Integral High-Order Sliding Modes[1]

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Abstract - Integral sliding mode approach is extended to arbitrary-order sliding modes. As a result any high-order sliding-mode transient dynamics can be chosen, the transient time can be prescribed in advance as a function of initial conditions. The resulting controller is robust and capable to control the output of any smooth uncertain SISO system of a known permanent relative degree. The control smoothness can be deliberately increased without loss of convergence.

Index Terms - high-order sliding mode, robustness, output feedback control, finite-time stability

I. Introduction

Sliding modes are used to control uncertain systems keeping a properly chosen constraint by means of high-frequency control switching. Let the constraint be given by the equation $\sigma = 0$, where s is the output of an uncertain single-input-single-output (SISO) dynamic system. Then the standard sliding-mode control of the form $u = - k \operatorname{sign} \sigma$ solves the problem, if the relative degree is 1, i.e. if $\dot\sigma$ explicitly depends on the control u and $k \frac{\partial}{\partial u}\dot\sigma > 0$. Featuring robustness and high accuracy [6, 28], the standard sliding mode usage is however restricted due to the chattering effect caused by the control switching [9, 10], and the necessity of the relative degree to be 1.

A number of approaches [27, 11] were developed to withstand the dangerous chattering effect. High-order sliding mode (HOSM) approach suggests to treat the input u of the system as a new state variable, using its time derivative $\dot u$ as a new control [2, 16, 5], thus increasing the relative degree and hiding the switching in the control derivative. HOSM [16, 19, 20] is applicable with arbitrary relative degree r, i.e. when u explicitly appears for the first time in $\sigma^{(r)}$ and $\frac{\partial}{\partial u}\sigma^{(r)}$ is separated

[1] The main results of this paper were presented at the Scientific Workshop VSS'2004 [16].

from zero. The corresponding finite-time-convergent controllers (r-sliding controllers) [17, 19, 20] produce control being a bounded discontinuous function of σ and its real-time-calculated successive derivatives $\dot{\sigma}$, $\ddot{\sigma}$, ..., $\sigma^{(r-1)}$. The control is predefined for any r, and provides in finite time for keeping the equalities $\sigma = \dot{\sigma} = ... = \sigma^{(r-1)} = 0$ (the r-sliding mode [16-20]). The lacking derivatives can be produced by the recently proposed robust exact finite-time-convergent differentiators [13, 17, 19, 24, 25] generating output-feedback controllers [17, 19, 20, 21]. The approach provides also for higher accuracy with discrete sampling [15, 19].

Some realization problems of higher-order sliding modes are caused by the complicated structure of the transient process, which is difficult to monitor with $r > 2$ [8, 15, 19, 21, 22]. Another specific problem concerns the above smoothing-control procedure, which supposes the control derivative to be treated as a new control. Due to the control interaction any applied (r + 1)-sliding controller is effective only in some vicinity of the (r + 1)-sliding mode $\sigma = \dot{\sigma} = ... = \sigma^{(r)} = 0$. The global convergence is so far assured only for the transfer from $r = 1$ to $r = 2$ by a suitable controller modification [16].

The above issues can be resolved eliminating the transient process. The corresponding technique is known for the (standard) 1-sliding mode, is called integral sliding mode and has a lot of successful applications [29, 30]. The idea is to construct such a smooth function $\Sigma(t, x)$ of the relative degree 1, which equals zero at the initial point, and provides for the solution of the original problem, if $\Sigma(t, x) \equiv 0$ is kept. Contrary to that approach its r-sliding generalization proposed in the paper, introduces the function Σ of the relative degree r and the equality $\Sigma(t, x) \equiv 0$ is kept in the r-sliding mode. As a result, any transient dynamics can be prescribed to HOSM, the transient time can be chosen as a function of initial conditions. The semi-global convergence is assured when the control smoothness is risen artificially increasing the relative degree.

II. The problem statement and the integral *r*-sliding mode conception

Consider a dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)u, \quad \sigma = \sigma(t, x), \tag{1}$$

where $x \in \mathbf{R}^n$, a, b and σ: $\mathbf{R}^{n+1} \to \mathbf{R}$ are unknown smooth functions, $u \in \mathbf{R}$, n can be also uncertain. The task is to establish and keep the equality $\sigma = 0$. All differential equations are understood in the Filippov sense [7], which means that discontinuous dynamics is acceptable.

The relative degree r of the system is assumed to be constant and known, which means [12] that the control appears first time explicitly in the rth total time derivative of σ and

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \tag{2}$$

where $h(t,x) = \sigma^{(r)}|_{u=0}$, $g(t,x) = \frac{\partial}{\partial u}\sigma^{(r)}$ are some unknown functions. It is supposed that

$$0 < K_{\mathrm{m}} \le \frac{\partial}{\partial u}\sigma^{(r)} \le K_{\mathrm{M}}, \ \ |\, \sigma^{(r)}|_{u=0}\,| \le C \tag{3}$$

for some K_{m}, K_{M}, $C > 0$. Trajectories of (2) are assumed infinitely extendible in time for any Lebesgue-measurable bounded control $u(t, x)$. It is not required, but often needed in practice that the system be weakly minimum phase.

The above problem statement is standard and is solved by known r-sliding controllers [19 - 22]

$$u = \alpha\Psi_r(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}), \tag{5}$$

which actually solve the problem for the differential inclusion $\sigma^{(r)} \in [-C, C] + [K_{\mathrm{m}}, K_{\mathrm{M}}]u$ instead of (2), (3). Here Ψ_r is a bounded discontinuous function. Only the control gain $\alpha > 0$ needs to be adjusted for the concrete values of C, K_{m}, K_{M}, providing for the finite-time convergence of the inclusion trajectories to zero.

Suppose that it is needed to avoid the uncertainty of the transient process, and/or some transient time restrictions or maximal admissible values of the coordinates during the transient are required. Let these requirements be fulfilled by a transient trajectory $\sigma(t, x(t)) = \varphi(t)$, which means that

$$\varphi(t_0) = \sigma(t_0), \quad \dot{\varphi}(t_0) = \dot{\sigma}(t_0), \quad ..., \quad \varphi^{(r-1)}(t_0) = \sigma^{(r-1)}(t_0), \qquad \varphi(t) = 0 \quad \text{with } t \ge t_f, \tag{6}$$

where t_0 and $t_f > t_0$ are respectively the initial and the final times. Alternatively, some dynamics

$$\sigma^{(i)} = z^{(i)}, \ \ z^{(r)} = V(z, \dot{z}, ..., z^{(r-1)}), \quad z^{(i)}(t_0) = \sigma^{(i)}(t_0), \quad i = 0, ..., r - 1, \tag{7}$$

can be required. Here and further, for the sake of brevity, $\sigma(t)$ is written instead of $\sigma(t, x(t))$ whenever the ambiguity is avoided.

Standard integral sliding mode. A stable polynomial $\mu^{r-1} + d_1 \mu^{r-2} + ... + d_{r-1}$ is chosen, and the auxiliary function $\Sigma = \sigma^{(r-1)} - \varphi^{(r-1)} + d_1 (\sigma^{(r-2)} - \varphi^{(r-2)}) + ... + d_{r-1} (\sigma - \varphi)$ is introduced. Alternatively, $\Sigma = \sigma^{(r-1)} - z^{(r-1)} + d_1 (\sigma^{(r-2)} - z^{(r-2)}) + ... + d_{r-1} (\sigma - z)$, if (7) is to be maintained. The problem is solved by means of the 1-sliding-mode control $u = -k(t, x)$ sign Σ. Theoretically, the equality $\Sigma = 0$ is kept from the very beginning and forever. In practice, nevertheless, any initial measurement error will lead to asymptotic convergence only. Moreover, the control inevitably depends on all derivatives of σ and is, therefore, not bounded. The final accuracy is proportional to the sampling time interval [].

Integral r-sliding mode. Let $\varphi^{(r-1)}(t)$ be a Lipschitz function, then almost everywhere it has a globally bounded derivative $\varphi^{(r)}(t)$, and the new output $\Sigma(t, x) = \sigma(t, x) - \varphi(t)$ satisfies conditions (2), (3) with some changed constants K_{m}, K_{M}, $C > 0$. Alternatively, $\Sigma = \sigma - z$ is taken, if (7) is to be maintained with any globally bounded function V. Thus, $\Sigma \equiv 0$ can be kept by the controller $u = \alpha\Psi_r(\Sigma, \dot{\Sigma}, ..., \Sigma^{(r-1)})$, and the considered problem is solved by means of a globally bounded control. Due to the finite-time convergence of the controller, any small initial measurement error is practically immediately compensated. The accuracy $|\sigma^{(i)} - \varphi^{(i)}| \leq \gamma_i \tau^{r-i}$ is provided, τ being the sampling time interval, $i = 0, ..., r - 1$ [].

The lacking derivatives of σ can be calculated on-line by means of the robust exact differentiators [13, 19, 24, 25] with finite-time convergence. Since they require the boundedness of $\sigma^{(r)}$, their convergence is global in the case of the bounded r-sliding control, and the inequalities of the form $|\Sigma^{(i)}| \leq \mu_i \varepsilon^{(r - i)/r}$ are assured with the sampling noise of the magnitude ε [21]. Only local convergence takes place in the case of the above standard integral-sliding-mode approach.

III. Applications of high-order integral sliding modes

Transient time assignment for *r*-sliding mode. Introduce few notions to be used further. Denote

$$\vec{\sigma} = (\sigma, \dot{\sigma}, ..., \sigma^{(r-1)})^t, \quad d_\kappa\vec{\sigma} = (\kappa^r\sigma, \kappa^{r-1}\dot{\sigma}, ..., \kappa\sigma^{(r-1)})^t.$$

The linear transformation d_κ: $\mathbf{R}^r \to \mathbf{R}^r$ is called *the homogeneity dilation* [1]. A function $f(\vec{\sigma})$ is called *r-sliding homogeneous with the homogeneity degree* (*weight*) *m* if the identity $f(d_\kappa\vec{\sigma}) = \kappa^m f(\vec{\sigma})$ holds for any $\vec{\sigma}$ and any positive κ. Controller (5) is called *r-sliding homogeneous* [20] if the identity $\Psi_r(\vec{\sigma}) = \Psi_r(d_\kappa\vec{\sigma})$ holds for any $\vec{\sigma}$ and any positive κ. In other words, $\Psi_r(\vec{\sigma})$ is a homogeneous function of the weight 0.

Let the (r-1)-smooth function $\varphi(t)$ satisfying (6) be represented in the form

$$\varphi = (t - t_f)^r f(t), \qquad f(t) = c_0 + c_1(t - t_0) + ...+ c_{r-1}(t - t_0)^{r-1}. \tag{8}$$

Parameters c_i are now to be found from the conditions (6), (7) after t_f is assigned. As a result, the function φ turns out to be a function of t and initial conditions $\vec{\sigma}(t_0)$. Obviously, any constant value of the transient time $t_f - t_0$ requires unacceptably large control values in order to steer the trajectory to the r-sliding mode from far distanced initial values and leads to very low convergence rate if $\vec{\sigma}(t_0)$ is close to zero. Thus, let $t_f - t_0$ be a continuous positive-definite r-sliding homogeneous function of the initial conditions $\vec{\sigma}(t_0)$ of the degree 1, i.e.

$$t_f - t_0 = T(\vec{\sigma}(t_0)), \qquad \forall\kappa > 0 \;\; T(d_\kappa\vec{\sigma}) \equiv \kappa\, T(\vec{\sigma}). \tag{9}$$

In particular, the choice $t_f - t_0 = T(\vec{\sigma}(t_0)) = \lambda\, (|\sigma(t_0)|^{p/r} + |\dot{\sigma}(t_0)|^{p/(r-1)} + ... + |\sigma^{(r-1)}(t_0)|^{p})^{1/p}$ is valid, where p is the least common multiple of 1, 2, ..., r, and $\lambda > 0$.

Theorem 1. *Conditions* (6), (8), (9) *define a unique function* $\varphi(t, \vec{\sigma}(t_0))$. *With sufficiently large* α *the controller*

$$u = \alpha\Psi_r(\Sigma, \dot{\Sigma}, ..., \Sigma^{(r-1)}), \quad \Sigma(t, x) = \begin{cases} \sigma(t, x) - \varphi(t, \vec{\sigma}(t_0)), & t_0 \le t \le t_f \\ \sigma(t, x), & t \ge t_f \end{cases} \tag{10}$$

establishes the globally finite-time-stable r-sliding mode $\sigma \equiv 0$ *with the transient time* (9). *The equality* $\sigma(t, x(t)) = \varphi(t, \vec{\sigma}(t_0))$ *is kept during the transient process.*

Proof. The Theorem actually means that the parameter α can be taken independently of the initial conditions, which follows from the global boundedness of the function $\frac{\partial^r}{\partial t^r}\varphi(t,\vec{\sigma})$. The proof is based on a few Lemmas.

Lemma 1. *The fractions* $\sigma^{(i)}/T^{r-i}(\vec{\sigma})$, $i = 0, ..., r$, *are uniformly bounded for all* $\vec{\sigma} \in \mathbf{R}^r$.

Proof. Consider the values of the function $\xi(\vec{\sigma}) = \sigma^{(i)}/T^{r-i}$ on the homogeneous sphere $\Omega = \{\vec{\sigma}\,|T(\vec{\sigma}) = 1\}$. Obviously Ω is a closed set. Its boundedness follows from the continuity and homogeneity of $T(\vec{\sigma})$, and from the equality $T(0) = 0$. Thus Ω is compact, which implies the boundedness of the continuous function $\xi(\vec{\sigma})$ on Ω. The function $\xi(\vec{\sigma})$ has the homogeneity degree 0, which means that it is invariant with respect to the dilation (13), i.e. $\xi(d_\kappa\vec{\sigma}) \equiv \xi(\vec{\sigma})$ is kept for any $\kappa > 0$. Since there is always $\mu > 0$ such that $T(d_\mu\vec{\sigma}) = 1$, the function $\xi(\vec{\sigma})$ is globally bounded.

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Differentiating (8) obtain

$$\varphi^{(i)} = \sum_{j=0}^{i} \binom{i}{j} \frac{d^{i-j}}{dt^{i-j}} (t - t_f)^r f^{(j)}, \qquad i = 0, ..., r .$$

Let $T(\vec{\sigma}(t_0))$ be the right-hand side of (13), and $\vec{\sigma}(t_0) \neq 0$. Denote $F_i = f^{(i)}(t_0) = i!c_i$. Taking $t = t_0$ obtain from (6)

$$\sigma^{(i)}(t_0) = \sum_{j=0}^{i} \binom{i}{j} \frac{r!}{(r-i+j)!} (-T)^{r-i+j} F_j, \qquad i = 0, ..., r. \qquad (11)$$

Lemma 2. *Equations* (11) *can be solved for* $F_i = F_i(\vec{\sigma}(t_0))$. *The functions* $F_i(\vec{\sigma})$ *are r-sliding homogeneous of the degree - i, and the expressions* $|F_i(\vec{\sigma})T^i(\vec{\sigma})|$ *are uniformly bounded.*

Proof. Solving the equations for F_i, obtain that F_i can be found recursively as

$$F_0 = (-T)^{-r}\sigma(t_0),$$

$$F_1 = (-T)^{-r}\,[\dot{\sigma}(t_0) - r\,(-T)^{r-1} F_0], ...,$$

$$F_i = (-T)^{-r}\left[\sigma^{(i)}(t_0) - \sum_{j=0}^{i-1}\binom{i}{j}\frac{r!}{(r-i+j)!}(-T)^{r-i+j}F_j\right],\ i = 0, ..., r - 1,$$

$$F_r = 0.$$

Thus, equations (11) are satisfied by some functions $F_i(\vec{\sigma})$ continuos everywhere except the origin $\vec{\sigma} = 0$. Applying the homogeneity dilation and taking into account that $\sigma^{(i)}$ and T are homogeneous functions of the weight $r - i$ and 1 respectively, obtain that $F_j(\vec{\sigma})$ are also homogeneous of the weight $-j$. Due to the continuity of $F_i(\vec{\sigma})$ on the compact set $T = 1$ obtain the global boundedness of the homogeneous function $F_i(\vec{\sigma})T^i(\vec{\sigma})$ of the weight 0 (see the proof of Lemma 1). ■

Lemma 3. *Conditions* (6), (8), (9) *define a unique function* $\varphi(t, \vec{\sigma}(t_0))$, *such that the expressions* $\varphi^{(i)}(t, \vec{\sigma}(t_0))\ T^{-(r-i)}(\vec{\sigma}(t_0))$ *are globally uniformly bounded with* $t_0 \le t \le t_f(\vec{\sigma}(t_0))$, $i = 0, ..., r$. *Functions* $\varphi^{(i)}(t, \vec{\sigma})$ *are homogeneous functions of the weight* $r - i$ *with respect to the extended homogeneity dilation* $\tilde{d}_\kappa : (t, \vec{\sigma}) \mapsto (\kappa t, d_\kappa \vec{\sigma})$.

Proof. As follows from the Taylor formula

$$f^{(i)}(t, \vec{\sigma}(t_0)) = F_i + F_{i+1}\frac{t-t_0}{1!} + ... + F_{r-1}\frac{(t-t_0)^{r-i-1}}{(r-i-1)!},\ i = 0, ..., r-1; \quad f^{(r)}(t, \vec{\sigma}(t_0)) \equiv 0. \qquad (12)$$

By virtue of the trivial inequality $|t - t_0|/T \le 1$ and Lemma 2, changing i to j, obtain that $f^{(j)}(t, \vec{\sigma}(t_0))T^{-j}(\vec{\sigma}(t_0))$ are uniformly bounded with $t - t_0 < T$ and $j = 0, ..., r$. As follows now from the formula

$$\varphi^{(i)}(t, \vec{\sigma}(t_0)) = \sum_{j=0}^{i}\binom{i}{j}\frac{r!}{(r-i+j)!}(-T(\vec{\sigma}(t_0)))^{r-i+j} f^{(j)}(t, \vec{\sigma}(t_0)), \quad i = 0, ..., r, \qquad (13)$$

$\varphi^{(i)}(t, \vec{\sigma}(t_0))$ is homogeneous and $\varphi^{(i)}(t, \vec{\sigma}(t_0))T^{-(r-i)}(\vec{\sigma}(t_0))$ is globally bounded. ■

The Theorem follows now from Lemma 3 with $i = r$. ■

The calculations of the coefficients $F_i(\vec{\sigma}(t_0))$ are easily performed by computer in real time at the moment t_0. Afterwards, the real-time calculation of $\varphi^{(r)}(t, \vec{\sigma}(t_0))$ is performed according to (12), (13). The corresponding formulas are listed in the simulation Section for $r = 4$.

Suppose that sampling noises are present being bounded Lebesgue-measurable functions of time of any nature, and sampling is carried out with some sampling intervals. Suppose that the noise magnitudes of the measurements of σ, $\dot{\sigma}$, ..., $\sigma^{(l)}$ do not exceed $\beta_0\varepsilon$, $\beta_1\varepsilon^{(r-1)/r}$, …, $\beta_k\varepsilon^{(r-l)/r}$ respectively with some constants $\beta_0, \ldots, \beta_l > 0$ and the rest of derivatives be estimated by means of the $(r - l)$th order differentiator [] with a proper parameter set. Let also sampling intervals not exceed $\tau = \varepsilon^{1/r} > 0$. Then, as follows easily from Theorem 2 in [20], *under the conditions of Theorem* 1 *the inequalities* $|\sigma| < \gamma_0\tau^r = \gamma_0\varepsilon$, $|\dot{\sigma}| < \gamma_1\tau^{r-1} = \gamma_1\varepsilon^{(r-1)/r}$, …, $|\sigma^{(r-1)}| < \gamma_{r-1}\tau = \gamma_{r-1}\varepsilon^{1/r}$ *are established with some positive constants* $\gamma_0, \ldots, \gamma_{r-1}$ *independent of* $\varepsilon > 0$.

Rising the control smoothness degree. Choose some integer $k > r$ and consider $u^{(k-r)}$ as a new control. The new relative degree is k. Introduce the function φ satisfying the conditions

$$\varphi(t_0) = \sigma(t_0), ..., \varphi^{(k-1)}(t_0) = \sigma^{(k-1)}(t_0)\ , \varphi(t) = (t - t_f)^k\ (c_0 + c_1(t - t_0) + ...+ c_{k-1}\,(t - t_0)^{k-1}) \qquad (14)$$

$$t_f - t_0 = \mathrm{T}(\sigma(t_0), \dot{\sigma}(t_0), ..., \sigma^{(k-1)}(t_0)), \mathrm{T}(\sigma, \dot{\sigma}, ..., \sigma^{(k-1)}) \equiv \mathrm{T}(\kappa^r\sigma, \kappa^{r-1}\dot{\sigma}, ..., \kappa\sigma^{(k-1)})\ , \qquad (15)$$

where $\kappa > 0$ and T is continuous and positive definite. Let the bounded feedback control be

$$u^{(k-r)} = \alpha\Psi_k(\Sigma, \dot{\Sigma}, ..., \Sigma^{(k-1)}), \quad \Sigma = \begin{cases} \sigma(t,x) - \varphi(t), & t_0 \le t \le t_f \\ \sigma(t,x), & t > t_f \end{cases} \qquad (16)$$

with arbitrary initial values $u(t_0)$, ..., $u^{(k-r-1)}(t_0)$, where Ψ_k is a finite-time convergent r-sliding homogeneous controller.

Define the smooth function $u_{eq}(t, x) = - h(t, x)/g(t, x)$ from (2) and the condition $\sigma^{(r)} = 0$. Denote by $\zeta(t, x)$ the $(k\text{-}r)$th total derivative of $u_{eq}(t, x)$ with respect to the equation

$$\dot{x} = a(t,x) + b(t,x)\, u_{eq}(t, x).$$

Theorem 2. *Let the initial conditions* t_0, $x(t_0)$, $u(t_0)$, ..., $u^{(k-r-1)}(t_0)$ *belong to some compact set in* $\mathbf{R}^{n+k-r+1}$ *and* $\zeta(t, x)$ *be uniformly bounded. Then with sufficiently large* α *controller* (16) *establishes the k-sliding mode* $\sigma \equiv 0$ *with the transient time* (15). *The equality* $\sigma(t, x(t)) = \varphi(t, \vec{\sigma}(t_0))$ *is kept* *during the transient process.*

Proof. Differentiating (2) obtain

$$\Sigma^{(k)} = \tilde{h}\,(t, x, u, ..., u^{(k\text{-}r\text{-}1)}) + g(t, x)\, u^{(k\text{-}r)} \tag{17}$$

where $\tilde{h}$ can be expressed using the Lie derivatives and is, therefore, a difference between a smooth function and the uniformly bounded function $\varphi^{(k)}(t, \vec{\sigma}(t_0))$, $t_0 \le t \le t_f(\vec{\sigma}(t_0))$. The motions in the k-sliding mode $\Sigma \equiv 0$ are described by the replacement of the new control $\xi = u^{(k\text{-}r)}$ by its *equivalent* value

$$u^{(k\text{-}r)} = \xi_{eq} = -\,\tilde{h}\,(t, x, u, ..., u^{(k\text{-}r\text{-}1)})/g(t, x).$$

From the equation $\Sigma^{(r)} = 0$ obtain that the k-sliding motion $\Sigma = 0$ implies the equality $u = \tilde{u}_{eq}\,(t, x)$, where

$$\tilde{u}_{eq}\,(t, x) = \begin{cases} u_{eq}(t,x) + \varphi^{(r)}(t, \vec{\sigma}(t_0)) / g(t, x(t)), & t_0 \le t \le t_f\,(\vec{\sigma}(t_0)) \\ u_{eq}(t,x), & t > t_f\,(\vec{\sigma}(t_0)) \end{cases}.$$

Differentiating $k - r - 1$ times the equation $\Sigma^{(r)} = 0$ obtain that

$$\xi_{eq} = \tilde{u}_{eq}^{(k-r)}\,(t, x),$$

where $\tilde{u}_{eq}^{(k-r)}$ is the $(k\text{-}r)$th total derivative of $\tilde{u}_{eq}$ with respect to the equation

$$\dot{x} = a(t,x) + b(t,x)\,\tilde{u}_{eq}\,(t, x). \tag{18}$$

The points of the trajectories of

$$\dot{x} = a(t,x) + b(t,x)u\,, \qquad u^{(k\text{-}r)} = \tilde{u}_{eq}^{(k-r)}\,(t, x)$$

starting from the given compact set Ω of initial values with $\min_{\Omega} t_0 \le t \le \max_{\Omega} t_f(\vec{\sigma}(t_0))$ form a compact set Φ in the space $t, x, u, ..., u^{(k\text{-}r\text{-}1)}$ [7]. Let $\tilde{\Phi}$ be some compact set containing Φ in its interior, $\tilde{C} = K_M \max\,\{\sup|\zeta(t, x)|+1, \max_{\tilde{\Phi}} |\,\xi_{eq}|\}$. Then, due to (18), inequalities $|\tilde{h}\,| \le \tilde{C}$ and $K_m \le g \le K_M$ hold on the trajectories of (1), (16), which provides for the local k-sliding convergence [19] with α large enough. ■

As follows from the Theorem, the problem of the interaction between the control and its derivatives is solved here. Indeed, due to the transient absence, control and its derivatives track the smooth function $\tilde{u}_{eq}$ and its successive total time derivatives calculated with respect to the corresponding zero-dynamics system.

IV. Simulation example

Consider a variable-length pendulum control problem. Friction is assumed absent. All motions are restricted to some vertical plane. A load of known mass m is moving along the pendulum rod (Fig. 1). Its distance from O equals $R(t)$ and is not measured. An engine transmits a torque w, which is considered as control. The task is to track some function x_c given in real time by the angular coordinate x of the rod.

The system is described by the equations

$$\ddot{x} = -2\,\frac{\dot{R}}{R}\,\dot{x} - g\frac{1}{R}\sin x + \frac{1}{mR^2}\,w, \qquad \dot{w} = u. \tag{19}$$

where g = 9.81 is the gravitational constant, m = 1. Let R, $\dot{R}$, $\ddot{R}$, $\dot{x}_c$ and $\ddot{x}_c$ be bounded, R be separated from 0, $\sigma = x - x_c$, $\dot{\sigma} = \dot{x} - \dot{x}_c$ be available. The initial conditions are $x(0) = \dot{x}(0) = 0$. The natural relative degree of the system is 2, but it is artificiallly raised to 4, using $\ddot{w} = u$ as the new control, in order to avoid dangerous control switching.

Let $w(0) = \dot{w}(0) = 0$. Since $\ddot{x}|_{u=0}$ linearly depends on $\dot{x}$, it is not uniformly bounded. Nevertheless all requirements of the Theorems are satisfied in any bounded vicinity of the origin $x = \dot{x} = 0$, which provides for the local application of the method [20]. Following are the functions R and x_c considered in the simulation:

$$R = 1 + 0.25\sin 4t + 0.5\cos t, \qquad x_c = 0.5\sin 0.5t + 0.5\cos t\,.$$

While parameters of the controllers demonstrated further may be evaluated with respect to the above-mentioned restrictions on unknown functions $R(t)$, $x_c(t)$, their derivatives and some chosen bound on $\dot{x}$, they are usually excessively large in this case. The best way is to tune the parameters during simulation. Surely, the controlled class is somewhat smaller, but it still allows significant disturbances of the considered realizations of R and x_c.

The transient dynamics is chosen according to (6), (8) - (9):

$$\varphi(t) = (t - t_f)^4(c_0 + c_1(t - t_0) + c_2(t - t_0)^2 + c_3(t - t_0)^3),\ T = t_f - t_0 = \lambda(|s_0(t_0)|^3 + s_1(t_0)^4 + |s_2(t_0)|^6 + s_3(t_0)^{12})^{1/12},$$

$$c_0 = \mathrm{s}(t_0)\, T^{-4}, \quad c_1 = s_1(t_0)\, T^{-4} + 4\, \mathrm{s}(t_0)\, T^{-5}, \qquad c_2 = [s_2(t_0)\, T^{-4} + 8\, s_1(t_0)\, T^{-5} + 20\, \mathrm{s}(t_0)\, T^{-6}]\, /2,$$

$$c_3 = [s_3(t_0)T^{-4} + 12\, s_2(t_0)T^{-5} + 60\, s_1(t_0)\, T^{-6} + 120\, \mathrm{s}(t_0)T^{-7}]\, /6.$$

Here s_0, s_1 are some noisy measured values of σ and $\dot{\sigma}$, s_2, s_3 are the outputs of the 2nd-order differentiator estimating $\ddot{\sigma}$, $\dddot{\sigma}$ respectively. The above parameters of the function φ are calculated at the moment t_0 chosen to provide sufficient time for the differentiator convergence. Function φ and its derivatives are calculated further analytically according to (13). The output-feedback controller takes now the form

$$\xi(t) = \begin{cases} 0 & \text{with } t \notin [t_0, t_f] \\ \varphi(t) & \text{with } t \in [t_0, t_f] \end{cases}; \quad u = \begin{cases} 0 & \text{with } t < t_0 \\ \alpha\Psi_4(s_0 - \xi, s_1 - \dot{\xi}, s_2 - \ddot{\xi}, s_3 - \dddot{\xi}) & \text{with } t \in [t_0, t_f] \end{cases},$$

where the function Ψ_4 is to be defined further. The 2nd-order finite-time convergent differentiator [19] supplies estimates of σ, $\dot{\sigma}$, $\ddot{\sigma}$, $\dddot{\sigma}$:

$$\dot{y} = v_1, \qquad v_1 = -\,2\, L^{1/3}\, |y - s_1|^{2/3}\, \mathrm{sign}(y - s_1) + s_2,$$

$$\dot{s}_2 = v_2, \qquad v_2 = -\,1.5\, L^{1/2}\, |s_2 - v_1|^{1/2}\, \mathrm{sign}(s_2 - v_1) + s_3,$$

$$\dot{s}_3 = -\,1.1\, L\, \mathrm{sign}(s_3 - v_2).$$

Here L is to be larger than $\sup|\sigma^{(4)}|$, which exists due to Theorem 4, y is an auxiliary variable approximating $\dot{\sigma}$. The initial values of the differentiator are taken $y(0) = s_1(0)$, $s_2(0) = s_3(0) = 0$.

Denote $z_i = s_i - \xi^{(i)}$, $i = 0, 1, 2, 3$. The 4-sliding homogeneous quasi-continuous controller [22]

$$H_1 = z_3 + 3[|z_2| + (|z_1| + 0.5|z_0|^{3/4})^{-1/3} |z_1 + 0.5|z_0|^{3/4} \mathrm{sign}\, z_0|] [|z_2| + (|z_1| + 0.5|z_0|^{3/4})^{2/3}]^{-1/2},$$

$$H_2 = |z_3| + 3[|z_2| + (|z_1| + 0.5|z_0|^{3/4})^{2/3}]^{1/2}, \quad u = \alpha\Psi_4(z_0, z_1, z_2, z_3) = -H_1 / H_2 .$$

was considered. It is continuous (or can be redefined according to continuity) everywhere except the set $z_0 = z_1 = z_2 = z_3 = 0$ [22]. The parameters which are still lacking are taken as follows: $\alpha = 70$, $L = 300$, $t_0 = 1$. The convergence-time parameter λ takes on values 2, 4, 6.

The integration was carried out according to the Euler method (the only reliable method with discontinuous dynamics). The 4-sliding deviations for $\lambda = 2, 6$, the tracking performance, the torque and the differentiator convergence in the absence of output noises with $\lambda = 2$ are shown in Fig. 1. The accuracies $|\sigma| \le 8.4\cdot10^{-15}$, $|\dot\sigma| \le 2.7\cdot10^{-11}$, $|\ddot\sigma| \le 2.0\cdot10^{-7}$, $|\dddot\sigma| \le 0.0036$ were obtained with the sampling step $\tau = 10^{-5}$. It is seen that the accurate differentiation is ensured already with $t = 0.15$.

Check now the robustness of the method. The accuracies changed to $|\sigma| \le 4.4\cdot10^{-4}$, $|\dot\sigma| \le 2.8\cdot10^{-3}$, $|\ddot\sigma| \le 0.054$, $|\dddot\sigma| \le 1.7$ with $\tau = 0.005$ (Fig. 2). The tracking accuracy $|\sigma| \le 0.013$ was obtained with $\tau = 10^{-5}$ in the presence of non-smooth non-centered noises with the same magnitude $\varepsilon = 0.01$, the accuracy $|\sigma| \le 0.052$ was obtained with $\tau = 0.005$, $\varepsilon = 0.01$ (Fig. 2). The graph of the torque $w(t)$ shows that, while noises and discrete sampling lead to the loss of accuracy, no significant chattering is observed. The results practically do not depend on the noise frequencies.

Following are the simulation results of the plain controllers [13, 15] given for the comparison of transient processes. In the absence of output noises the tracking accuracies $|\sigma| \le 6.8\cdot10^{-13}$, $|\dot\sigma| \le 5.8\cdot10^{-10}$, $|\ddot\sigma| \le 1.9\cdot10^{-6}$, $|\dddot\sigma| \le 0.013$ were attained after application of the quasi-sliding controller with $\lambda = 2$ (Fig. 3) and $\tau = 10^{-5}$. Similar accuracies were obtained in all cases. The nested-sliding-mode controller is slightly less precise. In particular, the tracking accuracies $|\sigma| \le 2.0\cdot10^{-12}$, $|\dot\sigma| \le 1.3\cdot10^{-9}$, $|\ddot\sigma| \le 2.3\cdot10^{-6}$, $|\dddot\sigma| \le 0.016$ were attained after application of the standard controller with $\lambda = 2$ and $\tau = 10^{-5}$.

VII. Conclusions

The integral sliding mode approach allows to prescribe any needed transient dynamics to high-order sliding-mode systems. In particular, any positive definite *r*-sliding homogeneous function of initial conditions in the space of the output and its derivatives can be realized as a settling-time function. The same approach allows to solve the long-lasting problem of the control interaction in the control smoothing based on the artificial increase of the relative degree. A robust output-feedback controller is obtained when combined with robust exact differentiator.

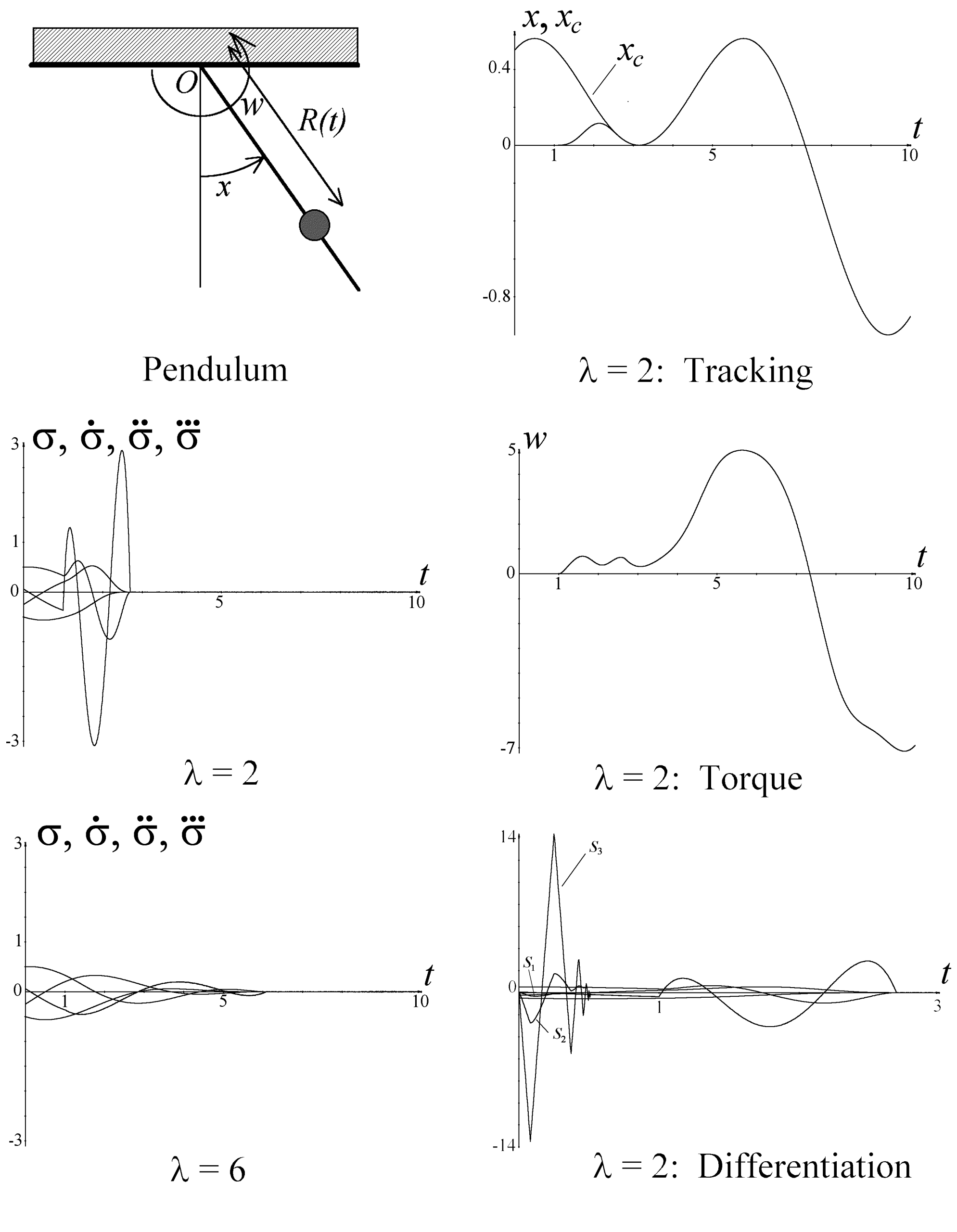

Fig. 1: 4-sliding pendulum control and transient adjustment

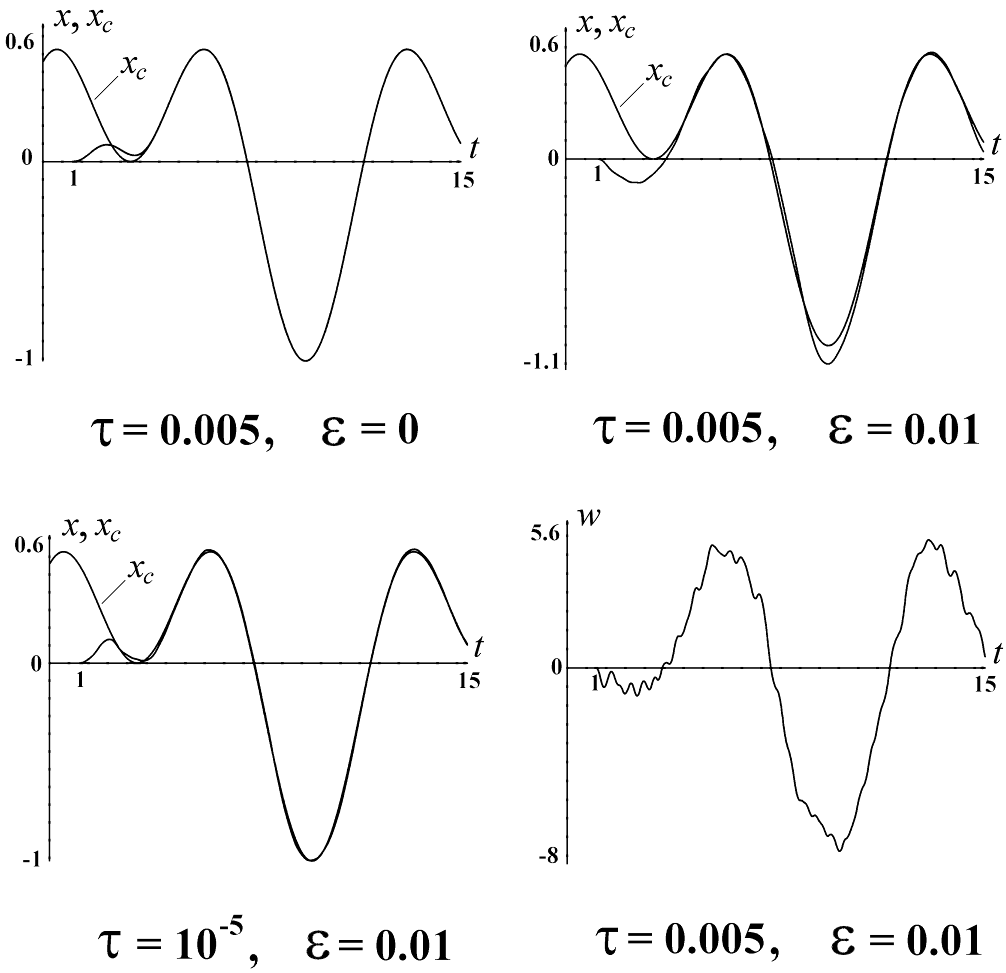

Fig. 2: Performance with various values of the noise amplitude ε and sampling step τ

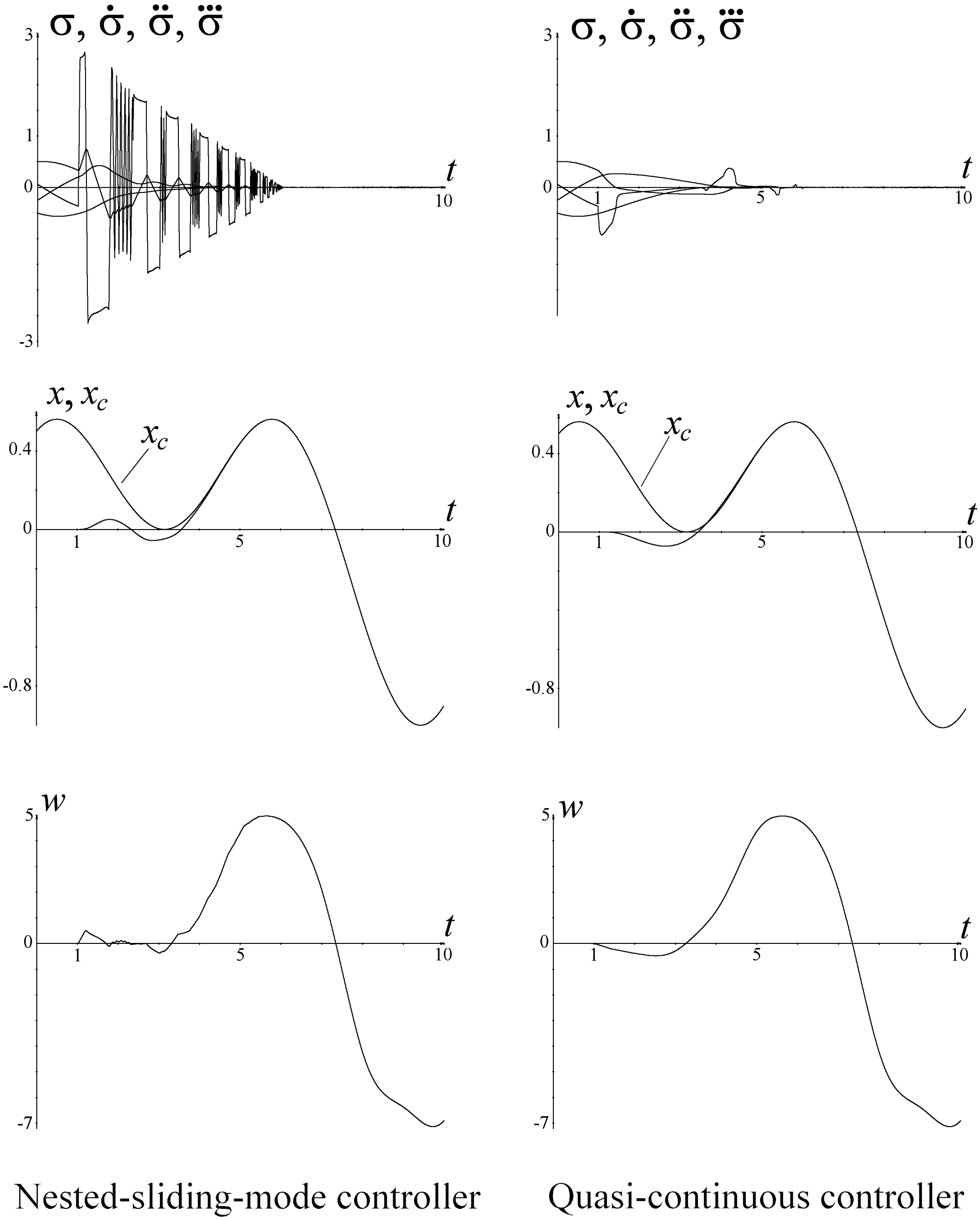

Fig. 3: Straight-forward 4-sliding pendulum control

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