\right) &= i\chi^{2\alpha}\overset{0}{E}_\alpha, \end{aligned} \tag{5.1.35}$$

From now on we will omit the index 0 in the notations $\overset{0}{E}$ and $\overset{0}{H}$, since the higher amplitudes ($\overset{m}{E}$ and $\overset{m}{H}$ for $m > 0$) are not used further in the current section.

Let us exclude the vector H from system (5.1.35). To this end we first transform this system to a more convenient form. Putting $\beta = 1$, we multiply the first equation of the system by $g_{\beta 1}$; the second, by $g_{\beta 2}$ and sum the so-obtained equalities. Then we perform the same procedure with $\beta = 2$. After simple transformations using (5.1.22) and (5.1.24), we arrive at the next system equivalent to (5.1.35):

$$\begin{aligned} &E_{1\,;\,3} - E_{3\,;\,1} + g^{1/2}\,(H_{3\,;}{}^{2} - H^{2}{}_{;\,3}) = in^{-2}\chi_1^{\alpha}E_\alpha, \\ &E_{2\,;\,3} - E_{3\,;\,2} - g^{1/2}\,(H_{3\,;}{}^{1} - H^{1}{}_{;\,3}) = in^{-2}\chi_2^{\alpha}E_\alpha. \end{aligned} \tag{5.1.36}$$

Since $E_3 \equiv 0$,

$$E_{\alpha\,;\,3} - E_{3\,;\,\alpha} = \frac{\partial E_\alpha}{\partial \tau}. \tag{5.1.37}$$

From (5.1.28) with the help of (3.6.9), we obtain

$$\begin{aligned} H^{1}{}_{;\,3} &= \frac{\partial (g^{-1/2}E_2)}{\partial \tau} + g^{-1/2}\left(\Gamma^1_{31}E_2 - \Gamma^1_{32}E_1\right) = \\ &= g^{-1/2}\left[\frac{\partial E_2}{\partial \tau} - \Gamma^1_{23}E_1 - \left(\Gamma^2_{23} + \Gamma^3_{33}\right)E_2\right], \end{aligned}$$

$$\begin{aligned} H^{2}{}_{;\,3} &= -\frac{\partial (g^{-1/2}E_1)}{\partial \tau} + g^{-1/2}\left(\Gamma^2_{31}E_2 - \Gamma^2_{32}E_1\right) = \\ &= g^{-1/2}\left[-\frac{\partial E_1}{\partial \tau} + \left(\Gamma^1_{13} + \Gamma^3_{33}\right)E_1 + \Gamma^2_{13}E_2\right]. \end{aligned}$$

Inserting values (5.1.21) of the Christoffel symbols into the last equalities, we have

$$\begin{aligned} H^{1}{}_{;\,3} &= g^{-1/2}\left[\frac{\partial E_2}{\partial \tau} - \frac{1}{2}g^{1\alpha}\frac{\partial g_{2\alpha}}{\partial \tau}E_1 - \left(\frac{1}{2}g^{2\alpha}\frac{\partial g_{2\alpha}}{\partial \tau} - n^{-1}\frac{\partial n}{\partial \tau}\right)E_2\right], \\ H^{2}{}_{;\,3} &= g^{-1/2}\left[-\frac{\partial E_1}{\partial \tau} + \left(\frac{1}{2}g^{1\alpha}\frac{\partial g_{1\alpha}}{\partial \tau} - n^{-1}\frac{\partial n}{\partial \tau}\right)E_1 + \frac{1}{2}g^{2\alpha}\frac{\partial g_{1\alpha}}{\partial \tau}E_2\right]. \end{aligned} \tag{5.1.38}$$

We now calculate the derivatives $H_{3\,;}{}^{\alpha}$. Using (5.1.27) and (5.1.28), we obtain

$$H_{3\,;\,\alpha} = -\Gamma^\beta_{\alpha 3}H_\beta = -g_{\beta\gamma}\Gamma^\beta_{\alpha 3}H^\gamma = g^{-1/2}\Gamma^\beta_{\alpha 3}\left(g_{2\beta}E_1 - g_{1\beta}E_2\right).$$

Inserting the Christoffel symbols (5.1.21), we come to

$$H_{3\,;\,\alpha} = \frac{1}{2}g^{-1/2}\left(\frac{\partial g_{\alpha 2}}{\partial \tau}E_1 - \frac{\partial g_{\alpha 1}}{\partial \tau}E_2\right).$$

It follows from this that

$$H_{3\,;}{}^{\alpha} = g^{\alpha\beta}H_{3\,;\,\beta} = \frac{1}{2}g^{-1/2}g^{\alpha\beta}\left(\frac{\partial g_{\beta 2}}{\partial \tau}E_1 - \frac{\partial g_{\beta 1}}{\partial \tau}E_2\right). \tag{5.1.39}$$

Inserting (5.1.37)–(5.1.39) into (5.1.36), we arrive at the system

$$2\frac{\partial E_\alpha}{\partial \tau} + aE_\alpha - \frac{\partial g_{\alpha\beta}}{\partial \tau}E^\beta = in^{-2}\chi^\beta_\alpha E_\beta \qquad (\alpha = 1, 2), \tag{5.1.40}$$

where the notation

$$a = \frac{1}{2} g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial \tau} + n^{-1} \frac{\partial n}{\partial \tau} \tag{5.1.41}$$

is used. Equations (5.1.40) are just the above-mentioned solvability conditions for equations (5.1.30)–(5.1.31) of the first approximation. Observe that these conditions turn out to be a system of *ordinary* differential equations along the ray $x^\alpha = x^\alpha_0$. Of course, it is not coincidental. The situation is similar when the method of geometrical optics is applied to hyperbolic equations of a sufficiently wide class; compare with Chapter 2 of the book [44]. We also draw the reader's attention to the feature mentioned in the introduction to the current chapter: system (5.1.40) contains only the components, of the tensor χ, which are orthogonal to the ray.

The *amplitude* A of an electromagnetic wave is defined by the equality

$$A^2 = |E|^2 = g^{\alpha\beta} E_\alpha \overline{E}_\beta \tag{5.1.42}$$

On the base of (5.1.40), we obtain an equation for A. To this end, we differentiate equality (5.1.42)

$$\frac{\partial A^2}{\partial \tau} = 2\mathrm{Re}\left(g^{\alpha\beta} \frac{\partial E_\alpha}{\partial \tau} \overline{E}_\beta \right) + \frac{\partial g^{\alpha\beta}}{\partial \tau} E_\alpha \overline{E}_\beta.$$

Inserting the expression for $\partial E_\alpha / \partial \tau$ which follows from (5.1.40) into the last equation, we obtain

$$\frac{\partial A^2}{\partial \tau} = \mathrm{Re}\left[\left(g^{\alpha\gamma} g^{\beta\delta} \frac{\partial g_{\gamma\delta}}{\partial \tau} - a g^{\alpha\beta} \right) E_\alpha \overline{E}_\beta + i n^{-2} \chi_{\alpha\beta} E^\alpha \overline{E}^\beta \right] + \frac{\partial g^{\alpha\beta}}{\partial \tau} E_\alpha \overline{E}_\beta.$$

From now on we assume that the tensor χ is Hermitian, i.e., that $\chi_{\alpha\beta} = \overline{\chi}_{\beta\alpha}$. In this case the second summand in the brackets is pure imaginary and, consequently, can be deleted. The expression in the parentheses is symmetric in α and β and, consequently, the first summand in the brackets is real. So the previous equality can be rewritten as:

$$\frac{\partial A^2}{\partial \tau} = \left(\frac{\partial g^{\alpha\beta}}{\partial \tau} + g^{\alpha\gamma} g^{\beta\delta} \frac{\partial g_{\gamma\delta}}{\partial \tau} - a g^{\alpha\beta} \right) E_\alpha \overline{E}_\beta$$

By differentiating the identity $g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma$, we obtain the relation

$$\frac{\partial g^{\alpha\beta}}{\partial \tau} + g^{\alpha\gamma} g^{\beta\delta} \frac{\partial g_{\gamma\delta}}{\partial \tau} = 0. \tag{5.1.43}$$

with help of which the previous formula takes the form

$$2A \frac{\partial A}{\partial \tau} = -a g^{\alpha\beta} E_\alpha \overline{E}_\beta = -a A^2$$

or

$$2 \frac{\partial A}{\partial \tau} + aA = 0. \tag{5.1.44}$$

By (5.1.23), equality (5.1.41) takes the form

$$a = \frac{1}{n} \frac{\partial n}{\partial \tau} + \frac{1}{J} \frac{\partial J}{\partial \tau}. \tag{5.1.45}$$

Inserting this value into (5.1.44), we arrive at the equation

$$\frac{\partial}{\partial \tau}\left[\ln\left(n^{1/2}J^{1/2}A\right)\right] = 0.$$

Integrating this equation, we obtain the classical formula for the amplitude of an electromagnetic wave:

$$A = \frac{C}{\sqrt{nJ}}, \tag{5.1.46}$$

where C is a constant depending on a ray. Observe that the tensor χ does not enter this formula, i.e., the amplitude of a wave in a quasi-isotropic medium coincides with that of the corresponding isotropic medium. As is known, the physical interpretation of formula (5.1.46) is as follows: within the scope of the zero approximation of geometrical optics, the energy of an electromagnetic wave propagates along a ray tube.

5.1.4 Rytov's law

We transform system (5.1.40), changing variables

$$E_\alpha = An^{-1}\eta_\alpha \tag{5.1.47}$$

where A is the amplitude defined by (5.1.42). Using (5.1.40) and (5.1.44), we obtain

$$2\frac{\partial \eta_\alpha}{\partial \tau} = 2\frac{\partial}{\partial \tau}\left(\frac{n}{A}E_\alpha\right) = 2\frac{n}{A}\frac{\partial E_\alpha}{\partial \tau} - 2\frac{n}{A^2}\frac{\partial A}{\partial \tau}E_\alpha + \frac{2}{A}\frac{\partial n}{\partial \tau}E_\alpha =$$
$$= \frac{n}{A}\left(g^{\beta\gamma}\frac{\partial g_{\alpha\beta}}{\partial \tau}E_\gamma - aE_\alpha + in^{-2}\chi^\beta_\alpha E_\beta\right) + \frac{na}{A}E_\alpha + \frac{2}{A}\frac{\partial n}{\partial \tau}E_\alpha =$$
$$= g^{\beta\gamma}\frac{\partial g_{\alpha\beta}}{\partial \tau}\eta_\gamma + \frac{2}{n}\frac{\partial n}{\partial \tau}\eta_\alpha + in^{-2}\chi^\beta_\alpha\eta_\beta.$$

Thus we have arrived at the system

$$\frac{\partial \eta_\alpha}{\partial \tau} - \frac{1}{2}g^{\beta\gamma}\frac{\partial g_{\alpha\beta}}{\partial \tau}\eta_\gamma - \frac{1}{n}\frac{\partial n}{\partial \tau}\eta_\alpha = \frac{i}{2n^2}\chi^\beta_\alpha\eta_\beta \qquad (\alpha = 1, 2). \tag{5.1.48}$$

Until now all our calculations were performed with respect to the Euclidean metric. Let us invoke the Riemannian metric (5.1.17) that assumes the next form in ray coordinates:

$$d\ell^2 = h_{\alpha\beta}dx^\alpha dx^\beta + (d\tau)^2, \quad g_{\alpha\beta} = n^{-2}h_{\alpha\beta}. \tag{5.1.49}$$

Note that the vector η has unit length with respect to this metric, which fact explains the meaning of the change (5.1.47). Inserting expression (5.1.49) for $g_{\alpha\beta}$ into (5.1.48), we transform this equation so as to have

$$\frac{\partial \eta_\alpha}{\partial \tau} - \frac{1}{2}h^{\beta\gamma}\frac{\partial h_{\alpha\beta}}{\partial \tau}\eta_\gamma = \frac{i}{2n^2}\chi^\beta_\alpha\eta_\beta. \tag{5.1.50}$$

Let $\tilde\Gamma^i_{jk}$ be the Christoffel symbols of metric (5.1.49); they are expressed from $h_{\alpha\beta}$ by formulas similar to (5.1.21). In particular, $\tilde\Gamma^\gamma_{\alpha 3} = \frac{1}{2}h^{\beta\gamma}\partial h_{\alpha\beta}/\partial\tau$. From this, by the rule for calculating a covariant derivative, we see that equation (5.1.50) can be rewritten as:

$$\eta_{\alpha\,;\,3} = \frac{i}{2n^2}\chi^\beta_\alpha\eta_\beta \qquad (\alpha = 1, 2). \tag{5.1.51}$$

As mentioned above, the field η is orthogonal to the ray $\gamma : x^\alpha = x_0^\alpha$, i.e.,

$$\langle \eta, \dot\gamma \rangle = \eta_j \dot\gamma^k = \eta_3 = 0, \tag{5.1.52}$$

and has unit length:

$$|\eta|^2 = h^{jk}\eta_j\overline{\eta}_k = 1. \tag{5.1.53}$$

It follows from (5.1.52) and (5.1.53) that

$$\eta_{3\,;\,3} = 0 \tag{5.1.54}$$

The index 3 is distinguished in (5.1.51) and (5.1.54), since these equations are written in ray coordinates. In an arbitrary coordinate system equalities (5.1.51) and (5.1.54) are replaced with the next ones:

$$\left(\frac{D\eta}{d\tau}\right)_j = \frac{i}{2n^2}\left(\delta_j^k - \frac{1}{|\dot\gamma|^2}\dot\gamma_j\dot\gamma^k\right)\chi_k^l\eta_l \tag{5.1.55}$$

where $D/d\tau = \dot\gamma^k\nabla_k$ is the operator of absolute differentiation along a ray γ. Validity of (5.1.55) follows from the fact that this formula is invariant under change of coordinates and coincides with (5.1.51), (5.1.54) in ray coordinates.

To find a coordinate-free form of equation (5.1.55), we note that the tensor $\chi = (\chi_l^k)$ can be considered as a linear operator $\chi : \mathbf{R}^3 \to \mathbf{R}^3$. For $0 \neq \xi \in \mathbf{R}^3$, we denote by $P_\xi : \mathbf{R}^3 \to \mathbf{R}^3$ the orthogonal projection onto the plane $\xi^\perp = \{\eta \in \mathbf{R}^3 \mid \langle \xi, \eta\rangle = 0\}$. In coordinate form this operator is given by the formulas

$$(P_\xi a)_j = \left(\delta_j^k - \frac{1}{|\xi|^2}\xi_j\xi^k\right)a_k.$$

Thus equation (5.1.55) is written in coordinate-free form as follows:

$$\frac{D\eta}{d\tau} = \frac{i}{2n^2}P_{\dot\gamma}\chi\eta. \tag{5.1.56}$$

The right-hand side of equation (5.1.56) is understood in the following sense: considered as a linear operator, the tensor χ is applied to the vector η; the operator $P_{\dot\gamma}$ is then applied to the so-obtained vector. In the next section we shall see that the right-hand side of (5.1.56) can be understood in another way: first the operator $P_{\dot\gamma}$ is applied to the tensor χ and the so-obtained linear operator is then applied to the vector η,

In the case $\chi = 0$, i.e., for an isotropic medium, equation (5.1.56) means that the vector field η is parallel along the ray γ in the sense of the Riemannian metric (5.1.17). This fact is well known to be equivalent to the Rytov law (we will discuss the Rytov law for isotropic media in detail in the beginning of the next chapter). Therefore we call equation (5.1.56) the *Rytov law for quasi-isotropic media.*

5.1.5 The Euclidean form of the Rytov law

We fix a point on the ray γ, and identify the origin of some Cartesian coordinate system $(x^1, x^2, x^3) = (x, y, z)$ with the point and its coordinate vectors with the Frenet frame λ, ν, β of the ray γ at the point. In this coordinate system metric (5.1.17) has the form

$$d\ell^{\,2} = n^2(dx^2 + dy^2 + dz^2). \tag{5.1.57}$$

At the coordinate origin the next relations are valid:

$$\begin{aligned}\frac{D\nu}{d\tau} &= \frac{1}{n}\nabla_1\nu = \frac{1}{n}\left(\frac{d\nu}{ds} + \Gamma^1_{12}\lambda + \Gamma^2_{12}\nu + \Gamma^3_{12}\beta\right),\\ \frac{D\beta}{d\tau} &= \frac{1}{n}\nabla_1\beta = \frac{1}{n}\left(\frac{d\beta}{ds} + \Gamma^1_{13}\lambda + \Gamma^2_{13}\nu + \Gamma^3_{13}\beta\right),\end{aligned} \tag{5.1.58}$$

where s is the Euclidean arc length of γ. The vector $\dot{\gamma} = n^{-1}\lambda$ is parallel along γ in the sense of metric (5.1.57) and, consequently, at the origin

$$0 = n^2\frac{D(n^{-1}\lambda)}{\partial\tau} = n\nabla_1(n^{-1}\lambda) = \nabla_1\lambda - \frac{n_x}{n}\lambda = \frac{d\lambda}{ds} + \Gamma^1_{11}\lambda + \Gamma^2_{11}\nu + \Gamma^3_{11}\beta - \frac{n_x}{n}\lambda.$$

Here $n_x = \partial n/\partial x$. Inserting the values of $d\lambda/ds,\ d\nu/ds$ and $d\beta/ds$ which follow from the Frenet formulas (K is the curvature and κ is the torsion of the ray)

$$\frac{d\lambda}{ds} = K\nu, \quad \frac{d\nu}{ds} = -K\lambda - \kappa\beta, \quad \frac{d\beta}{ds} = \kappa\nu \tag{5.1.59}$$

and the values

$$\begin{aligned}&\Gamma^1_{11} = \frac{n_x}{n}, \quad &&\Gamma^2_{11} = -\frac{n_y}{n}, \quad &&\Gamma^3_{11} = -\frac{n_z}{n},\\ &\Gamma^1_{12} = \frac{n_y}{n}, \quad &&\Gamma^2_{12} = \frac{n_x}{n}, \quad &&\Gamma^3_{12} = 0,\\ &\Gamma^1_{13} = \frac{n_z}{n}, \quad &&\Gamma^2_{13} = 0, \quad &&\Gamma^3_{13} = \frac{n_x}{n}\end{aligned} \tag{5.1.60}$$

for the Christoffel symbols of metric (5.1.57) into the previous equality, we see that the relations $n_y = Kn,\ n_z = 0$ hold at the origin. Now formulas (5.1.58) take the form

$$\frac{D\nu}{d\tau} = \frac{1}{n}\left(\frac{n_x}{n}\nu - \kappa\beta\right), \quad \frac{D\beta}{d\tau} = \frac{1}{n}\left(\kappa\nu + \frac{n_x}{n}\beta\right). \tag{5.1.61}$$

We change variables in equation (5.1.56) by the formula $\eta = n^{-1}F$. The vector field F is of unit length in the sense of the Euclidean metric (5.1.6) and is orthogonal to the tangent vector λ of the ray γ. Consequently, it can be represented as $F = F_\nu\nu + F_\beta\beta$ where F_ν and F_β are functions on the ray γ. We will obtain equations for these functions which follow from (5.1.56). To this end we introduce the matrix

$$\begin{pmatrix} \chi_{\lambda\lambda} & \chi_{\lambda\nu} & \chi_{\lambda\beta}\\ \chi_{\nu\lambda} & \chi_{\nu\nu} & \chi_{\nu\beta}\\ \chi_{\beta\lambda} & \chi_{\beta\nu} & \chi_{\beta\beta}\end{pmatrix}$$

of components of the tensor χ in the basis λ, ν, β. First we calculate the right-hand side of equation (5.1.56)

$$\begin{aligned}P_{\dot{\gamma}}\chi\eta &= P_\lambda\chi(n^{-1}(F_\nu\nu + F_\beta\beta)) = n^{-1}(F_\nu P_\lambda\chi\nu + F_\beta P_\lambda\chi\beta) =\\ &= n^{-1}\left[F_\nu(\chi_{\nu\nu}\nu + \chi_{\nu\beta}\beta) + F_\beta(\chi_{\beta\nu}\nu + \chi_{\beta\beta}\beta)\right] =\\ &= n^{-1}\left[(\chi_{\nu\nu}F_\nu + \chi_{\nu\beta}F_\beta)\nu + (\chi_{\beta\nu}F_\nu + \chi_{\beta\beta}F_\beta)\beta\right].\end{aligned} \tag{5.1.62}$$

Now we calculate the left-hand side of (5.1.56)

$$\frac{D\eta}{d\tau} = \frac{D}{d\tau}\left(\frac{F_\nu}{n}\nu + \frac{F_\beta}{n}\beta\right) =$$

$$= -\frac{1}{n^2}\frac{dn}{d\tau}(F_\nu\nu + F_\beta\beta) + \frac{1}{n}\frac{dF_\nu}{d\tau}\nu + \frac{1}{n}\frac{dF_\beta}{d\tau}\beta + \frac{F_\nu}{n}\frac{D\nu}{d\tau} + \frac{F_\beta}{n}\frac{D\beta}{d\tau},$$

Using the relation $d\tau = n\,ds$ and (5.1.61), at the origin we have

$$\frac{D\eta}{d\tau} = -\frac{n_x}{n^3}(F_\nu\nu + F_\beta\beta) + \frac{1}{n^2}\frac{dF_\nu}{ds}\nu + \frac{1}{n^2}\frac{dF_\beta}{ds}\beta + \frac{F_\nu}{n^2}\left(\frac{n_x}{n}\nu - \kappa\beta\right) + \frac{F_\beta}{n^2}\left(\kappa\nu + \frac{n_x}{n}\beta\right),$$

$$\frac{D\eta}{d\tau} = \frac{1}{n^2}\left[\left(\frac{dF_\nu}{ds} + \kappa F_\beta\right)\nu + \left(\frac{dF_\beta}{ds} - \kappa F_\nu\right)\beta\right]. \tag{5.1.63}$$

Inserting (5.1.62) and (5.1.63) into (5.1.56), we arrive at the system

$$\begin{aligned} \frac{dF_\nu}{ds} &= \frac{i}{2n}(\chi_{\nu\nu}F_\nu + \chi_{\nu\beta}F_\beta) - \kappa F_\beta \\ \frac{dF_\beta}{ds} &= \frac{i}{2n}(\chi_{\beta\nu}F_\nu + \chi_{\beta\beta}F_\beta) + \kappa F_\nu \end{aligned} \tag{5.1.64}$$

which is just the Euclidean form of the Rytov law. Although we have proved (5.1.64) only at the origin, these equations hold at each point of the ray, since they are independent of choice of coordinates. Equations (5.1.64) were obtained by Yu. A. Kravtsov.

5.1.6 The inverse problem

We denote $f = i\chi/(2n^2)$ and rewrite (5.1.56) as

$$\frac{D\eta}{d\tau} = P_{\dot\gamma} f\eta. \tag{5.1.65}$$

Assume a quasi-isotropic medium under investigation to be contained in a bounded domain $D \subset \mathbf{R}^3$. We also assume that the isotropic part $\varepsilon\delta_{\alpha\beta}$ of the dielectric permeability tensor (5.1.5) is known from some considerations, i.e., the metric (5.1.17) and its geodesics are known. Our problem consists in determining the anisotropic part $\chi_{\alpha\beta}$ of the dielectric permeability tensor. To this end we can to fulfil tomographic measurements of the following type: to examine the domain D along every geodesic of the metric (5.1.17) (for instance, with the help of a laser in the case of the optical band), choosing arbitrary polarization parameters for the incoming light and measuring them for the outcoming light.

The mathematical formulation of our problem is as follows. Metric (5.1.17) is known on a domain D. For every geodesic $\gamma : [0, 1] \to D$ with the endpoints in the boundary of D the value $\eta(1)$ of the solution to system (5.1.65) is known as a function of the initial value $\eta(0)$ and geodesic γ:

$$\eta(1) = U(\gamma)\eta(0). \tag{5.1.66}$$

In other words, the fundamental matrix $U(\gamma)$ of system (5.1.65) is known. We have to determine $f(x)$ from $U(\gamma)$.

With the problem formulated, we now distract ourselves from the initial physical situation and consider the above-stated problem for a domain $D \subset \mathbf{R}^n (n \geq 2)$ and for a Riemannian metric

$$dt^2 = g_{ij} dx^i dx^j \tag{5.1.67}$$

of the general type which is given on D. Of course, the factor $P_{\dot\gamma}$ in (5.1.65) is now the orthogonal projection in the sense of metric (5.1.67).

Note that the version of our problem obtained by omitting the factor $P_{\dot\gamma}$ in (5.1.65), i.e., by replacing (5.1.65) with the equation

$$\frac{D\eta}{dt} = f\eta,$$

is also sensible from a mathematical (and, perhaps, from an applied) standpoint. Such a problem of vector tomography is considered in [137], uniqueness of a solution and a stable estimate are obtained under some conditions on the smallness of the domain D and the desired field f. Until now there is no success in extending the methods of [137] to the case of equation (5.1.65).

The above-formulated inverse problem is nonlinear. Now linearize it. To this end we fix a geodesic $\gamma : [0,1] \to D$ whose endpoints belong to the boundary of D and introduce an orthonormal basis $e_1(t), \dots, e_{n-1}(t), e_n(t) = c\dot\gamma(t)$ that is parallel along γ. In this basis, system (5.1.65) looks like:

$$\dot\eta_i = \sum_{j=1}^{n-1} f_{ij} \eta^j \qquad (1 \leq i \leq n-1).$$

We represent a solution to the Cauchy problem of the system by the Neumann series

$$\eta(1) = \left[I + \int_0^1 F(t)dt + \int_0^1 F(t)dt \int_0^t F(t_1)dt_1 + \dots \right] \eta(0), \tag{5.1.68}$$

where $F = (f_{ij})_{i,j=1}^{n-1}$ and I is the identity matrix. Deleting the terms that are nonlinear in f, we obtain

$$\eta^i(1) - \eta^i(0) = \int_0^1 \sum_{j=1}^{n-1} f_{ij}(t) \eta^j(0) dt \qquad (1 \leq i \leq n-1). \tag{5.1.69}$$

Multiplying each of the equalities (5.1.69) by ξ^i and summing them up, we arrive at the relation

$$Jf(\gamma; \xi, \eta) = \int_0^1 \sum_{i,j=1}^{n-1} f_{ij}(t) \xi^i \eta^j(0) dt, \tag{5.1.70}$$

where $Jf(\gamma; \xi, \eta) = \sum_{i=1}^{n-1} \xi^i(\eta^i(1) - \eta^i(0))$ is a known function. To write equation (5.1.70) in invariant form, we put $\xi^n = \eta^n = 0$ and increase the upper summation limit in (5.1.70) to n. Then we note that the vector fields $\xi(t) = \xi^i e_i(t)$ and $\eta(t) = \eta^i(0) e_i(t)$ are parallel along γ and perpendicular to the vector $\dot\gamma$. Now equation (5.1.70) takes the form

$$Jf(\gamma; \xi, \eta) = \int_0^1 f_{ij}(\gamma(t)) \xi^i(t) \eta^j(t) \, dt. \tag{5.1.71}$$

The integrand in (5.1.71) is evidently independent of the choice of a local coordinate system in a neighbourhood of the point $\gamma(t)$. Thus we arrive at the next linear problem:

Problem 5.1.1 *Let a Riemannian metric* (5.1.67) *be given in a domain* $D \subset \mathbf{R}^n$. *Determine a tensor field* $f = (f_{ij})$ *that is defined on* D, *if for every geodesic* $\gamma : [0,1] \to D$ *whose endpoints are in the boundary of* D, *bilinear form* (5.1.71) *is known for all* $\xi, \eta \in \gamma^\perp$, *where* $\gamma^\perp$ *is the space of vector fields parallel along* γ *and orthogonal to* $\dot\gamma$.

In what follows we restrict ourselves to considering symmetric tensor fields $f_{ij} = f_{ji}$. In this case bilinear form (5.1.71) can be replaced with the quadratic one

$$Jf(\gamma,\eta) = \int\limits_0^1 f_{ij}(\gamma(t))\eta^i(t)\eta^j(t)\,dt \qquad (\eta \in \gamma^\perp). \tag{5.1.72}$$

The function $Jf(\gamma,\eta)$ defined by the last equality is called the *transverse ray transform* of the field f.

Like every tomography problem, the problem of inverting the transverse ray transform is overdetermined in dimensions $n > 2$: the desired field f depends on the n-dimensional variable $x \in D$ while the known function $Jf(\gamma,\eta)$ depends on the $2(n-1)$-parameter family of geodesics γ. At the same time our problem is underdetermined with respect to the number of the sought functions: we have to determine $n(n+1)/2$ components of the field f provided that only $n(n-1)/2$ coefficients of the quadratic form $Jf(\gamma,\eta) = a_{ij}(f,\gamma)\eta^i\eta^j$ are known (since $\dim\gamma^\perp = n-1$).

At the conclusion of the section we will show that, in the case when metric (5.1.67) coincides with the Euclidean one (i.e., $\varepsilon = \mathrm{const}$ in (5.1.5)), the problem of inverting the transverse ray transform is reduced to the classical tomography problem of inverting the ray transform for a scalar function (for $n = 3$, to the problem of inverting the Radon transform). Indeed, in this case geodesics coincide with straight lines that can be parameterized by a point $x \in \mathbf{R}^n$ and a direction vector $0 \neq \xi \in \mathbf{R}^n$. Extending f by zero outside D, in this case we can rewrite equation (5.1.72) as follows:

$$Jf(\gamma,\eta) = \int\limits_{-\infty}^{\infty} f_{ij}(x+\xi t)\eta^i\eta^j dt \qquad (x,\eta \in \mathbf{R}^n,\ 0 \neq \xi \in \mathbf{R}^n,\ \eta\perp\xi). \tag{5.1.73}$$

We fix $0 \neq \eta \in \mathbf{R}^n$ and define the function $\varphi(y) = f_{ij}(y)\eta^i\eta^j$ on the hyperplane $\mathbf{R}^{n-1}_{\eta,p} = \{y \in \mathbf{R}^n \mid \langle y,\eta\rangle = p\}$. By (5.1.73), we know the integrals of φ along all straight lines contained in $\mathbf{R}^{n-1}_{\eta,p}$. So we can determine the values of φ on every hyperplane $\mathbf{R}^{n-1}_{\eta,p}$, and, consequently, the field f on $\mathbf{R}^n$ too.

5.2 The transverse ray transform on a CDRM

Let (M,g) be a Riemannian manifold with boundary. By $S^m\pi_M = (S^m\Pi M, q, \partial_+\Omega M)$ we denote the bundle over $\partial_+\Omega M$ induced from $S^m\tau'_M$ by the restriction $p : \partial_+\Omega M \to M$ of the projection of the tangent bundle, i.e., $S^m\pi_M = p^*S^m\tau'_M$. Thus sections $\varphi \in C^\infty(S^m\pi_M)$ of this bundle are functions sending a point $(x,\xi) \in \partial_+\Omega M$ to some tensor $\varphi(x,\xi) \in S^mT'_xM$.

Recall that, for a vector $\xi \in T_xM$, by $i_\xi : S^mT'_xM \to S^{m+1}T'_xM$ and $j_\xi : S^{m+1}T'_xM \to S^mT'_xM$ we denote the operators of the symmetric multiplication by ξ and convolution with ξ. These operators are dual to one other and, consequently, we have the orthogonal decomposition $S^mT'_xM = \operatorname{Ker} j_\xi \oplus \operatorname{Im} i_\xi$. Let $P_\xi : S^mT'_xM \to S^mT'_xM$ be the orthogonal projection onto the first summand of the decomposition. As follows from Lemma 2.6.1, for $\xi \neq 0$, this operator is given in coordinate form by the equality

$$(P_\xi u)_{i_1\dots i_m} = \left(\delta_{i_1}^{j_1} - \frac{1}{|\xi|^2}\xi_{i_1}\xi^{j_1}\right)\dots\left(\delta_{i_m}^{j_m} - \frac{1}{|\xi|^2}\xi_{i_m}\xi^{j_m}\right)u_{j_1\dots j_m}. \tag{5.2.1}$$

The *transverse ray transform* on a CDRM (M, g) is the linear operator

$$J : C^\infty(S^m\tau'_M) \to C^\infty(S^m\pi_M), \tag{5.2.2}$$

defined by the equality

$$Jf(x,\xi) = \int\limits_{\tau_-(x,\xi)}^{0} I_\gamma^{t,0}(P_{\dot\gamma(t)}f(\gamma(t)))\,dt \qquad ((x,\xi) \in \partial_+\Omega M), \tag{5.2.3}$$

where $\gamma = \gamma_{x,\xi} : [\tau_-(x,\xi), 0] \to M$ is a maximal geodesic satisfying the initial conditions $\gamma(0) = x$ and $\dot\gamma(0) = \xi$; $I_\gamma^{t,0}$ is the parallel translation along γ from the point $\gamma(t)$ to the point $\gamma(0) = x$.

Let us explain the relationship between this definition and definition (5.1.72) given in Section 5.1. To this end we note that, for $\xi, \eta \in T_xM$, (5.2.1) implies the relation

$$P_\xi\eta^m = (P_\xi\eta)^m. \tag{5.2.4}$$

Let $(x,\xi) \in \partial_+\Omega M$, $\eta \in T_xM$, $\langle\xi,\eta\rangle = 0$; and $\eta(t)$ be a vector field parallel along $\gamma = \gamma_{x,\xi}$ and such that $\eta(0) = \eta$. We take the scalar product of equality (5.2.3) with $(\eta + a\xi)^m$, where $a \in \mathbf{R}$. Using (5.2.4), we transform the integrand of the so-obtained equality as follows:

$$\begin{aligned}\langle I_\gamma^{t,0}(P_{\dot\gamma(t)}f(\gamma(t))), (\eta + a\xi)^m\rangle &= \langle P_{\dot\gamma(t)}f(\gamma(t)), I_\gamma^{0,t}(\eta + a\xi)^m\rangle = \\ = \langle P_{\dot\gamma(t)}f(\gamma(t)), (\eta(t) + a\dot\gamma(t))^m\rangle &= \langle f(\gamma(t)), P_{\dot\gamma(t)}(\eta(t) + a\dot\gamma(t))^m\rangle = \\ = \langle f(\gamma(t)), (P_{\dot\gamma(t)}(\eta(t) + a\dot\gamma(t)))^m\rangle &= \langle f(\gamma(t)), (\eta(t))^m\rangle.\end{aligned}$$

We thus obtain the equality

$$\langle Jf(x,\xi), (\eta + a\xi)^m\rangle = \langle Jf(x,\xi), \eta^m\rangle = \int\limits_{\tau_-(x,\xi)}^{0} f_{i_1\dots i_m}(\gamma(t))\eta^{i_1}(t)\dots\eta^{i_m}(t)\,dt, \tag{5.2.5}$$

which coincides up to a factor with (5.1.72) for $m = 2$.

It is interesting to compare (5.2.5) with operator (4.2.2), which may be called, in the context of the current chapter, the longitudinal ray transform. First, for $m = 0$ these transforms coincide. Second, for $n = 2$ they reduce to one other by a diffeomorphism $TM \to TM$ mapping (x,ξ) into (x,ξ'), where ξ' is obtained from ξ by the rotation of $\pi/2$ in the positive direction (for simplicity, we assume M to be oriented). For the other values of m, n the operators I and J provide us with different information on a field f. Simultaneous consideration of these transforms seems to be of some interest, but in this book we leave aside this question.

The next claim is proved in exact analogy with Theorem 4.2.1.

Theorem 5.2.1 *For a CDRM M, the transverse ray transform* (5.2.3) *can be extended to a bounded operator*

$$J : H^k(S^m\tau'_M) \to H^k(S^m\pi_M) \tag{5.2.6}$$

for every integer $k \geq 0$.

For $x \in M$, we put

$$K(x) = \sup_\sigma |K(x,\sigma)|, \tag{5.2.7}$$

where $K(x,\sigma)$ is the sectional curvature defined by formula (3.2.14).

For a CDRM (M, g), we introduce the characteristic

$$k(M,g) = \sup_{(x,\xi)\in\partial_-\Omega M} \int\limits_0^{\tau_+(x,\xi)} tK(\gamma_{x,\xi}(t))\,dt, \tag{5.2.8}$$

where $\gamma = \gamma_{x,\xi}$ is a maximal geodesic defined by the initial conditions $\gamma_{x,\xi}(0) = x$, $\dot\gamma_{x,\xi}(0) = \xi$. Note distinctions of this characteristic from the quantity $k^+(M,g)$ introduced by formula (4.3.3). First, the integrand of (4.3.3) depends only on sectional curvatures in the two-dimensional directions containing the vector $\dot\gamma_{x,\xi}(t)$ while the integrand of (5.2.8) depends on all two-dimensional planes at the point $\gamma(t)$. Second, the quantity $k^+(M,g)$ is determined only by positive values of the sectional curvature while $k(M,g)$ depends on the negative values too.

We can now formulate the main result of the current chapter.

Theorem 5.2.2 *For integers m and n satisfying the inequalities $n \geq 3$ and $n > m$, there exists a positive number $\varepsilon(m,n)$ such that, for every compact dissipative Riemannian manifold (M,g) of dimension n satisfying the condition*

$$k(M,g) < \varepsilon(m,n), \tag{5.2.9}$$

the transverse ray transform

$$J : H^1(S^m\tau'_M) \to H^1(S^m\pi_M)$$

is injective. For a field $f \in H^1(S^m\tau'_M)$, the stability estimate

$$\|f\|_0 \leq C\|Jf\|_1 \tag{5.2.10}$$

holds with a constant C independent of f.

In the theorem the condition $n \geq 3$ is essential, since we have seen that, for $n = 2$, the operator J reduces to the operator I with nontrivial kernel. The second condition $n > m$ seems to be unessential; it is due to the method of the proof.

In what follows we assume that $m \geq 1$, since for $m = 0$, as we have seen, J coincides with I and Theorem 5.2.2 is a consequence of Theorem 4.3.3.

The remainder of the chapter is devoted to the proof of Theorem 5.2.2. It has much in common with the proof of Theorem 4.3.3, but at the same time it has some new principal moments. Now we only note that, with the help of Theorem 5.2.1 and in exact analogy with the argument presented just after the formulation of Lemma 4.3.4, one can easily show that it suffices to prove Theorem 5.2.2 for a real field $f \in C^\infty(S^m\tau'_M)$.

5.3 Reduction of Theorem 5.2.2 to an inverse problem for the kinetic equation

Let (M, g) be a CDRM, $f \in C^\infty(S^m\tau'_M)$. Define a semibasic tensor field $u = (u_{i_1\dots i_m}(x,\xi))$ on T^0M by the equality

$$u(x,\xi) = \int\limits_{\tau_-(x,\xi)}^{0} I_\gamma^{t,0}(P_{\dot\gamma(t)}f(\gamma(t)))\,dt \qquad ((x,\xi) \in T^0M), \tag{5.3.1}$$

where the same notations are used as in definition (5.2.3) of the transverse ray transform. The difference between formulas (5.2.3) and (5.3.1) is the fact that (5.2.3) is considered only for $(x,\xi) \in \partial_+\Omega M$ while (5.3.1), for all $(x,\xi) \in T^0M$. In particular, we have the boundary condition for u

$$u|_{\partial_+\Omega M} = Jf. \tag{5.3.2}$$

Since $\tau_-(x,\xi) = 0$ for $(x,\xi) \in \partial_-\Omega M$, we have the second boundary condition

$$u|_{\partial_-\Omega M} = 0. \tag{5.3.3}$$

The field u depends smoothly on $(x,\xi) \in T^0M$ except for the points of the set $T^0(\partial M)$ where some derivatives of u can be infinite. Consequently, some of the integrals considered below are improper and we have to verify their convergence. The verification is performed in the same way as in Section 4.6, since the singularities of the field u are due only to the singularities of the lower integration limit in (5.3.1). So we will not pay attention to these singularities in what follows.

The field $u(x,\xi)$ is homogeneous in its second argument

$$u(x, t\xi) = t^{-1}u(x,\xi) \qquad (t > 0) \tag{5.3.4}$$

and satisfies the differential equation

$$Hu(x,\xi) = P_\xi f(x) \tag{5.3.5}$$

on $T^0M \setminus T^0(\partial M)$. Relations (5.3.4) and (5.3.5) are derived from definition (5.3.1) by repeating the corresponding arguments of Section 4.6 almost word by word, we omit the proofs.

We note finally that the field $u = (u_{i_1\dots i_m}(x,\xi))$ is symmetric in all its indices and satisfies the relation

$$j_\xi u(x,\xi) = 0, \tag{5.3.6}$$

as follows immediately from the fact that the integrand of (5.3.1) has the same properties.

Observe a specific character of the right-hand side of equation (5.3.5). Conditionally speaking, it is a product of the two factors depending upon different arguments. This circumstance is crucial for what follows.

Written in coordinate form, equation (5.3.5) presents a *system* of differential equations for components of the field u. This constitutes a principal distinction with Chapter 4 where *one* equation was considered.

Thus we have arrived at the following inverse problem for the kinetic equation: one has to estimate the factor $f(x)$ on the right-hand side of (5.3.5) by boundary value (5.3.2) of a solution to the equation.

For a field $u(x,\xi)$ satisfying (5.3.2)–(5.3.6), we write down the Pestov differential identity (4.4.4), multiply it by the volume form $\partial\Sigma$ and integrate over ΩM. Transforming the divergent terms of the so-obtained equality in accord with the Gauss-Ostrogradskiĭ formulas (3.6.35) and (3.6.36), we arrive at the relation

$$\int\limits_{\Omega M}\Big[|\overset{h}{\nabla}u|^2+(n-2)\langle w,\xi\rangle\Big]\,d\Sigma=$$

$$=2\int\limits_{\Omega M}\langle\overset{h}{\nabla}u,\overset{v}{\nabla}(Hu)\rangle\,d\Sigma-\int\limits_{\partial\Omega M}\langle v,\nu\rangle\,d\Sigma^{2n-2}+\int\limits_{\Omega M}\mathcal{R}_1[u]\,d\Sigma, \tag{5.3.7}$$

where ν is the outer normal to ∂M, the semibasic vector fields v and w are defined by formulas (4.4.5), (4.4.6) and the notation

$$\mathcal{R}_1[u]=R_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^j u^{i_1\dots i_m}\cdot\overset{v}{\nabla}{}^l u_{i_1\dots i_m}+\sum_{k=1}^m R^{i_k}{}_{pqj}\xi^q u^{i_1\dots i_{k-1}pi_{k+1}\dots i_m}\overset{v}{\nabla}{}^j u_{i_1\dots i_m} \tag{5.3.8}$$

is used. The coefficient $n-2$ at the second summand of the integrand on the left-hand side of (5.3.7) is due to the fact that the field $w(x,\xi)$ is homogeneous of degree -1 in its second argument as follows from (4.4.6) and (5.3.4). Moreover, (4.4.6) implies that $\langle w,\xi\rangle=|Hu|^2$, and equality (5.3.7) takes the form

$$\int\limits_{\Omega M}\Big[|\overset{h}{\nabla}u|^2+(n-2)|Hu|^2\Big]\,d\Sigma=$$

$$=2\int\limits_{\Omega M}\langle\overset{h}{\nabla}u,\overset{v}{\nabla}(Hu)\rangle\,d\Sigma-\int\limits_{\partial\Omega M}\langle v,\nu\rangle\,d\Sigma^{2n-2}+\int\limits_{\Omega M}R_1[u]\,d\Sigma. \tag{5.3.9}$$

5.4 Estimation of the summand related to the right-hand side of the kinetic equation

Most of difficulties are caused by the first summand on the right-hand side of (5.3.9). As is seen from (4.6.7), in the previous chapter the integrand of the corresponding summand was in divergent form; in the present case this is not so.

By the definition of the operator P_ξ, the right-hand side of equation (5.3.5) can be represented as

$$P_\xi f(x)=f(x)-i_\xi y(x,\xi) \tag{5.4.1}$$

with some symmetric semibasic field y of degree $m-1$ which in turn can be decomposed into the sum

$$y(x,\xi)=\tilde{y}(x,\xi)+i_\xi h(x,\xi),\quad j_\xi\tilde{y}(x,\xi)=0. \tag{5.4.2}$$

From (5.3.5) and (5.4.1), we obtain

$$\overset{v}{\nabla}(Hu)=\overset{v}{\nabla}(f-i_\xi y)=-\overset{v}{\nabla}(i_\xi y).$$

Written in coordinate form, this relation gives

$$\overset{v}{\nabla}_i(Hu)_{i_1\dots i_m} = -\overset{v}{\nabla}_i(i_\xi y)_{i_1\dots i_m} = -\sigma(i_1\dots i_m)\overset{v}{\nabla}_i(\xi_{i_1}y_{i_2\dots i_m}) =$$

$$= -\sigma(i_1\dots i_m)(g_{ii_1}y_{i_2\dots i_m} + \xi_{i_1}\overset{v}{\nabla}_i y_{i_2\dots i_m}),$$

where $\sigma(i_1\dots i_m)$ is symmetrization in the indices $i_1,\dots,i_m$. This implies that

$$\langle\overset{h}{\nabla}u, \overset{v}{\nabla}(Hu)\rangle = \overset{h}{\nabla}{}^i u^{i_1\dots i_m}\cdot\overset{v}{\nabla}_i(Hu)_{i_1\dots i_m} = -\overset{h}{\nabla}{}^i u^{i_1\dots i_m}(g_{ii_1}y_{i_2\dots i_m} + \xi_{i_1}\overset{v}{\nabla}_i y_{i_2\dots i_m}) =$$

$$= -\overset{h}{\nabla}_i u^{i\,i_2\dots i_m}\cdot y_{i_2\dots i_m} - \overset{h}{\nabla}{}^i(\xi_{i_1}u^{i_1i_2\dots i_m})\cdot\overset{v}{\nabla}_i y_{i_2\dots i_m} = -\overset{h}{\nabla}_i u^{i\,i_2\dots i_m}\cdot y_{i_2\dots i_m}.$$

Writing the last equality of this chain, we used condition (5.3.6). Inserting expression (5.4.2) for y into this relation and using (5.3.6) again, we obtain

$$\langle\overset{h}{\nabla}u, \overset{v}{\nabla}(Hu)\rangle = -\overset{h}{\nabla}_i u^{i\,i_2\dots i_m}\cdot(\tilde y_{i_2\dots i_m} + \xi_{i_2}h_{i_3\dots i_m}) =$$

$$= -\overset{h}{\nabla}_i u^{ii_2\dots i_m}\cdot\tilde y_{i_2\dots i_m} - \overset{h}{\nabla}_i(\xi_{i_2}u^{i\,i_2\dots i_m})\cdot h_{i_3\dots i_m} = -\overset{h}{\nabla}_i u^{ii_2\dots i_m}\cdot\tilde y_{i_2\dots i_m}. \tag{5.4.3}$$

Introducing a semibasic tensor field $\overset{h}{\delta}u$ by the equality

$$(\overset{h}{\delta}u)_{i_1\dots i_{m-1}} = \overset{h}{\nabla}{}^p u_{pi_1\dots i_{m-1}}, \tag{5.4.4}$$

we rewrite (5.4.3) in coordinate-free form

$$\langle\overset{h}{\nabla}u, \overset{v}{\nabla}(Hu)\rangle = -\langle\overset{h}{\delta}u, \tilde y\rangle. \tag{5.4.5}$$

From (5.4.5) with the help of the inequality between the arithmetical and geometrical means, we obtain

$$2|\langle\overset{h}{\nabla}u, \overset{v}{\nabla}(Hu)\rangle| \le b|\overset{h}{\delta}u|^2 + \frac{1}{b}|\tilde y|^2, \tag{5.4.6}$$

where b is an arbitrary positive number. We transform the first summand on the right-hand side of inequality (5.4.6), extracting the divergent part from this term. To this end we write

$$|\overset{h}{\delta}u|^2 = (\overset{h}{\delta}u)^{i_2\dots i_m}(\overset{h}{\delta}u)_{i_2\dots i_m} = \overset{h}{\nabla}_p u^{p\,i_2\dots i_m}\cdot\overset{h}{\nabla}{}^q u_{q\,i_2\dots i_m} =$$

$$= \overset{h}{\nabla}_p(u^{p\,i_2\dots i_m}\overset{h}{\nabla}{}^q u_{q\,i_2\dots i_m}) - u^{p\,i_2\dots i_m}\overset{h}{\nabla}_p\overset{h}{\nabla}{}^q u_{q\,i_2\dots i_m} =$$

$$= \overset{h}{\nabla}_p(u^{p\,i_2\dots i_m}\overset{h}{\nabla}{}^q u_{q\,i_2\dots i_m}) - u^p_{\cdot\,i_2\dots i_m}\overset{h}{\nabla}_p\overset{h}{\nabla}_q u^{q\,i_2\dots i_m}. \tag{5.4.7}$$

By commutation formula (3.5.12) for the operator $\overset{h}{\nabla}$,

$$\overset{h}{\nabla}_p\overset{h}{\nabla}_q u^{q\,i_2\dots i_m} = \overset{h}{\nabla}_q\overset{h}{\nabla}_p u^{q\,i_2\dots i_m} - R^i{}_{jpq}\xi^j\overset{v}{\nabla}_i u^{q\,i_2\dots i_m} +$$

$$+ R^q{}_{jpq}u^{j\,i_2\dots i_m} + \sum_{s=2}^m R^{i_s}{}_{jpq}u^{q\,i_2\dots i_{s-1}j\,i_{s+1}\dots i_m}.$$

Inserting this expression into the second summand on the right-hand side of (5.4.7), we obtain

$$|\overset{h}{\delta}u|^2 = -u^p_{\cdot i_2\dots i_m}\overset{h}{\nabla}_q\overset{h}{\nabla}_p u^{q\,i_2\dots i_m} + \overset{h}{\nabla}_p(u^{p\,i_2\dots i_m}\overset{h}{\nabla}{}^q u_{qi_2\dots i_m}) + \mathcal{R}_2[u], \tag{5.4.8}$$

where the notation

$$\mathcal{R}_2[u] = R_{ijkl}\xi^j u^k_{\cdot i_2\dots i_m}\overset{v}{\nabla}{}^i u^{l\,i_2\dots i_m} -$$
$$- R^k{}_{ijk}u^j_{\cdot i_2\dots i_m}u^{i\,i_2\dots i_m} + \sum_{s=2}^{m} R^{i_s}{}_{jkl}u^k_{\cdot i_2\dots i_m}u^{l\,i_2\dots i_{s-1}j\,i_{s+1}\dots i_m}. \tag{5.4.9}$$

is used for brevity.

We now transform the first summand on the right-hand side of (5.4.8) in the order reverse to that used in (5.4.7):

$$|\overset{h}{\delta}u|^2 = -\overset{h}{\nabla}_q(u^p_{\cdot i_2\dots i_m}\overset{h}{\nabla}_p u^{q\,i_2\dots i_m}) + \overset{h}{\nabla}_q u^p_{\cdot i_2\dots i_m}\cdot\overset{h}{\nabla}_p u^{q\,i_2\dots i_m} +$$
$$+ \overset{h}{\nabla}_p(u^{p\,i_2\dots i_m}\overset{h}{\nabla}{}^q u_{q\,i_2\dots i_m}) + \mathcal{R}_2[u] = \overset{h}{\nabla}{}^p u^{q\,i_2\dots i_m}\cdot\overset{h}{\nabla}_q u_{p\,i_2\dots i_m} +$$
$$+ \overset{h}{\nabla}_i(u^{i\,i_2\dots i_m}\overset{h}{\nabla}{}^j u_{j\,i_2\dots i_m} - u_{ji_2\dots i_m}\overset{h}{\nabla}{}^j u^{i\,i_2\dots i_m}) + \mathcal{R}_2[u]. \tag{5.4.10}$$

Defining a semibasic vector field $\tilde{v}$ by the formula

$$\tilde{v}^i = u^{i\,i_2\dots i_m}\overset{h}{\nabla}{}^j u_{j\,i_2\dots i_m} - u_{j\,i_2\dots i_m}\overset{h}{\nabla}{}^j u^{i\,i_2\dots i_m}, \tag{5.4.11}$$

we write the above-obtained relation (5.4.10) in the form

$$|\overset{h}{\delta}u|^2 = \overset{h}{\nabla}{}^i u^{j\,i_2\dots i_m}\cdot\overset{h}{\nabla}_j u_{i\,i_2\dots i_m} + \overset{h}{\nabla}_i\tilde{v}^i + \mathcal{R}_2[u]. \tag{5.4.12}$$

We introduce one else semibasic tensor field (the last one!) z on T^0M by the equality

$$\overset{h}{\nabla}_i u_{i_1\dots i_m} = \frac{\xi_i}{|\xi|^2}(Hu)_{i_1\dots i_m} + z_{i\,i_1\dots i_m}. \tag{5.4.13}$$

The idea of this new notation is the fact that the tensor z is orthogonal to the vector ξ in all its indices:

$$\xi^i z_{i\,i_1\dots i_m} = \xi^{i_1}z_{i\,i_1\dots i_m} = 0, \tag{5.4.14}$$

while the tensor $\overset{h}{\nabla}u = (\overset{h}{\nabla}_i u_{i_1\dots i_m})$ has the mentioned property only in the last m indices. Indeed, the second of the equalities (5.4.14) follows from (5.3.6), and the first follows from (5.4.13) and the definition of the operator H.

From (5.4.14) we conclude that the summands on the right-hand side of (5.4.13) are orthogonal to one other and, consequently,

$$|\overset{h}{\nabla}u|^2 = |Hu|^2 + |z|^2. \tag{5.4.15}$$

The first summand on the right-hand side of (5.4.12) can by expressed through z. Indeed,

$$\overset{h}{\nabla}{}^i u^{j\,i_2\dots i_m}\cdot\overset{h}{\nabla}_j u_{i\,i_2\dots i_m} =$$

$$= \left(z^{i\,j\,i_2\dots i_m} + \frac{1}{|\xi|^2}\xi^i (Hu)^{j\,i_2\dots i_m}\right)\left(z_{j\,i\,i_2\dots i_m} + \frac{1}{|\xi|^2}\xi_j (Hu)_{i\,i_2\dots i_m}\right) =$$

$$= z^{i\,j\,i_2\dots i_m} z_{j\,i\,i_2\dots i_m} + \frac{1}{|\xi|^2}\xi_j z^{i\,j\,i_2\dots i_m}(Hu)_{i\,i_2\dots i_m} +$$

$$+\frac{1}{|\xi|^2}\xi^i z_{j\,i\,i_2\dots i_m}(Hu)^{j\,i_2\dots i_m} + \frac{1}{|\xi|^4}\xi^i\xi_j (Hu)^{j\,i_2\dots i_m}(Hu)_{i\,i_2\dots i_m}.$$

The last three summands on the right-hand side of the preceding formula are equal to zero, by (5.3.5) and (5.4.14), and we obtain the relation

$$\overset{h}{\nabla}{}^i u^{j\,i_2\dots i_m} \cdot \overset{h}{\nabla}_j u_{i\,i_2\dots i_m} = z^{i\,j\,i_2\dots i_m} z_{j\,i\,i_2\dots i_m}.$$

With the help of the above formula, (5.4.12) implies the inequality

$$|\overset{h}{\delta} u|^2 \le |z|^2 + \overset{h}{\nabla}_i \tilde v^i + \mathcal{R}_2[u]. \tag{5.4.16}$$

From (5.4.6) and (5.4.16), we conclude that

$$2|\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle| \le b|z|^2 + \frac{1}{b}|\tilde y|^2 + b\overset{h}{\nabla}_i \tilde v^i + b\mathcal{R}_2[u]. \tag{5.4.17}$$

Now express $\tilde y$ through f. To this end, recall relations (5.4.1) and (5.4.2) that, if combined, give

$$f = P_\xi f + i_\xi \tilde y + i_\xi^2 h, \quad j_\xi \tilde y = 0. \tag{5.4.18}$$

Applying the operator j_ξ to the first of the equalities (5.4.18), we obtain

$$j_\xi f = j_\xi i_\xi \tilde y + j_\xi i_\xi^2 h. \tag{5.4.19}$$

Now use the next

Lemma 5.4.1 *For an integer $k \ge 1$ the equality*

$$j_\xi i_\xi^k = \frac{k}{m+k}|\xi|^2 i_\xi^{k-1} + \frac{m}{m+k} i_\xi^k j_\xi$$

holds on the space of symmetric semibasic tensor fields of degree m.

In the case $k = 1$ this claim coincides with Lemma 3.3.3 (it does not matter that Lemma 3.3.3 is formulated for ordinary tensors and the present claim, for semibasic ones; the proof remains the same). In the general case the claim is easily proved by induction on k.

Transforming each of the summands on the right-hand side of (5.4.19) with the help of Lemma 5.4.1 and taking the second of the equalities (5.4.18) into account, we obtain

$$j_\xi f = \frac{|\xi|^2}{m}\tilde y + i_\xi\left(\frac{2}{m}|\xi|^2 + \frac{m-2}{m} i_\xi j_\xi\right) h.$$

The second summand on the right-hand side of the last relation is in the kernel of the operator P_ξ, so it implies that

$$\tilde y = \frac{m}{|\xi|^2} P_\xi j_\xi f.$$

Inserting this expression into (5.4.17), we obtain

$$2|\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle| \le b|z|^2 + \frac{m^2}{b|\xi|^4}|P_\xi j_\xi f|^2 + b\overset{h}{\nabla}_i \tilde v^i + b\mathcal{R}_2[u]. \tag{5.4.20}$$

Integrating inequality (5.4.20) and then transforming the third summand on the right-hand side of the so-obtained inequality with the help of the Gauss-Ostrogradskiĭ formula for the horizontal derivative, we find

$$2\int\limits_{\Omega M} |\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle|\, d\Sigma \le \int\limits_{\Omega M}\left(b|z|^2 + \frac{m^2}{b}|P_\xi j_\xi f|^2\right) d\Sigma +$$

$$+\, b\int\limits_{\partial\Omega M}\langle \tilde v, \nu\rangle\, d\Sigma^{2n-2} + b\int\limits_{\Omega M}\mathcal{R}_2[u]\, d\Sigma. \tag{5.4.21}$$

It follows from (5.3.9) and (5.4.21) that

$$\int\limits_{\Omega M}\left[|\overset{h}{\nabla} u|^2 + (n-2)|Hu|^2\right] d\Sigma \le \int\limits_{\Omega M}\left(b|z|^2 + \frac{m^2}{b}|P_\xi j_\xi f|^2\right) d\Sigma +$$

$$+ \int\limits_{\partial\Omega M}\langle b\tilde v - v, \nu\rangle\, d\Sigma^{2n-2} + \int\limits_{\Omega M}\mathcal{R}[u]\, d\Sigma, \tag{5.4.22}$$

where

$$\mathcal{R}[u] = \mathcal{R}_1[u] + b\mathcal{R}_2[u]. \tag{5.4.23}$$

Replacing the summands of the integrand on the left-hand side of (5.4.22) with their expressions by formulas (5.3.5) and (5.4.15) and using factorization (3.6.34) of the form $d\Sigma$, we rewrite inequality (5.4.22) as

$$\int\limits_M \left\{\int\limits_{\Omega_x M}\left[(1-b)|z|^2 + (n-1)|P_\xi f|^2 - \frac{m^2}{b}|P_\xi j_\xi f|^2\right] d\omega_x(\xi)\right\} dV^n(x) \le$$

$$\le \int\limits_{\partial\Omega M}\langle b\tilde v - v, \nu\rangle\, d\Sigma^{2n-2} + \int\limits_{\Omega M}\mathcal{R}[u]\, d\Sigma. \tag{5.4.24}$$

Now an intermediate summary is in order. For a solution u to equation (5.3.5) satisfying conditions (5.3.4) and (5.3.6), we obtained inequality (5.4.24) where b is an arbitrary positive number, the semibasic vector fields v and $\tilde v$ are defined by formulas (4.4.5) and (5.4.11), the function $\mathcal{R}[u]$ is defined by (5.3.8), (5.4.9) and (5.4.23). As far as the field z participating in (5.4.24) is concerned, in what follows we shall use only the equality

$$|\overset{h}{\nabla} u|^2 = |z|^2 + |P_\xi f|^2. \tag{5.4.25}$$

which is a consequence of (5.3.5) and (5.4.15). Note that boundary conditions (5.3.2) and (5.3.3) were not used until now.

5.5 Estimation of the boundary integral and summands depending on curvature

First we will estimate the last of the integrals on the right-hand side of inequality (5.4.24) through $\int_{\Omega M} |\overset{h}{\nabla} u|^2 d\Sigma$.

Agree to denote various constants that depend only on m and n by the same letter C.

For a Riemannian manifold (M, g), a point $x \in M$ and vectors $\xi, \eta, \lambda, \mu \in T_x M$, the inequality

$$|R_{ijkl}(x)\xi^i \eta^j \lambda^k \mu^l| \le CK(x)|\xi||\eta||\lambda||\mu|, \tag{5.5.1}$$

holds where $K(x)$ is defined by formula (5.2.7). It follows, for instance, from the explicit expression, of components of the curvature tensor through the sectional curvature, presented on p. 112 of the book [41].

From (5.3.8), (5.4.9) and (5.4.23) with the help of (5.5.1), we see that the inequality

$$|\mathcal{R}[u](x,\xi)| \le CK(x)(|u(x,\xi)|^2 + |\overset{v}{\nabla} u(x,\xi)|^2) \tag{5.5.2}$$

holds on ΩM under the assumption that the number b in (5.4.23) satisfies the condition $0 < b \le 1$.

By boundary condition (5.3.3), the fields u and $\overset{v}{\nabla} u$ vanish on $\partial_-\Omega M$. Integrating (5.5.2) and applying Lemma 4.5.1, we arrive at the relation

$$\int\limits_{\Omega M} |\mathcal{R}[u]| d\Sigma \le Ck \int\limits_{\Omega M} (|Hu|^2 + |H\overset{v}{\nabla} u|^2) d\Sigma, \tag{5.5.3}$$

where $k = k(M, g)$ is given by formula (5.2.8).

Estimate $\int_{\Omega M} |H\overset{v}{\nabla} u|^2 d\Sigma$ through $\int_{\Omega M} |\overset{h}{\nabla} u|^2 d\Sigma$. To this end, note that the definition of H implies the commutation formula $\overset{v}{\nabla} H - H\overset{v}{\nabla} = \overset{h}{\nabla}$. With the help of the last formula, we obtain

$$|H\overset{v}{\nabla} u|^2 \le 2(|\overset{h}{\nabla} u|^2 + |\overset{v}{\nabla} Hu|^2). \tag{5.5.4}$$

Moreover, the inequality

$$|Hu|^2 = |\langle \xi, \overset{h}{\nabla} \rangle u|^2 \le |\overset{h}{\nabla} u|^2. \tag{5.5.5}$$

holds on ΩM. Using (5.5.4) and (5.5.5), we transform (5.5.3) to find

$$\int\limits_{\Omega M} |\mathcal{R}[u]|\, d\Sigma \le Ck \left[\int\limits_{\Omega M} |\overset{h}{\nabla} u|^2 d\Sigma + \int\limits_{M} \left(\int\limits_{\Omega_x M} |\overset{v}{\nabla} Hu|^2 d\omega_x(\xi) \right) dV^n(x) \right]. \tag{5.5.6}$$

We fix a point $x \in M$ and choose coordinates in some of its neighbourhoods so that $g_{ij}(x) = \delta_{ij}$. Applying the operator $\overset{v}{\nabla}_i = \partial / \partial \xi^i$ to the equality

$$(Hu)_{i_1 \dots i_m} = \left(\delta^{j_1}_{i_1} - \frac{\xi_{i_1} \xi^{j_1}}{|\xi|^2} \right) \dots \left(\delta^{j_m}_{i_m} - \frac{\xi_{i_m} \xi^{j_m}}{|\xi|^2} \right) f_{j_1 \dots j_m}(x),$$

that follows from (5.3.5) and (5.2.1), we obtain the representation

$$(\overset{v}{\nabla} Hu)_{i_1 \dots i_{m+1}} = |\xi|^{-2m-2} P^{j_1 \dots j_m}_{i_1 \dots i_{m+1}}(\xi) f_{j_1 \dots j_m}(x)$$

with some polynomials $P^{j_1\dots j_m}_{i_1\dots i_{m+1}}(\xi)$ independent of x. The representation implies the estimate

$$\int\limits_{\Omega_x M} |\overset{v}{\nabla} Hu|^2 d\omega_x(\xi) \le C|f(x)|^2. \tag{5.5.7}$$

It follows from Lemma 5.6.1 given below that the quadratic form $\int_{\Omega_x M} |P_\xi f|^2 d\omega_x(\xi)$ is positive-definite on $S^m T'_x M$, and the estimate

$$|f|^2 \le C \int\limits_{\Omega_x M} |P_\xi f|^2 d\omega_x(\xi) \tag{5.5.8}$$

holds with a constant C independent of x.

Combining (5.5.7) and (5.5.8), we obtain

$$\int\limits_{\Omega_x M} |\overset{v}{\nabla} Hu|^2 d\omega_x(\xi) \le C \int\limits_{\Omega_x M} |Hu|^2 d\omega_x(\xi). \tag{5.5.9}$$

Inequalities (5.5.5) and (5.5.9) allows estimate (5.5.6) to take its final form:

$$\int\limits_{\Omega M} |\mathcal{R}[u]|\, d\Sigma \le Ck \int\limits_{\Omega M} |\overset{h}{\nabla} u|^2 d\Sigma. \tag{5.5.10}$$

With the help of (5.5.10), our main inequality (5.4.24) implies the next one:

$$\int\limits_M \left\{ \int\limits_{\Omega_x M} \left[(1-b)|z|^2 + (n-1)|P_\xi f|^2 - \frac{m^2}{b} |P_\xi j_\xi f|^2 \right] d\omega_x(\xi) \right\} dV^n(x) -$$

$$- Ck \int\limits_{\Omega M} |\overset{h}{\nabla} u|^2 d\Sigma \le \int\limits_{\partial\Omega M} \langle b\tilde v - v, \nu\rangle \, d\Sigma^{2n-2}. \tag{5.5.11}$$

Note that, deriving this estimate, we have used boundary condition (5.3.3), but (5.3.2) was not used until now.

We now estimate the right-hand side of (5.5.11) through Jf.

From (4.4.5) and (5.4.11) we obtain, on $\partial\Omega M$,

$$\langle b\tilde v - v, \nu\rangle = b(\nu_i u^{ii_2\dots i_m} \overset{h}{\nabla}{}^j u_{ji_2\dots i_m} - \nu^i u^{ji_2\dots i_m} \overset{h}{\nabla}_j u_{ii_2\dots i_m}) +$$

$$+ \nu_i \xi^j \overset{v}{\nabla}{}^i u^{i_1\dots i_m} \cdot \overset{h}{\nabla}_j u_{i_1\dots i_m} - \nu_i \xi^i \overset{h}{\nabla}{}^j u^{i_1\dots i_m} \cdot \overset{v}{\nabla}_j u_{i_1\dots i_m}. \tag{5.5.12}$$

Show that the right-hand side of (5.5.12) depends only on the restriction of u to $\partial\Omega M$. To this end we choose a coordinate system $x^1,\dots x^n$ in a neighbourhood of a point $x_0 \in \partial M$ such that the boundary ∂M is determined by the equation $x^n = 0$ and $g_{in} = \delta_{in}$. Then the vector ν has the coordinates $(0,\dots,0,1)$ and (5.5.12) is written as:

$$\langle b\tilde v - v, \nu\rangle = Lu \equiv b(u^{n\, i_2\dots i_m} \overset{h}{\nabla}{}^\alpha u_{\alpha i_2\dots i_m} - u^{\alpha i_2\dots i_m} \overset{h}{\nabla}_\alpha u_{n\, i_2\dots i_m}) +$$

$$+ \xi^\alpha \overset{v}{\nabla}{}^n u^{i_1\dots i_m} \cdot \overset{h}{\nabla}_\alpha u_{i_1\dots i_m} - \xi^n \overset{h}{\nabla}{}^\alpha u^{i_1\dots i_m} \cdot \overset{v}{\nabla}_\alpha u_{i_1\dots i_m}, \tag{5.5.13}$$

where the summation from 1 to $n-1$ over index α is performed.

It is essential that Lu does not contain the derivative $\overset{h}{\nabla}_n\, u$. So if $y^1, \dots, y^{2n-2}$ is a local coordinate system on $\partial\Omega M$, then Lu is expressible by a quadratic form in the components of the fields u, $\partial u/\partial y^i$ and $\partial u/\partial|\xi|$. By homogeneity (5.3.4), $\partial u/\partial|\xi| = -u$ and, consequently, L is a quadratic differential operator on the bundle $p^*(S^m\tau'_M)$ where $p : \partial\Omega M \to M$ is the restriction of the projection of the tangent bundle. So (5.5.13) implies the estimate

$$\left| \int\limits_{\partial\Omega M} \langle b\tilde{v} - v, \nu\rangle \, d\Sigma^{2n-2} \right| \le D\|u|_{\partial\Omega M}\|_1^2, \tag{5.5.14}$$

where a constant D depends on m and (M, g), unlike the constant C on (5.5.11). Dependence D on b can be extracted by assuming that $0 < b \le 1$.

Recalling boundary conditions (5.3.2) and (5.3.3), we rewrite estimate (5.5.14) as

$$\left| \int\limits_{\partial\Omega M} \langle b\tilde{v} - v, \nu\rangle \, d\Sigma^{2n-2} \right| \le D\|Jf\|_1^2. \tag{5.5.15}$$

With the help of (5.5.15), our main inequality (5.5.11) takes the form

$$\int\limits_M \left\{ \int\limits_{\Omega_x M} \left[(1-b)|z|^2 + (n-1)|P_\xi f|^2 - \frac{m^2}{b}|P_\xi j_\xi f|^2 \right] d\omega_x(\xi) \right\} dV^n(x) -$$

$$- Ck \int\limits_{\Omega M} |\overset{h}{\nabla} u|^2 d\Sigma \le D\|Jf\|_1^2. \tag{5.5.16}$$

5.6 Proof of Theorem 5.2.2

The next claim contains the main algebraic difficulties of our problem.

Lemma 5.6.1 *Let m, n be integers satisfying the conditions $n \ge 3$, $n > m \ge 1$. For every Riemannian manifold (M, g) of dimension n and every point $x \in M$, the quadratic form*

$$\langle Bf, f\rangle = \int\limits_{\Omega_x M} \left[(n-1)|P_\xi f|^2 - m^2|P_\xi j_\xi f|^2 \right] d\omega_x(\xi) \tag{5.6.1}$$

is positive-definite on $S^m T'_x M$ and the estimate

$$\int\limits_{\Omega_x M} \left[(n-1)|P_\xi f|^2 - m^2|P_\xi j_\xi f|^2 \right] d\omega_x(\xi) \ge \delta|f|^2 \tag{5.6.2}$$

holds with a positive coefficient δ depending only on m and n.

The proof of this lemma will be given later, and now we will finish the proof of Theorem 5.2.2 by making use of the lemma.

Proving Lemma 5.6.1, we will see that the estimate

$$\int\limits_{\Omega_x M} |P_\xi f|^2 d\omega_x(\xi) \le C|f|^2 \tag{5.6.3}$$

holds with a constant C depending only on m and n. From (5.6.2) and (5.6.3), we obtain the inequality

$$\int\limits_{\Omega_x M} \Big[(n-1)|P_\xi f|^2 - m^2|P_\xi j_\xi f|^2\Big]\, d\omega_x(\xi) \ge \delta \int\limits_{\Omega_x M} |P_\xi f|^2 d\omega_x(\xi), \qquad (5.6.4)$$

where the coefficient δ differs from that used in (5.6.2), but is also positive and depends only on m and n.

In particular, (5.6.2) implies that

$$\int\limits_{\Omega_x M} |P_\xi j_\xi f|^2 d\omega_x(\xi) \le C_1 \int\limits_{\Omega_x M} |P_\xi f|^2 d\omega_x(\xi) \qquad (5.6.5)$$

with a constant C_1 depending only on m and n.

Assuming that $0 < b \le 1$, it follows from (5.6.4) and (5.6.5) that

$$\int\limits_{\Omega_x M} \left[(n-1)|P_\xi f|^2 - \frac{m^2}{b}|P_\xi j_\xi f|^2\right] d\omega_x(\xi) =$$

$$= \int\limits_{\Omega_x M} \Big[(n-1)|P_\xi f|^2 - m^2|P_\xi j_\xi f|^2\Big]\, d\omega_x(\xi) - m^2(1/b-1) \int\limits_{\Omega_x M} |P_\xi j_\xi f|^2 d\omega_x(\xi) \ge$$

$$\ge \Big(\delta - m^2 C_1\,(1/b-1)\Big) \int\limits_{\Omega_x M} |P_\xi f|^2 d\omega_x(\xi).$$

With the help of this inequality, (5.5.16) implies that

$$\int\limits_{\Omega M} \left[(1-b)|z|^2 + \Big(\delta - m^2 C_1\,(1/b-1)\Big)\, |P_\xi f|^2 - Ck|\overset{h}{\nabla} u|^2\right] d\Sigma \le D\|Jf\|_1^2. \qquad (5.6.6)$$

Until now the number b was subordinate to the only condition $0 < b \le 1$. We choose it in such a way that

$$\delta - m^2 C_1\,(1/b-1) = \delta/2$$

and denote by δ_1 the minimum of the numbers $1-b$ and $\delta/2$. then (5.6.6) implies that

$$\int\limits_{\Omega M} \left[\delta_1(|z|^2 + |P_\xi f|^2) - Ck|\overset{h}{\nabla} u|^2\right] d\Sigma \le D\|Jf\|_1^2.$$

Recalling (5.4.25), we rewrite the last inequality as:

$$(\delta_1 - Ck) \int\limits_{\Omega M} |\overset{h}{\nabla} u|^2 d\Sigma \le D\|Jf\|_1^2. \qquad (5.6.7)$$

The constants δ_1 and C on (5.6.7) depend only on m and n, and the quantity $k = k(M,g)$ satisfies condition (5.2.9) of Theorem 5.2.2. Choosing the value of $\varepsilon(m,n)$ in the formulation of the theorem in such a way that $\delta_1 - C\varepsilon(m,n) = \delta_1/2$, (5.6.7) implies the estimate

$$\int\limits_{\Omega M} |\overset{h}{\nabla} u|^2 d\Sigma \le D_1\|Jf\|_1^2, \qquad (5.6.8)$$

where $D_1 = 2D/\delta_1$.

With the help of the inequality

$$|f(x)|^2 \le C \int\limits_{\Omega_x M} |P_\xi f(x)|^2 d\omega_x(\xi)$$

following from Lemma 5.6.1 and equation (5.3.5), we obtain

$$\|f\|_0^2 = \int\limits_M |f(x)|^2 dV^n(x) \le C \int\limits_M \left[\int\limits_{\Omega_x M} |P_\xi f(x)|^2 d\omega_x(\xi) \right] dV^n(x) = C \int\limits_{\Omega M} |Hu|^2 d\Sigma.$$

Using (5.5.5), we transform the last inequality to obtain:

$$\|f\|_0^2 \le C \int\limits_{\Omega M} |\overset{h}{\nabla} u|^2 d\Sigma. \tag{5.6.9}$$

Inequalities (5.6.8) and (5.6.9) imply estimate (5.2.10). Theorem 5.2.2 is proved.

5.7 Decomposition of the operators A_0 and A_1

The remainder of the chapter is devoted to the proof of Lemma 5.6.1.

Under the conditions of this lemma, we choose an orthonormal basis for $T_x M$ and, with its help, identify $T_x M$ and the space $\mathbf{R}^n$ provided with the standard scalar product $\langle , \rangle$. Then $S^m T'_x M$ is identified with $S^m = S^m \mathbf{R}^n$, the m-th symmetric power of the space $\mathbf{R}^n$; the sphere $\Omega_x M$ is identified with $\Omega = \{\xi \in \mathbf{R}^n \mid |\xi| = 1\}$; and the measure $d\omega_x$, with the standard angle measure $d\omega$ on Ω. Let $\omega = 2\pi^{n/2}/\Gamma(n/2)$ be the volume of Ω.

We define the operators $A_0, A_1 : S^m \to S^m$ by the equalities

$$A_0 = \frac{1}{\omega} \int\limits_\Omega P_\xi \, d\omega(\xi), \qquad A_1 = \frac{1}{\omega} \int\limits_\Omega i_\xi P_\xi j_\xi \, d\omega(\xi) \tag{5.7.1}$$

and put

$$B = (n-1)A_0 - m^2 A_1. \tag{5.7.2}$$

Then, for $f \in S^m$,

$$\langle Bf, f \rangle = \frac{1}{\omega} \int\limits_\Omega \left[(n-1)|P_\xi f|^2 - m^2 |P_\xi j_\xi f|^2 \right] d\omega(\xi).$$

Thus Lemma 5.6.1 is equivalent to positive definiteness of the operator B.

Let $\delta = (\delta_{ij}) \in S^2$ be the Kronecker tensor. By $i : S^m \to S^{m+2}$ we denote the operator of symmetric multiplying by δ and by $j : S^{m+2} \to S^m$, the operator of convolution with δ. The operators i and j are dual to one other.

Lemma 5.7.1 *For $m \ge 1$, the equalities*

$$A_\alpha = (m-\alpha)! \, \Gamma\left(\frac{n}{2}\right) \sum_{p=0}^{[m/2]} a_\alpha(p,m,n) i^p j^p \qquad (\alpha = 0, 1) \tag{5.7.3}$$

hold on S^m*, where* $[m/2]$ *is the integral part of* $m/2$ *and*

$$a_\alpha(p,m,n) = \frac{(-1)^\alpha}{2^{2p}(p!)^2}\sum_{k=2p}^{m}(-1)^k\frac{(1+\alpha k-\alpha)k!}{2^k(m-k)!(k-2p)!\Gamma\left(k+\frac{n}{2}\right)}. \qquad (5.7.4)$$

P r o o f. First of all we note that the set $\{\eta^m \mid \eta \in \mathbf{R}^n\}$ generates the vector space S^m, as follows from the equality

$$\frac{\partial^m}{\partial t^\beta}\bigg|_{t=0}\left(\sum_i t_i e_i\right)^m = m!\, e^\beta$$

valid for a basis $e_1,\dots,e_n$ for the space $\mathbf{R}^n$ and every multi-index β of length $|\beta| = m$. By the last observation, to prove equalities (5.7.3) it suffices to show that the relations

$$\langle A_\alpha\eta^m,\zeta^m\rangle = (m-\alpha)!\,\Gamma\left(\frac{n}{2}\right)\sum_{p=0}^{[m/2]} a_\alpha(p,m,n)\langle i^p j^p \eta^m,\zeta^m\rangle \qquad (5.7.5)$$

hold for every $\eta,\zeta \in \Omega$.

Note that, for $|\eta| = |\zeta| = 1$,

$$\langle i^p j^p \eta^m,\zeta^m\rangle = \langle j^p\eta^m, j^p\zeta^m\rangle = \langle \eta^{m-2p},\zeta^{m-2p}\rangle = \langle\eta,\zeta\rangle^{m-2p}.$$

Thus the desired equalities (5.7.5) can be rewritten as:

$$\langle A_\alpha\eta^m,\zeta^m\rangle = (m-\alpha)!\,\Gamma\left(\frac{n}{2}\right)\sum_{p=0}^{[m/2]} a_\alpha(p,m,n)\langle\eta,\zeta\rangle^{m-2p}. \qquad (5.7.6)$$

By definition (5.7.1) of the operators A_α,

$$\langle A_\alpha\eta^m,\zeta^m\rangle = \frac{1}{\omega}\int_\Omega \langle P_\xi j_\xi^\alpha \eta^m, j_\xi^\alpha\zeta^m\rangle\, d\omega(\xi) \qquad (\alpha = 0,1). \qquad (5.7.7)$$

Note that, for $|\xi| = |\eta| = 1$, the relation

$$j_\xi\eta^m = \langle\eta,\xi\rangle\eta^{m-1} \qquad (5.7.8)$$

is valid and, as one can easily see from (5.2.1),

$$P_\xi\eta^m = (P_\xi\eta)^m = (\eta - \langle\eta,\xi\rangle\xi)^m. \qquad (5.7.9)$$

By (5.7.8) and (5.7.9), the integrand of (5.7.7) can be transformed as follows:

$$\langle P_\xi j_\xi^\alpha\eta^m, j_\xi^\alpha\zeta^m\rangle = (\langle\eta,\xi\rangle\langle\zeta,\xi\rangle)^\alpha\langle P_\xi\eta^{m-\alpha},\zeta^{m-\alpha}\rangle =$$

$$= (\langle\eta,\xi\rangle\langle\zeta,\xi\rangle)^\alpha(\langle\eta,\zeta\rangle - \langle\eta,\xi\rangle\langle\zeta,\xi\rangle)^{m-\alpha}.$$

Inserting this expression into (5.7.7) and denoting $x = \langle\eta,\zeta\rangle$ for brevity, we obtain

$$\langle A_\alpha\eta^m,\zeta^m\rangle = (-1)^\alpha\sum_{l=0}^{m}(-1)^l\binom{m-\alpha}{l-\alpha}x^{m-l}\frac{1}{\omega}\int_\Omega(\langle\eta,\xi\rangle\,\langle\zeta,\xi\rangle)^l\,d\omega(\xi). \qquad (5.7.10)$$

From now on agree that $\binom{p}{q} = 0$ for $q < 0$.

Calculate the integral on the right-hand side of (5.7.10). To this end choose an orthonormal basis $e_1, \dots, e_n$ for $\mathbf{R}^n$ such that $\eta = e_1,\ \zeta = xe_1 + (1-x^2)^{1/2}e_2$. Then

$$\langle \eta, \xi \rangle = \xi_1, \qquad \langle \zeta, \xi \rangle = x\xi_1 + (1-x^2)^{1/2}\xi_2.$$

Consequently,

$$\frac{1}{\omega}\int\limits_{\Omega} (\langle \eta, \xi\rangle \langle \zeta, \xi\rangle)^l \, d\omega(\xi) = \frac{1}{\omega}\int\limits_{\Omega} \xi_1^l [x\xi_1 + (1-x^2)^{1/2}\xi_2]^l \, d\omega(\xi) =$$

$$= \sum_{r=0}^{[l/2]} \binom{l}{2r} x^{l-2r}(1-x^2)^r \frac{1}{\omega}\int\limits_{\Omega} \xi_1^{2l-2r}\xi_2^{2r}\, d\omega(\xi). \tag{5.7.11}$$

As is known [112],

$$\frac{1}{\omega}\int\limits_{\Omega} \xi_1^{2l-2r}\xi_2^{2r}\, d\omega(\xi) = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi\Gamma\left(l+\frac{n}{2}\right)}\Gamma\left(l-r+\frac{1}{2}\right)\Gamma\left(r+\frac{1}{2}\right).$$

Inserting this expression into (5.7.11), we obtain

$$\frac{1}{\omega}\int\limits_{\Omega} (\langle \eta, \xi\rangle\langle \zeta, \xi\rangle)^l \, d\omega(\xi) = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi\Gamma\left(l+\frac{n}{2}\right)}\sum_{r=0}^{[l/2]}\binom{l}{2r}\Gamma\left(l-r+\frac{1}{2}\right)\Gamma\left(r+\frac{1}{2}\right)x^{l-2r}(1-x^2)^r.$$

Replacing the integral of (5.7.10) with its value from the last equality, changing the summation limits in the so-obtained double sum and introducing the notation

$$c_\alpha(r,m,n) = \Gamma\left(r+\frac{1}{2}\right)\sum_{l=2r}^{m}(-1)^l\binom{m-\alpha}{l-\alpha}\binom{l}{2r}\frac{\Gamma\left(l-r+\frac{1}{2}\right)}{\Gamma\left(l+\frac{n}{2}\right)}, \tag{5.7.12}$$

we arrive at the relation

$$\langle A_\alpha \eta^m, \zeta^m\rangle = (-1)^\alpha \frac{\Gamma\left(\frac{n}{2}\right)}{\pi}\sum_{r=0}^{[m/2]} c_\alpha(r,m,n)x^{m-2r}(1-x^2)^r.$$

Expanding the factor $(1-x^2)^r$ on the right-hand side of this equality in powers of x and changing the summation limits in the so-obtained double sum, we obtain

$$\langle A_\alpha \eta^m, \zeta^m\rangle = (-1)^\alpha \frac{\Gamma\left(\frac{n}{2}\right)}{\pi}\sum_{p=0}^{[m/2]}(-1)^p\left[\sum_{r=p}^{[m/2]}(-1)^r\binom{r}{p}c_\alpha(r,m,n)\right]x^{m-2p}. \tag{5.7.13}$$

Recalling that $x = \langle \eta, \zeta\rangle$ and comparing the above-obtained relation (5.7.13) with the desired equality (5.7.5), we see that relations (5.7.5) hold with

$$a_\alpha(p,m,n) = \frac{(-1)^{p+\alpha}}{\pi(m-\alpha)!}\sum_{r=p}^{[m/2]}(-1)^r\binom{r}{p}c_\alpha(r,m,n). \tag{5.7.14}$$

Now to finish the proof we have to verify that equalities (5.7.14) are equivalent to formula (5.7.4).

Inserting expression (5.7.12) for $c_\alpha(r,m,n)$ into (5.7.14), changing the summation limits in the so-obtained double sum and introducing the notation

$$\gamma(p,l)=\sum_{r=p}^{[l/2]}(-1)^r\binom{r}{p}\binom{l}{2r}\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(l-r+\frac{1}{2}\right), \tag{5.7.15}$$

we arrive at the equality

$$a_\alpha(p,m,n)=\frac{(-1)^{p+\alpha}}{\pi(m-\alpha)!}\sum_{l=2p}^{m}(-1)^l\binom{m-\alpha}{l-\alpha}\frac{\gamma(p,l)}{\Gamma\left(l+\frac{n}{2}\right)}. \tag{5.7.16}$$

It turns out that sum (5.7.15) can be simplified. To this end, use the well-known relations

$$\Gamma\left(r+\frac{1}{2}\right)=\frac{(2r-1)!!}{2^r}\sqrt{\pi},\quad \Gamma\left(l-r+\frac{1}{2}\right)=\frac{(2l-2r-1)!!}{2^{l-r}}\sqrt{\pi}.$$

Inserting these expressions into (5.7.15), we obtain

$$\gamma(p,l)=\frac{\pi}{2^l}\sum_{r=p}^{[l/2]}(-1)^r\binom{r}{p}\binom{l}{2r}(2r-1)!!(2l-2r-1)!!.$$

After simple transformations the last equality takes the form:

$$\gamma(p,l)=\frac{\pi(l!)^2}{2^{2l}p!(l-p)!}\sum_{r=p}^{[l/2]}(-1)^r\binom{l-p}{r-p}\binom{2l-2r}{l}.$$

We change the summation index of this sum according to the formula $k=r-p$:

$$\gamma(p,l)=(-1)^p\frac{\pi(l!)^2}{2^{2l}p!(l-p)!}\sum_{k=0}^{[l/2]-p}(-1)^k\binom{l-p}{k}\binom{2l-2p-2k}{l}. \tag{5.7.17}$$

We now use the known relation ([103], p. 620, formula (62))

$$\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{k}\binom{2n-2k}{n+m}=2^{n-m}\binom{n}{m}.$$

Putting $n=l-p,\ m=p$ in this formula, we obtain

$$\sum_{k=0}^{\left[\frac{l-p}{2}\right]}(-1)^k\binom{l-p}{k}\binom{2l-2p-2k}{l}=2^{l-2p}\binom{l-p}{p}. \tag{5.7.18}$$

Note that the upper summation limit in this formula can be decreased to $[l/2]-p$, since $\binom{2l-2p-2k}{l}=0$ for $k>[l/2]-p$. Inserting (5.7.18) into (5.7.17), we obtain

$$\gamma(p,l)=(-1)^p\frac{\pi(l!)^2}{2^{l+2p}(p!)^2(l-2p)!}. \tag{5.7.19}$$

Finally, inserting the value of $\gamma(p,l)$ from the last equality into (5.7.16), we arrive at (5.7.4). The lemma is proved.

5.8 Proof of Lemma 5.6.1

We first shall show that Lemma 5.6.1 is a consequence of the next

Lemma 5.8.1 *For $n \geq 3, m \geq 1$ and $0 \leq p \leq [m/2]$, the numbers $a_0(p,m,n)$ defined by equality* (5.7.4) *are positive. The numbers $\tilde{a}(p,m,n)$ defined by the formula*

$$\tilde{a}(p,m,n) = \frac{1}{2^{2p}(p!)^2} \sum_{k=2p}^{m} (-1)^k \frac{(k+1)!}{2^k (m-k)!(k-2p)!\Gamma\left(k+\frac{n}{2}\right)} \tag{5.8.1}$$

are positive in each of the next two cases: 1) $n=3,\ 1 \leq m \leq 2,\ 0 \leq p \leq [m/2]$; *and* 2) $n \geq 4,\ m \geq 1,\ 0 \leq p \leq [m/2]$.

Indeed, by Lemma 5.7.1 the operator $B = (n-1)A_0 - m^2 A_1$ can be represented as:

$$B = m!\,\Gamma\left(\frac{n}{2}\right) \sum_{p=0}^{[m/2]} b(p,m,n) i^p j^p, \tag{5.8.2}$$

where

$$b(p,m,n) = (n-1)a_0(p,m,n) - m\,a_1(p,m,n). \tag{5.8.3}$$

On assuming that $n > m \geq 1$ and using positiveness of $a_0(p,m,n)$, from equality (5.8.3) it follows that

$$b(p,m,n) \geq m[a_0(p,m,n) - a_1(p,m,n)]. \tag{5.8.4}$$

Comparing (5.7.4) and (5.8.1), we see that

$$a_0(p,m,n) - a_1(p,m,n) = \tilde{a}(p,m,n). \tag{5.8.5}$$

Relations (5.8.4) and (5.8.5) show that positiveness of $\tilde{a}(p,m,n)$ implies the same for $b(p,m,n)$.

Thus, for $n > m \geq 1$ and $n \geq 3$ all coefficients of sum (5.8.2) are positive. So, for $0 \neq f \in S^m$,

$$\langle Bf, f\rangle = m!\Gamma\left(\frac{n}{2}\right) \sum_{p=0}^{[m/2]} b(p,m,n)\langle j^p f, j^p f\rangle \geq m!\,\Gamma\left(\frac{n}{2}\right) b(0,m,n)|f|^2 > 0.$$

This proves Lemma 5.6.1.

P r o o f of Lemma 5.8.1. We consider the integral

$$I_\alpha(p,m,n) = \int_0^1 x^{2p+\alpha}(1-x)^{n/2-2-\alpha}\left(1-\frac{x}{2}\right)^{m-2p} dx. \tag{5.8.6}$$

It converges under the next restrictions on the parameters:

$$0 \leq p \leq [m/2], \quad 0 \leq \alpha < \frac{n}{2} - 1 \tag{5.8.7}$$

and, evidently, it is a positive number.

We transform the right-hand side of (5.8.6), expanding the last factor of the integrand in powers of x :

$$I_\alpha(p,m,n) = (m-2p)! \sum_{r=0}^{m-2p} \frac{(-1)^r}{2^r r!(m-2p-r)!} \int_0^1 x^{r+2p+\alpha}(1-x)^{n/2-2-\alpha}dx.$$

Expressing the integral on the right-hand side of the last equality through the Γ-function, we obtain

$$I_\alpha(p,m,n) = \Gamma\left(\frac{n}{2}-1-\alpha\right)(m-2p)! \sum_{r=0}^{m-2p} (-1)^r \frac{\Gamma(r+2p+\alpha+1)}{2^r r!(m-2p-r)!\Gamma\left(r+2p+\frac{n}{2}\right)}.$$

Changing the summation index of the last sum by the formula $k = r+2p$, we arrive at the relation

$$I_\alpha(p,m,n) = 2^{2p}\Gamma\left(\frac{n}{2}-1-\alpha\right)(m-2p)! \sum_{k=2p}^{m} (-1)^k \frac{\Gamma(k+1+\alpha)}{2^k (m-k)!(k-2p)!\Gamma\left(k+\frac{n}{2}\right)}. \tag{5.8.8}$$

We put $\alpha = 0$ in (5.8.8), In this case restrictions (5.8.7) look as: $n \geq 3,\ 0 \leq p \leq [m/2]$. As the result, we have

$$I_0(p,m,n) = 2^{2p}\Gamma\left(\frac{n}{2}-1\right)(m-2p)! \sum_{k=2p}^{m} (-1)^k \frac{k!}{2^k (m-k)!(k-2p)!\Gamma\left(k+\frac{n}{2}\right)}.$$

The right-hand side of the last equality differs only by a positive factor from the right-hand side of (5.7.4) for $\alpha = 0$. So positiveness of $I_0(p,m,n)$ implies that of $a_0(p,m,n)$. Thus the first claim of the lemma is proved.

We now put $\alpha = 1$ in (5.8.8). In this case restrictions (5.8.7) look as: $n \geq 5,\ 0 \leq p \leq [m/2]$. As a result, we have

$$I_1(p,m,n) = 2^{2p}\Gamma\left(\frac{n}{2}-2\right)(m-2p)! \sum_{k=2p}^{m} (-1)^k \frac{(k+1)!}{2^k (m-k)!(k-2p)!\Gamma\left(k+\frac{n}{2}\right)}.$$

The right-hand side of the last equality differs only by a positive factor from the right-hand side of (5.8.1) and, consequently, positiveness of $I_1(p,m,n)$ implies that of $\tilde{a}(p,m,n)$. This proves the second claim of the lemma for $n \geq 5$.

For $n = 4$, equality (5.8.1) has the form

$$\tilde{a}(p,m,4) = \frac{1}{2^{2p}(p!)^2} \sum_{k=2p}^{m} (-1)^k \frac{1}{2^k (m-k)!(k-2p)!}.$$

Changing the summation index by the formula $k = 2p + r$, we obtain

$$\tilde{a}(p,m,4) = \frac{1}{2^{4p}(p!)^2(m-2p)!} \sum_{r=0}^{m-2p} (-1)^r \binom{m-2p}{r} \frac{1}{2^r} = \frac{(1-1/2)^{m-2p}}{2^{4p}(p!)^2(m-2p)!} > 0.$$

Finally, for $n = 3$, the second claim of the lemma is verified by the direct calculation according to formula (5.8.1).

5.9 Final remarks

As we have already noted, the restriction $n > m$ is most likely to be unessential for validity of Theorem 5.2.2, it is rather due to the method of the proof. But this restriction does not exhaust the possibilities of our method. Indeed, this restriction has been used only in the proof of Lemma 5.6.1. The claim of the lemma is equivalent to the positive definiteness of the operator B. Thus Theorem 5.2.2 is valid in all cases when the next question has a positive answer.

Problem 5.9.1 *For what values of m and n is the operator*

$$B = (n-1)A_0 - m^2 A_1 = \frac{1}{\omega}\int\limits_{\Omega}[(n-1)P_\xi - m^2 i_\xi P_\xi j_\xi]d\omega(\xi)$$

positive-definite on the space $S^m\mathbf{R}^n$?

Lemma 5.6.1 gives only a partial answer to this question. The author failed in finding a full answer.

If the condition $n > m$ is replaced with $n - 1 \ge \beta m$ for some $\beta < 1$, then inequality (5.8.4) becomes

$$b(p,m,n) \ge m[\beta a_0(p,m,n) - a_1(p,m,n)] = m\tilde a_\beta(p,m,n),$$

where

$$\tilde a_\beta(p,m,n) = \frac{1}{2^{2p}(p!)^2}\sum_{k=2p}^{m}(-1)^k\frac{(k+\beta)k!}{2^k(m-k)!(k-2p)!\Gamma\left(k+\frac{n}{2}\right)}.$$

Thus the problem reduces to the question: for what β are the numbers $\tilde a_\beta(p,m,n)$ positive in the domain $n-1 \ge \beta m$, $0 \le p \le [m/2]$?

Another possible approach to Problem 5.9.1 is in finding the eigenvalues of the operator B and examining their positiveness. As one can show, each of the operators A_α $(\alpha = 0, 1)$ has $[m/2]+1$ eigenspaces $S^m_k \subset S^m$ such that S^m_k $(0 \le k \le [m/2])$ consists of the tensors of the type $i^k f$, where $f \in S^{m-2k}$ satisfies the condition $jf = 0$. The space S^m_k belongs to the eigenvalue $\lambda_0(m-2k,m,n)$, of the operator A_0, which is expressed by the equality

$$\lambda_0(m-2k,m,n) = \frac{n-2}{n+2m-4k-2}\mu(m-2k,m,n)$$

through the coefficients of the Fourier expansion

$$\frac{t^m|t|^{n-2}}{(1-t^2)^{(n-2)/2}} = \sum_{k=-\infty}^{[m/2]}\mu(m-2k,m,n)C^{(n/2-1)}_{m-2k}(t) \tag{5.9.1}$$

in the Gegenbauer polynomials. At the same time S^m_k belongs to the eigenvalue $\lambda_1(m-2k,m,n)$ of the operator A_1 related, for $m-2k > 0$, to the coefficients of expansion (5.9.1) by the equality

$$\lambda_1(m-2k,m,n) = \frac{n-2}{m(n+2m-4k-2)}\times$$
$$\times[(m+n-2)\mu(m-2k,m,n) - (m+n-3)\mu(m-2k,m-2,n)]$$

If $m - 2k = 0$, then the correspondent eigenvalue of the operator A_1 is zero. Thus our problem reduces to verification, of positiveness of some linear combinations of the coefficients of expansion (5.9.1), which is connected with tremendous algebraic difficulties. In despair of finding a theoretical solution to Problem 5.9.1, the author took an experimental way and calculated the eigenvalues of the operator B for $1 \le m, n \le 50$. The results of the calculation allow one to formulate the conjecture that the problem has a positive answer for $n \ge (8m + 23)/13$.

Chapter 6

The truncated transverse ray transform

The reader, perhaps, has already paid attention to the distinction, between systems (2.16.1) and (5.1.64), which exists even in the case of a homogeneous medium when the torsion κ is equal to zero. Meanwhile, both systems describe the same physical process, evolution of the polarization ellipse along a light ray.

The mentioned distinction relates to polarization measurements performed in practice. In the previous chapter we assumed that fundamental matrix (5.1.66) of system (5.1.65) is known for every ray. In physical terms, it means that one has to measure not only polarization ellipse but also the phases of electric vector oscillations in two mutually perpendicular directions. However, only the difference of the phases is usually measured in practical polarimetry. This circumstance compels us to replace (5.1.65) with a system similar to (2.16.1) and the transverse ray transform, with some new integral geometry operator which is called the truncated transverse ray transform.

In the first section we discuss the transition from (5.1.65) to a system of the type of (2.16.1). This transition is well known in the physical literature (for instance, see [1]). In the remainder of the chapter our presentation mainly follows the paper [127].

In Section 6.2 we define the truncated transverse ray transform of a tensor field of degree m on a compact dissipative Riemannian manifold of dimension n and formulate the main result of the chapter, Theorem 6.2.2, which asserts that in sufficiently simple cases this transform is invertible on the subspace of fields orthogonal to the metric tensor. Sections 6.3–6.5 contain the proof of Theorem 6.2.2. It is interesting that, for $n > 3$ the proof repeats, almost word by word, the corresponding arguments of the previous chapter; a supplementary complication arises only in the case $n = 3, m = 2$ which is of the main interest from the standpoint of the physical interpretation. In Section 6.6 we obtain an inversion formula for the truncated transverse ray transform on Euclidean space in the case $n = 3, m = 2$.

6.1 The polarization ellipse

Let us return to considering equation (5.1.56) on assumption that the tensor $\chi = \chi_{ij}$ is real and symmetric. We fix a ray $\gamma : [0,1] \to \mathbf{R}^3$ and an orthonormal basis $e_1(\tau), e_2(\tau), e_3(\tau) = c\dot\gamma(\tau)$ that is parallel along γ in the sense of metric (5.1.17). Let $\eta(\tau) = \eta_1(\tau)e_1(\tau) +$

$\eta_2(\tau)e_2(\tau)$ be the expansion of a solution to equation (5.1.56) in this basis, and χ_{ij} be the components of the tensor χ in this basis. Equation (5.1.56) is equivalent to the system

$$\begin{aligned} \frac{d\eta_1}{d\tau} &= \frac{i}{2n^2}\left(\chi_{11}\eta_1 + \chi_{12}\eta_2\right), \\ \frac{d\eta_2}{d\tau} &= \frac{i}{2n^2}\left(\chi_{21}\eta_1 + \chi_{22}\eta_2\right). \end{aligned} \tag{6.1.1}$$

The vectors η and $E = An^{-1}\eta$ are complex. It is the real vector

$$\xi(\tau, t) = \mathrm{Re}\left[\eta(\tau)e^{i(k\tau-\omega t)}\right]$$

that has a physical meaning. We fix a point $\tau = \tau_0$ on the ray. The end of the vector

$$\xi(t) = \mathrm{Re}\left[(\eta_1 e_1 + \eta_2 e_2)\, e^{i(k\tau-\omega t)}\right] \tag{6.1.2}$$

runs an ellipse in the plane of the vectors e_1, e_2; it is called the *polarization ellipse*. Let us express its parameters through η_1, η_2.

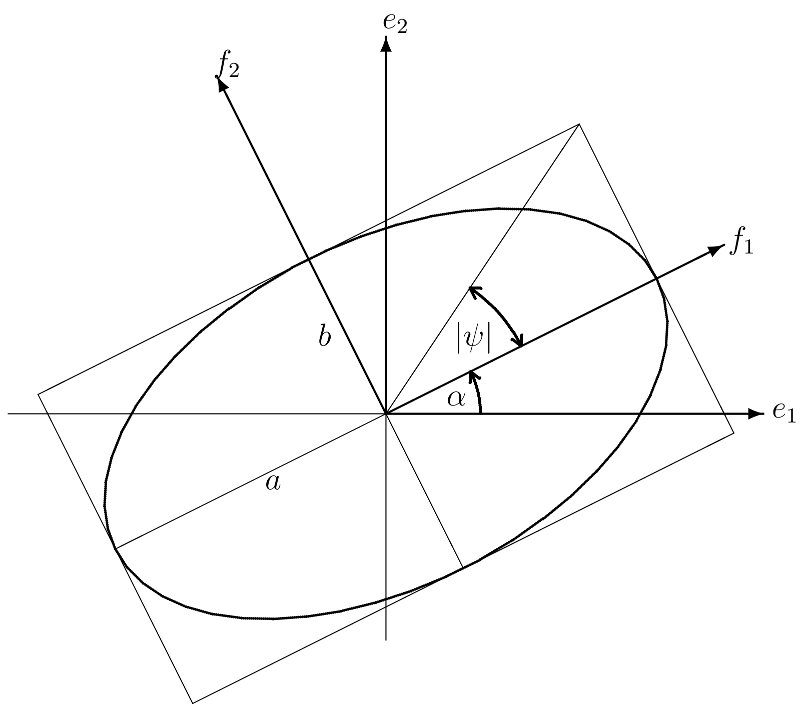

Fig. 3

Let f_1, f_2 be an orthonormal basis for the polarization plane whose orientation coincides with that of e_1, e_2 and the vector f_1 is directed along the major axis of the polarization ellipse. Let α be the angle of the rotation from the basis e_1, e_2 to the basis f_1, f_2, i.e.,

$$f_1 = e_1 \cos\alpha + e_2 \sin\alpha, \qquad f_2 = -e_1 \sin\alpha + e_2 \cos\alpha \tag{6.1.3}$$

If a, b are the semiaxes of the ellipse, then

$$\xi(t) = a\cos(\varphi_0 - \omega t) f_1 \pm b \sin(\varphi_0 - \omega t) f_2 = \mathrm{Re}\left[(af_1 \mp ibf_2)\, e^{i(\varphi_0-\omega t)}\right]. \tag{6.1.4}$$

The choice of the sign in these equalities depends on whether the polarization is right or left, i.e., on the direction in which the end of the vector $\xi(t)$ runs over the ellipse as the time increases. Inserting values (6.1.3) of the vectors f_1 and f_2 into the last formula, we obtain

$$\xi(t) = \mathrm{Re}\left[((a\cos\alpha \pm ib\sin\alpha)e_1 + (a\sin\alpha \mp ib\cos\alpha)e_2)\, e^{i(\varphi_0-\omega t)}\right].$$

Comparing this equality with (6.1.2), we conclude

$$\eta_1 = (a\cos\alpha \pm ib\sin\alpha)e^{i(\varphi_0-k\tau)}, \quad \eta_2 = (a\sin\alpha \mp ib\cos\alpha)e^{i(\varphi_0-k\tau)}. \tag{6.1.5}$$

The vector η has unit length in metric (5.1.17), i.e., $|\eta_1|^2 + |\eta_2|^2 = n^{-2}$. In view of the last equality from formulas (6.1.4) we find $a^2 + b^2 = n^{-2}$. We define the angle ψ by the conditions

$$a = n^{-2}\cos\psi, \quad b = n^{-2}\sin\psi, \quad -\pi/2 < \psi \le \pi/2. \tag{6.1.6}$$

Note that $|\psi|$ coincides with the angle between the major axis of the polarization ellipse and its diagonal, i.e., determines the shape of the ellipse. The sign of ψ depends on whether the polarization is right or left.

We define a complex number Φ by the conditions

$$\tan\Phi = \eta_2/\eta_1, \qquad -\pi/2 < \mathrm{Re}\,\Phi \le \pi/2. \tag{6.1.7}$$

From (6.1.5) and (6.1.6), we obtain

$$\tan\Phi = \frac{\tan\alpha - i\tan\psi}{1 + i\tan\alpha\tan\psi}. \tag{6.1.8}$$

This formula shows that Φ is uniquely determined by the angles α and ψ. To make this dependence clearer, we introduce some new quantity $\tilde\psi$ by the relations

$$\tan\psi = -\tanh\tilde\psi, \qquad -\infty < \tilde\psi \le \infty. \tag{6.1.9}$$

Note that (6.1.9) establishes a one-to-one correspondence between ψ and $\tilde\psi$. Now formula (6.1.8) takes the form:

$$\tan\Phi = \frac{\tan\alpha + i\tanh\tilde\psi}{1 - i\tan\alpha\tanh\tilde\psi} = \frac{\tan\alpha + \tan(i\tilde\psi)}{1 - \tan\alpha\tan(i\tilde\psi)} = \tan(\alpha + i\tilde\psi).$$

By the above-imposed restrictions on α and $\mathrm{Re}\,\Phi$, this implies that

$$\Phi = \alpha + i\tilde\psi \tag{6.1.10}$$

The next claim summarizes the above considerations: *the complex ratio η_2/η_1 of the components of the vector η is in one-to-one correspondence with the pair of the angles (α, ψ) that determine the shape and disposition of the polarization ellipse.*

In the case of an isotropic medium, when $\chi \equiv 0$, the vector $\eta(\tau)$ is parallel along the ray γ and, consequently, the ratio η_2/η_1 is constant on the ray. Thus we arrive to the *Rytov law for isotropic media*: the angles α and ψ are constant along a ray. To find a Euclidean version of the law, we address system (5.1.64) that has the next form in the case of an isotropic medium:

$$dF_\nu/ds = -\kappa F_\beta, \qquad dF_\beta/ds = \kappa F_\nu. \tag{6.1.11}$$

Introduce a complex number Ψ by the conditions

$$\tan\Psi = F_\beta/F_\nu, \qquad -\pi/2 < \operatorname{Re}\Psi \le \pi/2 \tag{6.1.12}$$

and denote by θ the angle between the vectors F and ν. As above, we obtain the equality

$$\Psi = \theta + i\tilde{\psi}. \tag{6.1.13}$$

From (6.1.12), we conclude

$$\frac{d\Psi}{ds} = \frac{d}{ds}\left(\arctan\frac{F_\beta}{F_\nu}\right) = \kappa.$$

Comparing the last equality with (6.1.13), we arrive at the Euclidean version of the Rytov law:

$$\frac{d\theta}{ds} = \kappa. \tag{6.1.14}$$

We return to considering a quasi-isotropic medium. In practical polarimetry one usually measures the angles α and ψ that are represented in Fig. 3. As far as the phase φ_0 in (6.1.5) is concerned, measuring this quantity is connected with measuring distances that are comparable with the wavelength; therefore the phase φ_0 is not usually measured.

Let us address the inverse problem of determining the tensor field χ from results of polarization measurements. We assume that a quasi-isotropic medium under investigation is contained in a bounded domain $D \subset \mathbf{R}^3$, metric (5.1.17) is known, and the angles α and ψ can be measured for outcoming light along every geodesic $\gamma : [0,1] \to D$ with the endpoints on the boundary of D. We denote by $U(\gamma)$ the fundamental matrix of system (6.1.1), i.e.,

$$\begin{pmatrix} \eta_1(1) \\ \eta_2(1) \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} \eta_1(0) \\ \eta_2(0) \end{pmatrix}, \qquad U(\gamma) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \tag{6.1.15}$$

In Section 5.1, considering the inverse problem, we assumed the matrix $U(\gamma)$ to be completely known. Now, by the above conclusion, we assume that the ratio $\eta_2(1)/\eta_1(1)$ is known as a function of the ratio $\eta_2(0)/\eta_1(0)$, for all solutions to system (6.1.1). As one can easily see, this is equivalent to the fact that the matrix $U(\gamma)$ is known up to an arbitrary scalar factor. In other words, the results of the measurement do not change if a solution $(\eta_1(\tau), \eta_2(\tau))$ is multiplied by $e^{i\lambda(\tau)}$, where $\lambda(\tau)$ is an arbitrary real function on the ray.

Using the last observation, we change the variables in system (6.1.1) as follows:

$$f = \frac{i}{2n^2}\chi, \qquad \zeta = \exp\left[-\frac{i}{4n^2}\int\limits_{\tau_0}^{\tau}(\chi_{11}+\chi_{22})\,d\tau\right]\eta.$$

Then system (6.1.1) is transformed to the next one

$$\begin{aligned} \frac{d\zeta_1}{d\tau} &= \frac{1}{2}\left(f_{11}-f_{22}\right)\zeta_1 + f_{12}\zeta_2, \\ \frac{d\zeta_2}{d\tau} &= f_{21}\zeta_1 + \frac{1}{2}\left(f_{22}-f_{11}\right)\zeta_2 \end{aligned} \tag{6.1.16}$$

that is similar to (2.16.1) in structure.

As compared with (6.1.1), system (6.1.16) has the next advantage: the results of the measurements allow us to completely determine the fundamental matrix $U(\gamma)$ of system (6.1.16). Indeed, note that the trace of the matrix of this system is equal to zero. Therefore the fundamental matrix of the system satisfies the condition

$$\det U(\gamma) = 1. \tag{6.1.17}$$

Assume that, for every solution $\zeta(\tau)$ to system (6.1.16), the ratio $\zeta_2(1)/\zeta_1(1)$ is known as a function on $\zeta_2(0)/\zeta_1(0)$. As above, this allows us to determine the matrix $U(\gamma)$ up to some factor. This factor is found from condition (6.1.17).

Equations (6.1.16) are written in a basis $e_1(\tau), e_2(\tau), e_3(\tau) = c\dot\gamma(\tau)$ related to the ray γ. To find an invariant form of the equations, we note that the matrix of system (6.1.16) considered as a symmetric tensor on $\mathbf{R}^3$

$$Q_{\dot\gamma} f = \begin{pmatrix} \frac{1}{2}(f_{11} - f_{22}) & f_{12} & 0 \\ f_{21} & \frac{1}{2}(f_{22} - f_{11}) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is the orthogonal projection of the tensor $f = (f_{ij})$ onto the subspace $\operatorname{Ker} j \bigcap \operatorname{Ker} j_{\dot\gamma}$, of the space $S^2\mathbf{R}^3$ of symmetric tensors of degree 2, which is defined by the equations

$$jf = h^{ij} f_{ij} = 0, \qquad (j_{\dot\gamma} f)_i = f_{ij}\dot\gamma^j = 0,$$

where (h_{ij}) is metric tensor (5.1.17). The subspace $\operatorname{Ker} j \bigcap \operatorname{Ker} j_{\dot\gamma} \subset S^2\mathbf{R}^3$ depends only on the vector $\dot\gamma(\tau)$ but not on the choice of the orthonormal basis $e_1(\tau), e_2(\tau), e_3(\tau) = c\dot\gamma(\tau)$. Consequently, the orthogonal projection $Q_{\dot\gamma} : S^2\mathbf{R}^3 \to S^2\mathbf{R}^3$ onto this subspace is well-defined. Thus system (6.1.16) takes the invariant form:

$$\frac{D\zeta}{d\tau} = (Q_{\dot\gamma} f)\zeta. \tag{6.1.18}$$

As in Section 5.1.6, we now distract ourselves from the initial physical situation and consider equation (6.1.18) for a domain $D \subset \mathbf{R}^n$ and for a Riemannian metric (5.1.67), of general type, which is given on D.

If the tensor field f is a multiple of the metric tensor, i.e., $f_{ij} = \lambda(x)g_{ij}$ $(x \in D)$, then $Q_{\dot\gamma} f = 0$ and equation (6.1.18) degenerates. Therefore it is natural to consider only the tensor fields that satisfy the condition $jf = f_{ij}g^{ij} = 0$. Thus we arrive at the next inverse problem.

A Riemannian metric g is given on a bounded domain $D \subset \mathbf{R}^n$; a symmetric tensor field f of degree 2 is defined on D and satisfies the condition $jf = 0$. For every geodesic $\gamma : [0,1] \to D$ with the endpoints in the boundary of D, the value $\zeta(1)$ of any solution to equation (6.1.18) is known as a function of the initial value $\zeta(0)$ and the geodesic γ. In other words, the fundamental matrix $U(\gamma)$ of system (6.1.18) is known. One has to determine the field $f(x)$ from $U(\gamma)$.

Linearizing this problem in accord with the scheme of Section 5.6.1, we arrive at the next

Problem 6.1.1 (the problem of polarization tomography) *Let a Riemannian metric g be given in a bounded domain $D \subset \mathbf{R}^n$. Determine a symmetric tensor field $f = (f_{ij})$ that is defined on D and satisfies the condition $jf = f_{ij}g^{ij} = 0$, if the tensor*

$$Kf(\gamma) = \int\limits_0^1 I_\gamma^{t,0}(Q_{\dot\gamma(t)} f(\gamma(t)))\, dt$$

is known for every geodesic $\gamma : [0,1] \to D$ with endpoints in the boundary of D. Here $I_\gamma^{t,0}$ is the parallel translation along γ from the point $\gamma(t)$ to the point $\gamma(0)$.

6.2 The truncated transverse ray transform

Recall that, for a Riemannian manifold (M,g) and a point $x \in M$, by $i : S^mT'_xM \to S^{m+2}T'_xM$, we denote the operator of symmetric multiplication by the metric tensor g and by $j : S^{m+2}T'_xM \to S^mT'_xM$, the operator of convolution with g. For $\xi \in T_xM$, let $i_\xi : S^mT'_xM \to S^{m+1}T'_xM$ and $j_\xi : S^{m+1}T'_xM \to S^mT'_xM$ be the symmetric multiplication by the vector ξ and the convolution with ξ. Let $Q_\xi : S^mT'_xM \to S^mT'_xM$ be the orthogonal projection onto the intersection of the kernels of the operators j and j_ξ. Recall also that $P_\xi : S^mT'_xM \to S^mT'_xM$ denotes the orthogonal projection onto $\operatorname{Ker} j_\xi$. The next claim is an analog of Lemma 2.6.1.

Lemma 6.2.1 *For $(x,\xi) \in TM$ and $f \in S^mT'_xM$ $(m \geq 0)$, there exist uniquely determined tensors $y \in S^{m-1}T'_xM$ and $a \in S^{m-2}T'_xM$ such that*

$$f = Q_\xi f + i_\xi y + ia, \tag{6.2.1}$$

$$j_\xi a = 0. \tag{6.2.2}$$

The tensor y can be represented as

$$y = \tilde y + i_\xi h, \qquad j_\xi \tilde y = 0, \tag{6.2.3}$$

where $\tilde y$ is expressed via f by the equality

$$\tilde y = mP_\xi j_\xi f. \tag{6.2.4}$$

The proof of the lemma will be given in Section 6.4.

The *truncated transverse ray transform* on a CDRM (M,g) is the linear operator

$$K : C^\infty(S^m\tau'_M) \to C^\infty(S^m\pi_M), \tag{6.2.5}$$

defined by the equality

$$Kf(x,\xi) = \int\limits_{\tau_-(x,\xi)}^0 I_\gamma^{t,0}(Q_{\dot\gamma(t)} f(\gamma(t)))\, dt \qquad ((x,\xi) \in \partial_+\Omega M), \tag{6.2.6}$$

where the notation is the same as in the definition (5.2.3) of the operator J.

As in the previous two chapters, one shows that (6.2.5) is extendible to a bounded operator

$$K : H^k(S^m\tau'_M) \to H^k(S^m\pi_M) \tag{6.2.7}$$

for every integer $k \geq 0$.

The main result of the current chapter is the next

Theorem 6.2.2 *For integers m and n satisfying the inequalities $n \geq 3$ and $n > m$, there exists a positive number $\varepsilon(m,n)$ such that, for every compact dissipative Riemannian manifold (M,g) of dimension n satisfying the condition*

$$k(M,g) < \varepsilon(m,n), \tag{6.2.8}$$

where $k(M,g)$ is defined by formula (5.2.8), every field $f \in H^1(S^m\tau'_M)$ satisfying the condition

$$jf = 0 \tag{6.2.9}$$

is uniquely determined by Kf. The stability estimate

$$\|f\|_0 \leq C\|Kf\|_1 \tag{6.2.10}$$

holds with a constant C independent of f.

In the theorem the condition $n \geq 3$ is essential, since $Q_\xi = 0$ and the operator K is equal to zero in the case $m \geq 2 = n$. As far as the second condition $n > m$ is concerned, remarks are possible that are similar to those of Section 5.9.

In what follows we assume that $m \geq 2$, since $Q_\xi = P_\xi$ and the operator K coincides with J for $m < 2$.

As in the previous two chapters, using the boundedness of operator (6.2.7), one can see that it suffices to prove Theorem 6.2.2 for a real field $f \in C^\infty(S^m\tau'_M)$. The proof is presented in the next section. Our exposition of arguments similar to those used in the previous chapter will be kept at a minimum.

6.3 Proof of Theorem 6.2.2

The next claim is an analog of Lemma 5.6.1.

Lemma 6.3.1 *Let (M,g) be a Riemannian manifold of dimension n. For every point $x \in M$ and every tensor $f \in S^mT'_xM$ $(m \geq 0)$ satisfying (6.2.9), the equality*

$$\langle Bf, f\rangle = \frac{1}{\omega}\int\limits_{\Omega_xM}\Big[(n-1)|Q_\xi f|^2 - m^2|P_\xi j_\xi f|^2\Big]\,d\omega_x(\xi) = \chi(m,n)|f|^2 \tag{6.3.1}$$

holds with a coefficient $\chi(m,n)$ depending only on m and n. The coefficient $\chi(m,n)$ is positive for $n > m \geq 3$ or $n-1 > m = 2$ and equal to zero for $n-1 = m = 2$.

The proof of the lemma is presented in Section 6.5.

Now we start proving Theorem 6.2.2. As above, we denote by the letter C various constants depending only on m and n. For a real field $f \in C^\infty(S^m\tau'_M)$ $(m \geq 2,\ n = \dim M \geq 3)$ satisfying (6.2.9), we define a semibasic tensor field $u = (u_{i_1\dots i_m}(x,\xi))$ on T^0M by the equality

$$u(x,\xi) = \int\limits_{\tau_-(x,\xi)}^{0} I_\gamma^{t,0}(Q_{\dot\gamma(t)}f(\gamma(t)))\,dt \qquad ((x,\xi) \in T^0M), \tag{6.3.2}$$

where the same notations are used as in (5.3.1). This field satisfies the equation

$$Hu(x,\xi) = Q_\xi f(x), \tag{6.3.3}$$

with boundary conditions (5.3.3) and

$$u|_{\partial_+\Omega M} = Kf. \tag{6.3.4}$$

The field u is symmetric in all its indices, satisfies the homogeneity relation (5.3.4), condition (5.3.6), and the next equality

$$ju(x,\xi) = 0. \tag{6.3.5}$$

Writing down the Pestov identity for u and integrating it, we arrive at relation (5.3.9). Using decomposition (6.2.1), we transform the integrand of the first integral on the right-hand side of (5.3.9) as follows:

$$\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = \langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Q_\xi f)\rangle = \langle \overset{h}{\nabla} u, \overset{v}{\nabla}(f - i_\xi y - ia)\rangle =$$

$$= -\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(i_\xi y)\rangle - \langle \overset{h}{\nabla} u, \overset{v}{\nabla}(ia)\rangle. \tag{6.3.6}$$

Let us show that the second summand on the right-hand side of (6.3.6) is equal to zero. Indeed,

$$\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(ia)\rangle = \overset{h}{\nabla}{}^i u^{i_1\dots i_m} \cdot (g_{i_1 i_2} \overset{v}{\nabla}_i a_{i_3\dots i_m}) = \overset{h}{\nabla}{}^i (ju)^{i_3\dots i_m} \cdot \overset{v}{\nabla}_i a_{i_3\dots i_m}.$$

By (6.3.5), the right-hand side of the last formula is equal to zero. Thus (6.3.6) gives

$$\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = -\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(i_\xi y)\rangle. \tag{6.3.7}$$

With the help of representations (6.2.3) and (6.2.4), we transform the right-hand side of equality (6.3.7) in the same way as in Section 5.4. As a result, we arrive at inequality (5.4.22) where some semibasic field $z = (z_{ii_1\dots i_m})$ is defined by formula (5.4.13), symmetric in the last m indices, satisfies (5.4.14), (5.4.15) and the relations

$$g^{i_1 i_2} z_{i\, i_1 i_2\dots i_m} = 0, \tag{6.3.8}$$

$$\overset{h}{\nabla}{}^i u^{j\, i_2\dots i_m} \cdot \overset{h}{\nabla}_j u_{i\, i_2\dots i_m} = z^{i\, j\, i_2\dots i_m} z_{j\, i\, i_2\dots i_m}. \tag{6.3.9}$$

Besides, for the field $\overset{h}{\delta} u$ defined by formula (5.4.4), equality (5.4.5) holds. Using (6.3.3) and (5.4.15), we transform inequality (5.4.22) to obtain

$$\int\limits_M \left\{ \int\limits_{\Omega_x M} \left[(1-b)|z|^2 + (n-1)|Q_\xi f|^2 - \frac{m^2}{b} |P_\xi j_\xi f|^2 \right] d\omega_x(\xi) \right\} dV^n(x) \le$$

$$\le \int\limits_{\partial\Omega M} \langle b\tilde{v} - v, \nu\rangle \, d\Sigma^{2n-2} + \int\limits_{\Omega M} \mathcal{R}[u]\, d\Sigma. \tag{6.3.10}$$

Recall that here b is an arbitrary positive number, the semibasic vector fields v and $\tilde v$ are defined by formulas (4.4.5) and (5.4.11), the function $\mathcal{R}[u]$ is defined by formulas (5.3.8), (5.4.9) and (5.4.23). The field z satisfies the equality

$$|\overset{h}{\nabla} u|^2 = |z|^2 + |Q_\xi f|^2 \tag{6.3.11}$$

that follows from (5.4.15) and (6.3.3).

The integrals on the right-hand side of (6.3.10) satisfy the estimates similar to those of Section 5.5. With the help of these estimates, (6.3.10) implies the next analog of inequality (5.5.16):

$$\int\limits_M \left\{ \int\limits_{\Omega_x M} \left[(1-b)|z|^2 + (n-1)|Q_\xi f|^2 - \frac{m^2}{b}|P_\xi j_\xi f|^2 \right] d\omega_x(\xi) \right\} dV^n(x) -$$

$$- Ck \int\limits_{\Omega M} |\overset{h}{\nabla} u|^2 d\Sigma \le D\|Kf\|_1^2. \tag{6.3.12}$$

For $n > m \ge 3$ or $n-1 > m = 2$, the quadratic form $\langle Bf, f\rangle$ in (6.3.12) is positive-definite, by Lemma 6.3.1, and the proof of Theorem 6.2.2 is completed in the same way as in Section 5.6.

In the case $n-1 = m = 2$ the quadratic form $\langle Bf, f\rangle$ is equal to zero, and therefore the first summand on the right-hand side of equality (5.3.9) must be investigated more carefully. The next circumstance turns out to be crucial here: the factors on the right-hand side of (6.3.9) differ from one other by the transposition of the first two indices. According to this observation, we decompose the field $z = (z_{ijk})$ into the sum of two fields

$$z = z^+ + z^-, \tag{6.3.13}$$

where

$$z^+_{ijk} = \frac{1}{2}(z_{ijk} + z_{jik}), \qquad z^-_{ijk} = \frac{1}{2}(z_{ijk} - z_{jik}). \tag{6.3.14}$$

Summands on the right-hand side of (6.3.13) are orthogonal to one other and, consequently,

$$|z|^2 = |z^+|^2 + |z^-|^2. \tag{6.3.15}$$

Moreover, (6.3.13) and (6.3.14) imply that

$$z^{ijk} z_{jik} = |z^+|^2 - |z^-|^2. \tag{6.3.16}$$

By (6.3.15) and (6.3.16), equalities (6.3.9) and (5.4.15) take the form

$$\overset{h}{\nabla}{}^i u^{jk} \cdot \overset{h}{\nabla}_j u_{ik} = |z^+|^2 - |z^-|^2, \tag{6.3.17}$$

$$|\overset{h}{\nabla} u|^2 = |Hu|^2 + |z^+|^2 + |z^-|^2. \tag{6.3.18}$$

In Section 5.4, for the field $\overset{h}{\delta} u$, the relation

$$|\overset{h}{\delta} u|^2 = \overset{h}{\nabla}{}^i u^{jk} \cdot \overset{h}{\nabla}_j u_{ik} + \overset{h}{\nabla}_i \tilde v^i + \mathcal{R}_2[u], \tag{6.3.19}$$

was obtained. Together with (6.3.17), it implies the equality

$$|\overset{h}{\delta}u|^2 = |z^+|^2 - |z^-|^2 + \overset{h}{\nabla}_i\tilde{v}^i + \mathcal{R}_2[u]. \tag{6.3.20}$$

From (5.4.5), using (6.3.20), we obtain the estimate

$$2|\langle\overset{h}{\nabla}u, \overset{v}{\nabla}(Hu)\rangle| \le |z^+|^2 - |z^-|^2 + |\tilde{y}|^2 + \overset{h}{\nabla}_i\tilde{v}^i + \mathcal{R}_2[u]. \tag{6.3.21}$$

Let us reproduce the arguments used in deriving (6.3.12) from inequality (5.3.9); but now, instead of relations (5.4.15) and (5.4.17), we use (6.3.18) and (6.3.21) respectively. As a result, instead of (6.3.12), we arrive at the next more precise inequality:

$$\int\limits_M \left\{ \int\limits_{\Omega_x M} \left[2|z^-|^2 + (n-1)|Q_\xi f|^2 - m^2|P_\xi j_\xi f|^2\right] d\omega_x(\xi) \right\} dV^n(x) \le$$

$$\le Ck \int\limits_{\Omega M} |\overset{h}{\nabla}u|^2 d\Sigma + D\|Kf\|_1^2.$$

By Lemma 6.3.1, in the case $n = 3$, $m = 2$ under consideration, the sum of the last two summands on the left-hand side vanishes after integration over $\Omega_x M$. Thus, the last inequality takes the form:

$$\int\limits_{\Omega M} |z^-|^2 d\Sigma \le Ck \int\limits_{\Omega M} |\overset{h}{\nabla}u|^2 d\Sigma + D\|Kf\|_1^2. \tag{6.3.22}$$

Introducing the notations

$$\|f\|^2 = \int\limits_M |f(x)|^2 dV^n(x), \qquad \|z^-\|^2 = \int\limits_{\Omega M} |z^-|^2 d\Sigma, \qquad \|\overset{h}{\nabla}u\|^2 = \int\limits_{\Omega M} |\overset{h}{\nabla}u|^2 d\Sigma,$$

we rewrite estimate (6.3.22) as

$$\|z^-\|^2 \le Ck\|\overset{h}{\nabla}u\|^2 + D\|Kf\|_1^2. \tag{6.3.23}$$

It turns out that the norm $\|\overset{h}{\nabla}u\|^2$ can be estimated from above by $\|f\| \cdot \|z^-\|$. To this end, we multiply the equality

$$z^-_{ijk} = \frac{1}{2}\left[\overset{h}{\nabla}_i u_{jk} - \frac{1}{|\xi|^2}\xi_i(Hu)_{jk} - \overset{h}{\nabla}_j u_{ik} + \frac{1}{|\xi|^2}\xi_j(Hu)_{ik}\right],$$

which follows from (6.3.14) and (5.4.13), by g^{jk} and perform the summation with respect to j, k. Taking the equalities $ju = j(Hu) = j_\xi(Hu) = 0$ into account, we obtain

$$(\overset{h}{\delta}u)_i = -2z^-_{ijk}g^{jk}.$$

Inserting this expression into (5.4.5), we conclude

$$\langle\overset{h}{\nabla}u, \overset{v}{\nabla}(Hu)\rangle = 2z^-_{ijk}\tilde{y}^i g^{jk}.$$

This relation implies the estimate

$$|\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle| \le C|z^-|\cdot|\tilde{y}|.$$

Multiplying the last inequality by the volume form $d\omega_x(\xi)$, integrating it and applying the Cauchy-Bunyakovskiĭ inequality, we arrive at the estimate

$$\int\limits_{\Omega_x M} |\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle|\, d\omega_x(\xi) \le C\left(\int\limits_{\Omega_x M} |z^-|^2 d\omega_x(\xi)\right)^{1/2}\left(\int\limits_{\Omega_x M} |\tilde{y}|^2 d\omega_x(\xi)\right)^{1/2} \tag{6.3.24}$$

with a constant C independent of x.

The expression (6.2.4) of the tensor $\tilde{y}(x,\xi)$ from $f(x)$ implies the estimate

$$\int\limits_{\Omega_x M} |\tilde{y}|^2 d\omega_x(\xi) \le C_1|f(x)|^2$$

with some constant C_1 independent of x. Comparing the last relation with (6.3.24), we obtain the inequality

$$\int\limits_{\Omega_x M} |\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle|\, d\omega_x(\xi) \le C|f(x)|\left(\int\limits_{\Omega_x M} |z^-|^2\, d\omega_x(\xi)\right)^{1/2}.$$

Multiplying this inequality by $dV^n(x)$ and integrating it over M, we obtain

$$\int\limits_{\Omega M} |\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle|\, d\Sigma \le C\|f\|\cdot\|z^-\|. \tag{6.3.25}$$

It follows from (5.3.9) and (6.3.25) that

$$\|\overset{h}{\nabla} u\|^2 \le C\|f\|\cdot\|z^-\| - \int\limits_{\partial\Omega M} \langle v,\nu\rangle\, d\Sigma^{2n-2} + \int\limits_{\Omega M} \mathcal{R}_1[u]\, d\Sigma.$$

The integrals on the right-hand side of this relation are estimated as above, and we arrive at the inequality

$$\|\overset{h}{\nabla} u\|^2 \le C\|f\|\cdot\|z^-\| + C_1 k\|\overset{h}{\nabla} u\|^2 + D_1\|Kf\|_1^2. \tag{6.3.26}$$

Recall that the quantity $k = k(M,g)$ is subordinate to condition (6.2.8) of the theorem. The constants C and C_1 on (6.3.26) are universal unlike the coefficient D_1 depending on (M,g). Choose the value of $\varepsilon(m,n)$ in the formulation of the theorem in such a way that $C_1 k < 1$. Then (6.3.26) implies the relation

$$\|\overset{h}{\nabla} u\|^2 \le C\|f\|\cdot\|z^-\| + D_1\|Kf\|_1^2,$$

which can be transformed as follows with the help of the inequality between arithmetical and geometrical means:

$$\|\overset{h}{\nabla} u\|^2 \le b\|f\|^2 + \frac{C_1}{b}\|z^-\|^2 + D_1\|Kf\|_1^2, \tag{6.3.27}$$

where b is an arbitrary positive number.

The final part of the proof consists in comparing estimates (6.3.23) and (6.3.27). Majorizing the second summand on the right-hand side of (6.3.27) with the help of (6.3.23), we obtain the inequality

$$\|\overset{h}{\nabla}u\|^2 \le b\|f\|^2 + \frac{C_1}{b}\left(Ck\|\overset{h}{\nabla}u\|^2 + D\|Kf\|_1^2\right) + D_1\|Kf\|_1^2,$$

which can be written as

$$\left(1 - \frac{Ck}{b}\right)\|\overset{h}{\nabla}u\|^2 \le b\|f\|^2 + \frac{D}{b}\|Kf\|_1^2. \tag{6.3.28}$$

Furthermore, we have the estimate

$$\|f\|^2 \le C_1\|\overset{h}{\nabla}u\|^2, \tag{6.3.29}$$

which is concluded in full analogy with inequality (5.6.9). From (6.3.28) and (6.3.29), we obtain

$$\left(1 - \frac{Ck}{b} - C_1 b\right)\|\overset{h}{\nabla}u\|^2 \le \frac{D}{b}\|Kf\|_1^2. \tag{6.3.30}$$

We choose the quantities b and $\varepsilon(m,n)$ in such a way that the number in the parentheses is positive. Then (6.3.30) gives

$$\|\overset{h}{\nabla}u\|^2 \le D_1\|Kf\|_1^2. \tag{6.3.31}$$

From (6.3.29) and (6.3.31), we finally obtain

$$\|f\|^2 \le D_2\|Kf\|_1^2.$$

The theorem is proved.

6.4 Decomposition of the operator Q_ξ

Before proving Lemma 6.2.1, we will expose a few auxiliary claims. For $\xi \in T_xM$, on $S^mT'_xM$ $(m \ge 0,\ n = \dim M)$ the next commutation formulas are valid:

$$j_\xi i_\xi^k = \frac{k}{m+k}|\xi|^2 i_\xi^{k-1} + \frac{m}{m+k} i_\xi^k j_\xi, \tag{6.4.1}$$

$$ji^k = \frac{2k(n+2m+2k-2)}{(m+2k-1)(m+2k)} i^{k-1} + \frac{m(m-1)}{(m+2k-1)(m+2k)} i^k j, \tag{6.4.2}$$

$$ji_\xi = \frac{2}{m+1} j_\xi + \frac{m-1}{m+1} i_\xi j, \tag{6.4.3}$$

$$j_\xi i = \frac{2}{m+2} i_\xi + \frac{m}{m+2} i j_\xi, \tag{6.4.4}$$

The first of these formulas have been proved above (Lemmas 3.3.3 and 5.4.1). Equalities (6.4.2) and (6.4.3) can be proved by arguments similar to those in the proof of Lemma 3.3.3; we omit them. Relation (6.4.4) is obtained from (6.4.3) by passing to the dual operators. Strictly speaking, (6.4.1) and (6.4.2) are meaningful only for $k > 0$; nevertheless we can assume these equalities to hold for $k = 0$ too, since the coefficient of the first summand on the right-hand side is equal to zero in this case.

Lemma 6.4.1 *For $\xi \neq 0$, the next commutation formulas are valid:*

$$P_\xi i = iP_\xi - \frac{1}{|\xi|^2} i_\xi^2 P_\xi, \tag{6.4.5}$$

$$jP_\xi = P_\xi j - \frac{1}{|\xi|^2} P_\xi j_\xi^2. \tag{6.4.6}$$

P r o o f. It suffices to verify the first formula, since it implies the second by passing to dual operators.

The operator $i_\xi j_\xi$ is nonnegative, since it is the product of two operators dual to one other. Therefore (6.4.1) implies that $j_\xi i_\xi$ is positive for $\xi \neq 0$ and, consequently, has an inverse operator $(j_\xi i_\xi)^{-1}$.

Recall that P_ξ is the orthogonal projection onto the first summand of the decomposition

$$S^m T'_x M = \operatorname{Ker} j_\xi \oplus \operatorname{Im} i_\xi. \tag{6.4.7}$$

Let $\overline{P}_\xi = E - P_\xi$ (E is the identity operator) be the projection onto the second summand of this decomposition. The next formula is valid:

$$E - P_\xi = \overline{P}_\xi = i_\xi (j_\xi i_\xi)^{-1} j_\xi. \tag{6.4.8}$$

Indeed, the operator defined by equality (6.4.8) vanishes on the first summand of decomposition (6.4.7) and satisfies the relations $\overline{P}_\xi^2 = \overline{P}_\xi, \overline{P}_\xi i_\xi = i_\xi$. These properties uniquely determine the projection onto the second summand.

Using (6.4.8) and (6.4.4), we obtain

$$\overline{P}_\xi i = i_\xi (j_\xi i_\xi)^{-1}(j_\xi i) = i_\xi (j_\xi i_\xi)^{-1} \left(\frac{2}{m+2} i_\xi + \frac{m}{m+2} i j_\xi \right),$$

$$\overline{P}_\xi i = \frac{2}{m+2} i_\xi (j_\xi i_\xi)^{-1} i_\xi + \frac{m}{m+2} i_\xi (j_\xi i_\xi)^{-1} i j_\xi. \tag{6.4.9}$$

By (6.4.4), taking permutability of the operators i and i_ξ into account, we write

$$(j_\xi i_\xi) i = (j_\xi i) i_\xi = \left(\frac{2}{m+3} i_\xi + \frac{m+1}{m+3} i j_\xi \right) i_\xi = \frac{2}{m+3} i_\xi^2 + \frac{m+1}{m+3} i (j_\xi i_\xi).$$

Multiplying the extreme terms of this chain of equalities by $(j_\xi i_\xi)^{-1}$ from left and right, we obtain

$$(j_\xi i_\xi)^{-1} i = \frac{m+3}{m+1} i (j_\xi i_\xi)^{-1} - \frac{2}{m+1} (j_\xi i_\xi)^{-1} i_\xi^2 (j_\xi i_\xi)^{-1}.$$

The last relation holds on $S^m T'_x M$; therefore we have to decrease the value of m by one, applying this equality for transformation of the second summand on the right-hand side of (6.4.9). We thus obtain

$$\overline{P}_\xi i = \frac{2}{m+2} i_\xi (j_\xi i_\xi)^{-1} i_\xi + \frac{m}{m+2} i_\xi \left[\frac{m+2}{m} i (j_\xi i_\xi)^{-1} - \frac{2}{m} (j_\xi i_\xi)^{-1} i_\xi^2 (j_\xi i_\xi)^{-1} \right] j_\xi.$$

Rewriting the last formula as

$$\overline{P}_\xi i = i_\xi i (j_\xi i_\xi)^{-1} j_\xi + \frac{2}{m+2} i_\xi (j_\xi i_\xi)^{-1} i_\xi (E - i_\xi (j_\xi i_\xi)^{-1} j_\xi)$$

and recalling (6.4.8), we obtain

$$\overline{P}_\xi i = i\overline{P}_\xi + \frac{2}{m+2} i_\xi (j_\xi i_\xi)^{-1} i_\xi P_\xi.$$

This implies that

$$P_\xi i = (E - \overline{P}_\xi) i = i - \overline{P}_\xi i = i - i\overline{P}_\xi - \frac{2}{m+2} i_\xi (j_\xi i_\xi)^{-1} i_\xi P_\xi,$$

i.e.,

$$P_\xi i = iP_\xi - \frac{2}{m+2} i_\xi (j_\xi i_\xi)^{-1} i_\xi P_\xi. \tag{6.4.10}$$

We put $k = 2$ in equality (6.4.1), multiply it by P_ξ from the right and use the relation $j_\xi P_\xi = 0$; as a result we have

$$i_\xi P_\xi = \frac{m+2}{2|\xi|^2} j_\xi i_\xi^2 P_\xi.$$

Multiplying the last equality by $(j_\xi i_\xi)^{-1}$ from the left, we obtain

$$(j_\xi i_\xi)^{-1} i_\xi P_\xi = \frac{m+2}{2|\xi|^2} i_\xi P_\xi. \tag{6.4.11}$$

(6.4.10) and (6.4.11) imply (6.4.5). The lemma is proved.

Lemma 6.4.2 *For $\xi \neq 0$, the operator $jP_\xi i$ maps isomorphically the space $\mathrm{Ker}\, j_\xi$ onto itself.*

P r o o f. We consider the operator $jP_\xi i : S^m T'_x M \to S^m T'_x M$. With the help of the equality $P_\xi j P_\xi = jP_\xi$ following from (6.4.6), we obtain $jP_\xi i = P_\xi j P_\xi i$. Consequently, $\mathrm{Im}(jP_\xi i) \subset \mathrm{Ker}\, j_\xi$. Let us prove that the restriction

$$jP_\xi i : \mathrm{Ker}\, j_\xi \to \mathrm{Ker}\, j_\xi \tag{6.4.12}$$

is an isomorphism. Let a tensor u of degree m be in the kernel of operator (6.4.12), then

$$j_\xi u = 0, \quad jP_\xi i u = 0. \tag{6.4.13}$$

Taking the scalar product of the second of these equalities with u, we obtain

$$0 = \langle jP_\xi i u, u\rangle = \langle P_\xi i u, iu\rangle = \langle P_\xi i u, P_\xi i u\rangle.$$

Thus, system (6.4.13) is equivalent to the next one:

$$j_\xi u = 0, \qquad P_\xi i u = 0. \tag{6.4.14}$$

We transform the second of equations (6.4.14) with the help of (6.4.5):

$$0 = P_\xi i u = iP_\xi u - \frac{1}{|\xi|^2} i_\xi^2 P_\xi u.$$

By the first of the equations (6.4.14), $P_\xi u = u$, and therefore the previous equality gives

$$iu = \frac{1}{|\xi|^2} i_\xi^2 u.$$

We take the scalar product of the last relation with iu and transform the so-obtained formula to the form

$$\langle jiu, u\rangle = \frac{1}{|\xi|^2}\langle i_\xi u, j_\xi iu\rangle. \tag{6.4.15}$$

By (6.4.2), (6.4.4) and the first of the equations (6.4.14), we obtain

$$jiu = \frac{2(n+2m)}{(m+1)(m+2)}u + \frac{m(m-1)}{(m+1)(m+2)}iju,$$

$$j_\xi iu = \frac{2}{m+2} i_\xi u.$$

Inserting these expressions into (6.4.15), we arrive at the equality

$$\frac{2(n+2m)}{m+1}|u|^2 + \frac{m(m-1)}{m+1}|ju|^2 = \frac{2}{|\xi|^2}|i_\xi u|^2. \tag{6.4.16}$$

Finally, using (6.4.1) and the first of equations (6.4.14), we obtain

$$|i_\xi u|^2 = \langle j_\xi i_\xi u, u\rangle = \frac{|\xi|^2}{m+1}|u|^2.$$

Inserting the last value into the right-hand side of (6.4.16), we conclude that

$$2(n+2m)|u|^2 + m(m-1)|ju|^2 = 2|u|^2.$$

This equality can hold only for $u = 0$. The lemma is proved.

P r o o f of Lemma 6.2.1. First we prove uniqueness of representation (6.2.1), (6.2.2). Successively applying the operators j and j_ξ to the first of these equalities, we obtain

$$ji_\xi y + jia = jf, \tag{6.4.17}$$

$$j_\xi i_\xi y + j_\xi ia = j_\xi f. \tag{6.4.18}$$

From (6.4.18), we infer

$$y = -(j_\xi i_\xi)^{-1} j_\xi ia + (j_\xi i_\xi)^{-1} j_\xi f \tag{6.4.19}$$

and substitute the obtained expression into (6.4.17)

$$jia - ji_\xi (j_\xi i_\xi)^{-1} j_\xi ia = jf - ji_\xi (j_\xi i_\xi)^{-1} j_\xi f.$$

Rewriting the last equation as

$$j(E - i_\xi (j_\xi i_\xi)^{-1} j_\xi) ia = j(E - i_\xi (j_\xi i_\xi)^{-1} j_\xi) f$$

and using (6.4.8), we obtain

$$jP_\xi ia = jP_\xi f. \tag{6.4.20}$$

The right-hand side of (6.4.20) belongs to $\operatorname{Ker} j_\xi$, since $j_\xi j P_\xi = j j_\xi P_\xi = 0$. Therefore, by Lemma 6.4.2, equation (6.4.20) has a unique solution satisfying (6.2.2). Together with (6.4.19), this proves uniqueness.

We now prove existence. Let $f \in S^m T'_x M$. By Lemma 6.4.2, equation (6.4.20) has a unique solution a satisfying (6.2.2). Defining y by formula (6.4.19) and $\tilde f$, by the equality

$$f = \tilde f + i_\xi y + ia, \tag{6.4.21}$$

one can easily see that $j\tilde f = j_\xi \tilde f = 0$ and the summands on the right-hand side of (6.4.21) are orthogonal to one other. Consequently, $\tilde f = Q_\xi f$.

Finally, we prove validity of (6.2.4). Inserting the value (6.2.3) of y into (6.4.18), we obtain

$$j_\xi i_\xi \tilde y + j_\xi i_\xi^2 h + j_\xi ia = j_\xi f. \tag{6.4.22}$$

We transform each of the summands on the left-hand side of this equality with the help of (6.4.1) and (6.4.4), taking (6.2.2) and the second of equations (6.2.3) into account:

$$j_\xi i_\xi \tilde y = \frac{|\xi|^2}{m}\tilde y, \quad j_\xi i_\xi^2 h = \frac{2|\xi|^2}{m} i_\xi h + \frac{m-2}{m} i_\xi^2 j_\xi h, \quad j_\xi ia = \frac{2}{m} i_\xi a.$$

Inserting these values into (6.4.22), we obtain

$$\frac{|\xi|^2}{m}\tilde y + i_\xi\left(\frac{2|\xi|^2}{m} h + \frac{m-2}{m} i_\xi j_\xi h + \frac{2}{m} a\right) = j_\xi f.$$

Applying the operator P_ξ to the last equality and taking the relation $P_\xi i_\xi = 0$ into account, we arrive at (6.2.4). The lemma is proved.

In Section 6.5 we will need the decomposition, of the operator Q_ξ, which is given by the next

Lemma 6.4.3 *For $0 \neq \xi \in T_x M$ and a tensor $f \in S^m T'_x M$ satisfying the condition $jf = 0$, the representation*

$$Q_\xi f = \sum_{k=0}^{[m/2]} c_{mk} \frac{1}{|\xi|^{2k}} P_\xi i^k j_\xi^{2k} f \tag{6.4.23}$$

is valid where

$$c_{mk} = \frac{m!}{2^k k!(m-2k)!} \frac{n+2m-3}{(n+2m-3)(n+2m-5)\dots(n+2m-2k-3)}. \tag{6.4.24}$$

First we will prove an auxiliary relation.

Lemma 6.4.4 *For $k \geq 0$, the equality*

$$P_\xi j_\xi^2 i^k = \frac{2k}{(m+2k-1)(m+2k)}|\xi|^2 P_\xi i^{k-1} + \frac{m(m-1)}{(m+2k-1)(m+2k)} P_\xi i^k j_\xi^2. \tag{6.4.25}$$

is valid on $S^m T'_x M$.

P r o o f proceeds by induction on k. For $k = 0$, validity of (6.4.25) is trivial, since the factor at the first summand on the right-hand side is equal to zero. Assume (6.4.25) to hold for $k = l$. Increasing the value of m on this equality by 2 and multiplying it by i from the right, we arrive at the formula

$$P_\xi j_\xi^2 i^{l+1} = \frac{2l}{(m+2l+1)(m+2l+2)}|\xi|^2 P_\xi i^l + \frac{(m+1)(m+2)}{(m+2l+1)(m+2l+2)} P_\xi i^l j_\xi^2 i. \quad (6.4.26)$$

valid on $S^m T'_x M$.

Using (6.4.2) and (6.4.4), we obtain

$$j_\xi^2 i = j_\xi(j_\xi i) = j_\xi\left(\frac{2}{m+2} i_\xi + \frac{m}{m+2} i j_\xi\right) = \frac{2}{m+2} j_\xi i_\xi + \frac{m}{m+2}(j_\xi i) j_\xi =$$

$$= \frac{2}{m+2}\left(\frac{1}{m+1}|\xi|^2 E + \frac{m}{m+1} i_\xi j_\xi\right) + \frac{m}{m+2}\left(\frac{2}{m+1} i_\xi + \frac{m-1}{m+1} i j_\xi\right) j_\xi =$$

$$= \frac{2}{(m+1)(m+2)}|\xi|^2 E + \frac{m(m-1)}{(m+1)(m+2)} i j_\xi^2 + \frac{4m}{(m+1)(m+2)} i_\xi j_\xi.$$

Multiplying the end terms of this chain of equalities by $P_\xi i^l$ from the left, we obtain

$$P_\xi i^l j_\xi^2 i = \frac{2}{(m+1)(m+2)}|\xi|^2 P_\xi i^l + \frac{m(m-1)}{(m+1)(m+2)} P_\xi i^{l+1} j_\xi^2 + \frac{4m}{(m+1)(m+2)} P_\xi i^l i_\xi j_\xi. \quad (6.4.27)$$

The last summand on the right-hand side of this formula is equal to zero, since i commutes with i_ξ and $P_\xi i_\xi = 0$. Deleting this summand and replacing the second term on the right-hand side of (6.4.26) with its value (6.4.27), we arrive at (6.4.25) for $k = l+1$. The lemma is proved.

P r o o f of Lemma 6.4.3. We denote the right-hand side of formula (6.4.23) by $\tilde f$. To prove the equality $Q_\xi f = \tilde f$ we have to show that

$$j_\xi \tilde f = 0, \quad j \tilde f = 0 \quad (6.4.28)$$

and

$$f - \tilde f \in \operatorname{Im} i_\xi + \operatorname{Im} i. \quad (6.4.29)$$

First we prove (6.4.29). By (6.4.24), $c_{m0} = 1$ and sum (6.4.23) can be written as:

$$\tilde f = P_\xi f + \sum_{k=1}^{[m/2]} c_{mk} \frac{1}{|\xi|^{2k}} P_\xi i^k j_\xi^{2k} f. \quad (6.4.30)$$

Since $f - P_\xi f \in \operatorname{Im} i_\xi$, to prove (6.4.29) it suffices to show that each summand of the sum on (6.4.30) belongs to $\operatorname{Im} i_\xi + \operatorname{Im} i$. But this fact follows evidently from (6.4.5).

Validity of the first of the relations (6.4.28) follows directly from (6.4.30), since $j_\xi P_\xi = 0$. To verify the second, we apply the operator j to equality (6.4.30):

$$j \tilde f = \sum_{k=0}^{[m/2]} c_{mk} \frac{1}{|\xi|^{2k}} j P_\xi i^k j_\xi^{2k} f. \quad (6.4.31)$$

Using (6.4.6), we obtain

$$jP_\xi i^k j_\xi^{2k} f = P_\xi j i^k j_\xi^{2k} f - \frac{1}{|\xi|^2} P_\xi j_\xi^2 i^k j_\xi^{2k} f.$$

Transposing the operators j and i^k in the first summand with the help of (6.4.2) and then using permutability of j and j_ξ and the condition $jf = 0$, we obtain

$$jP_\xi i^k j_\xi^{2k} f = \frac{2k(n+2m-2k-2)}{m(m-1)} P_\xi i^{k-1} j_\xi^{2k} f - \frac{1}{|\xi|^2} P_\xi j_\xi^2 i^k j_\xi^{2k} f.$$

Transforming the second summand with the help of Lemma 6.4.4, we arrive at the relation

$$jP_\xi i^k j_\xi^{2k} f = \frac{2k(n+2m-2k-3)}{m(m-1)} P_\xi i^{k-1} j_\xi^{2k} f - \frac{(m-2k-1)(m-2k)}{m(m-1)} \frac{1}{|\xi|^2} P_\xi i^k j_\xi^{2k+2} f.$$

Inserting the last value into (6.4.31), we obtain

$$j\tilde f = \frac{1}{m(m-1)} \sum_{k=1}^{[m/2]} [2k(n+2m-2k-3)c_{mk} -$$

$$- (m-2k+1)(m-2k+2)c_{m,k-1}] \frac{1}{|\xi|^{2k}} P_\xi i^{k-1} j_\xi^{2k} f.$$

Thus, to verify the second of equalities (6.4.28) it suffices to show that the numbers in brackets are equal to zero. This fact follows directly from (6.4.24). The lemma is proved.

6.5 Proof of Lemma 6.3.1

By Lemma 5.7.1, for a tensor $f \in S^m T'_x M$ satisfying (6.2.9), the equality

$$\frac{1}{\omega} \int\limits_{\Omega_x M} i_\xi P_\xi j_\xi f \, d\omega_x(\xi) = \mu(m,n) f \tag{6.5.1}$$

holds where

$$\mu(m,n) = -(m-1)!\,\Gamma\left(\frac{n}{2}\right) \sum_{k=0}^{m} (-1)^k \frac{k}{2^k (m-k)!\,\Gamma\left(k+\frac{n}{2}\right)}. \tag{6.5.2}$$

Consequently,

$$\frac{1}{\omega} \int\limits_{\Omega_x M} |P_\xi j_\xi f|^2 d\omega_x(\xi) = \left\langle \frac{1}{\omega} \int\limits_{\Omega_x M} i_\xi P_\xi j_\xi f \, d\omega_x(\xi), f \right\rangle = \mu(m,n)|f|^2. \tag{6.5.3}$$

Given $0 \le k \le [m/2]$, we consider the operator $A_k : S^m T'_x M \to S^m T'_x M$ defined by the formula

$$A_k = \frac{1}{\omega} \int\limits_{\Omega_x M} P_\xi i^k j_\xi^{2k} d\omega_x(\xi). \tag{6.5.4}$$

Let us show that it can be expanded as follows

$$A_k = \sum_{p=0}^{[m/2]} a_k(p,m,n)\, i^p j^p \tag{6.5.5}$$

and find the first coefficient of the expansion.

By the same arguments as in Section 5.7, we can assume that $S^mT'_xM = S^m\mathbf{R}^n$ and $\Omega_x M$ coincides with the unit sphere Ω of the space $\mathbf{R}^n$. To prove (6.5.5) it suffices to show that

$$\langle A_k\eta^m, \zeta^m\rangle = \sum_{p=0}^{[m/2]} a_k(p,m,n)\langle \eta,\zeta\rangle^{m-2p} \tag{6.5.6}$$

for $\eta,\zeta \in \Omega$.

We write

$$\langle P_\xi i^k j_\xi^{2k}\eta^m, \zeta^m\rangle = \langle j_\xi^{2k}\eta^m, j^k(P_\xi\zeta)^m\rangle = \langle \xi,\eta\rangle^{2k}|P_\xi\zeta|^{2k}\langle \eta, P_\xi\zeta\rangle^{m-2k} =$$
$$= \langle \xi,\eta\rangle^{2k}(1-\langle\xi,\zeta\rangle^2)^k(\langle\eta,\zeta\rangle - \langle\xi,\eta\rangle\langle\xi,\zeta\rangle)^{m-2k}.$$

Inserting this expression into (6.5.4) and introducing the notation $x = \langle\eta,\zeta\rangle$, we obtain

$$\langle A_k\eta^m, \zeta^m\rangle = \frac{1}{\omega}\int\limits_\Omega \langle\eta,\xi\rangle^{2k}(1-\langle\zeta,\xi\rangle^2)^k(x-\langle\eta,\xi\rangle\langle\zeta,\xi\rangle)^{m-2k}\,d\omega(\xi).$$

We expand the integrand in powers of x :

$$\langle A_k\eta^m, \zeta^m\rangle = \sum_{l=0}^{m-2k}(-1)^l\binom{m-2k}{l}x^{m-2k-l}\frac{1}{\omega}\int\limits_\Omega\langle\eta,\xi\rangle^{2k+l}\langle\zeta,\xi\rangle^l(1-\langle\zeta,\xi\rangle^2)^k\,d\omega(\xi). \tag{6.5.7}$$

The integral on (6.5.7) is found by arguments similar to the corresponding paragraph of Section 5.7:

$$\frac{1}{\omega}\int\limits_\Omega\langle\eta,\xi\rangle^{2k+l}\langle\zeta,\xi\rangle^l(1-\langle\zeta,\xi\rangle^2)^k d\omega(\xi) =$$
$$= \frac{\Gamma\left(\frac{n}{2}\right)}{\pi}\sum_{p=0}^{k}(-1)^p\binom{k}{p}\sum_{r=0}^{p+[l/2]}\binom{2p+l}{2r}\frac{\Gamma(k+l+p-r+\frac12)\Gamma(r+\frac12)}{\Gamma(k+l+p+\frac{n}{2})}x^{2p+l-r}(1-x^2)^r.$$

Inserting the last expression into (6.5.7) and expanding the right-hand side of the so-obtained equality in powers of x, we obtain

$$\langle A_k\eta^m, \zeta^m\rangle = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi}\sum_{l=0}^{m-2k}\sum_{p=0}^{k}\sum_{r=0}^{p+[l/2]}\sum_{s=0}^{r}(-1)^{l+p+s}\binom{m-2k}{l}\binom{k}{p}\binom{2p+l}{2r}\binom{r}{s}\times$$
$$\times\frac{\Gamma\left(k+l+p-r+\frac12\right)\Gamma\left(r+\frac12\right)}{\Gamma\left(k+l+p+\frac{n}{2}\right)}x^{m-2k+2p-2r+2s}. \tag{6.5.8}$$

Equality (6.5.8) implies, first, possibility of the representation

$$\langle A_k\eta^m, \zeta^m\rangle = \sum_{p=0}^{[m/2]} a_k(p,m,n)x^{m-2p}, \tag{6.5.9}$$

i.e., validity of (6.5.5)–(6.5.6) and, second, the fact that the coefficient of x^m on (6.5.9) depends only on the summands of (6.5.8) that relate to $p = k, s = r$. In other words,

$$a_k(0,m,n) =$$

$$= \frac{\Gamma\left(\frac{n}{2}\right)}{\pi}\sum_{l=0}^{m-2k}\sum_{r=0}^{k+[l/2]}(-1)^{k+l+r}\binom{m-2k}{l}\binom{2k+l}{2r}\frac{\Gamma\left(2k+l-r+\frac{1}{2}\right)\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(2k+l+\frac{n}{2}\right)}. \quad (6.5.10)$$

Transforming (6.5.10) by changing the summation index of the first sum with respect to the formula $2k+l=p$, we obtain

$$a_k(0,m,n) =$$

$$= (-1)^k\frac{\Gamma\left(\frac{n}{2}\right)}{\pi}\sum_{p=2k}^{m}\frac{(-1)^p}{\Gamma\left(p+\frac{n}{2}\right)}\binom{m-2k}{m-p}\sum_{r=0}^{[p/2]}(-1)^r\binom{p}{2r}\Gamma\left(p-r+\frac{1}{2}\right)\Gamma\left(r+\frac{1}{2}\right). \quad (6.5.11)$$

Note that the inner sum coincides with the quantity $\gamma(0,p)$ defined by formula (5.7.15). Inserting the value (5.7.19) of this quantity into (6.5.11), we arrive at the equality

$$a_k(0,m,n) = (-1)^k\Gamma\left(\frac{n}{2}\right)(m-2k)!\sum_{p=2k}^{m}(-1)^p\frac{p!}{2^p(m-p)!(p-2k)!\Gamma\left(p+\frac{n}{2}\right)}. \quad (6.5.12)$$

Thus we have shown that, for a tensor $f \in S^mT'_xM$ satisfying (6.2.9), the equality

$$A_kf = \frac{1}{\omega}\int\limits_{\Omega_xM} P_\xi i^k j_\xi^{2k} f\, d\omega_x(\xi) = a_k(0,m,n)f \qquad (0 \le k \le [m/2]) \quad (6.5.13)$$

holds with the coefficients $a_k(0,m,n)$ given by formula (6.5.12). From this with the help of Lemma 6.4.3, we obtain

$$\frac{1}{\omega}\int\limits_{\Omega_xM} Q_\xi f\, d\omega_x(\xi) = \lambda(m,n)f, \quad (6.5.14)$$

where

$$\lambda(m,n) = m!\Gamma\left(\frac{n}{2}\right)\sum_{p=0}^{[m/2]}\frac{(-1)^p}{2^pp!}\frac{n+2m-3}{(n+2m-3)(n+2m-5)...(n+2m-2p-3)}\times$$

$$\times\sum_{k=2p}^{m}(-1)^k\frac{k!}{2^k(m-k)!(k-2p)!\Gamma\left(k+\frac{n}{2}\right)}. \quad (6.5.15)$$

Relations (6.5.3) and (6.5.14) imply equality (6.3.1) with

$$\chi(m,n) = (n-1)\lambda(m,n) - m^2\mu(m,n). \quad (6.5.16)$$

We have now to demonstrate that the last claim of Lemma 6.3.1 is valid for the quantity $\chi(m,n)$ given by formulas (6.5.2), (6.5.15) and (6.5.16).

For $m = 2$, by the direct calculation relevant to formulas (6.5.2) and (6.5.15), we obtain the equality $\chi(2, n) = n - 3$, which proves the claim of Lemma 6.3.1 in the case $m = 2$. Therefore, we further assume that

$$n > m \geq 3. \tag{6.5.17}$$

The quantity $\lambda(m, n)$ is nonnegative, so the inequality

$$(n-1)\lambda(m,n) - m^2\mu(m,n) \geq m[\lambda(m,n) - m\mu(m,n)].$$

holds under assumption (6.5.17). By (6.5.2) and (6.5.15),

$$\frac{1}{m!\Gamma\left(\frac{n}{2}\right)}[\lambda(m,n) - m\mu(m,n)] =$$

$$= \sum_{p=0}^{[m/2]}(-1)^p \frac{1}{2^p p!}\frac{n+2m-3}{(n+2m-3)(n+2m-5)...(n+2m-2p-3)}\times$$

$$\times \sum_{k=2p}^{m}(-1)^k \frac{k!}{2^k(m-k)!(k-2p)!\Gamma\left(k+\frac{n}{2}\right)} + \sum_{k=0}^{m}(-1)^k \frac{k}{2^k(m-k)!\Gamma\left(k+\frac{n}{2}\right)}.$$

Extracting the summands that correspond to $p = 0$ and $p = 1$ from the first sum on the right-hand side and including them into the second sum, we rewrite the latter equality in the form

$$\frac{1}{m!\Gamma\left(\frac{n}{2}\right)}[\lambda(m,n) - m\mu(m,n)] = \nu(m,n) + \kappa(m,n), \tag{6.5.18}$$

where

$$\nu(m,n) = \sum_{k=0}^{m}(-1)^k \frac{k+1}{2^k(m-k)!\Gamma\left(k+\frac{n}{2}\right)} - \frac{1}{2}\frac{1}{n+2m-5}\sum_{k=2}^{m}(-1)^k \frac{k(k-1)}{2^k(m-k)!\Gamma\left(k+\frac{n}{2}\right)}, \tag{6.5.19}$$

$$\kappa(m,n) = \sum_{p=2}^{[m/2]}\frac{(-1)^p}{2^p p!}\frac{n+2m-3}{(n+2m-3)(n+2m-5)...(n+2m-2p-3)}\times$$

$$\times \sum_{k=2p}^{m}(-1)^k \frac{k!}{2^k(m-k)!(k-2p)!\Gamma\left(k+\frac{n}{2}\right)} \tag{6.5.20}$$

with $\kappa(3, n) = 0$. Let us prove that $\nu(m, n) > 0$ under condition (6.5.17) and $\kappa(m, n) > 0$ under the condition $n > m \geq 4$, thereby we shall complete the proof.

Recall that integral (5.8.6) was introduced in Section 5.8, and equality (5.8.8) was established for it. Taking $\alpha = 1$ and $p = 0$ in (5.8.8), we obtain

$$\sum_{k=0}^{m}(-1)^k \frac{k+1}{2^k(m-k)!\Gamma\left(k+\frac{n}{2}\right)} = \frac{1}{m!\Gamma\left(\frac{n}{2}-2\right)}I_1(0,m,n). \tag{6.5.21}$$

As is seen from (5.8.6), the integral $I_1(0, m, n)$ converges only for $n \geq 5$. So we assume for a while that $n \geq 5$. Taking $\alpha = 0$ and $p = 1$ in (5.8.8), we obtain

$$\sum_{k=2}^{m}(-1)^k \frac{k(k-1)}{2^k(m-k)!\Gamma\left(k+\frac{n}{2}\right)} = \frac{1}{4(m-2)!\Gamma\left(\frac{n}{2}-1\right)}I_0(1,m,n). \tag{6.5.22}$$

We insert values (6.5.21) and (6.5.22) into (6.5.19):

$$\nu(m,n) = \frac{1}{m!\Gamma\left(\frac{n}{2}-2\right)}\left[I_1(0,m,n) - \frac{m(m-1)}{4(n-4)(n+2m-5)}I_0(1,m,n)\right]. \tag{6.5.23}$$

From this, using positiveness of $I_0(1,m,n)$ and the inequality

$$\frac{m(m-1)}{4(n-4)(n+2m-5)} \le \frac{1}{2},$$

valid under the assumptions $n \ge 5$ and (6.5.17), we conclude

$$\nu(m,n) \ge \frac{1}{m!\Gamma\left(\frac{n}{2}-2\right)}\left[I_1(0,m,n) - \frac{1}{2}I_0(1,m,n)\right].$$

Inserting the expressions of $I_1(0,m,n)$ and $I_0(1,m,n)$ given by formula (5.8.6) into the last equality, we obtain

$$\nu(m,n) \ge \frac{1}{m!\Gamma\left(\frac{n}{2}-2\right)}\int\limits_0^1 x(1-x)^{n/2-3}\left(1-\frac{x}{2}\right)^{m-2}\left[\left(1-\frac{x}{2}\right)^2 - \frac{1}{2}x(1-x)\right]dx.$$

The integrand is positive for $0 < x < 1$. Thus positiveness of $\nu(m,n)$ is proved in the case $n \ge 5$. For $n = 4$, according to (6.5.17), we should only examine the case $m = 3$. Positiveness of $\nu(3,4)$ is checked by a direct calculation by formula (6.5.19).

For $m = 4$ or $m = 5$, the outer sum in (6.5.20) consists of a single positive term. Therefore, we further assume that

$$n > m \ge 6. \tag{6.5.24}$$

Collecting the terms of the outer sum in (6.5.20) in pairs corresponding to even values, $p = 2q$, and odd values, $p = 2q+1$, we rewrite (6.5.20) as

$$\kappa(m,n) \ge \sum_{2\le 2q<[m/2]} \frac{1}{2^{2q}(2q)!}\frac{n+2m-3}{(n+2m-3)(n+2m-5)\dots(n+2m-4q-3)}\kappa_q(m,n),$$

where

$$\kappa_q(m,n) = \sum_{k=4q}^{m} \frac{(-1)^k k!}{2^k(m-k)!(k-4q)!\Gamma\left(k+\frac{n}{2}\right)} -$$

$$- \frac{1}{2(2q+1)(n+2m-4q-5)}\sum_{k=4q+2}^{m}\frac{(-1)^k k!}{2^k(m-k)!(k-4q-2)!\Gamma\left(k+\frac{n}{2}\right)}. \tag{6.5.25}$$

We shall show that quantities (6.5.25) are positive. Putting in (5.8.8) first $\alpha = 0,\ p = 2q$, and then $\alpha = 0,\ p = 2q+1$, we have

$$\sum_{k=4q}^{m}(-1)^k\frac{k!}{2^k(m-k)!(k-4q)!\Gamma\left(k+\frac{n}{2}\right)} = \frac{1}{2^{4q}\Gamma\left(\frac{n}{2}-1\right)(m-4q)!}I_0(2q,m,n),$$

$$\sum_{k=4q+2}^{m}(-1)^k \frac{k!}{2^k(m-k)!(k-4q-2)!\Gamma\left(k+\frac{n}{2}\right)} =$$
$$= \frac{1}{2^{4q+2}\Gamma\left(\frac{n}{2}-1\right)(m-4q-2)!} I_0(2q+1,m,n).$$

Inserting these values into (6.5.25), we obtain

$$\kappa_q(m,n) =$$
$$= \frac{1}{2^{4q}\Gamma\left(\frac{n}{2}-1\right)(m-4q)!}\left[I_0(2q,m,n) - \frac{(m-4q-1)(m-4q)}{8(2q+1)(n+2m-4q-5)} I_0(2q+1,m,n)\right]. \tag{6.5.26}$$

Let us integrate

$$I_0(2q,m,n) = \int_0^1 x^{4q}(1-x)^{n/2-2}\left(1-\frac{x}{2}\right)^{m-4q} dx$$

by parts:

$$I_0(2q,m,n) = \frac{1}{4q+1}\int_0^1 \frac{dx^{4q+1}}{dx}(1-x)^{n/2-2}\left(1-\frac{x}{2}\right)^{m-4q} dx =$$
$$= \frac{n-4}{2(4q+1)}\int_0^1 x^{4q+1}(1-x)^{n/2-3}\left(1-\frac{x}{2}\right)^{m-4q} dx +$$
$$+ \frac{m-4q}{2(4q+1)}\int_0^1 x^{4q+1}(1-x)^{n/2-2}\left(1-\frac{x}{2}\right)^{m-4q-1} dx.$$

Inserting this expression for $I_0(2q,m,n)$ and value (5.8.6) for $I_0(2q+1,m,n)$ into (6.5.26), we obtain

$$\kappa_q(m,n) = \frac{1}{2^{4q+1}\Gamma\left(\frac{n}{2}-1\right)(m-4q)!}\int_0^1 x^{4q+1}(1-x)^{n/2-3}\left(1-\frac{x}{2}\right)^{m-4q-2} f(x)\,dx,$$

where

$$f(x) = \frac{n-4}{4q+1}\left(1-\frac{x}{2}\right)^2 + \frac{m-4q}{4q+1}(1-x)\left(1-\frac{x}{2}\right) -$$
$$- \frac{(m-4q-1)(m-4q)}{4(2q+1)(n+2m-4q-5)} x(1-x).$$

Thus, for completing the proof it suffices to show the function f to be positive on the interval $(0,1)$ under conditions (6.5.24) and

$$4 \le 4q < m. \tag{6.5.27}$$

First, since $4q+1 < 2(2q+1)$,

$$f(x) > \frac{1}{2(2q+1)}\left[(n-4)\left(1-\frac{x}{2}\right)^2 + (m-4q)(1-x)\left(1-\frac{x}{2}\right) -\right.$$

$$- \frac{(m-4q-1)(m-4q)}{2(n+2m-4q-5)}x(1-x)\Bigg].$$

Then, (6.5.24) and (6.5.27) imply the inequality $n-4 > m-4q$ from which we obtain

$$f(x) > \frac{m-4q}{2(2q+1)}\left[\left(1-\frac{x}{2}\right)^2 + (1-x)\left(1-\frac{x}{2}\right) - \frac{m-4q-1}{2(n+2m-4q-5)}x(1-x)\right].$$

Finally, (6.5.24) and (6.5.27) lead to

$$\frac{m-4q-1}{2(n+2m-4q-5)} < \frac{1}{2},$$

and, hence,

$$f(x) > \frac{m-4q}{2(2q+1)}\left[\left(1-\frac{x}{2}\right)^2 + (1-x)\left(1-\frac{x}{2}\right) - \frac{1}{2}x(1-x)\right] > 0.$$

Lemma 6.3.1 is proved.

6.6 Inversion of the truncated transverse ray transform on Euclidean space

As have been mentioned at the end of Section 5.1, inversion of the transverse ray transform on Euclidean space reduces to the problem of inverting the ray transform of a scalar function. The situation is different for the truncated transverse ray transform. In the current section we will obtain an inversion formula for the operator K in the case $m = 2, n = 3$, which is of profound interest for polarization tomography. In the general case this formula is not found yet. Our arguments will appreciably follow the scheme of Sections 2.11–2.13.

We will use only Cartesian coordinate systems on $\mathbf{R}^3$. We use lower indices for denoting tensor components; on repeating indices the summation from 1 to 3 is understood.

For a tensor field $f \in \mathcal{S}(S^2)$ (see the definition in Section 2.1), the truncated transverse ray transform is defined by the formula

$$Kf(x,\xi) = \int_{-\infty}^{\infty} Q_\xi f(x+t\xi)\,dt \quad (x \in \mathbf{R}^3,\ \xi \in \mathbf{R}^3_0). \tag{6.6.1}$$

According to one remark of Section 6.1, we consider the problem of inverting operator (6.6.1) on the subspace of $\mathcal{S}(S^2)$ defined by the condition

$$jf = f_{kk} = 0. \tag{6.6.2}$$

As is seen from (6.4.23), for such tensors, the projection Q_ξ is given in coordinate form as follows

$$(Q_\xi f)_{ij} = f_{ij} - \frac{1}{|\xi|^2}\left(f_{ip}\xi_p\xi_j + f_{jp}\xi_p\xi_i\right) + \frac{1}{2|\xi|^4} f_{pq}\xi_p\xi_q\xi_i\xi_j + \frac{1}{2|\xi|^2} f_{pq}\xi_p\xi_q\delta_{ij}. \tag{6.6.3}$$

We define the operator $\mu : C^\infty(S^2; \mathbf{R}^3 \times \mathbf{R}^3_0) \to C^\infty(S^2; \mathbf{R}^3)$ by the equality (an analog of formula (2.11.1))

$$\mu\varphi(x) = \frac{1}{4\pi} \int\limits_\Omega \varphi(x, \xi)\, d\omega(\xi), \tag{6.6.4}$$

where Ω is the unit sphere of the space $\mathbf{R}^3$.

Let us calculate the composition $\mu K : \mathcal{S}(S^2) \to C^\infty(S^2)$. Starting with (6.6.1) and (6.6.3), we repeat the correspondent arguments of Section 2.11. In such a way, for a field f satisfying (6.6.2), we obtain

$$\mu K f = B_1 f + B_2 f + B_3 f + B_4 f, \tag{6.6.5}$$

where

$$B_1 f = \frac{1}{2\pi} f * |x|^{-2}, \tag{6.6.6}$$

$$(B_2 f)_{ij} = -\frac{1}{2\pi}\left(f_{ik} * \frac{x_j x_k}{|x|^4} + f_{jk} * \frac{x_i x_k}{|x|^4} \right), \tag{6.6.7}$$

$$(B_3 f)_{ij} = \frac{1}{4\pi} f_{kl} * \frac{x_i x_j x_k x_l}{|x|^6}, \tag{6.6.8}$$

$$(B_4 f)_{ij} = \frac{1}{4\pi} f_{kl} * \frac{x_k x_l}{|x|^4} \delta_{ij}. \tag{6.6.9}$$

Denoting $h = \frac{2}{\pi}\mu K f$ and applying the Fourier transform to relations (6.6.5)–(6.6.9), we arrive at the algebraic system of equations:

$$\hat f_{ij} - \frac{1}{2}\left(\hat f_{ik}\varepsilon_{jk} + \hat f_{jk}\varepsilon_{ik}\right) + \frac{3}{16}\hat f_{kl}\varepsilon^2_{ijkl} + \frac{1}{4}\hat f_{kl}\varepsilon_{kl}\delta_{ij} = |y|\hat h_{ij}, \tag{6.6.10}$$

where the notations $e_i = y_i/|y|$ and $\varepsilon_{ij} = \delta_{ij} - e_i e_j$ are used.

System (6.6.10) is uniquely solvable, as the next claim shows.

Lemma 6.6.1 *If the right-hand side meets the condition $j\hat h = 0$, then system (6.6.10) has a unique solution satisfying the condition $j\hat f = 0$. The solution is expressed through the right-hand side by the formula*

$$\hat f_{ij} = 2|y|\left[4\hat h_{ij} - 3\left(\hat h_{ik}e_k e_j + \hat h_{jk}e_k e_i\right) + \hat h_{kl}e_k e_l e_i e_j + \frac{5}{3}\hat h_{kl}e_k e_l \delta_{ij}\right]. \tag{6.6.11}$$

P r o o f. One can directly verify that the operators $A : \hat f \mapsto \hat h$ and $B : \hat h \mapsto \hat f$ defined by formulas (6.6.10) and (6.6.11) map the space $\operatorname{Ker} j$ into itself. So, it suffices to establish validity on $\operatorname{Ker} j$ of the identity obtainable by inserting value (6.6.11) for f into the left-hand side of equation (6.6.10). This identity is also verified by a direct calculation. The lemma is proved.

Writing (6.6.11) in invariant form

$$\hat f(y) = 2|y|\left(4 - \frac{6}{|y|^2} i_y j_y + \frac{1}{|y|^4} i_y^2 j_y^2 + \frac{5}{3}\frac{1}{|y|^2} i j_y^2\right)\hat h(y),$$

applying the inverse Fourier transform to the last equality and using arguments sited before the formulation of Theorem 2.12.2, we arrive at the next

Theorem 6.6.2 *A tensor field* $f \in \mathcal{S}(S^2)$ *on* $\mathbf{R}^3$ *satisfying condition* (6.6.2) *is recovered from its truncated transverse ray transform by the formula*

$$f = \frac{4}{\pi}(-\Delta)^{1/2}\left(4 - 6\Delta^{-1}d\delta + \Delta^{-2}d^2\delta^2 + \frac{5}{3}i\Delta^{-1}\delta^2\right)\mu K f,$$

where the operator μ *is defined by equality* (6.6.4), Δ *is the Laplacian,* d *is the operator of inner differentiation,* δ *is the divergence and* i *is the operator of symmetric multiplication by the Kronecker tensor.*

Chapter 7

The mixed ray transform

Here we will investigate the equations of dynamic elasticity by the same scheme as that which was applied to the Maxwell system in Chapter 5.

In Section 7.1 the traditional method of geometrical optics is applied to quasi-isotropic elastic media. We restrict ourselves to considering the zero approximation. We show that, relative to the classical case of isotropic media, in our case formulas for the zero approximation have the next features. First, the formula for the amplitude of a compression wave contains some factor that describes the accumulation, due to anisotropy, of the wave phase along a ray. Second, the Rytov law for shear waves contains a term that depends linearly on the anisotropic part of the elasticity tensor. Then we discuss the inverse problems of determining the anisotropic part of the elasticity tensor. For compression waves, the inverse problem is equivalent to the problem of inverting the (longitudinal) ray transform I, of a tensor field of degree 4, which was investigated in Chapter 4. For shear waves, the inverse problem, after linearization, leads to a new operator L of integral geometry which is called the mixed ray transform.

In Section 7.2 the definition of the mixed ray transform is generalized to tensor fields of degree $m + l$ symmetric in the first m and last l indices. This generalization is given to emphasize that the already-investigated operators I (the ray transform) and J (the transverse ray transform) are special cases of the operator L.

The problem of inverting the operator L is considered in Section 7.3. Our exposition of arguments similar to those used in the previous chapters is kept at a minimum. As in the cases of the operators J and K, our problem reduces in the long run to some algebraic question generalizing Problem 5.9.1. The author could solve the latter only for $m = l = 2$, so the final result is obtained only in this case. The algebraic problem is considered in Section 7.4.

The material of the chapter is first published in this book.

7.1 Elastic waves in quasi-isotropic media

7.1.1 The equations of dynamic elasticity

Let (x^1, x^2, x^3) be a curvilinear coordinate system in $\mathbf{R}^3$ in which the Euclidean metric is given by quadratic form (5.1.6). Elastic oscillations are described by the *displacement vector* $u(x,t) = (u^1, u^2, u^3)$. Within the frames of linear elasticity theory the *strain tensor* is introduced by the equality

$$\varepsilon_{jk} = \frac{1}{2}\left(u_{j\,;\,k} + u_{k\,;\,j}\right), \tag{7.1.1}$$

while determining the *stress tensor* by the formula

$$\sigma_{jk} = a_{jklm}\varepsilon^{lm}, \tag{7.1.2}$$

where $a = (a_{jklm})$ is called the *elasticity tensor*. It is assumed to possess the symmetries

$$a_{jklm} = a_{kjlm} = a_{jkml} = a_{lmjk}. \tag{7.1.3}$$

System (7.1.1), (7.1.2) is closed with the *equilibrium equations*

$$\sigma_{jk\,;}{}^{k} - \rho\frac{\partial^2 u_j}{\partial t^2} = 0, \tag{7.1.4}$$

where ρ is the density of a medium. The covariant derivatives are taken in the Euclidean metric (5.1.6). We restrict ourselves to considering waves harmonic in the time: $u(x,t) = u(x)e^{-i\omega t}$. Equations (7.1.1) and (7.1.2) do not change, and (7.1.4) is replaced with

$$\sigma_{jk\,;}{}^{k} + \omega^2\rho u_j = 0. \tag{7.1.5}$$

We assume the elasticity tensor to be representable as

$$a_{jklm} = \lambda g_{jk}g_{lm} + \mu\left(\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl}\right) + \frac{1}{\omega}c_{jklm} \tag{7.1.6}$$

where λ and μ are positive functions called the *Lame parameters*, and c is referred to as the *anisotropic part* of the elasticity tensor. A medium satisfying (7.1.6) is called *quasi-isotropic* (the term "slightly anisotropic medium" is also used). In particular, in the case $c \equiv 0$ we have an isotropic medium. Equation (7.1.2) is rewritten as follows:

$$\sigma_{jk} = \lambda g^{lm}\varepsilon_{lm}g_{jk} + 2\mu\varepsilon_{jk} + \frac{1}{\omega}c_{jklm}\varepsilon^{lm}. \tag{7.1.7}$$

We assume λ, μ, ρ, c_{jklm} to be smooth real functions of a point $x = (x^1, x^2, x^3)$.

7.1.2 The eikonal equation

The method of geometrical optics consists in representating a solution to system (7.1.1), (7.1.5), (7.1.7) by the asymptotic series

$$u_j = e^{i\omega\tau}\sum_{m=0}^{\infty}\frac{\overset{m}{u}_j}{(i\omega)^m}, \quad \varepsilon_{jk} = e^{i\omega\tau}\sum_{m=-1}^{\infty}\frac{\overset{m}{\varepsilon}_{jk}}{(i\omega)^m}, \quad \sigma_{jk} = e^{i\omega\tau}\sum_{m=-1}^{\infty}\frac{\overset{m}{\sigma}_{jk}}{(i\omega)^m},$$

where $\tau = \tau(x)$ is a real function. We insert the series into the equations under consideration, implement differentiations and equate the coefficients of the same powers of the frequency ω on the left- and right-hand sides of the so-obtained equalities. In such a way we arrive at the infinite system of equations

$$\overset{m}{\varepsilon}_{jk} = \frac{1}{2}\left(\overset{m}{u}_{j\,;\,k} + \overset{m}{u}_{k\,;\,j} + \overset{m+1}{u}_{j}\tau_{;\,k} + \overset{m+1}{u}_{k}\tau_{;\,j}\right) \qquad (m = -1, 0, ...), \tag{7.1.8}$$

$$\overset{m}{\sigma}_{jk} = \lambda g^{pq} \overset{m}{\varepsilon}_{pq} g_{jk} + 2\mu \overset{m}{\varepsilon}_{jk} + i c_{jkpq} \overset{m-1}{\varepsilon}{}^{pq} \qquad (m = -1, 0, \ldots). \tag{7.1.9}$$

$$\overset{m}{\sigma}_{jk\,;}{}^{k} + \overset{m+1}{\sigma}_{jk} \tau_{;}{}^{k} - \rho \overset{m+2}{u}_{j} = 0 \qquad (m = -2, -1, \ldots), \tag{7.1.10}$$

where it is assumed that $\overset{-1}{u} = \overset{-2}{\varepsilon} = \overset{-2}{\sigma} = 0$.

Consider the initial terms of this chain. Putting $m = -1$ in (7.1.8), (7.1.9) and $m = -2$ in (7.1.10), we have

$$\overset{-1}{\varepsilon}_{jk} = \frac{1}{2}\left(\overset{0}{u}_{j} \tau_{;\,k} + \overset{0}{u}_{k} \tau_{;\,j} \right), \tag{7.1.11}$$

$$\overset{-1}{\sigma}_{jk} = \lambda g^{pq} \overset{-1}{\varepsilon}_{pq} g_{jk} + 2\mu \overset{-1}{\varepsilon}_{jk}, \tag{7.1.12}$$

$$\overset{-1}{\sigma}_{jk} \tau_{;}{}^{k} - \rho \overset{0}{u}_{j} = 0. \tag{7.1.13}$$

We see that c does not participate in (7.1.11)–(7.1.13). So the conclusions to be derived from these equations are identical for isotropic and quasi-isotropic media.

From (7.1.11) and (7.1.12), we obtain

$$\overset{-1}{\sigma}_{jk} = \lambda \overset{0}{u}_{p} \tau_{;}{}^{p} g_{jk} + \mu (\overset{0}{u}_{j} \tau_{;\,k} + \overset{0}{u}_{k} \tau_{;\,j}). \tag{7.1.14}$$

Inserting the last expression into (7.1.13), we arrive at the relation

$$(\lambda + \mu) \overset{0}{u}_{p} \tau_{;}{}^{p} \tau_{;\,j} + \mu \tau_{;\,p} \tau_{;}{}^{p} \overset{0}{u}_{j} - \rho \overset{0}{u}_{j} = 0$$

which can be written in invariant form

$$(\lambda + \mu) \langle \overset{0}{u}, \nabla \tau \rangle \nabla \tau + (\mu |\nabla \tau|^2 - \rho) \overset{0}{u} = 0. \tag{7.1.15}$$

If the equality

$$|\nabla \tau|^2 = \rho / \mu \tag{7.1.16}$$

holds, then (7.1.15) implies the relation

$$\langle \overset{0}{u}, \nabla \tau \rangle = 0. \tag{7.1.17}$$

If (7.1.16) does not hold, by taking the scalar product of (7.1.15) and $\nabla \tau$, we obtain $|\nabla \tau|^2 = \rho/(\lambda + 2\mu)$. Inserting the last expression into (7.1.15), we have

$$\overset{0}{u} = \frac{\lambda + 2\mu}{\rho} \langle \overset{0}{u}, \nabla \tau \rangle \nabla \tau = A_p \frac{\nabla \tau}{|\nabla \tau|}. \tag{7.1.18}$$

Thus, we have arrived at the next conclusion: the function τ satisfies the *eikonal equation*

$$|\nabla \tau|^2 = n^2 \tag{7.1.19}$$

whose right-hand side assumes one of the two values:

$$n^2 = n_p^2 = \rho / (\lambda + 2\mu) \tag{7.1.20}$$

or

$$n^2 = n_s^2 = \rho / \mu. \tag{7.1.21}$$

The reciprocals $v_p = 1/n_p$ and $v_s = 1/n_s$ are called the *velocities of compression and shear waves* respectively. For $n = n_p$ a wave has a longitudinal nature within the scope of the zero approximation: the displacement vector is a multiple of $\nabla \tau$. For $n = n_s$ a wave is of a transversal nature within the scope of the zero approximation: the displacement vector is orthogonal to $\nabla \tau$.

The characteristics (rays) of the eikonal equation are geodesics of the Riemannian metric (5.1.17).

7.1.3 The amplitude of a compression wave

The *amplitude* A_p *of a compression wave* is defined by (7.1.18). A formula for A_p is obtained from equilibrium equation (7.1.10) for $m = -1$. To simplify calculations we use ray coordinates.

We fix a solution τ to the eikonal equation (7.1.19) for $n = n_p$. In some neighbourhood of any point one can introduce ray coordinates $x^1, x^2, x^3 = \tau$ as was explained in Section 5.1.3. The Euclidean metric (5.1.6) has the form (5.1.19) in these coordinates. The gradient of τ is given by equalities (5.1.20). By (7.1.18), the vector field $\overset{0}{u}$ has the coordinates

$$\overset{0}{u}_\alpha = \overset{0}{u}{}^\alpha = 0, \quad \overset{0}{u}_3 = n^{-1}A_p, \quad \overset{0}{u}{}^3 = nA_p. \tag{7.1.22}$$

In the current section Greek indices assume the values 1,2; on repeating Greek indices, summation from 1 to 2 is understood. By (5.1.20) and (7.1.22), formulas (7.1.14) take the form

$$\overset{-1}{\sigma}_{\alpha\beta} = \lambda n g_{\alpha\beta} A_p, \quad \overset{-1}{\sigma}_{\alpha 3} = 0, \quad \overset{-1}{\sigma}_{33} = (\lambda + 2\mu) n^{-1} A_p. \tag{7.1.23}$$

In the derivation of a formula for A_p we will need only the following equations, of system (7.1.8)–(7.1.10), which are written down below, while taking (5.1.19) and (5.1.20) into account:

$$\overset{0}{\varepsilon}_{\alpha\beta} = \frac{1}{2}\left(\overset{0}{u}_{\alpha\,;\,\beta} + \overset{0}{u}_{\beta\,;\,\alpha}\right), \qquad \overset{0}{\varepsilon}_{33} = \overset{0}{u}_{3\,;\,3} + \overset{1}{u}_3, \tag{7.1.24}$$

$$\overset{0}{\sigma}_{33} = \lambda n^{-2} g^{\alpha\beta}\, \overset{0}{\varepsilon}_{\alpha\beta} + (\lambda + 2\mu)\overset{0}{\varepsilon}_{33} + i c_{33pq}\, \overset{-1}{\varepsilon}{}^{pq}, \tag{7.1.25}$$

$$\overset{-1}{\sigma}_{3k\,;}{}^{k} + n^2\, \overset{0}{\sigma}_{33} - \rho \overset{1}{u}_3 = 0. \tag{7.1.26}$$

Using (5.1.21) and (7.1.22)–(7.1.23), we obtain

$$\overset{0}{u}_{\alpha\,;\,\beta} = \frac{1}{2} n \frac{\partial g_{\alpha\beta}}{\partial \tau} A_p, \qquad \overset{0}{u}_{3\,;\,3} = n^{-1}\frac{\partial A_p}{\partial \tau}, \tag{7.1.27}$$

$$\overset{-1}{\sigma}_{3\alpha\,;\,\beta} = \mu n \frac{\partial g_{\alpha\beta}}{\partial \tau} A_p,$$

$$\overset{-1}{\sigma}_{33\,;\,3} = (\lambda + 2\mu) n^{-1} \frac{\partial A_p}{\partial \tau} + \left[n^{-1} \frac{\partial(\lambda + 2\mu)}{\partial \tau} + (\lambda + 2\mu) n^{-2} \frac{\partial n}{\partial \tau}\right] A_p.$$

The last two equalities imply

$$\overset{-1}{\sigma}_{3k\,;}{}^{k} = (\lambda + 2\mu) n \frac{\partial A_p}{\partial \tau} + \left[\mu n g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial \tau} + n \frac{\partial(\lambda + 2\mu)}{\partial \tau} + (\lambda + 2\mu) \frac{\partial n}{\partial \tau}\right] A_p. \tag{7.1.28}$$

From (7.1.24) and (7.1.27), we obtain

$$\overset{0}{\varepsilon}_{\alpha\beta} = \frac{1}{2} n \frac{\partial g_{\alpha\beta}}{\partial \tau} A_p, \qquad \overset{0}{\varepsilon}_{33} = n^{-1} \frac{\partial A_p}{\partial \tau} + \overset{1}{u}_3. \tag{7.1.29}$$

With the help of the last formulas, (7.1.25) implies

$$\overset{0}{\sigma}_{33} = (\lambda + 2\mu) n^{-1} \frac{\partial A_p}{\partial \tau} + \frac{1}{2} \lambda n^{-1} g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial \tau} A_p + i c_{33pq}\, \overset{-1}{\varepsilon}{}^{pq} + (\lambda + 2\mu) \overset{1}{u}_3.$$

Inserting the found value for $\overset{0}{\sigma}_{33}$ and value (7.1.28) for $\overset{-1}{\sigma}_{3k\,;}{}^{k}$ into (7.1.26), we arrive at the relation

$$2(\lambda+2\mu)n\frac{\partial A_p}{\partial \tau}+\left[\frac{1}{2}(\lambda+2\mu)ng^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial \tau}+n\frac{\partial(\lambda+2\mu)}{\partial \tau}+(\lambda+2\mu)\frac{\partial n}{\partial \tau}\right]A_p+$$

$$+in^2c_{33pq}\overset{-1}{\varepsilon}{}^{pq}+[(\lambda+2\mu)n^2-\rho]\overset{1}{u}_3=0.$$

By (7.1.20), the coefficient of $\overset{1}{u}_3$ in this formula is equal to zero. Thus, inserting into the last formula the expressions

$$\overset{-1}{\varepsilon}{}^{\alpha\beta}=\overset{-1}{\varepsilon}{}^{\alpha 3}=0,\qquad \overset{-1}{\varepsilon}{}^{33}=n^3A_p$$

that follows from (7.1.11), (5.1.20) and (7.1.22), we arrive at the equation for the amplitude A_p:

$$\frac{\partial A_p}{\partial \tau}+\left(\frac{1}{4}g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial \tau}+\frac{1}{2}\frac{1}{\lambda+2\mu}\frac{\partial(\lambda+2\mu)}{\partial \tau}+\frac{1}{2}\frac{1}{n}\frac{\partial n}{\partial \tau}+\frac{i}{2}\frac{n^4c_{3333}}{\lambda+2\mu}\right)A_p=0. \tag{7.1.30}$$

Using notation (5.1.22) and formula (5.1.23), we transform equation (7.1.30) so as to obtain

$$\frac{\partial}{\partial \tau}\left[\ln\left(A_pJ^{1/2}n^{1/2}(\lambda+2\mu)\right)^{1/2}\right]=-\frac{i}{2}\frac{n^4}{\lambda+2\mu}c_{3333},$$

which, with (7.1.20) and (5.1.22) taken into account, implies

$$\frac{\partial}{\partial \tau}\left[\ln\left(A_p\sqrt{J\rho v_p}\right)\right]=-\frac{i}{2\rho v_p^6}c_{3333}.$$

Integrating the last equation, we obtain the formula for the amplitude of a compression wave:

$$A_p=\frac{C}{\sqrt{J\rho v_p}}\exp\left[-i\int\limits_{\tau_0}^{\tau}\frac{1}{2\rho v_p^6}c_{3333}\,d\tau\right] \tag{7.1.31}$$

where C is a constant depending on a ray; $d\tau$ is the length element of the ray in metric (5.1.17); the integration is executed starting from a fixed point of the ray to the current point in which the amplitude is calculated.

Formula (7.1.31) is obtained in a ray coordinate system for the ray $x^\alpha=x_0^\alpha$. For an arbitrary ray $\gamma=(\gamma^1,\gamma^2,\gamma^3)$, in arbitrary coordinates this formula looks like

$$A_p=\frac{C}{\sqrt{J\rho v_p}}\exp\left[-i\int\limits_{\gamma}\frac{1}{2\rho v_p^6}c_{jklm}\dot\gamma^j\dot\gamma^k\dot\gamma^l\dot\gamma^m d\tau\right], \tag{7.1.32}$$

where $\dot\gamma^j=d\gamma^j/d\tau$. Validity of (7.1.32) follows from the fact that this formula is invariant under change of coordinates, and coincides with (7.1.31) in ray coordinates.

For $c\equiv 0$, (7.1.32) gives the classical formula for the amplitude of a compression wave in an isotropic elastic medium:

$$A_p=\frac{C}{\sqrt{J\rho v_p}} \tag{7.1.33}$$

Comparing (7.1.32) and (7.1.33), we conclude that slight elastic anisotropy leads to the accumulation, of the phase of a compression wave along a ray, which is expressed by the integral in (7.1.32). A formula close to (7.1.32) is obtained in [17] in another way.

7.1.4 The amplitude of a shear wave

We fix a solution τ to the eikonal equation (7.1.19) for $n = n_s$ and introduce a ray coordinate system as above. Formulas (5.1.20) are preserved and (7.1.22), (7.1.23) are replaced with:

$$\overset{0}{u}_3 = \overset{0}{u}{}^3 = 0, \tag{7.1.34}$$

$$\overset{-1}{\sigma}_{\alpha\beta} = 0, \qquad \overset{-1}{\sigma}_{\alpha 3} = \mu\overset{0}{u}_\alpha, \qquad \overset{-1}{\sigma}_{33} = 0. \tag{7.1.35}$$

From system (7.1.8)–(7.1.10), in this case we use only the next equations

$$\overset{0}{\varepsilon}_{\alpha 3} = \frac{1}{2}(\overset{0}{u}_{\alpha\,;\,3} + \overset{0}{u}_{3\,;\,\alpha} + \overset{1}{u}_\alpha), \tag{7.1.36}$$

$$\overset{0}{\sigma}_{\alpha 3} = 2\mu\overset{0}{\varepsilon}_{\alpha 3} + ic_{\alpha 3pq}\overset{-1}{\varepsilon}{}^{pq}, \tag{7.1.37}$$

$$\overset{-1}{\sigma}_{\alpha k\,;}{}^{k} + n^2\overset{0}{\sigma}_{\alpha 3} - \rho\overset{1}{u}_\alpha = 0. \tag{7.1.38}$$

Inserting value (7.1.36) for $\overset{0}{\varepsilon}_{\alpha 3}$ and the expressions

$$\overset{-1}{\varepsilon}{}^{\alpha\beta} = 0, \qquad \overset{-1}{\varepsilon}{}^{\alpha 3} = \frac{1}{2}n^2\overset{0}{u}{}^\alpha, \qquad \overset{-1}{\varepsilon}{}^{33} = 0$$

that follow from (7.1.11), (5.1.20) and (7.1.34) into (7.1.37), we obtain

$$\overset{0}{\sigma}_{\alpha 3} = \mu(\overset{0}{u}_{\alpha\,;\,3} + \overset{0}{u}_{3\,;\,\alpha} + \overset{1}{u}_\alpha) + in^2c_{\alpha 3\beta 3}\overset{0}{u}{}^\beta. \tag{7.1.39}$$

From (7.1.34) and (7.1.35), we find the covariant derivatives

$$\overset{0}{u}_{\alpha\,;\,3} = \frac{\partial\overset{0}{u}_\alpha}{\partial\tau} - \frac{1}{2}g^{\beta\gamma}\frac{\partial g_{\alpha\beta}}{\partial\tau}\overset{0}{u}_\gamma, \qquad \overset{0}{u}_{3\,;\,\alpha} = -\frac{1}{2}g^{\beta\gamma}\frac{\partial g_{\alpha\beta}}{\partial\tau}\overset{0}{u}_\gamma, \tag{7.1.40}$$

$$\overset{-1}{\sigma}_{\alpha\beta\,;\,\gamma} = \frac{1}{2}\mu n^2\left(\frac{\partial g_{\alpha\gamma}}{\partial\tau}\overset{0}{u}_\beta + \frac{\partial g_{\beta\gamma}}{\partial\tau}\overset{0}{u}_\alpha\right),$$

$$\overset{-1}{\sigma}_{\alpha 3\,;\,3} = \frac{\partial(\mu\overset{0}{u}_\alpha)}{\partial\tau} - \frac{1}{2}\mu g^{\beta\gamma}\frac{\partial g_{\alpha\beta}}{\partial\tau}\overset{0}{u}_\gamma + \mu n^{-1}\frac{\partial n}{\partial\tau}\overset{0}{u}_\alpha.$$

By (5.1.23), the last two equalities imply

$$\overset{-1}{\sigma}_{\alpha k\,;}{}^{k} = \mu n^2\left[\frac{\partial\overset{0}{u}_\alpha}{\partial\tau} + \left(\frac{1}{J}\frac{\partial J}{\partial\tau} + \frac{1}{n}\frac{\partial n}{\partial\tau} + \frac{1}{\mu}\frac{\partial\mu}{\partial\tau}\right)\overset{0}{u}_\alpha\right]. \tag{7.1.41}$$

From (7.1.39) and (7.1.40), we obtain the representation

$$\overset{0}{\sigma}_{\alpha 3} = \mu\left(\frac{\partial\overset{0}{u}_\alpha}{\partial\tau} - g^{\beta\gamma}\frac{\partial g_{\alpha\beta}}{\partial\tau}\overset{0}{u}_\gamma + \overset{1}{u}_\alpha\right) + in^2c_{\alpha 3\beta 3}\overset{0}{u}{}^\beta$$

and insert it into (7.1.38)

$$\overset{-1}{\sigma}_{\alpha k\,;}{}^{k} + \mu n^2\left(\frac{\partial\overset{0}{u}_\alpha}{\partial\tau} - g^{\beta\gamma}\frac{\partial g_{\alpha\beta}}{\partial\tau}\overset{0}{u}_\gamma\right) + (\mu n^2 - \rho)\overset{1}{u}_\alpha + in^4c_{\alpha 3\beta 3}\overset{0}{u}{}^\beta = 0.$$

By (7.1.21) the coefficient of $\overset{1}{u}_\alpha$ in the last formula is equal to zero. Inserting expression (7.1.41) for the first term in the last formula and introducing the notation

$$a = \frac{1}{J}\frac{\partial J}{\partial \tau} + \frac{1}{n}\frac{\partial n}{\partial \tau} + \frac{1}{\mu}\frac{\partial \mu}{\partial \tau}. \tag{7.1.42}$$

we arrive at the next system for the functions $\overset{0}{u}_\alpha$:

$$2\frac{\partial \overset{0}{u}_\alpha}{\partial \tau} + a\overset{0}{u}_\alpha - g^{\beta\gamma}\frac{\partial g_{\alpha\beta}}{\partial \tau}\overset{0}{u}_\gamma = -i\frac{n^2}{\mu}c_{\alpha 3\beta 3}\overset{0}{u}^\beta \qquad (\alpha = 1, 2). \tag{7.1.43}$$

From now on we will omit the index 0 in the notation $\overset{0}{u}$, since other amplitudes ($\overset{m}{u}$ for $m \neq 0$) are not used further in the current section.

The *amplitude* A_s *of a shear wave* is defined by the equality

$$A_s^2 = |u|^2 = g^{\alpha\beta}u_\alpha \overline{u}_\beta. \tag{7.1.44}$$

From (7.1.43), we will obtain an equation for A_s. To this end, we differentiate (7.1.44)

$$\frac{\partial A_s^2}{\partial \tau} = 2\mathrm{Re}\left(g^{\alpha\beta}\frac{\partial u_\alpha}{\partial \tau}\overline{u}_\beta\right) + \frac{\partial g^{\alpha\beta}}{\partial \tau}u_\alpha\overline{u}_\beta.$$

Inserting the expression for $\partial u_\alpha/\partial\tau$, which follows from (7.1.43), into the last formula, we obtain

$$\frac{\partial A_s^2}{\partial \tau} = \mathrm{Re}\left[\left(g^{\alpha\gamma}g^{\beta\delta}\frac{\partial g_{\gamma\delta}}{\partial \tau} - ag^{\alpha\beta}\right)u_\alpha\overline{u}_\beta - i\frac{n^2}{\mu}c_{\alpha 3\beta 3}u^\alpha\overline{u}^\beta\right] + \frac{\partial g^{\alpha\beta}}{\partial \tau}u_\alpha\overline{u}_\beta.$$

The tensor c has the same symmetries (7.1.3) as the tensor a. Therefore, the second summand in the brackets is pure imaginary and, consequently, can be omitted. The expression in the parentheses is symmetric in α and β and, consequently, the first summand in the brackets is real. So the last formula can be rewritten as:

$$\frac{\partial A_s^2}{\partial \tau} = \left(\frac{\partial g^{\alpha\beta}}{\partial \tau} + g^{\alpha\gamma}g^{\beta\delta}\frac{\partial g_{\gamma\delta}}{\partial \tau} - ag^{\alpha\beta}\right)u_\alpha\overline{u}_\beta.$$

In view of (5.1.43), this equation takes the form

$$2A_s\frac{\partial A_s}{\partial \tau} = -ag^{\alpha\beta}u_\alpha\overline{u}_\beta = -aA_s^2,$$

or

$$2\frac{\partial A_s}{\partial \tau} + aA_s = 0. \tag{7.1.45}$$

Inserting value (7.1.42) for a into the last equation and using (7.1.21), we obtain

$$\frac{\partial}{\partial \tau}\left[\ln\left(A_s\sqrt{J\rho v_s}\right)\right] = 0.$$

Integrating, we arrive at the classical formula for the amplitude of a shear wave:

$$A_s = \frac{C}{\sqrt{J\rho v_s}},$$

where C is a constant depending on a ray. Observe that this formula remains the same for an isotropic and quasi-isotropic media.

7.1.5 Rytov's law

We transform system (7.1.43) by changing variables

$$u_\alpha = A_s n^{-1} \eta_\alpha, \tag{7.1.46}$$

where A_s is the amplitude defined by formula (7.1.44). Using (7.1.43) and (7.1.45), we obtain

$$2\frac{\partial \eta_\alpha}{\partial \tau} = 2\frac{\partial}{\partial \tau}\left(\frac{n}{A_s}u_\alpha\right) = 2\frac{n}{A_s}\frac{\partial u_\alpha}{\partial \tau} - 2\frac{n}{A_s^2}\frac{\partial A_s}{\partial \tau}u_\alpha + \frac{2}{A_s}\frac{\partial n}{\partial \tau}u_\alpha =$$
$$= \frac{n}{A_s}\left(g^{\beta\gamma}\frac{\partial g_{\alpha\beta}}{\partial \tau}u_\gamma - a u_\alpha - i\frac{n^2}{\mu}c_{\alpha 3\beta 3}u^\beta\right) + \frac{na}{A_s}u_\alpha + \frac{2}{A_s}\frac{\partial n}{\partial \tau}u_\alpha =$$
$$= g^{\beta\gamma}\frac{\partial g_{\alpha\beta}}{\partial \tau}\eta_\gamma + \frac{2}{n}\frac{\partial n}{\partial \tau}\eta_\alpha - i\frac{n^4}{\mu}c_{\alpha 3\beta 3}\eta^\beta.$$

We thus arrive at the equation

$$\frac{\partial \eta_\alpha}{\partial \tau} - \frac{1}{2}g^{\beta\gamma}\frac{\partial g_{\alpha\beta}}{\partial \tau}\eta_\gamma - \frac{1}{n}\frac{\partial n}{\partial \tau}\eta_\alpha = -\frac{i}{2}\frac{n^2}{\mu}c_{\alpha 3\beta 3}g^{\beta\gamma}\eta_\gamma. \tag{7.1.47}$$

Until now all our calculations were performed in the Euclidean metric. Let us invoke the Riemannian metric (5.1.17), which assumes the form (5.1.49) in ray coordinates. Note that the vector η has unit length with respect to this metric. Inserting expression (5.1.49) for $g_{\alpha\beta}$ into (7.1.47), we transform this equation as follows

$$\frac{\partial \eta_\alpha}{\partial \tau} - \frac{1}{2}h^{\beta\gamma}\frac{\partial h_{\alpha\beta}}{\partial \tau}\eta_\gamma = -\frac{i}{2}\frac{n^4}{\mu}c_{\alpha 3\beta 3}\eta^\beta. \tag{7.1.48}$$

It is essential that raising the index of the vector η is performed with respect to metric (5.1.49), i.e., $\eta^\beta = h^{\beta\gamma}\eta_\gamma$. We convince ourselves in the same manner as in Section 5.1.4 that equation (7.1.48) is equivalent to the next one

$$\eta_{\alpha\,;\,3} = -i\frac{1}{2\rho v_s^6}c_{\alpha 3\beta 3}\eta^\beta \qquad (\alpha = 1, 2). \tag{7.1.49}$$

Besides, the relations $|\eta|^2 = 1$ and $\langle \eta, \dot\gamma\rangle = 0$ imply that

$$\eta_{3\,;\,3} = 0 \tag{7.1.50}$$

In (7.1.49) and (7.1.50) the covariant derivatives are taken in metric (5.1.49). The index 3 is distinguished in these equations, since the former are written in ray coordinates. In an arbitrary coordinate system these equations are replaced with the next:

$$\left(\frac{D\eta}{d\tau}\right)_j = -i\frac{1}{2\rho v_s^6}(\delta_j^q - \dot\gamma_j\dot\gamma^q)c_{qklm}\dot\gamma^k\dot\gamma^m\eta^l, \tag{7.1.51}$$

where δ_q^j is the Kronecker tensor. Validity of (7.1.51) follows from the fact that this formula is invariant under change of coordinates and coincides with (7.1.49), (7.1.50) in ray coordinates. We call equation (7.1.51) the *Rytov law for a quasi-isotropic elastic medium* (in Riemannian form). For an isotropic medium it takes the form

$$\frac{D\eta}{d\tau} = 0 \tag{7.1.52}$$

and means that the vector field η is parallel along a ray in the sense of metric (5.1.17).

7.1.6 The inverse problem for compression waves

First of all we emphasize that the inverse problems considered below are of a rather formal character due to the next two circumstances. First, we deal only with the problem of determining the anisotropic part of the elasticity tensor while the isotropic part is assumed to be known. In practice, the latter is usually found as a solution of an inverse problem. The information given in such an inverse problem is usually the travel times field. The anisotropic part of the elasticity tensor gives some distortion into the travel times field. Thus, it stands to reason to consider both inverse problems simultaneously. Second, we impose no boundary condition on the boundary of a domain under consideration. Thus, we treat the problems as if the waves propagate in an unbounded medium, and use the boundary only as a surface at which the sources and detectors of oscillations are disposed. In point of fact, due to the reflection effects on the boundary, the possibility of registrating information that is used below as the data for inverse problems, seems to be rather problematic. Here we will not settle this question but only attract the reader's attention to the fact of existence of it and similar questions.

Let an elastic medium be contained in a bonded domain $D \subset \mathbf{R}^3$. We assume that the coefficient n_p of refraction for compression waves is known, i.e., metric (5.1.17) and its geodesics are known. Our problem is that of determining the tensor field c in the Hooke law (7.1.7). To this end, assume that we can dispose a source of compression waves in every point of the boundary ∂D and measure the phase of a compression wave on the same surface ∂D. We introduce the notation $b_{jklm} = \sigma c_{jklm}/(2\rho v_p^6)$ where σ is symmetrization, as the reader would recall. By (7.1.32), the problem is equivalent to the next: one has to determine a tensor field b which is defined on D, provided the number $\exp[-iIb(\gamma)]$ is known for every geodesic γ whose endpoints are in ∂D; here I is the ray transform. By the conventional monodromy principle, the function $Ib(\gamma)$ can be uniquely recovered from $\exp[-iIb(\gamma)]$, if the boundary ∂D is connected. Thus, our problem reduces to inverting the operator I.

In the three-dimensional case the tensor c possessing symmetries (7.1.3) has 21 different components. The symmetric tensor $b = \sigma c$ has only 15 different components. Consequently, 6 components of the desired tensor are lost just in posing the problem. As we know, b is recovered from Ib up to a potential field of the type dv, where v is a symmetric tensor field of degree 3 depending on 10 arbitrary functions. Thus our problem allows us to determine $15 - 10 = 5$ independent local functionals of the desired field.

7.1.7 The inverse problem for shear waves

Introduce a tensor field f by the equality

$$f_{jklm} = -i\frac{1}{4\rho v_s^6}(c_{jlkm} + c_{jmkl}).$$

It follows from (7.1.3) that f possesses the symmetries

$$f_{jklm} = f_{kjlm} = f_{jkml}, \tag{7.1.53}$$

$$f_{jklm} = f_{lmjk}, \tag{7.1.54}$$

and c is recovered from f by the formula

$$c_{jklm} = 2i\rho v_s^6(f_{jlkm} + f_{jmkl} - f_{jklm}).$$

Further we do not use symmetry (7.1.54) and consider the inverse problem for tensor fields possessing only (7.1.53). The question is formulated as follows: for every geodesic $\gamma : [a, b] \to D$ with endpoints in ∂D, the value $\eta(b)$ of a solution to system (7.1.51) is known as a function of the initial value $\eta(a)$ and the geodesic, $\eta(b) = U(\gamma)\eta(a)$; one has to determine the field $f(x)$ from $U(\gamma)$. Linearizing this problem by the scheme of Section 5.1.6, we arrive at the next linear problem.

Problem 7.1.1 *Given is a Riemannian metric in a bounded domain $D \subset \mathbf{R}^n$. Determine a tensor field $f = (f_{jklm})$ that is defined on D and has symmetries* (7.1.53), *if for every geodesic $\gamma : [a, b] \to D$ whose endpoints are in ∂D, the quadratic form*

$$Lf(\gamma, \eta) = \int\limits_a^b f_{jklm}(\gamma(t))\eta^j(t)\eta^k(t)\dot\gamma^l(t)\dot\gamma^m(t)\,dt \qquad (\eta \in \gamma^\perp) \tag{7.1.55}$$

is known on the space $\gamma^\perp$ of vector fields parallel along γ and orthogonal to $\dot\gamma$.

The operator L defined by equality (7.1.55) is called the *mixed ray transform*, according to the fact that the integrand in (7.1.55) depends on the component of the field f, whose two indices are directed along a ray and two other indices are orthogonal to the ray.

Note that (7.1.55) is obtained from the equations of elasticity by two asymptotic passages. The first of them is the expansion into asymptotic series in ω, and the second one is the linearization of the inverse problem. They are applicable together provided that $\omega^{-1} \ll c_{jklm} \ll 1$.

7.2 The mixed ray transform

Given a Riemannian manifold (M, g), let $S^m\tau'_M \otimes S^l\tau'_M$ be the bundle of covariant tensors of degree $m + l$ symmetric in the first m and last l indices. For $(x, \xi) \in TM$, we define the operator

$$\Lambda_\xi : S^mT'_xM \otimes S^lT'_xM \to S^mT'_xM$$

by the equality

$$(\Lambda_\xi f)_{i_1\dots i_m} = f_{i_1\dots i_m j_1\dots j_l}\xi^{j_1}\dots\xi^{j_l}.$$

By the *mixed ray transform* on a CDRM (M, g) we mean the operator

$$L : C^\infty(S^m\tau'_M \otimes S^l\tau'_M) \to C^\infty(S^m\pi_M), \tag{7.2.1}$$

that is defined by the formula

$$Lf(x, \xi) = \int\limits_{\tau_-(x,\xi)}^0 I_\gamma^{t,0}(P_{\dot\gamma(t)}\Lambda_{\dot\gamma(t)}f(\gamma(t)))\,dt \qquad ((x, \xi) \in \partial_+\Omega M), \tag{7.2.2}$$

where $\gamma = \gamma_{x,\xi}$ and the notation of Section 5.2 is used.

The relationship between (7.2.2) and (7.1.55) is expressed by the equality

$$\langle Lf(x,\xi),(\eta+a\xi)^m\rangle = \int\limits_{\tau_-(x,\xi)}^{0} f_{i_1\dots i_m j_1\dots j_l}(\gamma(t))\eta^{i_1}(t)\dots\eta^{i_m}(t)\dot\gamma^{j_1}(t)\dots\dot\gamma^{j_l}(t)\,dt \qquad (7.2.3)$$

which holds for $\langle\xi,\eta\rangle=0$. This equality is verified in the same manner as (5.2.5).

As in Chapter 4, we convince ourselves that (7.2.1) is extendible to a unique bounded operator

$$L: H^k(S^m\tau'_M\otimes S^l\tau'_M)\to H^k(S^m\pi_M) \qquad (k\ge 0).$$

Note that in the case $m=0$ the operator L coincides with the (longitudinal) ray transform I investigated in Chapter 4; and in the case $l=0$, with the transverse ray transform J of Chapter 5.

To what extent does Lf determine the field f? In order to answer the question, we introduce the next two mutually dual algebraic operators

$$S^m\tau'_M\otimes S^l\tau'_M \underset{\lambda}{\overset{\mu}{\rightleftarrows}} S^{m-1}\tau'_M\otimes S^{l-1}\tau'_M \qquad (7.2.4)$$

by the equalities

$$(\mu u)_{i_1\dots i_{m-1}j_1\dots j_{l-1}} = u_{i_1\dots i_m j_1\dots j_l}g^{i_m j_l},$$
$$(\lambda u)_{i_1\dots i_m j_1\dots j_l} = \sigma(i_1\dots i_m)\sigma(j_1\dots j_l)(g_{i_1j_1}u_{i_2\dots i_m j_2\dots j_l}).$$

If $f=\lambda w$ for some $w\in C^\infty(S^{m-1}\tau'_M\otimes S^{l-1}\tau'_M)$, then the integrand in (7.2.3) is identical zero and, consequently, $L(\lambda w)=0$.

We also introduce some differential operators (analogs of d and δ of Section 3.3)

$$C^\infty(S^m\tau'_M\otimes S^l\tau'_M) \underset{d'}{\overset{\delta'}{\rightleftarrows}} C^\infty(S^m\tau'_M\otimes S^{l-1}\tau'_M) \qquad (7.2.5)$$

by the formulas

$$(d'u)_{i_1\dots i_m j_1\dots j_l} = \sigma(j_1\dots j_l)u_{i_1\dots i_m j_1\dots j_{l-1};\,j_l},$$
$$(\delta' u)_{i_1\dots i_m j_1\dots j_{l-1}} = u_{i_1\dots i_m j_1\dots j_l;\,j_{l+1}}g^{j_l j_{l+1}}.$$

If $f=d'v$ and the field $v\in C^\infty(S^m\tau'_M\otimes S^{l-1}\tau'_M)$ satisfies the boundary condition $v|_{\partial M}=0$, then the integrand in (7.2.3) is the total derivative with respect to t and, consequently, $L(d'v)=0$.

For the operators d' and λ considered together, the next analog of Theorem 3.3.2 is valid:

Lemma 7.2.1 *Let M be a compact Riemannian manifold, $k\ge 1$. For every field $f\in H^k(S^m\tau'_M\otimes S^l\tau'_M)$, there exist ${}^sf\in H^k(S^m\tau'_M\otimes S^l\tau'_M)$, $v\in H^{k+1}(S^m\tau'_M\otimes S^{l-1}\tau'_M)$ and $w\in H^k(S^{m-1}\tau'_M\otimes S^{l-1}\tau'_M)$ such that*

$$f={}^sf+d'v+\lambda w, \qquad (7.2.6)$$

$$\delta'\,{}^sf=\mu\,{}^sf=0,\quad \mu v=0,\quad v|_{\partial M}=0. \qquad (7.2.7)$$

The field ${}^sf$ is uniquely determined by the field f, and the estimate

$$\|{}^sf\|_k\le C\|f\|_k \qquad (7.2.8)$$

is valid with a constant C independent of f.

Note that uniqueness of v and w is not asserted. The author does not know whether the uniqueness holds. As before, we call ${}^s\!f$ the *solenoidal part* of the field f.

P r o o f. By directly calculating in coordinates, we verify that d' and λ commute. Let $\tilde H^{k+1}(S^m\tau'_M\otimes S^{l-1}\tau'_M)$ be the subspace, of the space $H^{k+1}(S^m\tau'_M\otimes S^{l-1}\tau'_M)$, which consists of all the fields v satisfying the boundary condition $v|_{\partial M}=0$. For the operators d' and δ', an analog of Theorem 3.3.2 holds which is formulated and proved by repeating, almost word by word, the content of Section 3.3. In particular, it implies that the operator

$$d':\tilde H^{k+1}(S^m\tau'_M\otimes S^{l-1}\tau'_M)\rightarrow H^k(S^m\tau'_M\otimes S^l\tau'_M)$$

has closed range. Since λ and μ are algebraic operators, the decomposition

$$H^k(S^m\tau'_M\otimes S^l\tau'_M)=\operatorname{Ker}\mu\oplus\operatorname{Im}\lambda, \tag{7.2.9}$$

holds and, consequently, the range $\operatorname{Im}\lambda$ is closed. Thus, $\operatorname{Im}d'+\operatorname{Im}\lambda$ is a closed subspace of $H^k(S^m\tau'_M\otimes S^l\tau'_M)$, and therefore the decomposition

$$H^k(S^m\tau'_M\otimes S^l\tau'_M)=(\operatorname{Ker}\delta'\cap\operatorname{Ker}\mu)\oplus(\operatorname{Im}d'+\operatorname{Im}\lambda). \tag{7.2.10}$$

holds. By ${}^s\!f$ we denote the result of projecting the field f onto the first summand of the decomposition (7.2.10). Since the projection is bounded, estimate (7.2.8) is valid. By (7.2.10), the representation

$$f={}^s\!f+d'v'+\lambda w', \tag{7.2.11}$$

exists with some field v' satisfying the boundary condition $v'|_{\partial M}=0$. Now, according to (7.2.9), we represent v' as

$$v'=v+\lambda w'',\qquad \mu v=0. \tag{7.2.12}$$

It follows from the condition $v'|_{\partial M}=0$ that $v|_{\partial M}=0$. Inserting (7.2.12) into (7.2.11) and using permutability of d' and λ, we arrive at (7.2.6) and (7.2.7) with $w=w'+d'w''$. The lemma is proved.

The main result of the current chapter is the next

Theorem 7.2.2 *For $n\ge 2$, there exists $\varepsilon(n)>0$ such that, for every CDRM (M,g) of dimension n satisfying the condition*

$$k(M,g)<\varepsilon(n) \tag{7.2.13}$$

whose left-hand side is defined by formula (5.2.8) and for every field $f\in H^1(S^2\tau'_M\otimes S^2\tau'_M)$, the solenoidal part ${}^s\!f$ of the field f is uniquely determined by Lf. The conditional stable estimate

$$\|{}^s\!f\|_0^2\le C_1\left(\|j_\nu\,{}^s\!f|_{\partial M}\|_0\cdot\|Lf\|_0+\|Lf\|_1^2\right)\le C_2\left(\|f\|_1\cdot\|Lf\|_0+\|Lf\|_1^2\right), \tag{7.2.14}$$

holds with constants C_1 and C_2 independent of f; here $(j_\nu{}^s\!f)_{i_1i_2j_1}={}^s\!f_{i_1i_2j_1j_2}\nu^{j_2}$ and ν is the unit normal vector to ∂M.

The proof of the theorem is presented in the next section. The proof is performed for arbitrary m,l. Only after reducing the proof to a purely algebraic question, we put $m=l=2$.

7.3 Proof of Theorem 7.2.2

First of all, repeating the arguments that are presented after the formulation of Lemma 4.3.4, we see that it suffices to prove Theorem 7.2.2 for a real field $f \in C^\infty(S^m\tau'_M \otimes S^l\tau'_M)$ satisfying the conditions

$$d'f = 0, \tag{7.3.1}$$

$$\mu f = 0. \tag{7.3.2}$$

The solenoidal part of such a field coincides with f.

We define a semibasic tensor field $u(x,\xi) = (u_{i_1\dots i_m})$ on T^0M by the equality

$$u(x,\xi) = \int\limits_{\tau_-(x,\xi)}^{0} I_\gamma^{t,0}(P_{\dot\gamma(t)}\Lambda_{\dot\gamma(t)} f(\gamma(t)))\,dt \qquad ((x,\xi) \in T^0M). \tag{7.3.3}$$

The field u is symmetric in all its indices, homogeneous in its second argument:

$$u(x,t\xi) = t^{l-1}u(x,\xi) \qquad (t > 0), \tag{7.3.4}$$

satisfies the algebraic condition

$$j_\xi u(x,\xi) = 0, \tag{7.3.5}$$

the differential equation

$$Hu(x,\xi) = P_\xi\Lambda_\xi f(x) \tag{7.3.6}$$

and the boundary conditions

$$u|_{\partial_-\Omega M} = 0, \tag{7.3.7}$$

$$u|_{\partial_+\Omega M} = Lf. \tag{7.3.8}$$

We write down the Pestov identity (4.4.4) for u, integrate it over ΩM and transform the divergent terms of the so-obtained equality by the Gauss-Ostrogradskiĭ formulas for the horizontal and vertical divergence. We thus arrive at the relation

$$\int\limits_{\Omega M}\Big[|\overset{h}{\nabla}u|^2 + (n+2l-2)|Hu|^2\Big]\,d\Sigma = 2\int\limits_{\Omega M}\langle\overset{h}{\nabla}u, \overset{v}{\nabla}Hu\rangle\,d\Sigma -$$

$$-\int\limits_{\partial\Omega M}\langle v,\nu\rangle\,d\Sigma^{2n-2} + \int\limits_{\Omega M}\mathcal{R}_1[u]\,d\Sigma, \tag{7.3.9}$$

where the semibasic vector fields v and w are defined by formulas (4.4.5) and (4.4.6); the expression $\mathcal{R}_1[u]$, by equality (5.3.8). We write down the second summand of the integrand on the left-hand side of (7.3.9), taking homogeneity (7.3.4) and the equality $\langle w,\xi\rangle = |Hu|^2$ into account.

By the definition of the operator P_ξ, the right-hand side of equation (7.3.6) can be represented as

$$P_\xi\Lambda_\xi f(x) = \Lambda_\xi f(x) - i_\xi y(x,\xi) \tag{7.3.10}$$

with some symmetric semibasic field y which, in turn, can be represented according to (5.4.2). The field $\tilde y$ participating in (5.4.2) is expressed through f as follows:

$$\tilde y = \frac{m}{|\xi|^2}P_\xi j_\xi\Lambda_\xi f. \tag{7.3.11}$$

We apply the operator $\overset{v}{\nabla}$ to equality (7.3.10) and take the scalar product of the so-obtained equality and $\overset{h}{\nabla}u$. In such a way, taking (7.3.6) into account, we obtain

$$\langle \overset{h}{\nabla}u, \overset{v}{\nabla}Hu\rangle = \langle \overset{h}{\nabla}u, \overset{v}{\nabla}\Lambda_\xi f\rangle - \langle \overset{h}{\nabla}u, \overset{v}{\nabla}i_\xi y\rangle. \tag{7.3.12}$$

Transforming the second summand on the right-hand side of (7.3.12) in the same manner as in the beginning of Section 5.4, we arrive at the equality

$$\langle \overset{h}{\nabla}u, \overset{v}{\nabla}Hu\rangle = \langle \overset{h}{\nabla}u, \overset{v}{\nabla}\Lambda_\xi f\rangle - \langle \overset{h}{\delta}u, \tilde{y}\rangle, \tag{7.3.13}$$

where $\overset{h}{\delta}u$ is defined by formula (5.4.4).

Now address the first summand on the right-hand side of (7.3.13). By the definition of Λ_ξ,

$$\langle \overset{h}{\nabla}u, \overset{v}{\nabla}\Lambda_\xi f\rangle = \overset{h}{\nabla}{}^k u^{i_1\dots i_m} \cdot \overset{v}{\nabla}_k \left(f_{i_1\dots i_m j_1\dots j_l}\xi^{j_1}\dots\xi^{j_l}\right) =$$

$$= l\overset{h}{\nabla}{}^k u^{i_1\dots i_m} \cdot f_{i_1\dots i_m k j_2\dots j_l}\xi^{j_2}\dots\xi^{j_l} =$$

$$= \overset{h}{\nabla}{}^k \left(l u^{i_1\dots i_m} f_{i_1\dots i_m k j_2\dots j_l}\xi^{j_2}\dots\xi^{j_l}\right) - l u^{i_1\dots i_m}\left(g^{kp} f_{i_1\dots i_m k j_2\dots j_l\,;\,p}\right)\xi^{j_2}\dots\xi^{j_l}.$$

The second term on the right-hand side of the last formula is equal to zero, by (7.3.1). Thus, introducing a semibasic vector field $\tilde{w}$ by the equality

$$\tilde{w}_k = l u^{i_1\dots i_m} f_{i_1\dots i_m k j_2\dots j_l}\xi^{j_2}\dots\xi^{j_l}, \tag{7.3.14}$$

we have the representation $\langle \overset{h}{\nabla}u, \overset{v}{\nabla}\Lambda_\xi f\rangle = \overset{h}{\nabla}_i\tilde{w}^i$. In view of this representation, (7.3.13) takes the form

$$\langle \overset{h}{\nabla}u, \overset{v}{\nabla}Hu\rangle = \overset{h}{\nabla}_i\tilde{w}^i - \langle \overset{h}{\delta}u, \tilde{y}\rangle. \tag{7.3.15}$$

Estimate the last summand on the right-hand side of (7.3.15). To this end, repeat the arguments of the second half of Section 5.4. As a result, arrive at the inequality (an analog of (5.4.17))

$$2|\langle \overset{h}{\nabla}u, \overset{v}{\nabla}Hu\rangle| \le b|z|^2 + \frac{1}{b}|\tilde{y}|^2 + b\overset{h}{\nabla}_i\tilde{v}^i + \overset{h}{\nabla}_i\tilde{w}^i + b\mathcal{R}_2[u], \tag{7.3.16}$$

where b is an arbitrary positive number; the quantities z, $\tilde{v}$ and $\mathcal{R}_2[u]$ are defined by formulas (5.4.9), (5.4.11) and (5.4.13). Equality (5.4.15) holds.

By (7.3.11), (7.3.16) and (5.4.15); formula (7.3.9) implies the inequality (an analog of (5.4.24))

$$\int\limits_M \left\{ \int\limits_{\Omega_x M} \left[(1-b)|z|^2 + (n+2l-1)|P_\xi\Lambda_\xi f|^2\frac{m^2}{b}|P_\xi j_\xi\Lambda_\xi f|^2\right] d\omega_x(\xi)\right\} dV^n(x) \le$$

$$\le \int\limits_{\partial\Omega M} \langle b\tilde{v} + \tilde{w} - v, \nu\rangle\, d\Sigma^{2n-2} + \int\limits_{\Omega M} \mathcal{R}[u]\, d\Sigma, \tag{7.3.17}$$

where notation (5.4.23) is used.

By repeating the corresponding arguments of Sections 5.5 and 4.7, we establish that estimate (5.5.10) holds for the last integral on the right-hand side of (7.3.17) and the absolute value of the first integral on the right-hand side of (7.3.17) does not exceed the quantity $D(l\|j_\nu f|_{\partial M}\|_0 \cdot \|Lf\|_0 + \|Lf\|_1^2)$, where D is a constant depending on (M,g). Using these estimates, (7.3.17) implies the next analog of inequality (5.5.16):

$$\int\limits_M \left\{ \int\limits_{\Omega_x M} \left[(1-b)|z|^2 + (n+2l-1)|P_\xi \Lambda_\xi f|^2 - \frac{m^2}{b}|P_\xi j_\xi \Lambda_\xi f|^2\right] d\omega_x(\xi) \right\} dV^n(x) -$$

$$- Ck \int\limits_{\Omega M} |\overset{h}{\nabla} u|^2 d\Sigma \le D\left(l\|j_\nu f|_{\partial M}\|_0 \|Lf\|_0 + \|Lf\|_1^2\right).$$

Finally, repeating the arguments of Section 5.6, we find out that Theorem 7.2.2 holds in the case of those l, m, n, that answer the next question:

Problem 7.3.1 *Let $S^m = S^m(\mathbf{R}^n)$. For what values of l, m, n is the quadratic form*

$$\langle Bf, f\rangle = \frac{1}{\omega}\int\limits_\Omega \left[(n+2l-1)|P_\xi \Lambda_\xi f|^2 - m^2|P_\xi j_\xi \Lambda_\xi f|^2\right] d\omega_x(\xi) \tag{7.3.18}$$

positive-definite on the space $\operatorname{Ker}\mu \subset S^m \otimes S^l$, *where the operator* μ *is defined by formula* (7.2.4)?

This problem is a generalization of Problem 5.9.1. Addressing the problem, the author failed in applying the method that was used in the previous two chapters for studying similar questions. In the next section we use another approach to answering the question in the case $m = l = 2$.

7.4 The algebraic part of the proof

In this section $S^m = S^m(\mathbf{R}^n)$ (see Section 2.1). Only the Cartesian coordinates in $\mathbf{R}^n$ are in use; all tensor indices are written in the lower position; on repeating indices summation from 1 to n is assumed. We restrict ourselves to considering real tensors only. For $f \in S^m$, alongside with the norm $|f|$ generated by the scalar product (2.1.3), we will need the norm $\|f\|$ that is defined by the equality

$$\|f\|^2 = \frac{1}{\omega}\int\limits_\Omega |\langle f, \xi^m\rangle|^2 d\omega(\xi); \tag{7.4.1}$$

recall, that here Ω is the unit sphere of $\mathbf{R}^n$ and ω is its volume. We also recall that the operators, $i : S^m \to S^{m+2}$ of symmetric multiplication by the Kronecker tensor and $j : S^m \to S^{m-2}$ of convolution with this tensor, were introduced in Section 2.1.

Lemma 7.4.1 *For $f \in S^m$ satisfying the condition $jf = 0$, the equality*

$$\|f\|^2 = \frac{m!}{n(n+2)\dots(n+2m-2)}|f|^2. \tag{7.4.2}$$

holds.

P r o o f. We will show that this claim follows from a known property of spherical harmonics. Note that in coordinates the polynomial $\varphi(\xi)=\langle f,\xi^m\rangle$ can be written twofold:

$$\varphi(\xi)=f_{i_1\dots i_m}\xi_{i_1}\dots\xi_{i_m}=\sum_{|\alpha|=m} f_\alpha\xi^\alpha. \tag{7.4.3}$$

In the last sum $\alpha=(\alpha_1,\dots,\alpha_n)$ is a multi-index, $\xi^\alpha=\xi_1^{\alpha_1}\dots\xi_n^{\alpha_n}$. The coefficients of these two sums are related by the equality $f_\alpha=\frac{m!}{\alpha!}f_{i(\alpha)}$ where $i(\alpha)=(1\dots1\,2\dots2\dots n\dots n)$; the index 1 is repeated α_1 times, 2 is repeated α_2 times, and so on. For $|\alpha|=m$, (7.4.3) implies that $D^\alpha\varphi=\alpha!f_\alpha$ and, consequently,

$$|f|^2=\frac{1}{m!}\sum_{|\alpha|=m}\alpha!|f_\alpha|^2=\frac{1}{m!}\sum_{|\alpha|=m}\frac{1}{\alpha!}|D^\alpha\varphi|^2. \tag{7.4.4}$$

By (7.4.3), the condition $jf=0$ means that $\Delta\varphi=0$, i.e., that φ is a solid spherical harmonic of degree m. By Lemma XI.1 of [131], the equality

$$\sum_{|\alpha|=m}\frac{m!}{\alpha!}\int\limits_\Omega|D^\alpha\varphi(\xi)|^2d\omega(\xi)=\frac{2^m\Gamma\left(\frac{n}{2}+m\right)\Gamma(m+1)}{\Gamma\left(\frac{n}{2}\right)}\int\limits_\Omega|\varphi(\xi)|^2d\omega(\xi),$$

holds. Comparing it with (7.4.1) and (7.4.4), we arrive at (7.4.2). The lemma is proved.

Lemma 7.4.2 *The next equalities are valid:*

$$\|f\|^2=\frac{1}{n(n+2)}(2|f|^2+|jf|^2),\quad for\quad f\in S^2, \tag{7.4.5}$$

$$\|f\|^2=\frac{3}{n(n+2)(n+4)}(2|f|^2+3|jf|^2),\quad for\quad f\in S^3, \tag{7.4.6}$$

$$\|f\|^2=\frac{3}{n(n+2)(n+4)(n+6)}(8|f|^2+24|jf|^2+3|j^2f|^2),\quad for\quad f\in S^4. \tag{7.4.7}$$

P r o o f. We will only prove (7.4.6), the other two relations are proved in a similar way. First, with the help of commutation formula (6.4.2) for i and j, possibility of the representation

$$f=\tilde f+iv,\qquad j\tilde f=0 \tag{7.4.8}$$

is established for $f\in S^3$, where $\tilde f$ and v are expressed through f as follows:

$$\tilde f=f-\frac{3}{n+2}ijf,\qquad v=\frac{3}{n+2}jf. \tag{7.4.9}$$

The summands of expansion (7.4.8) are orthogonal to one another in the sense of the scalar product generating norm (7.4.1), since the functions $\langle\tilde f,\xi^3\rangle$ and $\langle iv,\xi^3\rangle|_\Omega=\langle v,\xi\rangle$ are spherical harmonics of different degrees. Therefore, by Lemma 7.4.1, (7.4.8) and (7.4.9) imply the equality

$$\begin{aligned}\|f\|^2=\|\tilde f\|^2+\|v\|^2&=\frac{6}{n(n+2)(n+4)}|\tilde f|^2+\frac{1}{n}|v|^2=\\&=\frac{6}{n(n+2)(n+4)}|\tilde f|^2+\frac{9}{n(n+2)^2}|jf|^2.\end{aligned} \tag{7.4.10}$$

Taking the scalar product of the first of the relations (7.4.9) and $\tilde f$ and using the fact that the operators i and j are dual to one other, we obtain $|\tilde f|^2 = |f|^2 - \frac{3}{(n+2)}|jf|^2$. Inserting the last expression into (7.4.10), we arrive at (7.4.6). The lemma is proved.

The operator $\Lambda_\xi : S^2 \otimes S^2 \to S^2$ introduced at the beginning of Section 7.2, for $m = l = 2$ is given in coordinate form by the equality

$$(\Lambda_\xi f)_{ij} = f_{ijkl}\xi_k\xi_l. \tag{7.4.11}$$

For $f \in S^2 \otimes S^2$ and $1 \le i, j \le n$, we define a tensor $f_{ij} \in S^2$ by putting $(f_{ij})_{kl} = f_{ijkl}$ and a tensor $f_i \in S^3$, by putting $(f_i)_{jkl} = \sigma(jkl)f_{ijkl}$.

Lemma 7.4.3 *For $f \in S^2 \otimes S^2$, the quadratic form B defined by formula* (7.3.18) *satisfies the relation*

$$\langle Bf, f\rangle = (n+3)\sum_{i,j}\|f_{ij}\|^2 - 2(n+5)\sum_i \|f_i\|^2 + (n+7)\|\sigma f\|^2. \tag{7.4.12}$$

P r o o f. Given $\xi \in \Omega$, by using (7.4.11) and coordinate representation (5.2.1) of the operator P_ξ, we obtain

$$(P_\xi\Lambda_\xi f)_{ij} = f_{pqkl}(\delta_{ip} - \xi_i\xi_p)(\delta_{jq} - \xi_j\xi_q)\xi_k\xi_l.$$

This implies the equality

$$|P_\xi\Lambda_\xi f|^2 = (P_\xi\Lambda_\xi f)_{ij}(P_\xi\Lambda_\xi f)_{ij} = f_{ijpq}f_{ijrs}\xi_p\xi_q\xi_r\xi_s -$$

$$- 2f_{ipqr}f_{istu}\xi_p\xi_q\xi_r\xi_s\xi_t\xi_u + f_{pqrs}f_{tuvw}\xi_p\xi_q\xi_r\xi_s\xi_t\xi_u\xi_v\xi_w$$

which can be written in the form

$$|P_\xi\Lambda_\xi f|^2 = \sum_{i,j}\langle f_{ij}, \xi^2\rangle^2 - 2\sum_i\langle f_i, \xi^3\rangle^2 + \langle \sigma f, \xi^4\rangle^2.$$

Integrating the last relation, we obtain

$$\frac{1}{\omega}\int\limits_\Omega |P_\xi\Lambda_\xi f|^2 d\omega(\xi) = \sum_{i,j}\|f_{ij}\|^2 - 2\sum_i\|f_i\|^2 + \|\sigma f\|^2. \tag{7.4.13}$$

In a similar way we obtain the coordinate representation

$$(P_\xi j_\xi \Lambda_\xi f)_i = f_{pjkl}(\delta_{ip} - \xi_i\xi_p)\xi_j\xi_k\xi_l,$$

which implies

$$\frac{1}{\omega}\int\limits_\Omega |P_\xi j_\xi\Lambda_\xi f|^2 d\omega(\xi) = \sum_i\|f_i\|^2 - \|\sigma f\|^2. \tag{7.4.14}$$

Inserting (7.4.13) and (7.4.14) into (7.3.18), we arrive at (7.4.12). The lemma is proved.

Recall that in Section 7.2 the operator μ was introduced. For $m = l = 2$ it is given in coordinate form as follows: $(\mu f)_{ik} = f_{ipkp}$.

Lemma 7.4.4 *For a tensor $f \in S^2 \otimes S^2$ satisfying the condition $\mu f = 0$, the inequality*

$$\frac{n(n+2)(n+4)}{2}\langle Bf, f\rangle \ge (n^2+5n+2)|f|^2 - 4(n+5)f_{ijkl}f_{ikjl} + \frac{12(n+7)}{n+6}|\sigma f|^2. \quad (7.4.15)$$

holds.

P r o o f. Expressing the norms $\|\cdot\|$ that participate in (7.4.12) through $|\cdot|$ with the help of Lemma 7.4.2, we arrive at the relation

$$n(n+2)\langle Bf, f\rangle = 2(n+3)\sum_{i,j}|f_{ij}|^2 + (n+3)\sum_{i,j}|jf_{ij}|^2 - \frac{12(n+5)}{n+4}\sum_i |f_i|^2 -$$

$$- \frac{18(n+5)}{n+4}\sum_i |jf_i|^2 + \frac{3(n+7)}{(n+4)(n+6)}(8|\sigma f|^2 + 24|j\sigma f|^2 + 3|j^2\sigma f|^2). \quad (7.4.16)$$

The quantities on the right-hand side of this equality are expressed as:

$$\sum_{i,j}|f_{ij}|^2 = f_{ijkl}f_{ijkl} = |f|^2, \quad (7.4.17)$$

$$\sum_{i,j}|jf_{ij}|^2 = \sum_{i,j}(f_{ijpp})^2 = |jf|^2, \quad (7.4.18)$$

$$(jf_i)_j = (f_i)_{jkl}\delta_{kl} = \frac{1}{3}(f_{ijkl} + f_{ikjl} + f_{iljk})\delta_{kl} = \frac{1}{3}f_{ijpp} + \frac{2}{3}f_{ipjp}.$$

Under the condition $\mu f = 0$, the second summand on the right-hand side of the last formula is equal to zero, and we obtain

$$\sum_i |jf_i|^2 = \frac{1}{9}|jf|^2. \quad (7.4.19)$$

Next,

$$\sum_i |f_i|^2 = \frac{1}{3}(f_{ijkl} + f_{ikjl} + f_{iljk})f_{ijkl} = \frac{1}{3}|f|^2 + \frac{1}{3}f_{ikjl}f_{ijkl} + \frac{1}{3}f_{iljk}f_{ijkl}.$$

Observing that the last two terms on the right-hand side of this equality coincide, we obtain

$$\sum_i |f_i|^2 = \frac{1}{3}|f|^2 + \frac{2}{3}f_{ijkl}f_{ikjl}. \quad (7.4.20)$$

Inserting (7.4.17)–(7.4.20) into (7.4.16), we arrive at the relation that implies inequality (7.4.15). The lemma is proved.

To estimate the second summand on the right-hand side of (7.4.15), we represent the tensor f as the sum

$$f = f^+ + f^-, \quad (7.4.21)$$

where

$$f^+_{ijkl} = \frac{1}{2}(f_{ijkl} + f_{ikjl}), \qquad f^-_{ijkl} = \frac{1}{2}(f_{ijkl} - f_{ikjl}). \quad (7.4.22)$$

The summands on the right-hand side of (7.4.21) are orthogonal to one another and, consequently,

$$|f|^2 = |f^+|^2 + |f^-|^2. \tag{7.4.23}$$

It follows from (7.4.22) that f^+ and f^- possess the symmetries

$$f^+_{ijkl} = f^+_{ikjl}, \qquad f^-_{ijkl} = -f^-_{ikjl},$$

with whose help we find the second summand on the right-hand side of (7.4.15):

$$\begin{aligned} f_{ijkl}f_{ikjl} &= (f^+_{ijkl} + f^-_{ijkl})(f^+_{ikjl} + f^-_{ikjl}) = \\ &= (f^+_{ijkl} + f^-_{ijkl})(f^+_{ijkl} - f^-_{ijkl}) = |f^+|^2 - |f^-|^2. \end{aligned} \tag{7.4.24}$$

Inserting (7.4.23) and (7.4.24) into (7.4.15), we arrive at the inequality

$$\frac{n(n+2)(n+4)}{2}\langle Bf, f\rangle \geq (n^2+n-18)|f|^2 + 8(n+5)|f^-|^2 + \frac{12(n+7)}{n+6}|\sigma f|^2. \tag{7.4.25}$$

The tensor $f \in S^2 \otimes S^2$ is expressed through f^- and σf by the relation

$$f_{ijkl} = (\sigma f)_{ijkl} + \frac{5}{12}(f^-_{ijkl} + f^-_{ijlk} + f^-_{jikl} + f^-_{jilk}) - \frac{1}{12}(f^-_{klij} + f^-_{klji} + f^-_{lkij} + f^-_{lkji}) \tag{7.4.26}$$

that is verified by inserting the expressions (7.4.22) and

$$(\sigma f)_{ijkl} = \frac{1}{6}(f_{ijkl} + f_{ikjl} + f_{iljk} + f_{jkil} + f_{jlik} + f_{klij})$$

into the right-hand side of (7.4.26). The first summand on the right-hand side of (7.4.26) is orthogonal to the others and, consequently,

$$|f|^2 = |\sigma f|^2 + |h|^2 \tag{7.4.27}$$

where

$$h_{ijkl} = \frac{5}{12}(f^-_{ijkl} + f^-_{ijlk} + f^-_{jikl} + f^-_{jilk}) - \frac{1}{12}(f^-_{klij} + f^-_{klji} + f^-_{lkij} + f^-_{lkji}).$$

The last formula implies the inequality

$$|h| \leq \frac{5}{3}|f^-| + \frac{1}{3}|f^-| = 2|f^-|$$

which, together with (7.4.27), gives

$$|f^-|^2 \geq \frac{1}{4}|f|^2 - \frac{1}{4}|\sigma f|^2.$$

With the help of the last inequality, (7.4.25) implies

$$\frac{n(n+2)(n+4)}{2}\langle Bf, f\rangle \geq (n^2+3n-8)|f|^2 - 2\frac{(n+5)(n+6)-6(n+7)}{n+6}|\sigma f|^2.$$

Finally, using the evident relation $|\sigma f| \leq |f|$, we arrive at the estimate

$$\frac{n(n+2)(n+4)(n+6)}{2}\langle Bf, f\rangle \geq (n^3+7n^2-24)|f|^2$$

that implies positive definiteness of the form $\langle Bf, f\rangle$ for $n \geq 2$.

Chapter 8

The exponential ray transform

Since recently the so-called problem of emission tomography gains popularity in mathematical tomography. In physical terms the problem reads as follows: given is a bounded domain which contains a medium that can absorb particles (or radiation); one has to determine a distribution of sources of particles inside the domain from the known flux on the boundary of the domain. Stated mathematically, the problem consists in inverting the operator that differs from the ray transform (4.2.2) by the presence of the factor $\exp\Big[-\int_t^0 \varepsilon(s;\dot\gamma_{x,\xi}(s))\,ds\Big]$ in the integrand. The function $\varepsilon(x,\xi)$ is called the absorption (or the attenuation), and the corresponding integral geometry operator is called the exponential ray transform (the term "attenuated ray transform" is also in use). We will denote this operator by I^ε.

Statements of problems of emission tomography can vary considerably. For instance, the problem of simultaneously determining the source f and the absorption ε is of great practical import. We will here deal with a more modest problem of determining the source f on condition that the absorption ε is known. Moreover, we will assume the absorption $\varepsilon(x,\xi)$ to be isotropic, i.e., independent of the second argument. Emphasize that the absorption $\varepsilon(x)$ can be complex. In this connection recall that, for wave processes, the imaginary part of the absorption describes the frequency dispersion.

In the current chapter we will restrict ourselves to considering the exponential ray transform of scalar functions, i.e., the case $m = 0$ in (4.2.2). In the case $m > 0$ investigation of the exponential ray transform comes across the next fundamental question: does there exist, for I^ε, an analog of the operator d of inner differentiation?

In the first section we give a definition of the exponential ray transform on a compact dissipative Riemannian manifold and formulate the main result of the chapter, Theorem 8.1.1, which asserts that this operator is invertible under some condition on the metric and absorption. This condition requires absence of conjugate points, on every geodesic, of some system of equations whose coefficients depend on the curvature tensor and absorption. We give also three consequences of the theorem which concern the cases of the zero absorption and zero curvature.

Section 8.2 contains the exposition of some new notion of tensor analysis, the so-called modified horizontal derivative, which plays the main role in the proof of Theorem 8.1.1. In the author's opinion, this notion will find other applications in integral geometry in future. So the exposition of the second section is rather detailed.

Section 8.3 contains the proof of Theorem 8.1.1. In Sections 8.4 and 8.5 we apply the modified horizontal derivative to two questions related to the nonlinear problem of determining a metric from its hodograph. First, we obtain a formula which expresses the volume of a simple compact Riemannian manifold through its hodograph. Then we prove that a simple metric is uniquely determined by its hodograph in a prescribed conformal class.

In Section 8.6 we discuss the relationship of our theorems to previous results on the subject.

The material of the current chapter is published in this book for the first time.

8.1 Formulation of the main definitions and results

Let (M, g) be a CDRM. We fix a (complex) function $\varepsilon \in C^\infty(M)$ which is further called the *absorption*. Given the absorption ε, by the *exponential ray transform* we mean the linear operator

$$I^\varepsilon : C^\infty(M) \to C^\infty(\partial_+\Omega M) \tag{8.1.1}$$

defined by the equality

$$I^\varepsilon f(x,\xi) = \int\limits_{\tau_-(x,\xi)}^{0} f(\gamma_{x,\xi}(t)) \exp\left[-\int\limits_{t}^{0} \varepsilon(\gamma_{x,\xi}(s))\,ds\right] dt \qquad ((x,\xi) \in \partial_+\Omega M), \tag{8.1.2}$$

where $\gamma_{x,\xi} : [\tau_-(x,\xi), 0] \to M$ is a maximal geodesic defined by the initial conditions $\gamma_{x,\xi}(0) = x$ and $\dot\gamma_{x,\xi}(0) = \xi$.

As in Section 4.2, one can show that operator (8.1.1) has some bounded extension

$$I^\varepsilon : H^k(M) \to H^k(\partial_+\Omega M) \tag{8.1.3}$$

for any integer $k \geq 0$.

Recall that a system of linear equations

$$\frac{d^2y}{dt^2} + A(t)y = 0 \qquad (a \leq t \leq b;\ y = (y^1, \dots, y^n))$$

is said to be *without conjugate points on the segment* $[a, b]$ if there is no nontrivial solution to this system which vanishes in two distinct points of the segment $[a, b]$.

Theorem 8.1.1 *Let a compact dissipative Riemannian manifold (M, g) and a function $\varepsilon \in C^\infty(M)$ be such that, for every $(x,\xi) \in \Omega M$, the equation*

$$\frac{D^2y}{dt^2} + \overset{\varepsilon}{R}(\dot\gamma, y)\dot\gamma = 0 \tag{8.1.4}$$

has no pair of conjugate points on the geodesic $\gamma = \gamma_{x,\xi} : [\tau_-(x,\xi), \tau_+(x,\xi)] \to M$. Here D/dt is the total derivative along γ (see Section 3.2), and $\overset{\varepsilon}{R}(\dot\gamma, y) : T_{\gamma(t)}M \to T_{\gamma(t)}M$ is the linear operator that in coordinate form is defined by the equalities

$$(\overset{\varepsilon}{R}(\dot\gamma, y))^i_l = \overset{\varepsilon}{R}{}^i{}_{jkl}\dot\gamma^j y^k,$$

$$\overset{\varepsilon}{R}_{ijkl} = R_{ijkl} + |\varepsilon|^2(g_{ik}g_{jl} - g_{il}g_{jk}), \tag{8.1.5}$$

and (R_{ijkl}) *is the curvature tensor. Then the operator*

$$I^\varepsilon : H^1(M) \to H^1(\partial_+\Omega M) \tag{8.1.6}$$

is injective and, for $f \in H^1(M)$, *the stability estimate*

$$\|f\|_0 \le C\|I^\varepsilon f\|_1 \tag{8.1.7}$$

holds with a constant C *independent of* f.

For practical applications, the case is important in which the absorption ε is not smooth. Of course, our assumption $\varepsilon \in C^\infty(M)$ is excessive. It suffices to assume that $\varepsilon \in C^1(M)$. In this case, operator (8.1.6) is bounded and depends continuously on $\varepsilon \in C^1(M)$. In the case when ε is a piecewise smooth function, the possibility of introducing bounded operator (8.1.6) depends on the geometry of the surfaces of discontinuity of the function ε and its first-order derivatives.

If $\varepsilon \equiv 0$, then the operator $I^\varepsilon = I$ coincides with the ray transform (4.2.1) for $m = 0$. In this case $\overset{\varepsilon}{R} = R$ and (8.1.4) coincides with the classical Jacobi equation [41]. A Riemannian manifold (M, g) is said to have no pair of conjugate points if the Jacobi equation has no pair of conjugate points on every geodesic. Such a CDRM is simple in the sense of the definition given in Section 1.1.

Corollary 8.1.2 *Let* (M, g) *be a CDRM without conjugate points. The ray transform*

$$I : H^1(M) \to H^1(\partial_+\Omega M)$$

is injective and, for $f \in H^1(M)$, *the stability estimate*

$$\|f\|_0 \le C\|If\|_1$$

holds with a constant C *independent of* f.

We will now briefly discuss the role of the curvature tensor in Theorem 8.1.1 and Corollary 8.1.2. It is well known that if all sectional curvatures are nonpositive, then the Jacobi equation has no pair of conjugate points on every geodesic. Of course, when the right-hand side of (8.1.5) is added with the summand containing $|\varepsilon|^2$, the last claim can become wrong; but a general tendency remains valid: the more negative is the sectional curvature, the more values of $|\varepsilon|$ are allowed without violating the assumptions of Theorem 8.1.1. In other words: large values of the absorption can be compensated by negative values of the curvature.

We now consider the case in which M is a bounded domain in $\mathbf{R}^n$ and the metric g coincides with the Euclidean one. The integration in (8.1.2) is performed over straight lines, i.e., this definition is replaced with the next

$$I^\varepsilon f(x,\xi) = \int\limits_{\tau_-(x,\xi)}^{0} f(x+t\xi)\exp\left[-\int\limits_{t}^{0} \varepsilon(x+s\xi)\,ds\right]dt \qquad ((x,\xi) \in \partial_+\Omega M). \tag{8.1.8}$$

System (8.1.4) reduces to the scalar equation

$$\frac{d^2y}{dt^2} + |\varepsilon|^2 y = 0.$$

We thus obtain

Corollary 8.1.3 *Let M be a bounded domain in $\mathbf{R}^n$ with smooth strictly convex boundary ∂M; $\varepsilon \in C^1(M)$ be a function such that the equation*

$$\frac{d^2y}{dt^2} + |\varepsilon|^2(\gamma(t))y = 0 \tag{8.1.9}$$

has no pair of conjugate points on every straight-line segment $\gamma : [a,b] \to M$. Then operator (8.1.6)*, which is defined by formula* (8.1.8)*, is injective and estimate* (8.1.7) *holds.*

A few conditions are known ensuring the absence of conjugate points for scalar equation (8.1.9). Some of them are based on the Sturm comparison theorems, while the others, on Lyapunov's integral estimates [46]. The simplest of them guarantees the absence of conjugate points under the assumption that the inequality

$$\varepsilon_0 \,\mathrm{diam} M < \pi \tag{8.1.10}$$

holds, where

$$\varepsilon_0 = \sup_{x \in M} |\varepsilon(x)|, \qquad \mathrm{diam} M = \sup_{x,y \in M} |x - y|. \tag{8.1.11}$$

We thus arrive at

Corollary 8.1.4 *Let M be a closed domain in $\mathbf{R}^n$ with smooth strictly convex boundary, and let a function $\varepsilon \in C^1(M)$ be such that the quantities ε_0 and* diam$M$ *defined by equalities* (8.1.11) *satisfy condition* (8.1.10)*. Then operator* (8.1.6)*, which is defined by* (8.1.8)*, is injective and estimate* (8.1.7) *holds.*

8.2 The modified horizontal derivative

Before starting with a formal exposition, we will informally discuss an idea that underlines the proof of Theorem 8.1.1.

The Pestov identity (4.4.4) played the main role in our proofs of all the main results of the previous four chapters. For the sake of convenience, we write down the identity here in the simplest case when $m = 0$:

$$2\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = |\overset{h}{\nabla} u|^2 + \overset{h}{\nabla}_i v^i + \overset{v}{\nabla}_i w^i - R_{ijkl}\xi^i\xi^k \overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u. \tag{8.2.1}$$

Here v and w are the semibasic vector fields determined through the function u by formulas (4.4.10) and (4.4.11). We applied this identity to inverse problems for various kinds of the kinetic equation. Let us reproduce here one of these equations, namely equation (4.6.4) in the simplest case $m = 0$:

$$Hu(x, \xi) = f(x). \tag{8.2.2}$$

Let us now discuss the question: has the Pestov identity (8.2.1) any modifications appropriate for investigating inverse problems for equation (8.2.2) as well as identity (8.2.1) itself?

The application of the operator $\overset{v}{\nabla}$ to equation (8.2.2) seems to be justified, since the operator annihilates the right-hand side of the equation. On the other hand, the multiplication of the vector $\overset{v}{\nabla}(Hu)$ just by $\overset{h}{\nabla}u$, which is performed on the left-hand side of identity (8.2.1), is not dictated by equation (8.2.2). Appropriateness of such the multiplication is rather explained by the form of the right-hand side of identity (8.2.1) that consists of the positive-definite quadratic form $|\overset{h}{\nabla}u|^2$ and divergent terms $\overset{h}{\nabla}_i v^i$ and $\overset{v}{\nabla}_i w^i$. The right-hand side of (8.2.1) also contains the fourth summand $R_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u$, but the very term is just undesirable for us. Using a rather remote analogy, our observation can be expressed by the next, leading but nonrigorous statement: $\overset{h}{\nabla}u$ is an integrating factor for the equation $\overset{v}{\nabla}(Hu) = 0$. Does this equation admit other integrating factors?

To answer the question, let us analyze the proof of the Pestov identity which is presented in Section 4.4. What properties of the horizontal derivative $\overset{h}{\nabla}$ were used in the proof? First of all, such were commutation formulas (3.5.11) and (3.5.12). As far as the properties of the operator $\overset{h}{\nabla}$ listed in Theorem 3.5.1 are concerned, they were used in a rather incomplete manner. Namely, the first of the properties listed in Theorem 3.5.1 was not utilized at all, and instead of the second we made use of a weaker claim:

$$Hu = \xi^i \overset{h}{\nabla}_i u. \tag{8.2.3}$$

If the first hypothesis of Theorem 3.5.1 is omitted and the second is replaced by (8.2.3); then, of course, the claim of the theorem about uniqueness of the operator $\overset{h}{\nabla}$ is not valid. It turns out that in this case the operator $\overset{h}{\nabla}$ is determined up to an arbitrary semibasic tensor field of degree 2. This arbitrariness can be used for compensating the last term on the right-hand side of (8.2.1); this is the main idea of the proof of Theorem 8.1.1.

Now we turn to the formal exposition.

Let (M, g) be a Riemannian manifold. We fix a real semibasic tensor field $a = (a^{ij}) \in C^\infty(\beta_0^2 M)$ such that it is symmetric

$$a^{ij} = a^{ji}, \tag{8.2.4}$$

positive-homogeneous in its second argument

$$a^{ij}(x, \lambda\xi) = \lambda a^{ij}(x, \xi) \qquad (\lambda > 0) \tag{8.2.5}$$

and orthogonal to the vector ξ:

$$a^{ij}(x, \xi)\xi_j = 0. \tag{8.2.6}$$

With the help of this tensor field, we define the *modified horizontal derivative*

$$\overset{a}{\nabla} : C^\infty(\beta_s^r M) \to C^\infty(\beta_s^{r+1} M) \tag{8.2.7}$$

by the equality

$$\overset{a}{\nabla}u = \overset{a}{\nabla}{}^k u^{i_1\dots i_r}_{j_1\dots j_s} \frac{\partial}{\partial \xi^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes \frac{\partial}{\partial \xi^k} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s} \tag{8.2.8}$$

where

$$\overset{a}{\nabla}{}^k u^{i_1\dots i_r}_{j_1\dots j_s} = \overset{h}{\nabla}{}^k u^{i_1\dots i_r}_{j_1\dots j_s} + A^k u^{i_1\dots i_r}_{j_1\dots j_s} \tag{8.2.9}$$

and

$$A^k u^{i_1\dots i_r}_{j_1\dots j_s} = a^{kp}\overset{v}{\nabla}_p u^{i_1\dots i_r}_{j_1\dots j_s} - \sum_{m=1}^{r} \overset{v}{\nabla}_p a^{ki_m}\cdot u^{i_1\dots i_{m-1}pi_{m+1}\dots i_r}_{j_1\dots j_s} + \sum_{m=1}^{s} \overset{v}{\nabla}_{j_m} a^{kp}\cdot u^{i_1\dots i_r}_{j_1\dots j_{m-1}pj_{m+1}\dots j_s}. \tag{8.2.10}$$

The operator $\overset{a}{\nabla}$ is evidently independent of the choice of a natural coordinate system presenting in formulas (8.2.8)–(8.2.10) and, consequently, the operator is globally defined. Let us consider its main properties.

First of all, $\overset{a}{\nabla}$ commutes with the convolution operators C^k_l and is a derivative relative to the tensor product in the next sense:

$$\overset{a}{\nabla}(u\otimes v) = \rho^{r+1}(\overset{a}{\nabla}u\otimes v) + u\otimes\overset{a}{\nabla}v \qquad (u\in C^\infty(\beta^r_s(M))$$

where ρ^{r+1} is the permutation of upper indices which translates the $(r+1)$-th index to the last position (compare with Theorem 3.5.1). Both of these properties are verified by direct calculations in coordinates and thus omitted.

Let us find out the interrelation between $\overset{a}{\nabla}$ and H. To this end we multiply equality (8.2.10) by ξ_k and perform the summation over k. Using (8.2.5) and (8.2.6), we obtain

$$\xi_k A^k u^{i_1\dots i_r}_{j_1\dots j_s} = -\sum_{m=1}^{r} \xi_k\overset{v}{\nabla}_p a^{ki_m}\cdot u^{i_1\dots i_{m-1}pi_{m+1}\dots i_r}_{j_1\dots j_s} + \sum_{m=1}^{s} \xi_k\overset{v}{\nabla}_{j_m} a^{kp}\cdot u^{i_1\dots i_r}_{j_1\dots j_{m-1}pj_{m+1}\dots j_s} =$$

$$= \sum_{m=1}^{r} a^{i_m}_p u^{i_1\dots i_{m-1}pi_{m+1}\dots i_r}_{j_1\dots j_s} - \sum_{m=1}^{s} a^p_{j_m} u^{i_1\dots i_r}_{j_1\dots j_{m-1}pj_{m+1}\dots j_s}.$$

Here, as usual, the rule $a^i_j = g_{jk}a^{ik}$ of lowering indices is used. Consequently,

$$\xi_k\overset{a}{\nabla}{}^k u^{i_1\dots i_r}_{j_1\dots j_s} = (Hu)^{i_1\dots i_r}_{j_1\dots j_s} + \sum_{m=1}^{r} a^{i_m}_p u^{i_1\dots i_{m-1}pi_{m+1}\dots i_r}_{j_1\dots j_s} - \sum_{m=1}^{s} a^p_{j_m} u^{i_1\dots i_r}_{j_1\dots j_{m-1}pj_{m+1}\dots j_s}. \tag{8.2.11}$$

In particular, for a scalar function $u\in C^\infty(TM)$,

$$Hu = \xi_i\overset{a}{\nabla}{}^i u. \tag{8.2.12}$$

From (8.2.6) and the equality $\xi^p\overset{v}{\nabla}_p a^{ij} = a^{ij}$ following from (8.2.5), we obtain

$$\overset{a}{\nabla}{}^i\xi^j = \overset{a}{\nabla}{}^i\xi_j = 0. \tag{8.2.13}$$

In the case of a general field a, the metric tensor is not parallel with respect to $\overset{a}{\nabla}$. Indeed,

$$\overset{a}{\nabla}{}^i g_{jk} = \overset{v}{\nabla}_j a^{ip}\cdot g_{pk} + \overset{v}{\nabla}_k a^{ip}\cdot g_{jp} = \overset{v}{\nabla}_j a^i_k + \overset{v}{\nabla}_k a^i_j.$$

In view of this fact one must exercise some care while raising and lowering indices in expressions that contain $\overset{a}{\nabla}$. It explains why we preferred to start with defining the operator $\overset{a}{\nabla}{}^i$ with the upper index.

We will now obtain a commutation formula for the operators $\overset{a}{\nabla}$ and $\overset{v}{\nabla}$. Using (3.5.11), we derive

$$\left(\overset{a}{\nabla}{}^i\overset{v}{\nabla}_j - \overset{v}{\nabla}_j\overset{a}{\nabla}{}^i\right) u^{i_1\dots i_r}_{j_1\dots j_s} = \left(A^i\overset{v}{\nabla}_j - \overset{v}{\nabla}_j A^i\right) u^{i_1\dots i_r}_{j_1\dots j_s}.$$

Transforming the right-hand side of this equality in accord with definition (8.2.10), after simple calculations we arrive at the formula

$$\left(\overset{a}{\nabla}{}^i\overset{v}{\nabla}_j - \overset{v}{\nabla}_j\overset{a}{\nabla}{}^i\right) u^{i_1\dots i_r}_{j_1\dots j_s} =$$

$$= \sum_{m=1}^{r} \overset{v}{\nabla}_j\overset{v}{\nabla}_p a^{ii_m} \cdot u^{i_1\dots i_{m-1}pi_{m+1}\dots i_r}_{j_1\dots j_s} - \sum_{m=1}^{s} \overset{v}{\nabla}_j\overset{v}{\nabla}_{j_m} a^{ip} \cdot u^{i_1\dots i_r}_{j_1\dots j_{m-1}pj_{m+1}\dots j_s}. \tag{8.2.14}$$

In particular, for a scalar function $u \in C^\infty(TM)$,

$$\left(\overset{a}{\nabla}{}^i\overset{v}{\nabla}_j - \overset{v}{\nabla}_j\overset{a}{\nabla}{}^i\right) u = 0. \tag{8.2.15}$$

We will now obtain a commutation formula for $\overset{a}{\nabla}{}^i$ and $\overset{a}{\nabla}{}^j$. First, consider the case of a scalar function $u \in C^\infty(TM)$. By (8.2.9),

$$\left(\overset{a}{\nabla}{}^i\overset{a}{\nabla}{}^j - \overset{a}{\nabla}{}^j\overset{a}{\nabla}{}^i\right) u = \left(\overset{h}{\nabla}{}^i\overset{h}{\nabla}{}^j - \overset{h}{\nabla}{}^j\overset{h}{\nabla}{}^i\right) u +$$

$$+ \left(\overset{h}{\nabla}{}^i A^j - A^j\overset{h}{\nabla}{}^i\right) u + \left(A^i\overset{h}{\nabla}{}^j - \overset{h}{\nabla}{}^j A^i\right) u + \left(A^iA^j - A^jA^i\right) u. \tag{8.2.16}$$

Calculate the last term on the right-hand side of (8.2.16):

$$\left(A^iA^j - A^jA^i\right) u =$$

$$= a^{ip}\overset{v}{\nabla}_p(A^ju) - \overset{v}{\nabla}_p a^{ij} \cdot A^pu - a^{jp}\overset{v}{\nabla}_p(A^iu) + \overset{v}{\nabla}_p a^{ji} \cdot A^pu =$$

$$= a^{ip}\overset{v}{\nabla}_p(A^ju) - a^{jp}\overset{v}{\nabla}_p(A^iu) = a^{ip}\overset{v}{\nabla}_p(a^{jq}\overset{v}{\nabla}_q u) - a^{jp}\overset{v}{\nabla}_p(a^{iq}\overset{v}{\nabla}_q u).$$

Using permutability of $\overset{v}{\nabla}_p$ and $\overset{v}{\nabla}_q$, obtain

$$\left(A^iA^j - A^jA^i\right) u = \left(a^{ip}\overset{v}{\nabla}_p a^{jq} - a^{jp}\overset{v}{\nabla}_p a^{iq}\right)\overset{v}{\nabla}_q u. \tag{8.2.17}$$

We now calculate the second term on the right-hand side of (8.2.16):

$$\left(\overset{h}{\nabla}{}^i A^j - A^j\overset{h}{\nabla}{}^i\right) u = \overset{h}{\nabla}{}^i(a^{jp}\overset{v}{\nabla}_p u) - a^{jp}\overset{v}{\nabla}_p\overset{h}{\nabla}{}^i u + \overset{v}{\nabla}_p a^{ij} \cdot \overset{h}{\nabla}{}^p u.$$

Using permutability of $\overset{h}{\nabla}{}^i$ and $\overset{v}{\nabla}_p$, we infer

$$\left(\overset{h}{\nabla}{}^i A^j - A^j\overset{h}{\nabla}{}^i\right) u = \overset{h}{\nabla}{}^i a^{jp} \cdot \overset{v}{\nabla}_p u + \overset{v}{\nabla}_p a^{ij} \cdot \overset{h}{\nabla}{}^p u.$$

By alternating the last equality with respect to i and j, derive

$$\left(\overset{h}{\nabla}{}^i A^j - A^j\overset{h}{\nabla}{}^i\right) u + \left(A^i\overset{h}{\nabla}{}^j - \overset{h}{\nabla}{}^j A^i\right) u = \left(\overset{h}{\nabla}{}^i a^{jp} - \overset{h}{\nabla}{}^j a^{ip}\right)\overset{v}{\nabla}_p u. \tag{8.2.18}$$

Inserting (8.2.17), (8.2.18) and (3.5.12) into the right-hand side of (8.2.16), we obtain

$$\left(\overset{a}{\nabla}{}^i\overset{a}{\nabla}{}^j - \overset{a}{\nabla}{}^j\overset{a}{\nabla}{}^i\right) u = -\left(R^{pqij}\xi_q + \overset{h}{\nabla}{}^j a^{ip} - \overset{h}{\nabla}{}^i a^{jp} + a^{jq}\overset{v}{\nabla}_q a^{ip} - a^{iq}\overset{v}{\nabla}_q a^{jp}\right)\overset{v}{\nabla}_p u. \tag{8.2.19}$$

We introduce the semibasic tensor field

$$\overset{a}{R}_{ijkl} = R_{ijkl} + \overset{h}{\nabla}_l \overset{v}{\nabla}_j a_{ik} - \overset{h}{\nabla}_k \overset{v}{\nabla}_j a_{il} +$$

$$+ a_{lp} \overset{v}{\nabla}{}^p \overset{v}{\nabla}_j a_{ik} - a_{kp} \overset{v}{\nabla}{}^p \overset{v}{\nabla}_j a_{il} + \overset{v}{\nabla}{}^p a_{ik} \cdot \overset{v}{\nabla}_j a_{lp} - \overset{v}{\nabla}{}^p a_{il} \cdot \overset{v}{\nabla}_j a_{kp}. \tag{8.2.20}$$

Performing the convolution of this equality with ξ^j and taking homogeneity (8.2.5) into account, we obtain

$$\overset{a}{R}_{ijkl} \xi^j = R_{ijkl} \xi^j + \overset{h}{\nabla}_l a_{ik} - \overset{h}{\nabla}_k a_{il} + a_{lp} \overset{v}{\nabla}{}^p a_{ik} - a_{kp} \overset{v}{\nabla}{}^p a_{il}. \tag{8.2.21}$$

In view of (8.2.21), formula (8.2.19) takes the final form:

$$\left(\overset{a}{\nabla}{}^i \overset{a}{\nabla}{}^j - \overset{a}{\nabla}{}^j \overset{a}{\nabla}{}^i \right) u = - \overset{a}{R}{}^{pqij} \xi_q \overset{v}{\nabla}_p u. \tag{8.2.22}$$

Similar but more cumbersome calculations show that, for a semibasic tensor field of arbitrary degree, the next commutation formula is valid:

$$\left(\overset{a}{\nabla}{}^k \overset{a}{\nabla}{}^l - \overset{a}{\nabla}{}^l \overset{a}{\nabla}{}^k \right) u^{i_1 \dots i_r}_{j_1 \dots j_s} = - \overset{a}{R}{}^{pqkl} \xi_q \overset{v}{\nabla}_p u^{i_1 \dots i_r}_{j_1 \dots j_s} +$$

$$+ \sum_{m=1}^{r} g_{pq} \overset{a}{R}{}^{i_m qkl} u^{i_1 \dots i_{m-1} p i_{m+1} \dots i_r}_{j_1 \dots j_s} - \sum_{m=1}^{s} g_{j_m q} \overset{a}{R}{}^{pqkl} u^{i_1 \dots i_r}_{j_1 \dots j_{m-1} p j_{m+1} \dots j_s}. \tag{8.2.23}$$

The semibasic tensor field $(\overset{a}{R}_{ijkl})$ defined by (8.2.21) will be called the *curvature tensor for the modified horizontal derivative* $\overset{a}{\nabla}$. As follows from (8.2.21), it is skew-symmetric in the indices k and l but, in general, it is not skew-symmetric in i and j in contrast to an ordinary curvature tensor. We shall need the following properties of the tensor:

$$\overset{a}{R}_{ipkq} \xi^p \xi^q = R_{ipkq} \xi^p \xi^q + \xi^p \overset{h}{\nabla}_p a_{ik} + a_{ip} a^p_j = \overset{a}{R}_{piqk} \xi^p \xi^q, \tag{8.2.24}$$

$$\overset{a}{R}_{ipkq} \xi^p \xi^q = \overset{a}{R}_{kpiq} \xi^p \xi^q, \qquad \overset{a}{R}_{pqir} \xi^p \xi^q \xi^r = \overset{a}{R}_{ipqr} \xi^p \xi^q \xi^r = 0. \tag{8.2.25}$$

Relations (8.2.25) follow from (8.2.24) with the help of (8.2.6). To prove the first of equalities (8.2.24), we multiply (8.2.21) by ξ^l and summarize over l. Using (8.2.6), we obtain

$$\overset{a}{R}_{ijkl} \xi^j \xi^l = R_{ijkl} \xi^j \xi^l + \xi^l \overset{h}{\nabla}_l a_{ik} - a_{kp} \xi^l \overset{v}{\nabla}{}^p a_{il}. \tag{8.2.26}$$

It follows from (8.2.6) that

$$\xi^i \overset{v}{\nabla}{}^j a_{ik} = -a_{ik}. \tag{8.2.27}$$

Transforming the last summand on the right-hand side of (8.2.26) with the help of (8.2.27), we obtain the first of equalities (8.2.24). To prove the second one, we multiply (8.4.20) by $\xi^i \xi^k$ and summarize over i and k:

$$\overset{a}{R}_{ijkl} \xi^i \xi^k = R_{ijkl} \xi^i \xi^k + \overset{h}{\nabla}_l (\xi^i \xi^k \overset{v}{\nabla}_j a_{ik}) - \xi^k \overset{h}{\nabla}_k (\xi^i \overset{v}{\nabla}_j a_{il}) +$$

$$+ a_{lp} \xi^i \xi^k \overset{v}{\nabla}{}^p \overset{v}{\nabla}_j a_{ik} + \xi^i \xi^k \overset{v}{\nabla}{}^p a_{ik} \cdot \overset{v}{\nabla}_j a_{lp} - \xi^i \overset{v}{\nabla}{}^p a_{il} \cdot \xi^k \overset{v}{\nabla}_j a_{kp}.$$

Transforming the summands on the right-hand side of the last formula with the help of (8.2.27), we arrive at the second of equalities (8.2.24).

Since the above-obtained properties of the operator $\overset{a}{\nabla}$ are quite similar to the corresponding properties of $\overset{h}{\nabla}$, we can assert that an analog of the Pestov identity (4.4.9) is valid for $\overset{a}{\nabla}$. Nevertheless, we will give the detailed proof of this identity, since in Section 4.4 it was obtained only for real functions while here we need it for complex functions.

Lemma 8.2.1 *Let M be a Riemannian manifold. For a function $u \in C^\infty(TM)$, the identity*

$$2\mathrm{Re}\,\langle \overset{a}{\nabla}u, \overset{v}{\nabla}(Hu)\rangle = |\overset{a}{\nabla}u|^2 + \overset{a}{\nabla}{}^i v_i + \overset{v}{\nabla}_i w^i - \overset{a}{R}_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l\overline{u} \tag{8.2.28}$$

holds, with

$$v_i = \mathrm{Re}\left(\xi_i\overset{a}{\nabla}{}^j u \cdot \overset{v}{\nabla}_j\overline{u} - \xi_j\overset{v}{\nabla}_i u \cdot \overset{a}{\nabla}{}^j\overline{u}\right), \tag{8.2.29}$$

$$w^i = \mathrm{Re}\left(\xi_j\overset{a}{\nabla}{}^i u \cdot \overset{a}{\nabla}{}^j\overline{u}\right). \tag{8.2.30}$$

P r o o f. By (8.2.12),

$$2\langle \overset{a}{\nabla}u, \overset{v}{\nabla}(Hu)\rangle = 2\overset{a}{\nabla}{}^i u \cdot \overset{v}{\nabla}_i(\xi_j\overset{a}{\nabla}{}^j\overline{u}) = 2|\overset{a}{\nabla}u|^2 + 2\xi_j\overset{a}{\nabla}{}^i u \cdot \overset{v}{\nabla}_i\overset{a}{\nabla}{}^j\overline{u}.$$

Therefore

$$4\mathrm{Re}\langle \overset{a}{\nabla}u, \overset{v}{\nabla}(Hu)\rangle = 4|\overset{a}{\nabla}u|^2 + 2\xi_j\overset{a}{\nabla}{}^i u \cdot \overset{v}{\nabla}_i\overset{a}{\nabla}{}^j\overline{u} + 2\xi_j\overset{a}{\nabla}{}^i\overline{u} \cdot \overset{v}{\nabla}_i\overset{a}{\nabla}{}^j u. \tag{8.2.31}$$

We introduce a function φ by the equality

$$2\xi_j\overset{a}{\nabla}{}^i u \cdot \overset{v}{\nabla}_i\overset{a}{\nabla}{}^j\overline{u} + 2\xi_j\overset{a}{\nabla}{}^i\overline{u} \cdot \overset{v}{\nabla}_i\overset{a}{\nabla}{}^j u = \overset{v}{\nabla}_i\left(\xi_j\overset{a}{\nabla}{}^i u \cdot \overset{a}{\nabla}{}^j\overline{u} + \xi_j\overset{a}{\nabla}{}^i\overline{u} \cdot \overset{a}{\nabla}{}^j u\right) +$$

$$+\,\overset{a}{\nabla}{}^j\left(\xi_j\overset{a}{\nabla}{}^i u \cdot \overset{v}{\nabla}_i\overline{u} + \xi_j\overset{a}{\nabla}{}^i\overline{u} \cdot \overset{v}{\nabla}_i u\right) - \overset{a}{\nabla}{}^i\left(\xi_j\overset{v}{\nabla}_i u \cdot \overset{a}{\nabla}{}^j\overline{u} + \xi_j\overset{v}{\nabla}_i\overline{u} \cdot \overset{a}{\nabla}{}^j u\right) - \varphi. \tag{8.2.32}$$

Show that φ is independent of the second-order derivatives of the function u. To this end express the derivatives of the products taking part in the last equality through the derivatives of the factors. Using (8.2.13), obtain

$$\varphi = -2\xi_j\overset{a}{\nabla}{}^i u \cdot \overset{v}{\nabla}_i\overset{a}{\nabla}{}^j\overline{u} - 2\xi_j\overset{a}{\nabla}{}^i\overline{u} \cdot \overset{v}{\nabla}_i\overset{a}{\nabla}{}^j u +$$

$$+\,2|\overset{a}{\nabla}u|^2 + \xi_j\overset{v}{\nabla}_i\overset{a}{\nabla}{}^i u \cdot \overset{a}{\nabla}{}^j\overline{u} + \xi_j\overset{a}{\nabla}{}^i u \cdot \overset{v}{\nabla}_i\overset{a}{\nabla}{}^j\overline{u} + \xi_j\overset{v}{\nabla}_i\overset{a}{\nabla}{}^i\overline{u} \cdot \overset{a}{\nabla}{}^j u + \xi_j\overset{a}{\nabla}{}^i\overline{u} \cdot \overset{v}{\nabla}_i\overset{a}{\nabla}{}^j u +$$

$$+\,\xi_j\overset{a}{\nabla}{}^j\overset{a}{\nabla}{}^i u \cdot \overset{v}{\nabla}_i\overline{u} + \xi_j\overset{a}{\nabla}{}^i u \cdot \overset{a}{\nabla}{}^j\overset{v}{\nabla}_i\overline{u} + \xi_j\overset{a}{\nabla}{}^j\overset{a}{\nabla}{}^i\overline{u} \cdot \overset{v}{\nabla}_i u + \xi_j\overset{a}{\nabla}{}^i\overline{u} \cdot \overset{a}{\nabla}{}^j\overset{v}{\nabla}_i u -$$

$$-\,\xi_j\overset{a}{\nabla}{}^i\overset{v}{\nabla}_i u \cdot \overset{a}{\nabla}{}^j\overline{u} - \xi_j\overset{v}{\nabla}_i u \cdot \overset{a}{\nabla}{}^i\overset{a}{\nabla}{}^j\overline{u} - \xi_j\overset{a}{\nabla}{}^i\overset{v}{\nabla}_i\overline{u} \cdot \overset{a}{\nabla}{}^j u - \xi_j\overset{v}{\nabla}_i\overline{u} \cdot \overset{a}{\nabla}{}^i\overset{a}{\nabla}{}^j u.$$

After evident transformations, this equality takes the form

$$\varphi = 2|\overset{a}{\nabla}u|^2 + \xi_j\overset{a}{\nabla}{}^i u \cdot \left(\overset{a}{\nabla}{}^j\overset{v}{\nabla}_i - \overset{v}{\nabla}_i\overset{a}{\nabla}{}^j\right)\overline{u} + \xi_j\overset{a}{\nabla}{}^i\overline{u} \cdot \left(\overset{a}{\nabla}{}^j\overset{v}{\nabla}_i - \overset{v}{\nabla}_i\overset{a}{\nabla}{}^j\right)u +$$

$$+\,\xi_j\overset{a}{\nabla}{}^j\overline{u}\cdot\left(\overset{v}{\nabla}_i\overset{a}{\nabla}{}^i-\overset{a}{\nabla}{}^i\overset{v}{\nabla}_i\right)u+\xi_j\overset{a}{\nabla}{}^ju\cdot\left(\overset{v}{\nabla}_i\overset{a}{\nabla}{}^i-\overset{a}{\nabla}{}^i\overset{v}{\nabla}_i\right)\overline{u}+$$

$$+\,\xi_j\overset{v}{\nabla}_i\overline{u}\cdot\left(\overset{a}{\nabla}{}^j\overset{a}{\nabla}{}^i-\overset{a}{\nabla}{}^i\overset{a}{\nabla}{}^j\right)u+\xi_j\overset{v}{\nabla}_iu\cdot\left(\overset{a}{\nabla}{}^j\overset{a}{\nabla}{}^i-\overset{a}{\nabla}{}^i\overset{a}{\nabla}{}^j\right)\overline{u}.$$

Applying commutation formulas (8.2.15) and (8.2.22), obtain

$$\varphi=2|\overset{a}{\nabla}u|^2+\overset{a}{R}{}^{pqij}\xi_q\xi_j\overset{v}{\nabla}_i\overline{u}\cdot\overset{v}{\nabla}_pu+\overset{a}{R}{}^{pqij}\xi_q\xi_j\overset{v}{\nabla}_iu\cdot\overset{v}{\nabla}_p\overline{u}.$$

The last two summands on the right-hand side of this formula are equal. Thus we derive

$$\varphi=2|\overset{a}{\nabla}u|^2+2\overset{a}{R}_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^ju\cdot\overset{v}{\nabla}{}^l\overline{u}.$$

Inserting the last expression into (8.2.32) and then (8.2.32) into (8.2.31), we arrive at the claim of the lemma.

We will now try to compensate the last term on the right-hand side of Pestov identity (8.2.28). Since additional terms of the same kind can arise in applications of the identity, we formulate our result in the following form.

Theorem 8.2.2 *Let (M,g) be a CDRM and $S\in C^\infty(\beta^0_4M;T^0M)$ be a real semibasic tensor field on T^0M possessing the properties*

$$S_{ipjq}\xi^p\xi^q=S_{jpiq}\xi^p\xi^q=S_{piqj}\xi^p\xi^q,\qquad S_{ipqr}\xi^p\xi^q\xi^r=0$$

and positive homogeneous of degree zero in ξ:

$$S_{ijkl}(x,t\xi)=S_{ijkl}(x,\xi)\qquad(t>0).$$

Assume that, for every $(x,\xi)\in\Omega M$, the equation

$$\frac{D^2y}{dt^2}+(R+S)(t)y=0 \tag{8.2.33}$$

has no pair of conjugate points on the geodesic $\gamma=\gamma_{x,\xi}\,:\,[\tau_-(x,\xi),\tau_+(x,\xi)]\to M$; here

$$[(R+S)(t)]^p_k=g^{ip}(R_{ijkl}+S_{ijkl})(\gamma(t),\dot\gamma(t))\dot\gamma^j(t)\dot\gamma^l(t).$$

Then there exists a real semibasic tensor field $a=(a^{ij})\in C^\infty(\beta^2_0M;T^0M)$ satisfying conditions (8.2.4)–(8.2.6) *and the equation*

$$(\overset{a}{R}_{ijkl}+S_{ijkl})\xi^i\xi^k=0. \tag{8.2.34}$$

P r o o f. By (8.2.24), equation (8.2.34) is equivalent to the next:

$$(Ha)_{ij}+a_{ip}a^p_j+\tilde R_{ipjq}\xi^p\xi^q=0, \tag{8.2.35}$$

where we put $\tilde R=R+S$ for brevity.

If the field $a(x,\xi)$ is positively-homogeneous of degree 1 in ξ, then the left-hand side of equation (8.2.35) is homogeneous of degree 2. Consequently, if we find a solution to equation (8.2.35) on ΩM, then we obtain a solution on the whole T^0M with the help of extension by homogeneity. Therefore we further consider equation (8.2.35) on ΩM.

We represent ΩM as the union of disjoint one-dimensional submanifolds, the orbits of the geodesic flow. Restricted to an orbit, (8.2.35) gives a system of ordinary differential equations. For distinct orbits, these systems does not relate to one other. We have to find a solution on every orbit in such a way that the family of the solutions forms a field smooth on the whole ΩM. The last requirement can be satisfied by appropriate choosing the initial values for the orbits. We start to realize our plan.

Given $(x,\xi) \in \partial_-\Omega M$, we consider a maximal geodesic $\gamma = \gamma_{x,\xi} : [0,\tau_+(x,\xi)] \to M$ defined by the initial conditions $\gamma(0) = x,\ \dot\gamma(0) = \xi$. Taking $x = \gamma(t)$ and $\xi = \dot\gamma(t)$ in (8.2.35), we obtain the system of the ordinary differential equations of the Riccati type:

$$\left(\frac{Da}{dt}\right)_{ij} + a_{ip}a^p_j + \tilde R_{ipjq}\dot\gamma^p\dot\gamma^q = 0. \tag{8.2.36}$$

To prove the theorem, it suffices to establish existence of a symmetric solution $(a_{ij}(t))$ to system (8.2.36) on the segment $[0,\tau_+(x,\xi)]$, which depends smoothly on $(x,\xi) \in \partial_-\Omega M$ and satisfies the additional condition

$$a_{ij}(t)\dot\gamma^j(t) = 0. \tag{8.2.37}$$

Taking the convolution of (8.2.36) with $\dot\gamma^j$, we see that an arbitrary solution to system (8.2.36) satisfies (8.2.37), provided this condition is satisfied for $t = 0$. Let us show that a similar assertion is valid for symmetry. Indeed, a solution $a_{ij}(t)$ to system (8.2.36) is represented as $a_{ij} = a^+_{ij} + a^-_{ij}$ where a^+_{ij} is symmetric and a^-_{ij} is skew-symmetric. Inserting this expression into (8.2.36), we obtain

$$\left[\left(\frac{Da^+}{dt}\right)_{ij} + g^{pq}\left(a^+_{ip}a^+_{qj} + a^-_{ip}a^-_{qj}\right) + \tilde R_{ipjq}\dot\gamma^p\dot\gamma^q\right] +$$

$$+ \left[\left(\frac{Da^-}{dt}\right)_{ij} + g^{pq}\left(a^+_{ip}a^-_{qj} + a^-_{ip}a^+_{qj}\right)\right] = 0.$$

The expression in the first brackets is symmetric and the expression in the second brackets is skew-symmetric. Consequently,

$$\left(\frac{Da^-}{dt}\right)_{ij} + g^{pq}\left(a^+_{ip}a^-_{qj} + a^-_{ip}a^+_{qj}\right) = 0.$$

The last equalities can be considered as a homogeneous linear system on a^-_{ij}. This system, together with the initial condition $a^-(0) = 0$, implies that $a^- \equiv 0$.

Thus, symmetry of the field (a_{ij}) and its orthogonality to the vector ξ are insured by the choice of the initial value. We now consider the question of existence of a solution to system (8.2.36). Raising the index i, we rewrite this system as:

$$\left(\frac{Da}{dt}\right)^i_j + a^i_p a^p_j + \tilde R^i_j = 0 \qquad (\tilde R^i_j = \tilde R^i{}_{pjq}\dot\gamma^p\dot\gamma^q)$$

or in matrix form

$$\frac{Da}{dt} + a^2 + \tilde R = 0. \tag{8.2.38}$$

We look for a solution to this equation in the form

$$a = \frac{Db}{dt} b^{-1} \tag{8.2.39}$$

Inserting (8.2.39) into (8.2.38), we arrive at the equation

$$\frac{D^2 b}{dt^2} + \tilde{R} b = 0. \tag{8.2.40}$$

Conversely, if equation (8.2.40) has a nondegenerate solution b, then equation (8.2.38) is satisfied by the matrix a defined by formula (8.2.39).

We denote by $z = (z^i_j(x,\xi;t))$ the solution to equation (8.2.40) which satisfies the initial conditions

$$z(0) = 0, \qquad \left(\frac{Dz}{dt}(0)\right)^i_j = \delta^i_j. \tag{8.2.41}$$

Observe that the field $z(x,\xi;t)$ is smooth with respect to the set of its arguments. By the condition of the theorem about the absence of conjugate points, the matrix $z^i_j(x,\xi;t)$ is nondegenerate for $0 < t \le \tau_+(x,\xi)$. By initial conditions (8.2.41), there is $t_0 > 0$ such that the matrices $z(x,\xi;t)$ and $\frac{Dz}{dt}(x,\xi;t)$ are positive-definite for $0 < t \le \tau(x,\xi) = \min(t_0, \tau_+(x,\xi))$. Consequently, the matrix

$$\tilde{b}(x,\xi;t) = \frac{Dz}{dt} + \lambda z \tag{8.2.42}$$

is nondegenerate for $0 \le t \le \tau(x,\xi)$ and any $\lambda > 0$. The determinant of the matrix $z(x,\xi;t)$ is bounded from below by some positive constant on the segment $t_0 \le t \le \tau_+(x,\xi)$. Consequently, choosing a sufficiently large positive constant λ in (8.2.42), we can guarantee that the matrix $\tilde{b}(x,\xi;t)$ is nondegenerate for all $(x,\xi;t)$ belonging to the set

$$G = \{(x,\xi;t) \mid (x,\xi) \in \partial_+\Omega M,\ 0 \le t \le \tau_+(x,\xi)\}.$$

Thus, we have found a nondegenerate solution $\tilde{b} = (\tilde{b}^i_j(x,\xi;t))$ to equation (8.2.40) depending smoothly on $(x,\xi;t) \in G$ and satisfying the initial conditions

$$\tilde{b}^i_j(0) = \delta^i_j, \qquad \left(\frac{D\tilde{b}}{dt}(0)\right)^i_j = \lambda\delta^i_j. \tag{8.2.43}$$

We now define

$$b^i_j(x,\xi;t) = \tilde{b}^i_j(x,\xi;t) - \lambda t \dot{\gamma}^i_{x,\xi}(t)\dot{\gamma}_{j,x,\xi}(t) \qquad ((x,\xi;t) \in G). \tag{8.2.44}$$

The matrix $b = (b^i_j)$ meets equation (8.3.24) and the initial conditions

$$b^i_j(0) = \delta^i_j, \qquad \left(\frac{Db}{dt}(0)\right)^i_j = \lambda(\delta^i_j - \xi^i\xi_j). \tag{8.2.45}$$

Let us show that the matrix $b(x,\xi;t)$ is nondegenerate for all $(x,\xi;t) \in G$. Indeed, let $\gamma = \gamma_{x,\xi}$ and $\eta \in T_{\gamma(t)}M$ be a nonzero vector. We represent it as $\eta = \tilde{\eta} + \mu\dot{\gamma}(t)$ where $\tilde{\eta} \perp \dot{\gamma}(t)$ and $|\tilde{\eta}|^2 + \mu^2 > 0$. Then

$$b^i_j(t)\eta^j = (\tilde{b}^i_j - \lambda t\dot{\gamma}^i\dot{\gamma}_j)(\tilde{\eta}^j + \mu\dot{\gamma}^j) = \tilde{b}^i_j\tilde{\eta}^j + \mu(\tilde{b}^i_j\dot{\gamma}^j - \lambda t\dot{\gamma}^i). \tag{8.2.46}$$

Equation (8.2.40) and initial conditions (8.2.43) imply that

$$\tilde{b}^i_j(t)\dot{\gamma}^j(t) = (1+\lambda t)\dot{\gamma}^i(t).$$

Inserting this expression into (8.2.46), we obtain

$$b^i_j(t)\eta^j = \tilde{b}^i_j(t)\left[\tilde{\eta}^j + \frac{\mu}{1+\lambda t}\dot{\gamma}^j(t)\right]. \tag{8.2.47}$$

The vector in the brackets is nonzero, since $\lambda > 0,\ t \geq 0$ and $\tilde{\eta}\perp\dot{\gamma}(t)$. Since the matrix $(\tilde{b}^i_j(t))$ is nondegenerate, the right-hand side of equality (8.2.47) is not equal to zero for $\eta \neq 0$. Since this is true for any t, the matrix $b = (b^i_j(x,\xi;t))$ is nondegenerate. Thus we have constructed the nondegenerate solution $b = (b^i_j(x,\xi;t))$ to equation (8.2.40), which satisfies initial conditions (8.2.45). Consequently, the matrix $a = (a^i_j(x,\xi;t))$ that is defined by formula (8.2.39) satisfies equation (8.2.38) and the initial conditions

$$a^i_j(x,\xi;0) = \lambda(\delta^i_j - \xi^i\xi_j).$$

Lowering the index i, we obtain

$$a_{ij}(x,\xi;0) = \lambda(g_{ij} - \xi_i\xi_j).$$

Whence we see that the tensor $a_{ij}(x,\xi;0)$ is symmetric and orthogonal to the vector ξ. As was noted, validity of these properties for $t = 0$ implies their validity for all t. The theorem is proved.

In conclusion of the section we will obtain a Gauss-Ostrogradskiĭ formula for the divergence $\overset{a}{\nabla}{}^i u_i$, where $u \in C^\infty(\beta^0_1 M)$ is a semibasic covector field. By definition (8.2.8)–(8.2.10),

$$\overset{a}{\nabla}{}^i u_i = \overset{h}{\nabla}{}^i u_i + A^i u_i = \overset{h}{\nabla}{}^i u_i + a^{ip}\overset{v}{\nabla}_p u_i + \overset{v}{\nabla}_i a^{ip}\cdot u_p = \overset{h}{\nabla}{}^i u_i + \overset{v}{\nabla}_p(a^{ip}u_i). \tag{8.2.48}$$

We now assume the field u to be homogeneous in its second argument:

$$u(x,t\xi) = t^\lambda u(x,\xi) \qquad (t>0).$$

We multiply equality (8.2.29) by $d\Sigma^{2n-1}$, integrate it and transform the integrals on the right-hand side of the so-obtained equality by the Gauss-Ostrogradskiĭ formulas for the horizontal and vertical divergences. As a result, we obtain

$$\int\limits_{\Omega M} \overset{a}{\nabla}{}^i u_i\, d\Sigma^{2n-1} = \int\limits_{\partial\Omega M} \nu^i u_i\, d\Sigma^{2n-2} + (\lambda+n-1)\int\limits_{\Omega M} \xi_p a^{ip} u_i\, d\Sigma^{2n-1}.$$

We observe that the integrand of the second integral on the right-hand side of this equality is equal to zero by (8.2.6). We thus obtain the next Gauss-Ostrogradskiĭ formula:

$$\int\limits_{\Omega M} \overset{a}{\nabla}{}^i u_i\, d\Sigma^{2n-1} = \int\limits_{\partial\Omega M} \langle\nu,u\rangle\, d\Sigma^{2n-2}. \tag{8.2.49}$$

8.3 Proof of Theorem 8.1.1

First of all we note that to prove the theorem it suffices to establish estimate (8.1.7) for a smooth function f.

For $f \in C^\infty(M)$, we define a function $u \in C^\infty(T^0M)$ by the equality

$$u(x,\xi) = \int\limits_{\tau_-(x,\xi)}^{0} f(\gamma_{x,\xi}(t)) \exp\left[-|\xi| \int\limits_{t}^{0} \varepsilon(\gamma_{x,\xi}(s))\,ds\right] dt. \tag{8.3.1}$$

This function satisfies the equation

$$Hu(x,\xi) + \varepsilon(x)|\xi|u(x,\xi) = f(x), \tag{8.3.2}$$

the boundary conditions

$$u|_{\partial_-\Omega M} = 0, \qquad u|_{\partial_+\Omega M} = I^\varepsilon f \tag{8.3.3}$$

and the condition of homogeneity

$$u(x,\lambda\xi) = \lambda^{-1}u(x,\xi) \qquad (\lambda > 0). \tag{8.3.4}$$

Let $a = (a^{ij}) \in C^\infty(\beta_0^2 M)$ be some semibasic tensor field satisfying conditions (8.2.4)–(8.2.6); a choice of this field will be specified later. By $\overset{a}{\nabla}$ we denote the modified horizontal derivative that is defined with the help of a. We define a semibasic covector field $y = (y_i)$ and a semibasic vector field $z = (z^i)$ on T^0M by the equalities

$$\overset{v}{\nabla}_i u = -\frac{u}{|\xi|^2}\xi_i + y_i, \tag{8.3.5}$$

$$\overset{a}{\nabla}{}^i u = \frac{Hu}{|\xi|^2}\xi^i + z^i. \tag{8.3.6}$$

The summands on the right-hand side of each of these equalities are orthogonal to one other. Indeed, taking the scalar product of each of these equalities and ξ, we obtain

$$\xi^i \overset{v}{\nabla}_i u = -u + \langle y, \xi\rangle, \qquad \xi_i \overset{a}{\nabla}{}^i u = Hu + \langle z, \xi\rangle. \tag{8.3.7}$$

By (8.3.4) and (8.2.12), $\xi^i \overset{v}{\nabla}_i u = -u$ and $\xi_i \overset{a}{\nabla}{}^i u = Hu$. Therefore (8.3.7) implies that $\langle y, \xi\rangle = \langle z, \xi\rangle = 0$. Thus, for $|\xi| = 1$, it follows from (8.3.6) that

$$|\overset{a}{\nabla} u|^2 = |Hu|^2 + |z|^2. \tag{8.3.8}$$

Applying the operator $\overset{v}{\nabla}$ to equation (8.3.2), we obtain

$$\overset{v}{\nabla} Hu + \varepsilon|\xi|\overset{v}{\nabla} u + \frac{\varepsilon u}{|\xi|}\xi = 0.$$

Together with (8.3.5) this implies that

$$\overset{v}{\nabla} Hu = -\varepsilon|\xi|y.$$

Taking the scalar product of the last equality with $\overset{a}{\nabla} u$ and using (8.3.6), we obtain

$$\langle \overset{a}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = -\overline{\varepsilon}|\xi|\langle z, y\rangle.$$

From the last equality, we derive the estimate

$$2|\langle \overset{a}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle| \le 2(|\varepsilon|\,|\xi|\,|y|)\,|z| \le |\varepsilon|^2|\xi|^2|y|^2 + |z|^2. \tag{8.3.9}$$

Let us write down the Pestov identity (8.2.28) for the function u. Estimating the left-hand side of the identity by (8.3.9) and replacing the first summand on its right-hand side with expression (8.3.8), we obtain the inequality

$$|Hu|^2 \le |\varepsilon|^2|y|^2 - \overset{a}{\nabla}{}^i v_i - \overset{v}{\nabla}_i w^i + \overset{a}{R}_{ijkl}\xi^i\xi^k \overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l \overline{u}, \tag{8.3.10}$$

which is valid for $|\xi| = 1$; here the fields (v_i) and (w^i) are defined by formulas (8.2.29), (8.2.30).

We transform the last term on the right-hand side of (8.3.10) by using (8.3.5) and the symmetries of the tensor $\overset{a}{R}_{ijkl}$:

$$\overset{a}{R}_{ijkl}\xi^i\xi^k \overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l \tilde{u} = \overset{a}{R}_{ijkl}(\xi \wedge \overset{v}{\nabla} u)^{ij}(\xi \wedge \overset{v}{\nabla}\overline{u})^{kl} = \overset{a}{R}_{ijkl}(\xi \wedge y)^{ij}(\xi \wedge \overline{y})^{kl} = \overset{a}{R}_{ijkl}\xi^i\xi^k y^j \overline{y}^l.$$

Inserting the last expression into (8.3.10), we obtain

$$|Hu|^2 \le -\overset{a}{\nabla}{}^i v_i - \overset{v}{\nabla}_i w^i + \left(\overset{a}{R}_{ijkl}\xi^i\xi^k + |\varepsilon|^2 g_{jl}\right) y^j \overline{y}^l.$$

Using the relations $|\xi|^2 = g_{ij}\xi^i\xi^j = 1$ and $\langle y, \xi\rangle = g_{ij}\xi^i\overline{y}^j = 0$, the last inequality takes the form:

$$|Hu|^2 \le -\overset{a}{\nabla}{}^i v_i - \overset{v}{\nabla}_i w^i + \left[\overset{a}{R}_{ijkl} + |\varepsilon|^2(g_{ik}g_{jl} - g_{il}g_{jk})\right]\xi^i\xi^k y^j \overline{y}^l. \tag{8.3.11}$$

Taking a in (8.3.11) to be some tensor field that exists by Theorem 8.2.2 with $S_{ijkl} = |\varepsilon|^2(g_{ik}g_{jl} - g_{il}g_{jk})$, we transform this inequality as follows

$$|Hu|^2 \le -\overset{a}{\nabla}{}^i v_i - \overset{v}{\nabla}_i w^i.$$

We multiply the last inequality by the volume form $d\Sigma = d\Sigma^{2n-1}$, integrate it over ΩM and apply Gauss-Ostrogradskiĭ formulas (8.2.49) and (3.6.35). As a result, we obtain the inequality

$$\int\limits_{\Omega M} |Hu|^2\, d\Sigma \le -\int\limits_{\partial\Omega M} \langle v, \nu\rangle\, d\Sigma^{2n-2} - (n-2)\int\limits_{\Omega M} \langle w, \xi\rangle\, d\Sigma.$$

The coefficient of the second integral on the right-hand side is due to the homogeneity of the field w which ensues from (8.2.26) and (8.3.4). It also follows from (8.2.26) that $\langle w, \xi\rangle = |Hu|^2$ and, consequently, the last inequality takes the form

$$(n-1)\int\limits_{\Omega M} |Hu|^2\, d\Sigma \le -\int\limits_{\partial\Omega M} \langle v, \nu\rangle\, d\Sigma^{2n-2}. \tag{8.3.12}$$

In the same way as in the previous chapters (see the arguments at the end of Section 4.6 and in the beginning of Section 4.7) we show that boundary conditions (8.3.3) and formula (8.2.29) imply the estimate

$$\left| \int\limits_{\partial\Omega M} \langle v, \nu \rangle \, d\Sigma^{2n-2} \right| \le C \| I^\varepsilon f \|_1^2$$

with a constant C dependent on (M, g) and the tensor field a but independent of f. In view of the estimate, (8.3.12) implies the inequality

$$\| Hu \|^2 = \int\limits_{\Omega M} |Hu|^2 \, d\Sigma \le C \| I^\varepsilon f \|_1^2. \tag{8.3.13}$$

It remains to observe that equation (8.3.2) implies the estimate

$$\| f \|_0^2 \le 2 \| Hu \|^2 + 2\varepsilon_0^2 \| u \|^2 \tag{8.3.14}$$

where ε_0 is defined by formula (8.1.11). With the help of the Poincaré inequality (Lemma 4.5.1), the first of the boundary conditions (8.3.3) implies the estimate

$$\| u \|^2 \le C_1 \| Hu \|^2.$$

Whence and from (8.3.14) it follows that

$$\| f \|_0^2 \le C_2 \| Hu \|^2. \tag{8.3.15}$$

Finally, (8.3.13) and (8.3.15) imply (8.1.7). The theorem is proved.

8.4 The volume of a simple compact Riemannian manifold

Recall that in Section 4.8 we accepted the next definition: a compact Riemannian manifold (M, g) is called *simple* if the boundary ∂M is strictly convex and every two points $x, y \in M$ are joint by a unique geodesic which depends smoothly on x, y. The definition implies that (M, g) is a CDRM without conjugate points. The converse claim is also valid: a CDRM without conjugate points is simple; we will not prove this assertion here (compare with one of remarks after the formulation of Theorem 4.3.3). Given a simple Riemannian manifold, the distance function $\rho(x, y)$ is smooth for $x \ne y$.

As was established in Section 4.8, for a simple compact Riemannian manifold (M, g), the function $\tau_+ : \partial_-\Omega M \to \mathbf{R}$ (which can be called the *angle hodograph*) is uniquely determined by the hodograph $\Gamma_g : \partial M \times \partial M \to \mathbf{R}$.

Recall that, for $0 \ne \xi \in T_x M$, by P_ξ we denote the orthogonal projection onto the hyperplane orthogonal to ξ, i.e., $P_\xi \eta = \eta - \langle \xi, \eta \rangle \xi / |\xi|^2$.

The main result of the current section is the next

Theorem 8.4.1 *Let (M,g) be a simple compact Riemannian manifold of dimension n. The Riemannian volume $V^n(M)$ is expressed through the angle hodograph $\tau_+ : \partial\Omega M \to \mathbf{R}$ by the formula*

$$V^n(M) = -\frac{\Gamma(n/2)}{2(n-1)\pi^{n/2}} \int\limits_{\partial M} \int\limits_{\Omega_x^- M} \langle P_\xi \overset{v}{\nabla} \tau_+, \nu\rangle \, d\omega_x(\xi)\, dV^{n-1}(x), \tag{8.4.1}$$

where ν is the unit vector of the outer normal to the boundary, dV^{n-1} is the Riemannian volume of ∂M, $\Omega_x^- M = \{\xi \in \Omega_x M \mid \langle \xi, \nu\rangle \le 0\}$ and $d\omega_x$ is the angle measure on the sphere $\Omega_x M$.

Possibly, formula (8.4.1) will be clearer after the next remark: the vector $P_\xi \overset{v}{\nabla}\tau_+(x,\xi)$ is the gradient of the restriction of the function $\tau_+(x,\cdot)$ to the unit sphere $\Omega_x M$, where the latter is considered as a Riemannian manifold whose metric is induced by the Euclidean structure of $T_x M$ (which in turn is induced by the Riemannian metric g).

We introduce the notation $\partial_- TM = \{(x,\xi) \in TM \mid x \in \partial M,\ \langle \xi, \nu\rangle \le 0\}$. Our proof of Theorem 8.4.1 is based on the next claim which will also be used in the next section.

Lemma 8.4.2 *Let (M,g) be a simple compact Riemannian manifold. There exists a semibasic tensor field $a \in C^\infty(\beta_0^2 M; T^0 M \setminus \partial_- TM)$ satisfying* (8.2.4)–(8.2.6) *and possessing the next two properties:*

(1) *The corresponding curvature tensor satisfies the relation*

$$\overset{a}{R}_{ijkl}\xi^i\xi^k = 0. \tag{8.4.2}$$

(2) *Let us fix a point $y_0 \in \partial M$ and define the unit vector field η on $M \setminus \{y_0\}$ by putting $\eta(x) = \dot\gamma_x(0)/|\dot\gamma_x(0)|$, where $\gamma_x : [-1,0] \to M$ is the geodesic satisfying the boundary conditions $\gamma_x(-1) = y_0,\ \gamma_x(0) = x$. Then the equality*

$$\nabla[f(x,\eta(x))] = (\overset{a}{\nabla} f)(x,\eta(x)). \tag{8.4.3}$$

holds for every function $f \in C^\infty(TM)$.

P r o o f. It suffices to define $a^{ij}(x,\xi)$ in the case $|\xi| = 1$; for the other values of ξ, the field $a(x,\xi)$ is extendible by homogeneity (compare with the arguments after formula (8.2.35)). Let $(x_0,\xi_0) \in \Omega M \setminus \partial_- \Omega M$ and $\gamma = \gamma_{x_0,\xi_0} : [\tau_-(x_0,\xi_0), 0] \to M$ be the geodesic satisfying the initial conditions $\gamma(0) = x_0,\ \dot\gamma(0) = \xi_0$. Then $y_0 = \gamma(\tau_-(x_0,\xi_0)) \in \partial M$. We construct the unit vector field $\eta(x)$ on $M \setminus \{y_0\}$ as described in the formulation of the lemma and define

$$a^i_j(x_0,\xi_0) = \eta^i{}_{;j}(x_0). \tag{8.4.4}$$

As usual, we put $a^{ij} = g^{jk}a^i_k$. Let us show the field $a = (a^{ij})$ to satisfy all claims of the lemma.

First of all $a \in C^\infty(\Omega M \setminus \partial_- \Omega M)$, by definition. Since the vector field $\eta(x)$ is of unit length,

$$\eta^i{}_{;j}(x)\eta_i(x) = 0. \tag{8.4.5}$$

Noting that

$$\eta(x_0) = \xi_0 \tag{8.4.6}$$

and putting $x = x_0$ in (8.4.5), we obtain

$$a^i_j \xi_i = 0. \tag{8.4.7}$$

To prove symmetry of the field a^{ij}, we fix (x_0, ξ_0), introduce the function $\tilde\rho(x) = \rho(x, y_0)$, where ρ is the distance function in the metric g, and note that $\eta(x) = \nabla\tilde\rho(x)$. Consequently,

$$g_{ik}\eta^k{}_{;\,j} = \eta_{i\,;\,j} = \tilde\rho_{\,;\,ij}$$

Thus, the tensor $a(x_0, \xi_0)$ coincides with the value of the Hessian of the function $\tilde\rho$ at the point x_0 and, consequently, is symmetric.

To prove property (8.4.3), we differentiate $f(x, \eta(x))$ as a composite function:

$$\nabla^i[f(x,\eta(x))] = g^{ij}\frac{\partial}{\partial x^j}[f(x,\eta(x))] = g^{ij}\left[\frac{\partial f}{\partial x^j}(x,\eta(x)) + \frac{\partial f}{\partial \xi^k}(x,\eta(x))\frac{\partial \eta^k}{\partial x^j}(x)\right]. \tag{8.4.8}$$

On the other hand, by the definition of covariant derivatives,

$$\eta^k{}_{;\,j}(x) = \frac{\partial \eta^k}{\partial x^j}(x) + \Gamma^k_{jl}(x)\eta^l(x),$$

$$(\overset{a}{\nabla}{}^i f)(x,\eta(x)) = g^{ij}\left[\frac{\partial f}{\partial x^j}(x,\eta(x)) - \Gamma^k_{jl}(x)\eta^l(x)\frac{\partial f}{\partial \xi^k}(x,\eta(x))\right] + a^{ik}(x,\eta(x))\frac{\partial f}{\partial \xi^k}(x,\eta(x)).$$

Expressing the partial derivatives $\partial\eta^k/\partial x^j$ and $g^{ij}\partial f/\partial x^j$ from the last two equalities and inserting these values into (8.4.8), we obtain

$$\nabla^i\left[f(x,\eta(x))\right] = (\overset{a}{\nabla}{}^i f)(x,\eta(x)) + \left[g^{ij}\eta^k{}_{;\,j}(x) - a^{ik}(x,\eta(x))\right]\frac{\partial f}{\partial \xi^k}(x,\eta(x)).$$

For $x = x_0$, the expression in the brackets is equal to zero, and we thus arrive at (8.4.3).

Finally, we prove (8.4.2). As has been noted, the field $\eta_{i\,;\,j} = g_{ik}\eta^k{}_{;\,j}$ is symmetric and, consequently, equality (8.4.5) can be rewritten as follows:

$$\eta^i{}_{;\,j}\eta^j = 0.$$

Differentiating this equality, we obtain

$$\eta^i{}_{;\,jk}\eta^j + \eta^i{}_{;\,j}\eta^j{}_{;\,k} = 0.$$

Changing the limits of differentiation in the first factor with the help of (3.2.11), we transform this equation as follows

$$\eta^i{}_{;\,kj}\eta^j + \eta^i{}_{;\,j}\eta^j{}_{;\,k} + R^i{}_{lkj}\eta^l\eta^j = 0. \tag{8.4.9}$$

Formulas (8.4.4) and (8.4.6) imply the similar equality at any point of the geodesic γ, i.e.,

$$\eta^i{}_{;\,j}(\gamma(t)) = a^i_j(\gamma(t), \dot\gamma(t)), \tag{8.4.10}$$

$$\eta^i(\gamma(t)) = \dot\gamma^i(t), \tag{8.4.11}$$

Putting $x = \gamma(t)$ in (8.4.9) and substituting values (8.4.10)–(8.4.11) for η^i and $\eta^i{}_{;j}$, we see that the tensor field $a = a(\gamma(t), \dot\gamma(t))$ along the geodesic γ satisfies the equation

$$\left(\frac{Da}{dt}\right)^i_k + a^i_j a^j_k + R^i{}_{lkj}\dot\gamma^l\dot\gamma^j = 0.$$

Validity of this equation for every geodesic γ is equivalent to condition (8.4.2), as has been established in the proof of Theorem 8.2.2. The lemma is proved.

Note that the field a has a singularity of the type τ_-^{-1} near $\partial_- TM$.

Let us apply formula (8.4.3) in the case $f = \tau_-$. Given $(x_0, \xi_0) \in \Omega M \setminus \partial_-\Omega M$, let $y_0 = \gamma_{x_0,\xi_0}(\tau_-(x_0, \xi_0))$ stands for the point y_0 participating in the formulation of the lemma. Then $\tau_-(x, \eta(x)) = -\rho(x, y_0)$ and $\eta(x_0) = \xi_0 = (\nabla_x\rho)(x_0, y_0)$. Putting $f = \tau_-$ and $x = x_0$ in (8.4.3), we obtain

$$-\xi_0 = -(\nabla_x\rho)(x_0, y_0) = (\overset{a}{\nabla}\tau_-)(x_0, \xi_0).$$

From this, taking the homogeneity of τ_- into account, we arrive at the relation

$$\overset{a}{\nabla}\tau_- = -\frac{\xi}{|\xi|} \tag{8.4.12}$$

which holds on $T^0M \setminus \partial_- TM$.

P r o o f of Theorem 8.4.1. Let a be the tensor field constructed in Lemma 8.4.2. By (8.4.2), for a real function $u \in C^\infty(T^0M \setminus \partial_- TM)$, the Pestov identity (8.2.28) has the form

$$2\langle\overset{a}{\nabla}u, \overset{v}{\nabla}(Hu)\rangle = |\overset{a}{\nabla}u|^2 + \overset{a}{\nabla}{}^i v_i + \overset{v}{\nabla}_i w^i, \tag{8.4.13}$$

where

$$v_i = \xi_i \overset{a}{\nabla}{}^j u \cdot \overset{v}{\nabla}_j u - \xi_j \overset{v}{\nabla}_i u \cdot \overset{a}{\nabla}{}^j u, \tag{8.4.14}$$

$$w^i = \xi_j \overset{a}{\nabla}{}^i u \cdot \overset{a}{\nabla}{}^j u. \tag{8.4.15}$$

By (8.4.12), for $u = \tau_-$ these formulas look like

$$\frac{1}{|\xi|^2} = -\overset{a}{\nabla}{}^i v_i - \overset{v}{\nabla}_i w^i, \tag{8.4.16}$$

$$v_i = -\frac{1}{|\xi|^2}\xi_i\xi^j\overset{v}{\nabla}_j\tau_- + \overset{v}{\nabla}_i\tau_- = (P_\xi\overset{v}{\nabla}\tau_-)_i, \tag{8.4.17}$$

$$w^i = \frac{\xi^i}{|\xi|^2}. \tag{8.4.18}$$

Let ε be a small positive number and $M_\varepsilon = \{x \in M \mid \rho(x, \partial M) \ge \varepsilon\}$. We integrate equality (8.4.16) over ΩM_ε and transform the divergent terms according to the Gauss-Ostrogradskiĭ formulas (4.6.35) and (8.2.49). As a result we arrive at the equality

$$\int\limits_{\Omega M_\varepsilon} d\Sigma = -\int\limits_{\partial\Omega M_\varepsilon}\langle v, \nu\rangle\, d\Sigma^{2n-2} - (n-2)\int\limits_{\Omega M_\varepsilon}\langle w, \xi\rangle\, d\Sigma. \tag{8.4.19}$$

By (8.4.18), $\langle w, \xi\rangle = 1$ for $|\xi| = 1$. So (8.4.19) is rewritten as:

$$(n-1)\int\limits_{\Omega M_\varepsilon} d\Sigma = -\int\limits_{\partial\Omega M_\varepsilon} \langle P_\xi \overset{v}{\nabla}\tau_-, \nu\rangle\, d\Sigma^{2n-2}. \tag{8.4.20}$$

The integrand on the right-hand side of this equality has no singularity near $\partial_-\Omega M$ and, moreover, it vanishes on $\partial_-\Omega M$. So we can pass to the limit in (8.4.20) as $\varepsilon \to \infty$. Passing to the limit and using representations (3.6.34) for $d\Sigma = d\Sigma^{2n-1}$ and $d\Sigma^{2n-2}$, we obtain

$$(n-1)\int\limits_M \int\limits_{\Omega_x M} d\omega_x(\xi)\, dV^n(x) = (-1)^{n+1}\int\limits_{\partial M}\int\limits_{\Omega_x^+ M} \langle P_\xi \overset{v}{\nabla}\tau_-, \nu\rangle\, d\omega_x(\xi)\, dV^{n-1}(x). \tag{8.4.21}$$

The left-hand side of (8.4.21) is equal to $(n-1)\omega_n V^n(M)$, where $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the volume of the unit sphere in $\mathbf{R}^n$. We change the integration variable on the right-hand side of (8.4.21) by the formula $\xi = -\eta$. Under this change the form $d\omega_x(\xi)$ transfers into $(-1)^n d\omega_x(\eta)$ and the field $P_\xi \overset{v}{\nabla}\tau_-$, into the field $P_\eta \overset{v}{\nabla}\tau_+$. As a result we arrive at (8.4.1). The theorem is proved.

8.5 Determining a Riemannian metric in a prescribed conformal class

Let M be a compact manifold. We introduce a Riemannian metric on M and denote the distance between points $x, y \in M$ in this metric by $\rho(x, y)$. Given a function $f : M \times M \to \mathbf{C}$, we introduce the notation

$$\|f\|_{W_1^1(M\times M,\,k)} =$$

$$= \int\limits_M \int\limits_M \Big[\rho^{1-k}(x,y)\,(|\nabla_x f(x,y)| + |\nabla_y f(x,y)|) + \rho^{-k}(x,y)|f(x,y)|\Big]\, dV(x)\, dV(y), \tag{8.5.1}$$

where dV is the Riemannian volume form. Under any change of the metric, norm (8.5.1) is replaced with an equivalent one.

Recall that, given a Riemannian manifold (M, g) and a point $x \in M$, the *exponential mapping* $\exp_x : T_x M \supset U \to M$ is defined by the equality $\exp_x(\xi) = \gamma_{x,\xi}(1)$, where $\gamma_{x,\xi}(t)$ is the geodesic satisfying the initial conditions $\gamma_{x,\xi}(0) = x,\ \dot\gamma_{x,\xi}(0) = \xi$. The exponential mapping is defined for $\xi \in T_x M$ such that the geodesic $\gamma_{x,\xi}(t)$ is defined for $t \in [0, 1]$. Given $\xi \in U$, by $d_\xi \exp_x : T_x M \to T_{\exp_x(\xi)} M$ we denote the differential of the exponential mapping.

Theorem 8.5.1 *Let (M, g_0) be an n-dimensional compact Riemannian manifold with boundary. Given positive numbers λ_0 and k_0, by $\Lambda(\lambda_0, k_0)$ we denote the set of real functions $\lambda \in C^\infty(M)$ satisfying the inequalities*

$$\lambda_0 \le \lambda \le \lambda_0^{-1} \tag{8.5.2}$$

and the next two conditions:

(1) *the Riemannian metric*

$$g = \lambda^2 g_0 \tag{8.5.3}$$

is such that (M, g) is a simple Riemannian manifold;

(2) *if $(x, \xi) \in TM$ is such that the exponential mapping $\exp_x$ of metric (8.5.3) is defined at the point ξ, then its differential satisfy the inequalities*

$$k_0|\eta| \le |(d_\xi \exp_x)\eta| \le k_0^{-1}|\eta| \qquad (\eta \in T_xM). \tag{8.5.4}$$

Then, for every two functions $\lambda, \tilde\lambda \in \Lambda(\lambda_0, k_0)$ and the corresponding metrics $g = \lambda^2 g_0$ and $\tilde g = \tilde\lambda^2 g_0$, the estimate

$$\|\lambda - \tilde\lambda\|^2_{L_2(M)} \le C\|\Gamma_g - \Gamma_{\tilde g}\|_{W_1^1(\partial M\times\partial M,\, n-1)} \tag{8.5.5}$$

holds with a constant C depending only on (M, g_0), λ_0 and k_0. Here Γ_g and $\Gamma_{\tilde g}$ are the hodographs of the metrics g and $\tilde g$.

Note that the norm

$$\|\Gamma_g\|_{W_1^1(\partial M\times\partial M,\, n-1)} = \int\limits_{\partial M}\int\limits_{\partial M}\Big[\rho_0^{2-n}\left(|\nabla_x\Gamma_g| + |\nabla_y\Gamma_g|\right) + \rho_0^{1-n}|\Gamma_g|\Big]\, dV_0^{n-1}(x)\, dV_0^{n-1}(y)$$

is finite for every metric g, since its hodograph satisfies the estimate $\Gamma_g(x, y) \le C_g\, \rho_0(x, y)$ with some constant C_g, where ρ_0 is the distance function of the metric g_0.

P r o o f. Let us fix a function $\lambda \in \Lambda(\lambda_0, k_0)$ and denote $g = \lambda^2 g_0$. Let a be the semibasic tensor field constructed in Lemma 8.4.2 for the metric g. By Lemma 8.2.1, for every real function $u \in C^\infty(T^0M)$, the Pestov identity (8.4.13) holds. Both parts of this identity are quadratic forms in the function u. Equating the correspondent symmetric bilinear forms, we obtain the identity

$$\langle \overset{a}{\nabla} u, \overset{v}{\nabla}(H\tilde u)\rangle + \langle \overset{a}{\nabla}\tilde u, \overset{v}{\nabla}(Hu)\rangle = \langle \overset{a}{\nabla}\tilde u, \overset{a}{\nabla} u\rangle + \overset{a}{\nabla}{}^i v_i + \overset{v}{\nabla}_i w^i \tag{8.5.6}$$

which holds for every real functions $u, \tilde u \in C^\infty(T^0M)$; here

$$v_i = \frac{1}{2}\left(\xi_i \overset{a}{\nabla}{}^j u \cdot \overset{v}{\nabla}_j \tilde u + \xi_i \overset{a}{\nabla}{}^j \tilde u \cdot \overset{v}{\nabla}_j u - \xi_j \overset{v}{\nabla}_i u \cdot \overset{a}{\nabla}{}^j \tilde u - \xi_j \overset{v}{\nabla}_i \tilde u \cdot \overset{a}{\nabla}{}^j u\right), \tag{8.5.7}$$

$$w^i = \frac{1}{2}\left(\xi_j \overset{a}{\nabla}{}^i u \cdot \overset{a}{\nabla}{}^j \tilde u + \xi_j \overset{a}{\nabla}{}^i \tilde u \cdot \overset{a}{\nabla}{}^j u\right). \tag{8.5.8}$$

Let $\tau_-(x, \xi)$ be the function corresponding to the metric g. We put $u = \tau_-$ in (8.5.6)–(8.5.8), and assume the function $\tilde u$ to be positively homogeneous of degree -1 in its second argument:

$$\tilde u(x, t\xi) = t^{-1}\tilde u(x, \xi) \qquad (t > 0). \tag{8.5.9}$$

The left-hand side of (8.5.6) is equal to zero. Indeed, by (8.4.12),

$$\langle \overset{a}{\nabla}\tau_-, \overset{v}{\nabla}(H\tilde u)\rangle + \langle \overset{a}{\nabla}\tilde u, \overset{v}{\nabla}(H\tau_-)\rangle = -\frac{1}{|\xi|^2}\xi^i \overset{v}{\nabla}_i(H\tilde u) - \langle \overset{a}{\nabla}\tilde u, \overset{v}{\nabla}(1)\rangle = 0,$$

since the function $H\tilde u$ is homogeneous of degree zero. In view of (8.4.12) into account, formulas (8.5.6)–(8.5.8) take the form

$$\frac{1}{|\xi|^2}H\tilde u + \overset{v}{\nabla}_i w^i = \overset{a}{\nabla}{}^i v_i, \tag{8.5.10}$$

$$v_i = \frac{1}{2}\left(\xi_i \overset{v}{\nabla}_j \tau_- \cdot \overset{a}{\nabla}{}^j \tilde{u} - \xi_j \overset{v}{\nabla}_i \tau_- \cdot \overset{a}{\nabla}{}^j \tilde{u} + \frac{\xi_i}{|\xi|^2}\tilde{u} + \overset{v}{\nabla}_i \tilde{u}\right), \tag{8.5.11}$$

$$w^i = \frac{1}{2}\left(\overset{a}{\nabla}{}^i \tilde{u} + \frac{\xi^i}{|\xi|^2} H\tilde{u}\right). \tag{8.5.12}$$

We choose a small $\varepsilon > 0$ and denote $M_\varepsilon = \{x \in M \mid \rho(x, \partial M) \geq \varepsilon\}$, where ρ is the distance function in the metric g. Integrating equality (8.5.10) over ΩM_ε and transforming the divergent terms by the Gauss-Ostrogradskiĭ formulas, we obtain

$$\int\limits_{\Omega M_\varepsilon} (H\tilde{u} + (n-2)\langle w, \xi\rangle)\, d\Sigma = \int\limits_{\partial\Omega M_\varepsilon} \langle v, \nu\rangle \; d\Sigma^{2n-2}, \tag{8.5.13}$$

where ν is the unit vector of the outer normal to $\partial\Omega M_\varepsilon$. By (8.5.11) and (8.5.12),

$$\langle w, \xi\rangle = H\tilde{u},$$

$$\langle v, \nu\rangle = \frac{1}{2}\left[\nu^i \left(\xi_i \overset{v}{\nabla}_j \tau_- - \xi_j \overset{v}{\nabla}_i \tau_-\right) \overset{a}{\nabla}{}^j \tilde{u} + \frac{\langle \xi, \nu\rangle}{|\xi|^2}\tilde{u} + \langle \nu, \overset{v}{\nabla}\tilde{u}\rangle\right].$$

Inserting these expressions into (8.5.13), we obtain

$$\int\limits_{\Omega M_\varepsilon} H\tilde{u}\, d\Sigma = \frac{1}{2(n-1)} \int\limits_{\partial\Omega M_\varepsilon} \left[\nu^i \left(\xi_i \overset{v}{\nabla}_j \tau_- - \xi_j \overset{v}{\nabla}_i \tau_-\right) \overset{a}{\nabla}{}^j \tilde{u} + \langle \nu, \overset{v}{\nabla}\tilde{u}\rangle + \langle \xi, \nu\rangle \tilde{u}\right] d\Sigma^{2n-2}. \tag{8.5.14}$$

Formula (8.5.14) holds for every function $\tilde{u}$ satisfying the homogeneity condition (8.5.9). In particular, putting $\tilde{u} = \tau_-$ and taking (8.4.12) into account, we conclude

$$-\int\limits_{\Omega M_\varepsilon} d\Sigma = \frac{1}{2(n-1)} \int\limits_{\partial\Omega M_\varepsilon} \left[\nu^i \left(\xi_i \overset{v}{\nabla}_j \tau_- - \xi_j \overset{v}{\nabla}_i \tau_-\right) \overset{a}{\nabla}{}^j \tau_- + \langle \nu, \overset{v}{\nabla}\tau_-\rangle + \langle \xi, \nu\rangle \tau_-\right] d\Sigma^{2n-2}. \tag{8.5.15}$$

Taking the difference of (8.5.14) and (8.5.15) and introducing the notation

$$w = \tilde{u} - \tau_-, \tag{8.5.16}$$

we obtain

$$\int\limits_{\Omega M_\varepsilon} (1 + H\tilde{u})\, d\Sigma =$$

$$= \frac{1}{2(n-1)} \int\limits_{\partial\Omega M_\varepsilon} \left[\nu^i \left(\xi_i \overset{v}{\nabla}_j \tau_- - \xi_j \overset{v}{\nabla}_i \tau_-\right) \overset{a}{\nabla}{}^j w + \langle \nu, \overset{v}{\nabla} w\rangle + \langle \xi, \nu\rangle w\right] d\Sigma^{2n-2}. \tag{8.5.17}$$

We specialize now the choice of the function $\tilde{u}$ in (8.5.16). Let $\tilde{\lambda}$ be a function belonging to the class $\Lambda(\lambda_0, k_0)$ and $\tilde{g} = \tilde{\lambda}^2 g_0$ be the corresponding metric. Given $(x, \xi) \in T^0 M$, by $\gamma_{x,\xi} : [\tau_-(x,\xi), 0] \to M$ we denote the geodesic of the metric g which satisfies the initial conditions $\gamma_{x,\xi}(0) = x,\ \dot{\gamma}_{x,\xi}(0) = \xi$ and put

$$\tilde{u}(x, \xi) = -\frac{\tilde{\rho}(x, \gamma_{x,\xi}(\tau_-(x,\xi)))}{|\xi|}, \tag{8.5.18}$$

where $\tilde{\rho}$ is the distance in the metric $\tilde{g}$ and $|\xi| = (g_{ij}\xi^i\xi^j)^{1/2}$ is the modulus of the vector ξ in the metric g. Let us show that under such choice the integrand on the right-hand side of (8.5.17) can be estimated as follows:

$$1 + H\tilde{u} \geq 1 - \frac{\tilde{\lambda}}{\lambda}. \tag{8.5.19}$$

Indeed, let us fix $(x, \xi) \in T^0M$ and denote $y = \gamma_{x,\xi}(\tau_-(x, \xi))$. We join the points y and x by a geodesic $\tilde{\gamma}$ of the metric $\tilde{g}$ and denote by η the tangent vector, at the point x to this geodesic, which has the unit length in the metric $\tilde{g}$, i.e., $|\eta|^{\sim} = 1$. By the formula for the first variation for the length of a geodesic [41],

$$\left.\frac{d}{dt}\right|_{t=0} [\tilde{\rho}(\gamma_{x,\xi}(t), y)] = \langle \xi, \eta \rangle^{\sim}, \tag{8.5.20}$$

where $\langle\ ,\ \rangle^{\sim}$ is the scalar product in the metric $\tilde{g}$.

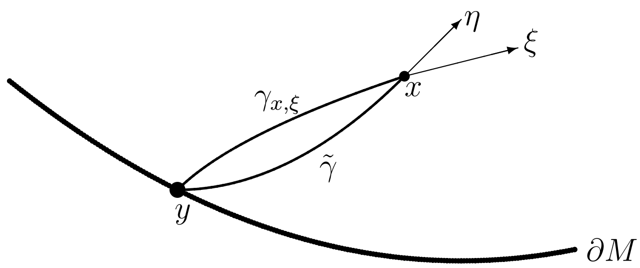

Fig. 4

Introducing the notation

$$\tilde{\mu}(x, \xi) = \tilde{\rho}(x, \gamma_{x,\xi}(\tau_-(x, \xi))), \tag{8.5.21}$$

we can write

$$H\tilde{\mu} = \xi^i \overset{h}{\nabla}_i \tilde{\mu} = \xi^i \left(\frac{\partial \tilde{\mu}}{\partial x^i} - \Gamma^k_{ij}\xi^j \frac{\partial \tilde{\mu}}{\partial \xi^k} \right). \tag{8.5.22}$$

On the other hand, differentiating the equality

$$\tilde{\rho}(\gamma_{x,\xi}(t), y) = \tilde{\mu}(\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t)) \qquad (y = \gamma_{x,\xi}(\tau_-(x, \xi)))$$

with respect to t, we obtain

$$\frac{d}{dt}[\tilde{\rho}(\gamma_{x,\xi}(t), y)] = \frac{\partial \tilde{\mu}}{\partial x^i}(\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t))\dot{\gamma}^i_{x,\xi}(t) + \frac{\partial \tilde{\mu}}{\partial \xi^i}(\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t))\ddot{\gamma}^i_{x,\xi}(t).$$

Putting $t = 0$ here and using equation (1.2.5) for geodesics, we conclude

$$\left.\frac{d}{dt}\right|_{t=0} [\tilde{\rho}(\gamma_{x,\xi}(t), y)] = \frac{\partial \tilde{\mu}}{\partial x^i}(x, \xi)\xi^i - \frac{\partial \tilde{\mu}}{\partial \xi^i}(x, \xi)\Gamma^i_{jk}(x)\xi^j\xi^k.$$

Comparing the last equality with (8.5.22), we see that

$$H\tilde{\mu}(x,\xi) = \frac{d}{dt}\Big|_{t=0}\left[\tilde{\rho}(\gamma_{x,\xi}(t),y)\right].$$

Together with (8.5.20), the last formula gives

$$H\tilde{\mu}(x,\xi) = \langle\xi,\eta\rangle^{\sim}.$$

Combining this equality with (8.5.18) and (8.5.21), we obtain

$$H\tilde{u}(x,\xi) = -\frac{1}{|\xi|}H\tilde{\mu}(x,\xi) = -\frac{\langle\xi,\eta\rangle^{\sim}}{|\xi|} \tag{8.5.23}$$

We now recall that $\tilde{g} = \dfrac{\tilde{\lambda}^2}{\lambda^2}g$ and $|\eta|^{\sim} = 1$. So (8.5.23) implies the inequality

$$|H\tilde{u}(x,\xi)| \le \frac{|\xi|^{\sim}|\eta|^{\sim}}{|\xi|} = \frac{\tilde{\lambda}}{\lambda}|\eta|^{\sim} = \frac{\tilde{\lambda}}{\lambda}.$$

Thus (8.5.19) is proved.

With the help of (8.5.19), formula (8.5.17) gives the inequality

$$\int\limits_{M_\varepsilon}\left(1-\frac{\tilde{\lambda}}{\lambda}\right)dV^n \le$$

$$\le \frac{1}{2\omega_n(n-1)}\int\limits_{\partial\Omega M_\varepsilon}\left[\nu^i\left(\xi_i\overset{v}{\nabla}_j\tau_- - \xi_j\overset{v}{\nabla}_i\tau_-\right)\overset{a}{\nabla}{}^j w + \langle\nu,\overset{v}{\nabla}w\rangle + \langle\xi,\nu\rangle w\right]d\Sigma^{2n-2}. \tag{8.5.24}$$

Here dV^n is the Riemannian volume form of the metric g and ω_n is the volume of the unit sphere in $\mathbf{R}^n$.

Let us now make use of the next claim.

Lemma 8.5.2 *For the function w given by formulas* (8.5.16) *and* (8.5.18)*, the right-hand side of* (8.5.24) *can be estimated as follows:*

$$\overline{\lim_{\varepsilon\to 0}}\int\limits_{\partial\Omega M_\varepsilon}\left[\nu^i\left(\xi_i\overset{v}{\nabla}_j\tau_- - \xi_j\overset{v}{\nabla}_i\tau_-\right)\overset{a}{\nabla}{}^j w + \langle\nu,\overset{v}{\nabla}w\rangle + \langle\xi,\nu\rangle w\right]d\Sigma^{2n-2} \le$$

$$\le C\|\Gamma_g - \Gamma_{\tilde{g}}\|_{W_1^1(\partial M\times\partial M,\,n-1)}, \tag{8.5.25}$$

where a constant C depends only on (M,g_0), λ_0 and k_0.

The proof of the lemma will be given at the end of the section, and now we finish proving the theorem with its help. From now on we agree to denote various constants depending only on (M,g_0), λ_0 and k_0 by the same letter C.

Estimating the right-hand side of (8.5.24) with the help of (8.5.25) and passing to the limit as $\varepsilon\to 0$, we obtain

$$\int\limits_{M}\left(1-\frac{\tilde{\lambda}}{\lambda}\right)dV^n \le C\|\Gamma_g - \Gamma_{\tilde{g}}\|_{W_1^1(\partial M\times\partial M,\,n-1)}. \tag{8.5.26}$$

The volume forms dV^n and dV_0^n of the metrics g and g_0 are connected by the equality $dV^n = \lambda^n dV_0^n$; so (8.5.26) can be rewritten in the form

$$\int\limits_M (\lambda - \tilde\lambda)\lambda^{n-1} dV_0^n \le C\|\Gamma_g - \Gamma_{\tilde g}\|_{W_1^1(\partial M\times\partial M,\,n-1)}.$$

The metrics g and $\tilde g$ are equal in rights, so we can write the second inequality

$$\int\limits_{M_\varepsilon} (\tilde\lambda - \lambda)\tilde\lambda^{n-1} dV_0^n \le C\|\Gamma_g - \Gamma_{\tilde g}\|_{W_1^1(\partial M\times\partial M,n-1)}.$$

Summing the last two inequalities, we obtain

$$\int\limits_M (\lambda - \tilde\lambda)(\lambda^{n-1} - \tilde\lambda^{n-1})\, dV_0^n \le C\|\Gamma_g - \Gamma_{\tilde g}\|_{W_1^1(\partial M\times\partial M,\,n-1)}. \tag{8.5.27}$$

Condition (8.5.2) implies the relation

$$(\lambda - \tilde\lambda)(\lambda^{n-1} - \tilde\lambda^{n-1}) = (\lambda - \tilde\lambda)^2(\lambda^{n-2} + \lambda^{n-3}\tilde\lambda + \ldots + \tilde\lambda^{n-2}) \ge (n-1)\lambda_0^{n-2}(\lambda - \tilde\lambda)^2.$$

Therefore (8.5.27) implies the estimate

$$\int\limits_M (\lambda - \tilde\lambda)^2\, dV_0^n \le C\|\Gamma_g - \Gamma_{\tilde g}\|_{W_1^1(\partial M\times\partial M,\,n-1)}$$

which coincides with (8.5.5). The theorem is proved.

P r o o f of Lemma 8.5.2 is based on the next two observations. First, it follows from definition (8.5.16), (8.5.18) of the function w that the equality

$$w(x,\xi) = \Gamma_g(x,y) - \Gamma_{\tilde g}(x,y) \tag{8.5.28}$$

holds for $(x,\xi) \in \partial\Omega M$, where $y = \gamma_{x,\xi}(\tau_-(x,\xi))$. Second, the integrand in (8.5.25) depends only on values of the function w on $\partial\Omega M_\varepsilon$, since the vector $\beta_j = \nu^i(\xi_i \overset{v}{\nabla}_j \tau_- - \xi_j \overset{v}{\nabla}_i \tau_-)$ is tangent to ∂M_ε. To make use of these observations, we will change the integration variables of integral (8.5.25) in two steps.

First of all, we represent integral (8.5.25) as

$$\int\limits_{\partial\Omega M_\varepsilon} \left[\nu^i\left(\xi_i \overset{v}{\nabla}_j \tau_- - \xi_j \overset{v}{\nabla}_i \tau_-\right) \overset{a}{\nabla}{}^j w + \langle \nu, \overset{v}{\nabla} w\rangle + \langle \xi, \nu\rangle w\right] d\Sigma^{2n-2} =$$

$$= \int\limits_{\partial M_\varepsilon} \left(I_1(x) + I_2(x)\right)\, dV^{n-1}(x), \tag{8.5.29}$$

where

$$I_1(x) = \int\limits_{\Omega_x M} \nu^i \left(\xi_i \overset{v}{\nabla}_j \tau_- - \xi_j \overset{v}{\nabla}_i \tau_-\right) \overset{a}{\nabla}{}^j w\, d\omega_x(\xi), \tag{8.5.30}$$

$$I_2(x) = \int\limits_{\Omega_x M} \left(\langle \nu, \overset{v}{\nabla} w\rangle + \langle \frac{\xi}{|\xi|^2}, \nu\rangle w\right) d\omega_x(\xi). \tag{8.5.31}$$

We change the integration variable in (8.5.30) and (8.5.31) by the formula

$$\eta = \frac{-\tau_-(x,\xi)}{|\xi|}\xi. \tag{8.5.32}$$

Since the function τ_- is homogeneous of degree -1, it follows from (8.5.32) that

$$-\tau_-(x,\eta) = |\xi|, \qquad -\tau_-(x,\xi) = |\eta| \tag{8.5.33}$$

and the inverse of mapping (8.5.32) coincides with (8.5.32):

$$\xi = \frac{-\tau_-(x,\eta)}{|\eta|}\eta. \tag{8.5.34}$$

Mapping (8.5.32) transfers the sphere $\Omega_x M$ into the hypersurface $\Upsilon_x M = \{\eta \in T_x M \mid -\tau_-(x,\eta) = 1\}$ of $T_x M$. Let dS_x^{n-1} be the volume form on $\Upsilon_x M$ induced by the Euclidean structure of the space $T_x M$ (which, in turn, is induced by the Riemannian metric g). The Jacobian of the mapping $\Omega_x M \to \Upsilon_x M,\ \xi \mapsto \eta$ is easily seen to be equal to

$$\frac{d\omega_x(\xi)}{dS_x^{n-1}(\eta)} = \frac{|\cos\alpha|}{|\eta|^{n-1}} \tag{8.5.35}$$

where α is the angle between the vectors $\overset{v}{\nabla}\tau_-(x,\xi)$ and ξ. Using homogeneity of τ_- and equalities (8.5.33), we obtain for $\xi \in \Omega_x M$

$$\cos\alpha = \frac{\langle \overset{v}{\nabla}\tau_-(x,\xi), \xi\rangle}{|\overset{v}{\nabla}\tau_-(x,\xi)|} = \frac{-\tau_-(x,\xi)}{|\overset{v}{\nabla}\tau_-(x,\xi)|} = \frac{|\eta|}{|\eta|^2|\overset{v}{\nabla}\tau_-(x,\eta)|} = \frac{1}{|\eta||\overset{v}{\nabla}\tau_-(x,\eta)|}.$$

Inserting this expression into (8.5.35), we find

$$\frac{d\omega_x(\xi)}{dS_x^{n-1}(\eta)} = \frac{1}{|\eta|^n|\overset{v}{\nabla}\tau_-(x,\eta)|}.$$

Since the integrands in (8.5.30) and (8.5.31) are homogeneous of degree -2, they are multiplied by $|\eta|^2$ under the change (8.5.32). Thus, after the change, integrals (8.5.30) and (8.5.31) take the form

$$I_1(x) = \int\limits_{\Upsilon_x M} |\eta|^{2-n}|\overset{v}{\nabla}\tau_-(x,\eta)|^{-1}\nu^j\left(\eta_j\overset{v}{\nabla}_i\tau_- - \eta_i\overset{v}{\nabla}_j\tau_-\right)\overset{a}{\nabla}{}^i w\,dS_x^{n-1}(\eta), \tag{8.5.36}$$

$$I_2(x) = \int\limits_{\Upsilon_x M} |\eta|^{2-n}|\overset{v}{\nabla}\tau_-(x,\eta)|^{-1}\left(\langle\nu,\overset{v}{\nabla}w\rangle + \langle\nu,\frac{\eta}{|\eta|^2}\rangle w\right)dS_x^{n-1}(\eta). \tag{8.5.37}$$

For x sufficiently close to ∂M and not belonging to ∂M, the semibasic covector field

$$\alpha_i(x,\eta) = |\eta|^{-1}|\overset{v}{\nabla}\tau_-|^{-1}\nu^j\left(\eta_j\overset{v}{\nabla}_i\tau_- - \eta_i\overset{v}{\nabla}_j\tau_-\right) \tag{8.5.38}$$

is smooth and bounded: $|\alpha| \le 2$. Consequently, formula (8.5.36) assumes the form (we return to denoting the integration variable by ξ)

$$I_1(x) = \int\limits_{\Upsilon_x M} |\xi|^{3-n}\alpha_i\overset{a}{\nabla}{}^i w\,dS_x^{n-1}(\xi). \tag{8.5.39}$$

Let us restrict the function $w(x,\eta)$ to the hypersurface $\Upsilon_x M \subset T_x M$ considered as a Riemannian manifold with the metric induced by the Euclidean structure of $T_x M$, and let $\mathrm{grad}_\eta w$ denotes the gradient of the restriction. We will estimate the integrand in (8.5.37) through $|w| + |\mathrm{grad}_\eta w|$. To this end we introduce the notation

$$\zeta = \nu - \langle \nu, \overset{v}{\nabla}\tau_- \rangle \eta. \tag{8.5.40}$$

Using homogeneity of τ_-, one can easily see that the vector ζ is orthogonal to $\overset{v}{\nabla}\tau_-$ and, consequently, is tangent to the hypersurface $\Upsilon_x M = \{\eta \mid -\tau_-(x,\eta) = 1\}$. Therefore the inequality

$$\langle \zeta, \overset{v}{\nabla} w \rangle \le |\zeta| \cdot |\mathrm{grad}_\eta w| \tag{8.5.41}$$

holds. Using (8.5.40) and (8.5.41), we estimate the integrand of (8.5.37):

$$|\eta|^{2-n} |\overset{v}{\nabla}\tau_-|^{-1} \left(\langle \nu, \overset{v}{\nabla} w \rangle + \langle \nu, \frac{\eta}{|\eta|^2} \rangle w \right) =$$

$$= |\eta|^{2-n} |\overset{v}{\nabla}\tau_-|^{-1} \left(\langle \zeta, \overset{v}{\nabla} w \rangle + \langle \nu, \overset{v}{\nabla}\tau_- \rangle w + \langle \nu, \frac{\eta}{|\eta|^2} \rangle w \right) \le$$

$$\le \ |\eta|^{2-n} |\overset{v}{\nabla}\tau_-|^{-1} \left(|\zeta| \cdot |\mathrm{grad}_\eta w| + |\overset{v}{\nabla}\tau_-| \cdot |w| + \frac{1}{|\eta|} |w| \right) =$$

$$= |\zeta| \cdot |\eta|^{2-n} |\overset{v}{\nabla}\tau_-|^{-1} |\mathrm{grad}_\eta w| + |\eta|^{2-n} \left(1 + |\eta|^{-1} |\overset{v}{\nabla}\tau_-|^{-1} \right) |w|. \tag{8.5.42}$$

By the relation $\langle \eta, \overset{v}{\nabla}\tau_- \rangle = -\tau_-$, the inequality

$$1 \le |\eta| \cdot |\overset{v}{\nabla}\tau_-(x,\eta)| \tag{8.5.43}$$

holds for $\eta \in \Upsilon_x M$. Besides, (8.5.40) implies the estimate

$$|\zeta| \le 1 + |\eta| \cdot |\overset{v}{\nabla}\tau_-(x,\eta)|.$$

which together with (8.5.43) gives

$$|\zeta| \le 2|\eta| \cdot |\overset{v}{\nabla}\tau_-|, \qquad |\eta|^{-1} |\overset{v}{\nabla}\tau_-^{-1}| \le 1. \tag{8.5.44}$$

With the help of (8.5.44), from (8.5.42) we obtain

$$|\eta|^{2-n} |\overset{v}{\nabla}\tau_-|^{-1} \left(\langle \nu, \overset{v}{\nabla} w \rangle + \langle \nu, \frac{\eta}{|\eta|^2} \rangle w \right) \le 2|\eta|^{3-n} |\mathrm{grad}_\eta w| + 2|\eta|^{2-n} |w|.$$

On using this inequality, (8.5.37) implies the estimate (we return to denoting the integration variable by ξ)

$$I_2(x) \le 2 \int\limits_{\Upsilon_x M} \left(|\xi|^{3-n} |\mathrm{grad}_\xi w| + |\xi|^{2-n} |w| \right) dS_x^{n-1}(\xi). \tag{8.5.45}$$

We now implement the second change of the integration variable $\xi \mapsto y$ in (8.5.39) and (8.5.45) by the formula

$$y = \exp_x(-\xi) = e_x(\xi) \tag{8.5.46}$$

(the second equality is the definition of the right-hand side). Since the hypersurface $\Upsilon_x M$ is defined by the equation $\tau_-(x,\xi) = -1$, it is transferred just into ∂M under mapping (8.5.46). By the same reason,

$$|\xi| = \rho(x,y) \tag{8.5.47}$$

where ρ is the distance in the metric g. Thus, after the change (8.5.46), relations (8.5.39) and (8.5.45) transform to the next:

$$I_2(x) \le 2 \int\limits_{\partial M} \left(\rho^{3-n}|(\mathrm{grad}_\xi w)\circ e_x^{-1}| + |\rho|^{2-n}|w\circ e_x^{-1}|\right) \frac{dS_x^{n-1}(\xi)}{dV^{n-1}(y)}\, dV^{n-1}(y),$$

$$I_1(x) = \int\limits_{\partial M} \rho^{3-n}\left(\alpha_i \overset{a}{\nabla}{}^i w\right)\circ e_x^{-1} \frac{dS_x^{n-1}(\xi)}{dV^{n-1}(y)}\, dV^{n-1}(y).$$

where dV^{n-1} is the Riemannian volume form corresponding to the metric g. By condition (8.5.4), the Jacobian $dS_x^{n-1}(\xi)/dV^{n-1}(y)$ is bounded by some constant depending only on λ_0 and k_0, and the estimate

$$|(\mathrm{grad}_\xi w)\circ e_x^{-1}| \le C|\mathrm{grad}_y(w\circ e_x^{-1})|$$

holds where grad_y stands for the gradient on the Riemannian manifold ∂M. Therefore the previous relations give

$$I_2(x) \le C \int\limits_{\partial M} \left(\rho^{3-n}|\mathrm{grad}_y(w\circ e_x^{-1})| + |\rho|^{2-n}|w\circ e_x^{-1}|\right) dV^{n-1}(y), \tag{8.5.48}$$

$$I_1(x) \le C \int\limits_{\partial M} \rho^{3-n}\left|\left(\alpha_i \overset{a}{\nabla}{}^i w\right)\circ e_x^{-1}\right| dV^{n-1}(y). \tag{8.5.49}$$

Equality (8.5.47) and definition (8.5.16), (8.5.18) of the function w imply the relation

$$(w\circ e_x^{-1})(y) = 1 - \frac{\tilde\rho(x,y)}{\rho(x,y)}, \tag{8.5.50}$$

where $\tilde\rho$ is the distance in the metric $\tilde g$. Consequently,

$$\mathrm{grad}_y(w\circ e_x^{-1}) = \rho^{-1}\mathrm{grad}_y(\rho-\tilde\rho) - \rho^{-2}(\rho-\tilde\rho)\mathrm{grad}_y\rho. \tag{8.5.51}$$

The gradient $\mathrm{grad}_y\rho$ of the function $\rho(x,y)$ on the manifold ∂M is the orthogonal projection of the gradient $\nabla_y\rho$ to the hyperplane $T_y(\partial M)$. Since $|\nabla_y\rho| = 1$, the inequality $|\mathrm{grad}_y\rho| \le 1$ holds. Consequently, (8.5.51) implies the estimate

$$|\mathrm{grad}_y(w\circ e_x^{-1})| \le \rho^{-1}|\mathrm{grad}_y(\rho-\tilde\rho)| + \rho^{-2}|\rho-\tilde\rho|. \tag{8.5.52}$$

With the help of (8.5.52), inequality (8.5.48) gives

$$I_2(x) \le C \int\limits_{\partial M} \left(\rho^{2-n}|\mathrm{grad}_y(\rho-\tilde\rho)| + \rho^{1-n}|\rho-\tilde\rho|\right) dV^{n-1}(y).$$

Integrating this inequality over $x \in \partial M_\varepsilon$, we obtain

$$\int\limits_{\partial M_\varepsilon} I_2(x)\, dV^{n-1}(x) \le C \int\limits_{\partial M_\varepsilon}\int\limits_{\partial M} \left(\rho^{2-n}|\mathrm{grad}_y(\rho-\tilde\rho)| + \rho^{1-n}|\rho-\tilde\rho|\right) dV^{n-1}(y)\, dV^{n-1}(x).$$

Passing to the limit here as $\varepsilon \to 0$, we conclude

$$\overline{\lim_{\varepsilon\to 0}} \int\limits_{\partial M_\varepsilon} I_2(x)\, dV^{n-1}(x) \le$$

$$\le C \int\limits_{\partial M}\int\limits_{\partial M} \left(\rho^{2-n}\left|\mathrm{grad}_y(\Gamma_g - \Gamma_{\tilde g})\right| + \rho^{1-n}\left|\Gamma_g - \Gamma_{\tilde g}\right|\right) dV^{n-1}(y)\, dV^{n-1}(x).$$

It remains to note that, by condition (8.5.2), ρ and dV^{n-1} can be replaced with ρ_0 and dV_0^{n-1} corresponding to the metric g_0 as well as grad_y can be understood in the sense of g_0. We thus obtain

$$\overline{\lim_{\varepsilon\to 0}} \int\limits_{\partial M_\varepsilon} I_2(x)\, dV^{n-1}(x) \le C\left\|\Gamma_g - \Gamma_{\tilde g}\right\|_{W_1^1(\partial M\times\partial M,\, n)}. \tag{8.5.53}$$

We now address integral (8.5.49). Let us fix a point $y_0 \in \partial M$ and construct the unit vector field $\eta(x)$ on $M \setminus \{y_0\}$ as is described in claim (2) of Lemma 8.4.2. The field is connected with mapping (8.5.46) by the equality

$$\eta(x) = \frac{e_x^{-1}(y_0)}{|e_x^{-1}(y_0)|} = \frac{e_x^{-1}(y_0)}{\rho(x, y_0)}. \tag{8.5.54}$$

By Lemma 8.4.2, the relation

$$(\overset{a}{\nabla} w)(x, \eta(x)) = \nabla[w(x, \eta(x))]$$

holds for every function $w \in C^\infty(TM)$. Assuming $w(x,\xi)$ to be homogeneous of degree -1 in its second argument, the last relation together with (8.5.54) implies that

$$(\overset{a}{\nabla} w)\circ e_x^{-1} = \rho^{-1}\nabla_x[\rho\cdot(w\circ e_x^{-1})].$$

Recalling that (8.5.50) is satisfied by the function w participating in (8.5.49), we write the last equality in the form

$$(\overset{a}{\nabla} w)\circ e_x^{-1} = \rho^{-1}\nabla_x(\rho-\tilde\rho).$$

Therefore the integrand of (8.5.49) can be represented as:

$$\rho^{3-n}\left|\left(\alpha_i \overset{a}{\nabla}{}^i w\right)\circ e_x^{-1}\right| = \rho^{2-n}\left|\left(\alpha_i\circ e_x^{-1}\right)\nabla_x^i(\rho-\tilde\rho)\right|.$$

By (8.5.38), the vector field $\alpha_i \circ e_x^{-1}$ is tangent to ∂M_ε and bounded. So the previous equality implies the estimate

$$\rho^{3-n}\left|\left(\alpha_i \overset{a}{\nabla}{}^i w\right)\circ e_x^{-1}\right| \le 2\rho^{2-n}\left|\mathrm{grad}_x(\rho-\tilde\rho)\right|,$$

where grad_x stands for the gradient on the Riemannian manifold ∂M_ε. With the help of the last inequality, integral (8.5.49) is estimated as follows:

$$I_1(x) \le C \int\limits_{\partial M} \rho^{2-n} \left|\text{grad}_x(\rho - \tilde\rho)\right| dV^{n-1}(y).$$

As above, from this we obtain the inequality

$$\overline{\lim_{\varepsilon\to 0}} \int\limits_{\partial M_\varepsilon} I_1(x)\, dV^{n-1}(x) \le C\, \|\Gamma_g - \Gamma_{\tilde g}\|_{W_1^1(\partial M\times\partial M,\, n-1)}\,. \tag{8.5.55}$$

Relations (8.5.53), (8.5.55) and (8.5.29) imply (8.5.25). The lemma is proved.

8.6 Bibliographical remarks

The crucial point of this chapter is the Pestov identity (8.4.13) that does not explicitly contain the curvature tensor in contrast to its initial version (4.4.9). The first identity of such kind was obtained by R. G. Mukhometov [86]. The reader familiar with this paper would probably agree that our proof of the identity is simpler. Using his identity, R. G. Mukhometov proved the uniqueness in the linear problem of integral geometry for scalar functions, obtained a formula for the volume of a simple Riemannian manifold and the fact that a metric is uniquely determined by its hodograph in a prescribed conformal class. In Sections 8.4–8.5 we followed some ideas of [86].

Corollary 8.1.2 is almost coincident with one of the results of [86]. The only distinction is that our definition of CDRM includes the assumption of strict convexity of the boundary, while the Mukhometov theorem assumes the boundary to be convex (without the modifier "strict").

Our formula (8.4.1) for the volume is linear in the hodograph in contrast to Mukhometov's formula which, moreover, is painfully noninvariant. In two-dimensional case this formula was known earlier [110].

It is interesting to note a difference between Theorem 8.4.1 and the corresponding Mukhometov's result. Instead of (8.5.5), he obtained the estimate

$$\|\lambda - \tilde\lambda\|_{L_2(M)} \le C\|\Gamma_g - \Gamma_{\tilde g}\|_{\mathcal{L}_2^1(\partial M\times\partial M)},$$

where

$$\|f\|^2_{\mathcal{L}_2^1(\partial M\times\partial M)} =$$

$$= \int\limits_{\partial M}\int\limits_{\partial M} \left[\rho^{2-n}(x,y)\left(|\nabla_x f|^2 + |\nabla_y f|^2\right) + \rho^{4-n}(x,y)|\nabla_x\nabla_y f)|^2\right] dV^{n-1}(x)\, dV^{n-1}(y).$$

The problem of emission tomography is thoroughly investigated in the case when the metric is Euclidean and the absorption ε is constant [95, 125, 133]. In the case of the Euclidean metric and nonconstant absorption (Corollaries 8.1.3 and 8.1.4 relate to this case), as far as the author knows, all results are obtained under some assumptions on smallness of the absorption ε or the domain M [106, 73, 52]. Of particular interest is the paper [27] by D. Finch, where Corollary 8.1.4 is proved in which the right-hand side of inequality (8.1.9) is replaced with 5.37.

Bibliography

[1] Aben, H.K. (1979). *Integrated Photoelasticity.* McGrow-Hill, New York.

[2] Aben, H.K., Josepson, J.I., Kell, K.-J.E. (1989). The case of weak birefringence in integrated photoelasticity. *Optics and Lasers in Engineering,* **11,** 145–157.

[3] Amirov, A.Kh. (1983). A class of multidimensional inverse problems. *Soviet Math. Dokl.,* **28,** no. 2, 341–343.

[4] Amirov, A.Kh. (1986). Existence and uniqueness theorems for the solution of an inverse problem for the transfer equation. *Siberian Math. J.,* **27,** no. 6, 785–800.

[5] Amirov, A.Kh. (1987). The question of the solvability of inverse problems. *Soviet Math. Dokl.,* **34,** no. 2, 258–259.

[6] Amirov, A.Kh. (1988). A class of inverse problems for a special kinetic equation. *Soviet Math. Dokl.,* **36,** no. 1, 35–37.

[7] Anikonov, Yu.E. (1978). *Some Methods for the Study of Multidimensional Inverse Problems for Differential Equations*. Nauka, Sibirsk. Otdel., Novosibirsk (in Russian).

[8] Anikonov, Yu.E. (1984). A criterion for two-dimensional Riemannian spaces of zero curvature. *Math. Notes,* **35**, no. 5-6, 441–443.

[9] Anikonov, Yu.E., Amirov, A.Kh. (1983). A uniqueness theorem for the solution of an inverse problem for the kinetic equation. *Soviet Math. Dokl.*, **28,** no. 2, 510–511.

[10] Anikonov, Yu.E., Romanov, V.G. (1976). On uniqueness of definition of first-order form by its integrals over geodesics. In: *Ill-posed Math. and Geophys. Problems.* Novosibirsk, 22–27 (in Russian).

[11] Berger, M., Ebin, D. (1969). Some decompositions of the space of symmetric tensors on a Riemannian manifold. *J. Differential Geom.,* **3**, 379–392.

[12] Bernstein, I.N., Gerver, M.L. (1980). Conditions of distinguishability of metrics by godographs. *Methods and Algoritms of Interpretation of Seismological Information.* Computerized Seismology. Vol. **13.** Nauka, Moscow, 50–73 (in Russian).

[13] Besse, A.L. (1978). *Manifolds All of Whose Geodesics are Closed.* Springer-Verlag, Berlin - Geidelberg - New Work.

[14] Blagoveshenskii, A.S. (1990). On the reconstruction of functions from the integrals along linear manifolds. *Math. Notices,* **39,** 841–849.

[15] Boman, J. (1990). Helgason's support theorem for Radon transforms – a new proof and a generalization. In: *Mathematical Methods in Tomography.* Springer-Verlag. Berlin - Geidelberg - New Work, 1–5.

[16] Boman, J., Quinto, E.T. (1987). Support theorems for real-analytic Radon transforms. *Duke Math. J.,* **55,** 943–948.

[17] Červeny, V., Jech, J. (1982). Linearized solutions of kinematic problems of seismic body waves in inhomogeneous slightly anisotropic media. *J. Geophys.,* **51**, 96–104.

[18] Červeny, V., Molotkov, I.A., Pšenčik, I. (1977). *Ray Method in Seismology.* Praha: Univ Karlova.

[19] Cormack, A.M. (1964). Representation of a function by its line integrals with some radiological applications. *J. Appl. Phys.,* **43,** 2908–2913.

[20] Cormack, A.M. (1981). The Radon transform on a family of curves in the plane, *Proceedings of the Amer. Math. Soc.,* **83,** 325–330.

[21] Cormack, A.M., Quinto, E.T. (1980). A Radon transform on spheres through the origin in $\mathbf{R}^n$ and applications to the Darboux equation. *Trans. Amer. Math. Soc.,* **260,** 575–581.

[22] Croke, C.B. (1991). Rigidity and the distance between boundary points. *J. of Differential Geometry,* **33,** no. 2, 445–464.

[23] Davison, M.E., Grünbaum, F.A. (1981). Tomographic reconstructions with arbitrary directions. *Comm. Pure Appl. Math.,* **34,** 77–120.

[24] Deans, S.R. (1983). *The Radon Transform and Some of its Applications.* John Wiley & Sons. New York etc.

[25] Eisenhart, L.P. (1927) *Riemannian Geometry.* Princeton University Publ. Princeton.

[26] Finch, D. (1985). Cone beam reconstructions with sources on a curve. *SIAM J. Appl. Math.,* **45,** 665–673.

[27] Finch, D. (1986). Uniqueness for the attenuated X-ray transform in the physical range. *Inverse problems,* **2,** 197–203.

[28] Finch, D.V., Solmon, D.C. (1980). Stability and consistency for the divergent beam X-ray transform. In: Herman, G.T., Natterer, F. *Mathematical Aspects of Computerized Tomography.* Springer-Verlag, 110–111.

[29] Finch, D.V., Solmon, D.C. (1983). A characterization of the range of the divergent beam X-ray transform. *SIAM. J. Math. Anal.,* **14(4),** 767–771.

[30] Friedrichs, K.O. (1958). Symmetric positive differential equations. *Comm. Pure and Appl. Math.,* **11,** no. 3, 333–418.

[31] Gel'fand, I.M., Gindikin, S., Graev, M.I. (1982). Integral geometry in affine and projective spaces. *J. Sov. Math.,* **18**, 39–167.

[32] Gel'fand, I.M., Goncharov, A.B. (1986). Reconstruction of finite functions from integrals over straigt lines through given points in space. *Dokl. Acad. Nauk SSSR,* **290,** 1037–1040 (in Russian).

[33] Gel'fand, I.M., Graev, M.I., Shapiro, Z.Ya. (1966). Integral geometry on manifolds of k-dimensional planes. *Soviet Math. Dokl.,* **7,** 801–804.

[34] Gel'fand, I.M., Graev, M.I., Shapiro, Z.Ya. (1967). Integral geometry on k-dimensional planes. *Funkcional. Anal. i Priložen.,* **1,** 15–31.

[35] Gel'fand, I.M., Graev, M.I., Vilenkin, N.Ya. (1966). *Generalised Functions.* Vol.5: *Integral Geometry and Representation Theory.* Academic Press, New Work-London.

[36] Gel'fand, I.M., Shilov, G.E. (1964). *Generalized Functions.* Vol.1: *Properties and Operations.* Acad. Press, New Work - London.

[37] Gerver, M.L., Nadirashvili, N.S. (1983). Nonscattering media. *Soviet Phys. Dokl.,* **28,** no. 6, 460–461.

[38] Gerver, M.L., Nadirashvili, N.S. (1984). A condition for isometry of Riemannian metrics in the disk. *Soviet Math. Dokl.,* **29,** no. 2, 199–203.

[39] Godbillon, C. (1969). *Géométrie Différentielle et Mécanique Analitique Collection Methodes.* Hermann, Paris.

[40] Greenleaf, A., Uhlmann, G. (1989). Nonlocal inversion formulas for the X-ray transform. *Duke Math. J.,* **58,** 205–240.

[41] Gromoll, D., Klinqenberg, W., Meyer, W. (1968). *Riemannsche Geometrie im Grossen.* Springer-Verlag, Berlin - Geidelberg - New York.

[42] Gromov, M. (1983). Filling Riemannian manifolds. *J. of Differential Geometry,* **18,** no. 1, 1–148.

[43] Guillemin, V. (1985). On some results of Gelfand in integral geometry. *Proc. Symp. Pure Math.,* **43,** 149–155.

[44] Guillemin, V., Sternberg, S. (1977). *Geometric Asymptotics.* Amer. Math. Soc., Providence, Rhode Island.

[45] Hamaker, C., Smith, K.T., Solmon, D.C., Wagner, S.L. (1980). The divergent beam X-ray transform. *The Rocky Mountain J. of Math.,* **10,** no. 1, 253–283.

[46] Hartman, Ph. (1964). *Ordinary Differential Equations.* John Wiley & Sons, New York etc.

[47] Helgason, S. (1965). The Radon transform on Eucledian spaces, compact two-point homogeneous spaces and Grassman manifolds. *Acta Math.,* **113,** 153–158.

[48] Helgason, S. (1980). *The Radon Transform.* Birkhäuser, Basel, Stuttgart.

[49] Helgason, S. (1980). Support of Radon transforms. *Adv. Math.,* **38,** 91–100.

[50] Herman, G.T. (1980). *Image Reconstruction from Projections. The Fundamentals of Computerized Tomography.* Academic Press, New York.

[51] Herman, G.T., Tuy, H., Landenberg, K., Sabatier, P. (1987). *Basic Methods of Tomography and Inverse Problems.* Adam Hilger, Bristol, England.

[52] Hertle, A. (1984). On the injectivity of the attenuated Radon transform. *Proc. AMS,* **92,** 201–206.

[53] Hörmander, L. (1983). *The Analysis of Linear Partial Differential Operators.* 1. *Distribution Theory and Fourier Analysis.* Springer-Verlag, Berlin - Geidelberg - New Work - Tokyo.

[54] Husmoller, D. (1966). *Fibre Bundles.* McGraw-Hill Book Company. New York etc.

[55] Jano, K., Bochner, S. (1953). *Curvative and Betti Numbers.* Princeton University Press, Princeton.

[56] John, F. (1938). The ultrahyperbolic differential equation with four independent variables. *Duke Math. J.,* **4,** 300–322.

[57] John, F. (1955). *Plane Waves and Spherical means Applied to Partial Differential Equations.* Interscience Publishers, New Work - London .

[58] Kireĭtov, V.R. (1980). Photometric quantities on a Riemannian manifold. *Soviet Math. Dokl.,* **21,** no. 3, 665–670.

[59] Kireĭtov, V.R. (1983). *Inverse Problems of the Photometry.* Acad. Nauk. SSSR, Sibirsk. Otdel., Vychisl. Tsentr. Novosibirsk (in Russian).

[60] Kirillov, A.A. (1961). On a problem of I.M. Gelfand. *Soviet Math. Dokl.,* **2,** 268–269.

[61] Kobayashi, S., Nomizu, K. (1963). *Foundations of Differential Geometry.* V.1. Interscience Publishers, New Work - London.

[62] Kravtsov, Yu.A. (1968). "Quasi-isotropic" approximation of geometric optics. *Dokl. Acad. Nauk. SSSR,* **183,** no. 1, 74–76 (in Russian).

[63] Kravtsov, Yu.A., Orlov, Yu.I. (1980). *Geometric Optics of Inhomogeneous Media.* Nauka, Moscow (in Russian).

[64] Lax, P.D., Philips, R.S. (1970). The Payley-Wiener theorem for the Radon transform. *Comm. Pure Appl. Math.,* **23,** 409–424.

[65] Leray, J., Scheibling, G. (1961). Composition des biréfringences; cas de milieux de faible anisotropic optique. *J. Chim. Phys.(France),* **58**, no. 9, 797–802.

[66] Louis, A.K. (1981). *Analitische Methoden in der Computer Tomographie.* Habilitationsschrift, Fachbereich Mathematik der Universität Münster.

[67] Louis, A.K. (1984). Orthogonal function series expansions and the nullspace of the Radon transform. *SIAM J. Math. Anal.,* **15,** 621–633.

[68] Louis, A.K. (1986). Incomplete data problems in X-ray computerized tomography. In: Singular value decomposition of the limited angle transform. *Numer. Math.,* **48,** 251–262.

[69] Ludwig, D. (1966). The Radon transform on Eucledian space. *Comm. Pure App. Math.,* **19,** 49–81.

[70] L'vov, I.V., Sharafutdinov, V.A. (1989). The tangential component of a tensor field. *Siberian Math. J.,* **30**, no. 5, 759–770.

[71] Maass, P. (1990). Singular value decompositions for Radon transforms. In: *Mathematical Methods in Tomography.* Springer-Verlag, Berlin - Geidelberg - New York, 6–14.

[72] Madych, W.R. (1990). Summability and approximate reconstruction from Radon transform data. In: *Integral Geometry and Tomography.* Amer. Math. Soc., Providence.

[73] Markoe, A., Quinto, E.T. (1985). An elementary proof of local invertibility for generalized and attenuated Radon transform. *SIAM J. Math. Anal.,* **16,** 1114–1119.

[74] Marr, R.B. (1974). On the reconstruction of a function on a circular domain from a sampling of its line integrals. *J. Math. Anal. Appl.,* **45,** 357–374.

[75] Michel, R. (1973). Problèmes d'analyse géométrique liés à la conjecture de Blasche. *Bull. Soc. Math. France,* **101,** 17–69.

[76] Michel, R. (1974). Les tenseurs symétriques dont l'énergie est nulle sur toutes les géodésiques des espaces symétriques de rang 1. *J. Math. Pures Appl.,* **53**, 271–278.

[77] Michel, R. (1977). Un problème d'exactitude concernant les tenseurs symétriques et les géodésiques. *C.R. Acad. Sci. Sér. I Math.,* **284,** 183–186.

[78] Michel, R. (1977). Tenseurs symétrques et géodésiques. *C.R. Acad. Sci. Sér. I Math.,* **284,** 1065–1068.

[79] Michel, R. (1978). Sur quelques problèmes de géométrie globale des géodésiques. *Bol. Soc. Brasil. Mat.,* **9,** 19–38.

[80] Michel, R. (1981). Sur la rigidité imposee par la longueur des géodesiques. *Invent. Math.,* **65,** no. 1, 71–84.

[81] Milnor, J. (1963). *Morse Theory.* Annals of Mathematical Studies, 51, Princeton University.

[82] Mizohata, S. (1977). *Theory of Equations with Partial Derivatives.* Mir Publ., Moscow (in Russian, translated from Japanese).

[83] Mukhometov, R.G. (1975). On the problem of integral geometry. In: *Math. Problems of Geophysics.* Akad. Nauk. SSSR, Sibirsk. Otdel., Vychisl. Tsentr, Novosibirsk, **6**, no 2, 212–242 (in Russian).

[84] Mukhometov, R.G. (1975). Inverse kinematic problem of seismic on the plane. *Math. Problems of Geophysics.* Akad. Nauk. SSSR, Sibirsk.Otdel., Vychisl. Tsentr, Novosibirsk, **6**, no 2, 243–252 (in Russian).

[85] Mukhometov, R.G. (1977). The reconstruction problem of a two-dimensional Riemannian metric, and integral geometry. *Soviet Math. Dokl.,* **18,** no. 1, 27–37.

[86] Mukhometov, R.G. (1982). On a problem of reconstructing Riemannian metrics. *Siberian Math. J.,* **22,** no. 3, 420–433.

[87] Mukhometov, R.G. (1988). Problems of integral geometry in a domain with a reflecting part of the boundary, *Soviet Math. Dokl.,* **36,** no. 2, 260–264.

[88] Mukhometov, R.G. (1989). Stability estimate of solution to a problem of computer tomography. In: *Correctness of Problems in Analysis.* Novosibirsk, Math. Inst. (in Russian).

[89] Mukhometov, R.G. (1990). A problem of integral geometry for a family of rays with multiple reflections. In: *Mathematical Methods in Tomography.* Springer-Verlag, Berlin - Geidelberg - New York, 46–51.

[90] Mukhometov, R.G., Romanov, V.G. (1979). On the problem of finding an isotropic Riemannian metric in n-dimensional space. *Soviet Math. Dokl.,* **19,** no. 6, 1330–1333.

[91] Muskhelishvili, N.I. (1975). *Some Basic Problems of the Mathematical Theory of Elasticity.* Leyden: Noordhoff Intern. Publ.

[92] Natterer, F. (1979). On the inversion of the attenuated Radon transform. *Numer. Math.,* **32,** 271–278.

[93] Natterer, F. (1981). The identificational problem in emission tomography. In: *Mathematical Aspects of Computerized Tomography Proceedings.* Springer-Verlag, Berlin - Geidelberg - New York, 45–46.

[94] Natterer, F. (1985). Fourier reconstruction in tomography. *Numer. Math.,* **47,** 343–353.

[95] Natterer, F. (1986). *The Mathematics of Computerized Tomography.* B.G. Teubner, Stuttgart, and John Wiley & Sons Ltd.

[96] Orlov, S.S. (1975). Theory of three-dimensional reconstruction. The recovery operator. *Kristallographiya,* **20,** 701–709.

[97] Palais, R.S. (1965). *Seminar on the Atiyah-Singer Index Theorem.* Princeton, New Jersey, Princeton University Press.

[98] Palamodov, V.P. (1990). Inversion formulas for the three-dimensional ray transform. In: *Mathematical Methods in Tomography.* Springer-Verlag, Berlin - Geidelberg - New York, 53–62.

[99] Palamodov, V.P. (1990). Some singular problems in tomography. In: *Mathematical Problems of Tomography.* Translation of Mathematical Monographs, **81,** 123–140.

[100] Pantjukhina, E.Yu. (1990). Description of the image of X-ray transform in two-dimensional case. In: *Methods of Solution of Inverse Problems.* Novosibirsk, 80–89 (in Russian).

[101] Perry, R.M. (1977). On reconstructing a function on the exterior of a disk from its Radon transform. *J. Math. Anal. Appl.,* **59,** 324–341.

[102] Pestov, L.N., Sharafutdinov, V.A. (1988). Integral geometry of tensor fields on a manifold of negative curvative. *Siberian Math. J.,* **29,** no. 3, 427–441.

[103] Prudnikov, A.P., Brychkov, Y.A., Marichev, O.I. (1986). *Integrals and Series.* Vol.1. *Elementary Functions.* New York a.o.: London and Breach Publ.

[104] Quinto, E.T. (1978). *On the locality and invertibility of Radon transforms.* Thesis, M.I.T., Cambridge.

[105] Quinto, E.T. (1980). The dependence of the generalized Radon transforms on the defining measures. *Trans. Amer. Math. Soc.,* **257,** 331–346.

[106] Quinto, E.T. (1983). The invertibility of rotation invariant Radon transforms. *J. Math. Anal. Appl.,* **91,** 510–522.

[107] Quinto, E.T. (1985). Singular value decompositions and inversion methods for the exterior Radon transform. *J. Math. Anal. Appl.,* **95,** 437–448.

[108] Romanov, V.G. (1979). Integral geometry on geodesics of an isotropic Riemannian metric. *Soviet Math. Dokl.,* **19,** no. 4, 847–851.

[109] Romanov, V.G. (1987). *Inverse Problems of Mathematical Physics.* NVU: Science Press, Utrecht, the Netherlands.

[110] Santaló, L.A. (1953). *Introduction to Integral Geometry.* Paris, Hermann & Co Editeurs, Rue de la Sorbonne.

[111] Schwartz, L. (1956). *Variedades Analiticas Complejas. Ecuaciones Differenciales Parciales Elipticas.* Bogota, D.E., Colombia.

[112] Schwartz, L. (1964). *Mathematical Methods for Physical Sciences.* Mir Publ., Moscow (in Russian).

[113] Sharafutdinov, V.A. (1983). A problem of integral geometry for tensor fields and Saint-Venan equation. *Siberian Math. J.,* **24,** no. 6, 968–977.

[114] Sharafutdinov, V.A. (1984). *Symmetric tensor fields on a Riemannian manifold.* Preprint, 84-539, Akad. Nauk. SSSR, Sibirsk. Otdel, Vychisl. Tsentr, Novosibirsk (in Russian).

[115] Sharafutdinov, V.A. (1986). A problem of integral geometry for generalized tensor fields on $\mathbf{R}^n$. *Soviet Math. Dokl.,* **33,** no. 1, 100–102.

[116] Sharafutdinov, V.A. (1989). Integral geometry of a tensor field along geodesics of a metric that is close to the Euclidean metric, *Soviet Math. Dokl.,* **39,** no. 1, 221–224.

[117] Sharafutdinov, V.A. (1989). The method of integral photo-elasticity in the case of weak optical anisotropy. *Eesti NSV Tead. Akad. Toimetised Füüs.-Mat.,***38,** no. 4, 379–389, 468 (Russian, English and Estonian summaries).

[118] Sharafutdinov, V.A. (1990). On the method of integrated photo-elasticity in the case of weak optic anisotropy. *Soviet Phys. Dokl.,* **35,** no. 3, 243–244.

[119] Sharafutdinov, V.A. (1991). The ray transform of symmetric tensor fields. *Siberian Advances in Math.,* **1,** no. 1, 160–184.

[120] Sharafutdinov, V.A. (1991). Tomographic problem of photoelasticity. *Proceedings of SPIE. Analytical Method for Optical Tomography.* Ed. G. Levin. Zvenigorod, Russia, 234–243.

[121] Sharafutdinov, V.A. (1992). Integral geometry of a tensor field on a manifold whose curvative is bounded above. *Siberian Math. J.,* **33,** no 3, 524–533.

[122] Sharafutdinov, V.A. (1992). On the emission tomography problem for inhomogeneous media. *Russian Acad. Sci. Dokl. Math.,* **336,** no 3, 446–448 (in Russian).

[123] Sharafutdinov, V.A. (1993). Quasi-isotropic approximation in dynamic elasticity and some problems of geotomography. *Dokl. of Russian Acad.,* **329,** no 6, 723–725 (in Russian).

[124] Sharafutdinov, V.A. (1993). The problem of polarization tomography, *J. Inverse Ill-posed Problems,* **1**, no. 1, 53–72.

[125] Sharafutdinov, V.A. (1993). Uniqueness theorems for the exponential X-ray transform. *J. Inverse Ill-Posed Problems*, **1**, no. 4, 355–372.

[126] Sharafutdinov, V.A. (1994). Quasi-isotropic approximation in dynamic elasticity and some problems of geotomography. *Geology and Geophysics,* **35**, no. 5, 68–82 (in Russian).

[127] Sharafutdinov, V.A. (1994). Mathematical aspects of polarization tomography. In: *Applications of the Radon Transform.* Amer. Math. Soc., Princeton, 120–140.

[128] Smirnov, V.I. (1964). *A Course of Higher Mathematics.* Vol.2: *Advanced calculus.* Oxford a.o.: Addison-Wesley Publ.

[129] Smith, K.T. (1984). Inversion of the X-ray transform. *SIAM-AMS Proceedings,* **14,** 41–52.

[130] Smith, K.T., Keinert, F. (1985). Mathematical foundations of computed tomography. *Applied Optics,* **24,** 3950–3957.

[131] Sobolev, S.L. (1974). *Introduction to the Theory of Cubature Formulas.* Nauka, Moscow (in Russian).

[132] Solmon, D.C. (1976). The X-ray transform. *Journ. Math. Anal. Appl.,* **56,** 61–83.

[133] Tretiak O., Metz, C. (1980). The exponential Radon transform. *SIAM J. Appl. Math.,* **39,** 341–354.

[134] Tuy, H.K. (1983). An inversion formula for cone-beam reconstruction. *SIAM J. Appl. Math.,* **43,** 546–552.

[135] Uhlenbeck, G.K., Ford, G.V. (1983). *Lectures in Statistical Mechanics.* Amer. Math. Soc., Providence.

[136] Vertgeim, L.B. (1991). Integral geometry of symmetric tensor fields in complex space. In: *Methods of Solution of the Ill-posed Problems.* Novosibirsk, Math. Inst. (in Russian).

[137] Vertgeim, L.B. (1992). Integral geometry with matrix weight and one nonlinear problem of matrix reconstruction. *Soviet Math. Dokl.,* **44,** no. 1, 132–135.

[138] Vladimirov, V.S. (1979). *Generalized functions in mathematical physics.* Mir Publ., Moscow.

[139] Volevič, L.R. (1965). Solvebility of boundary value problems for general elliptic systems. *Mat. Sb.(N.S.),* **68(110)**, 373–416.

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