Integral inequalities for a fractional operator of a function with respect to another function with nonsingular kernel

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Abstract

At first, we construct a connection between the Atangana–Baleanu and the Riemann–Liouville fractional integrals of a function with respect to a monotone function with nonsingular kernel. By examining this relationship and the iterated form of Prabhakar fractional model, we are able to find some new Hermite–Hadamard inequalities and related results on integral inequalities for the two models of fractional calculus which are defined using monotone functions with nonsingular kernels.

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1 Introduction

As we know, the fractional calculus is a fundamental model considered as a powerful tool in many fields, for example, rheology, dynamic systems, biophysics, electrical networks, blood flow phenomena; see for details [6, 32, 43, 52]. On September 1695 the fractional calculus was considered by Leibniz as a new model of mathematical analysis which involves derivatives and integrals taken to fractional orders or orders outside of the integer numbers. After Leibniz, many famous mathematicians and physicians have studied theory and application of this model for fractional derivatives and integrals and fractional differential equations; see for detail [12, 13, 35, 55]. Some of them were Grunwald, Liouville, Riemann–Letnikov, Caputo, and Atangana–Baleanu which we will consider in more detail in the next section. The inequality is a modern model of mathematical analysis that describes the growth rate of competing mathematical analysis. This model is also used in various fields such as integral equations, ordinary and partial differential equations [29, 59]. In particular, they have been successfully used in the study of fractional differential equations and sometimes called fractional integral inequalities. They have been vital in proving the uniqueness or existence of solutions for some well-known fractional differential equations and in providing boundedness to solve certain fractional boundary and initial value problems, see for detail [1, 7, 19, 24, 26, 33, 34, 41, 48–51]. The rest of the article is organized as follows: In Sect. 2, we recall some basic definitions and tools about fractional derivatives

with Mittag-Leffler and power law kernels of a function with respect to another function in the sense of Caputo and Riemann and their corresponding integrals. The existence of the strictly increasing function in the kernel allows us to include a large class of fractional operators, and hence our results will generalize and unify many existing results in the literature treating Hermite–Hadamard inequalities. We start Sect. 3 by introducing the corresponding fractional integral of the fractional operators with Mittag-Leffler kernels depending on a function and proving it for the first time. Then, we proceed to proving the Hermite–Hadamard fractional inequalities in the setting of these generalized fractional integrals. An example with supporting figures is presented in this section as well to clarify the proved main result. We devote Sect. 4 to presenting a related trapezoidal equality for such a class of fractional integrals. Finally, in Sect. 5, we outline our conclusions about the novel results we have proved in this short interesting study.

2 Definitions and preliminaries

Always, it is important and necessary to specify which model or definition is being used because there are many different ways of defining fractional integrals and derivatives. The most frequent definition of fractional integrals and derivatives is the Riemann-Liouville one, in which fractional integrals and derivatives are defined as follows.

Definition 2.1 ([32, 43, 52]) For any L^1 function $f(x)$ on an interval $[v_1, v_2]$ with $x \in [v_1, v_2]$, the η_1th left-RL fractional integral of $f(x)$ is defined by

$$^{RL}\mathrm{J}^{\eta_1}_{v_1+}f(x) := \frac{1}{\Gamma(\eta_1)} \int_{v_1}^{x} (x - \xi)^{\eta_1 - 1} f(\xi) \, \mathrm{d}\xi \tag{1}$$

for $\mathrm{Re}(\eta_1) > 0$. Also, for any C^n function $f(x)$ on an interval $[v_1, v_2]$ with $x \in [v_1, v_2]$, the η_1th left-RL fractional derivative of $f(x)$ is defined by

$$^{RL}\mathrm{D}^{\eta_1}_{v_1+}f(x) := \frac{\mathrm{d}^n}{\mathrm{d}x^n} {}^{RL}\mathrm{J}^{n-\eta_1}_{v_1+}f(x) \tag{2}$$

for $n - 1 \leq \mathrm{Re}(\eta_1) < n$.

Observe that since $^{RL}\mathrm{D}^{-\eta_1}_{v_1+}f(x) = {}^{RL}\mathrm{J}^{\eta_1}_{v_1+}f(x)$ and the derivative formula (2) is the analytic continuation of the integral formula (1) in η_1, thus differentiation and integration can be unified in a single operator that we call *differintegration*.

Definition 2.2 ([52]) For any L^1 function $f(x)$ on an interval $[v_1, v_2]$ with $x \in [v_1, v_2]$, the η_1th right-RL fractional integral of $f(x)$ is defined by

$$^{RL}\mathrm{J}^{\eta_1}_{v_2-}f(x) := \frac{1}{\Gamma(\eta_1)} \int_{x}^{v_2} (\xi - x)^{\eta_1 - 1} f(\xi) \, \mathrm{d}\xi \tag{3}$$

for $\mathrm{Re}(\eta_1) > 0$. Also, for any C^n function $f(x)$ on an interval $[v_1, v_2]$ with $x \in [v_1, v_2]$, the η_1th right-RL fractional derivative of $f(x)$ is defined by

$$^{RL}\mathrm{D}^{\eta_1}_{v_2-}f(x) := (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} {}^{RL}\mathrm{J}^{n-\eta_1}_{v_2-}f(x) \tag{4}$$

for $n - 1 \leq \mathrm{Re}(\eta_1) < n$.

In the recent decades, a strong modern direction of research in fractional calculus has brought the attention of interested researchers in various disciplines to investigate various possible ways to define fractional integrals and derivatives, often with different properties from the classical RL definition. Some of them are more effective than the RL model that can be used to model real-life data [8, 23]. One of the most recent models of fractional calculus discussed in this paper is the fractional calculus of a function with respect to another function (which is often called ψ-RL fractional calculus) that was firstly defined in the classical RL model by Osler [44] and the concept has been extended by many authors; e.g. [17, 56]. As shown in many papers cited below, it has been especially useful in real-world modeling.

Definition 2.3 ([17, 44, 56]) Suppose that $(v_1, v_2) \subseteq \mathfrak{R}$ is a finite or infinite interval of the real numbers $\mathfrak{R}$ and $\psi(x)$ is an increasing positive monotone function on the interval $[v_1, v_2]$ with $\psi'(x) \in L^1(v_1, v_2)$. Then the ψ-RL fractional integrals of a function f with respect to another function $\psi(x)$ on $[v_1, v_2]$ is defined by

$$\mathfrak{J}^{\eta_1}_{v_1+} f(x) = \frac{1}{\Gamma(\eta_1)} \int_{v_1}^{x} \psi'(\xi) \big(\psi(x) - \psi(\xi)\big)^{\eta_1 - 1} f(\xi) \, d\xi \tag{5}$$

for $\eta_1 \in \mathbb{C}$ and $\mathrm{Re}(\eta_1) > 0$. Furthermore, the ψ-R fractional derivatives of a function f with respect to another function $\psi(x)$ is defined by

$$_{\psi(x)}^{RL}\mathrm{D}^{\eta_1}_{v_1+} f(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \left(_{\psi(x)}^{RL}\mathfrak{J}^{n-\eta_1}_{v_1+} f(x) \right) \tag{6}$$

for $\eta_1 \in \mathbb{C}$ with $\mathrm{Re}(\eta_1) \geq 0$ and $n - 1 \leq \mathrm{Re}(\eta_1) < n$.

Special functions have a strong relationship with fractional calculus [28, 32], and the Mittag-Leffler function is a particular important one in this area, see e.g. [21, 22, 31]. There are two important models of fractional calculus which have been defined by integrals similar to (1) but with Mittag-Leffler (ML) functions in the kernel, namely the *Atangana–Baleanu* and *Prabhakar* models [9, 10].

Definition 2.4 ([17]) For any function $f(x)$ which is differentiable with $f' \in L^1[v_1, v_2]$ and $x \in [v_1, v_2]$, the η_1th Atangana–Baleanu (AB) fractional derivative of $f(x)$ with respect to another function $\psi(x)$ in the RL sense is defined by

$$_{\psi(x)}^{ABR}\mathrm{D}^{\eta_1}_{v_1+} f(x) := \frac{B(\eta_1)}{(1 - \eta_1)\psi'(x)} \cdot \frac{d}{dx} \int_{v_1}^{x} \psi'(\xi) E_{\eta_1}\left(\frac{-\eta_1}{1 - \eta_1} \big(\psi(x) - \psi(\xi)\big)^{\eta_1} \right) f(\xi) \, d\xi, \tag{7}$$

and in the Caputo sense it is defined by

$$_{\psi(x)}^{ABR}\mathrm{D}^{\eta_1}_{v_1+} f(x) := \frac{B(\eta_1)}{1 - \eta_1} \int_{v_1}^{x} E_{\eta_1}\left(\frac{-\eta_1}{1 - \eta_1} \big(\psi(x) - \psi(\xi)\big)^{\eta_1} \right) f'(\xi) \, d\xi \tag{8}$$

for $0 < \eta_1 < 1$, where $\psi(x)$ is as before, $E_{\eta_1}(z)$ is the standard one-parameter ML function, and $B(\eta_1) > 0$ is a normalization function that satisfies $B(0) = B(1) = 1$.

It is clear that the above definitions are for *left* AB integrals and derivatives. In a natural way similar to Definition 2.2, we can define *right* AB integrals and derivatives possibly.

Definition 2.5 ([27, 46]) For any L^1 function $f(x)$ on an interval $[\nu_1, \nu_2]$ with $x \in [\nu_1, \nu_2]$, the Prabhakar fractional integral operator is defined by

$$
{}^P\mathfrak{J}_{\nu_1+}^{\eta_1,\eta_2,\gamma,\omega}f(x) := \int_{\nu_1}^{x}(x-\xi)^{\eta_2-1}E_{\eta_1,\eta_2}^{\gamma}\big(\omega(x-\xi)^{\eta_1}\big)f(\xi)\,\mathrm{d}\xi \tag{9}
$$

for $\mathrm{Re}(\eta_1) > 0$, $\mathrm{Re}(\eta_2) > 0$, and $\gamma, \omega \in \mathbb{C}$, where $E_{\eta_1,\eta_2}^{\gamma}(z)$ is the three-parameter ML function.

Also, for any C^n function $f(x)$ on an interval $[\nu_1, \nu_2]$ with $x \in [\nu_1, \nu_2]$, the Prabhakar fractional derivative operator is defined as follows:

$$
{}^P D_{\nu_1+}^{\eta_1,\eta_2,\gamma,\omega}f(x) := \frac{\mathrm{d}^n}{\mathrm{d}x^n}\big({}^P\mathfrak{J}_{\nu_1+}^{\eta_1,n-\eta_2,-\gamma,\omega}f(x)\big) \tag{10}
$$

for $\mathrm{Re}(\eta_1) > 0$, $n - 1 \le \mathrm{Re}(\eta_2) < n$, and $\gamma, \omega \in \mathbb{C}$.

Again, the above definitions are for *left* Prabhakar integrals and derivatives. In a natural way similar to Definition 2.2, we can define *right* Prabhakar integrals and derivatives possibly.

Remark 2.1 Observe that Definition 2.5 can be seen as a special case of the generalized Prabhakar model defined in [57].

Definition 2.6 ([17]) For any L^1 function $f(x)$ on an interval $[\nu_1, \nu_2]$ with $x \in [\nu_1, \nu_2]$, the generalized Prabhakar fractional integral operator of a function $f(x)$ with respect to another function $\psi(x)$ is defined by

$$
{}_{\psi(x)}^{P}\mathfrak{J}_{\nu_1+}^{\eta_1,\eta_2,\gamma,\omega}f(x) := \int_{\nu_1}^{x}\big(\psi(x)-\psi(\xi)\big)^{\eta_2-1}E_{\eta_1,\eta_2}^{\gamma,\kappa}\big(\omega\big(\psi(x)-\psi(\xi)\big)^{\eta_1}\big)f(\xi)\,\mathrm{d}\xi \tag{11}
$$

for $\mathrm{Re}(\eta_1) > 0$, $\mathrm{Re}(\eta_2) > 0$, $\mathrm{Re}(\kappa) > 0$, $\mathrm{Re}(\kappa - \eta_1) < 1$, and $\gamma, \omega \in \mathbb{C}$, where $\psi(x)$ is as before and $E_{\eta_1,\eta_2}^{\gamma,\kappa}(z)$ is the generalized ML function defined by

$$
E_{\eta_1,\eta_2}^{\gamma,\kappa}(z) := \sum_{n=0}^{\infty}\frac{\Gamma(\gamma+\kappa n)z^n}{\Gamma(\gamma)\Gamma(\eta_1 n+\eta_2)}. \tag{12}
$$

Also, for any C^n function $f(x)$ on an interval $[\nu_1, \nu_2]$ with $x \in [\nu_1, \nu_2]$, the generalized Prabhakar fractional derivative operator of a function $f(x)$ with respect to another function $\psi(x)$ is defined by

$$
{}_{\psi(x)}^{P}D_{\nu_1+}^{\eta_1,\eta_2,\gamma,\omega}f(x) := \frac{\mathrm{d}^n}{\mathrm{d}x^n}\big({}_{\psi(x)}^{P}\mathfrak{J}_{\nu_1+}^{\eta_1,n-\eta_2,-\gamma,\omega}f(x)\big) \tag{13}
$$

for $\mathrm{Re}(\eta_1) > 0$ and $n - 1 \le \mathrm{Re}(\eta_2) < n$ and $\gamma, \omega \in \mathbb{C}$.

It is important to see Definitions 2.4 and 2.6 as interesting analogues of the basic Definition 2.1 from a pure mathematical point of view. Also, they have their own properties consistent with those of the ML functions [5, 15, 18, 27]. On the other hand, if we look carefully from an applied science point of view, we can deduce that these models of fractional calculus can be matched with real data where power law behavior disappears [14, 20, 58].

In the current study, we point out the uniting topic of integral inequalities involving the aforementioned novel models of fractional calculus. Particularly, we consider the well-known Hermite–Hadamard (HH) inequality, which has been successfully studied for RL fractional integrals. Also, we prove some new results analogous to the classical results which are valid in the generalized AB and Prabhakar models of fractional calculus. Finally, a related inequality of trapezoidal type is pointed out.

3 The new fractional HH-inequality

Our next findings are based on the following lemma.

Lemma 3.1 *For any* L^1 *function* $f(x)$ *on an interval* $[\nu_1, \nu_2]$ *with* $x \in [\nu_1, \nu_2]$, *the* $\eta_1 th$ *Atangana–Baleanu* (*AB*) *fractional integral of a function* $f(x)$ *with respect to another function can be represented as follows*:

$$\,^{AB}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\nu_1+}f(x) := \frac{\eta_1}{B(\eta_1)}\,^{RL}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\nu_1+}f(x) + \frac{1-\eta_1}{B(\eta_1)}f(x), \quad 0 < \eta_1 < 1, \tag{14}$$

where $B(\eta_1)$ *and* $\psi(x)$ *are as before*.

Proof The expression (14) is well defined iff the ψ-RL integral $\,^{RL}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\nu_1+}f(x)$ is well defined, which matches our assumptions on f for the η_1th ψ-ABR fractional derivative to be well defined. Following [17] where the AB fractional derivatives of a function $f(x)$ with respect to another function $\psi(x)$ can be represented in the series form

$$\,^{ABR}_{\psi(x)}D^{\eta_1}_{\nu_1+}f(x) = \frac{B(\eta_1)}{1-\eta_1}\sum_{n=0}^{\infty}\left(\frac{-\eta_1}{1-\eta_1}\right)^n\,^{RL}_{\psi(x)}\mathfrak{J}^{n\eta_1}_{\nu_1+}f(x). \tag{15}$$

It is sufficient if we show that

$$\,^{ABR}_{\psi(x)}D^{\eta_1}_{\nu_1+}\left(\,^{AB}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\nu_1+}f(x)\right) = f(x)$$

and

$$\,^{AB}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\nu_1+}\left(\,^{ABR}_{\psi(x)}D^{\eta_1}_{\nu_1+}f(x)\right) = f(x)$$

for each a, η_1, f, ψ as stated. By using (15), we deduce that

$$\,^{ABR}_{\psi(x)}D^{\eta_1}_{\nu_1+}\left(\,^{AB}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\nu_1+}f(x)\right)$$

$$= \,^{ABR}_{\psi(x)}D^{\eta_1}_{\nu_1+}\left(\frac{\eta_1}{B(\eta_1)}\,^{RL}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\nu_1+}f(x) + \frac{1-\eta_1}{B(\eta_1)}f(x)\right)$$

$$= \frac{\eta_1}{B(\eta_1)}\,^{ABR}_{\psi(x)}D^{\eta_1}_{\nu_1+}\left(\,^{RL}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\nu_1+}f(x)\right) + \frac{1-\eta_1}{B(\eta_1)}\,^{ABR}_{\psi(x)}D^{\eta_1}_{\nu_1+}f(x)$$

$$= \frac{\eta_1}{1-\eta_1}\sum_{n=0}^{\infty}\left(\frac{-\eta_1}{1-\eta_1}\right)^n\,^{RL}_{\psi(x)}\mathfrak{J}^{n\eta_1}_{\nu_1+}\left(\,^{RL}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\nu_1+}f(x)\right)$$

$$+ \sum_{n=0}^{\infty}\left(\frac{-\eta_1}{1-\eta_1}\right)^n\,^{RL}_{\psi(x)}\mathfrak{J}^{n\eta_1}_{\nu_1+}f(x)$$

$$= \sum_{n=0}^{\infty} \left(\frac{-\eta_1}{1-\eta_1} \right)^n {}_{\psi(x)}^{RL} \mathbb{J}_{\nu_1+}^{n\eta_1} f(x) - \sum_{n=0}^{\infty} \left(\frac{-\eta_1}{1-\eta_1} \right)^{n+1} {}_{\psi(x)}^{RL} \mathbb{J}_{\nu_1+}^{n\eta_1+\eta_1} f(x)$$

$$= f(x);$$

$${}_{\psi(x)}^{AB} \mathbb{J}_{\nu_1+}^{\eta_1} \left({}_{\psi(x)}^{ABR} D_{\nu_1+}^{\eta_1} f(x) \right)$$

$$= \frac{\eta_1}{B(\eta_1)} {}_{\psi(x)}^{RL} \mathbb{J}_{\nu_1+}^{\eta_1} \left({}_{\psi(x)}^{ABR} D_{\nu_1+}^{\eta_1} f(x) \right) + \frac{1-\eta_1}{B(\eta_1)} {}_{\psi(x)}^{ABR} D_{\nu_1+}^{\eta_1} f(x)$$

$$= \frac{\eta_1}{1-\eta_1} {}_{\psi(x)}^{RL} \mathbb{J}_{\nu_1+}^{\eta_1} \left(\sum_{n=0}^{\infty} \left(\frac{-\eta_1}{1-\eta_1} \right)^n {}_{\psi(x)}^{RL} \mathbb{J}_{\nu_1+}^{n\eta_1} f(x) \right)$$

$$\quad + \sum_{n=0}^{\infty} \left(\frac{-\eta_1}{1-\eta_1} \right)^n {}_{\psi(x)}^{RL} \mathbb{J}_{\nu_1+}^{n\eta_1} f(x)$$

$$= \sum_{n=0}^{\infty} \left(\frac{-\eta_1}{1-\eta_1} \right)^n {}_{\psi(x)}^{RL} \mathbb{J}_{\nu_1+}^{n\eta_1} f(x) - \sum_{n=0}^{\infty} \left(\frac{-\eta_1}{1-\eta_1} \right)^{n+1} {}_{\psi(x)}^{RL} \mathbb{J}_{\nu_1+}^{n\eta_1+\eta_1} f(x)$$

$$= f(x);$$

which completes the proof. □

Remark 3.1 The discrete version of (15) was obtained in [4] and its generalization to the case of generalized ML function case together with its discrete counterpart can be found in [2] and [3], respectively.

Now, we recall the standard Hermite–Hadamard (HH) inequality for an L^1 convex function $f : [\nu_1, \nu_2] \to \mathfrak{R}$ as follows:

$$f\left(\frac{\nu_1 + \nu_2}{2} \right) \leq \frac{1}{\nu_2 - \nu_1} \int_{\nu_1}^{\nu_2} f(x) \, dx \leq \frac{f(\nu_1) + f(\nu_2)}{2}. \tag{16}$$

In 2013, HH-inequality (16) was generalized to fractional integrals of RL type by Sarikaya et al. [53], which is as follows:

$$f\left(\frac{\nu_1 + \nu_2}{2} \right) \leq \frac{\Gamma(\eta_1 + 1)}{2(\nu_2 - \nu_1)^{\eta_1}} \left[{}^{RL} \mathbb{J}_{\nu_1+}^{\eta_1} f(\nu_2) + {}^{RL} \mathbb{J}_{\nu_2-}^{\eta_1} f(\nu_1) \right] \leq \frac{f(\nu_1) + f(\nu_2)}{2}, \tag{17}$$

where f is as before and $\eta_1 > 0$.

Most recently, in 2019, HH-inequality (16) was generalized to fractional integrals of ψ-RL type by Liu et al. [30] for an L^1 convex and positive function $f : [\nu_1, \nu_2] \to \mathfrak{R}$ and for an increasing positive monotone function ψ on $(\nu_1, \nu_2]$ with $\psi'(x) \in L^1(\nu_1, \nu_2)$:

$$f\left(\frac{\nu_1 + \nu_2}{2} \right)$$

$$\leq \frac{\Gamma(\eta_1 + 1)}{2(\nu_2 - \nu_1)^{\eta_1}} \left[{}_{\psi(x)}^{RL} \mathbb{J}_{\psi^{-1}(\nu_1)+}^{\eta_1} (f \circ \psi) \left(\psi^{-1}(\nu_2) \right) + {}_{\psi(x)}^{RL} \mathbb{J}_{\psi^{-1}(\nu_2)-}^{\eta_1} (f \circ \psi) \left(\psi^{-1}(\nu_1) \right) \right]$$

$$\leq \frac{f(\nu_1) + f(\nu_2)}{2}, \quad \eta_1 > 0. \tag{18}$$

Many further results have been derived from inequality (17); for details, see [16, 25, 34, 38, 47]. But so far such inequalities have not been investigated for other models of fractional

calculus involving Mittag-Leffler kernels. We do so here, presenting the main results in the following three theorems.

Theorem 3.1 *If $f : [v_1, v_2] \to \mathfrak{R}$ is an L^1 convex function, ψ is an increasing positive function on $(v_1, v_2]$ with $\psi'(x) \in L^1(v_1, v_2)$, and $\eta_1 \in (0, 1)$, then we have*

$$f\left(\frac{v_1 + v_2}{2}\right) \leq \frac{B(\eta_1)\Gamma(\eta_1)}{2[(v_2 - v_1)^{\eta_1} + (1 - \eta_1)\Gamma(\eta_1)]}\left[{}^{AB}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\psi^{-1}(v_1)+}(f \circ \psi)\left(\psi^{-1}(v_2)\right)\right.$$
$$\left. + {}^{AB}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\psi^{-1}(v_2)-}(f \circ \psi)\left(\psi^{-1}(v_1)\right)\right]$$
$$\leq \frac{f(v_1) + f(v_2)}{2}. \tag{19}$$

Proof From (14), we see that the AB fractional integral of a function $f(x)$ is clearly a linear combination of $f(x)$ itself with its ψ-RL fractional integral. Then we have

$${}^{AB}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\psi^{-1}(v_1)+}(f \circ \psi)\left(\psi^{-1}(v_2)\right) + {}^{AB}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\psi^{-1}(v_2)-}(f \circ \psi)\left(\psi^{-1}(v_1)\right)$$
$$= \left[\frac{\eta_1}{B(\eta_1)}{}^{RL}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\psi^{-1}(v_1)+}(f \circ \psi)\left(\psi^{-1}(v_2)\right) + \frac{1 - \eta_1}{B(\eta_1)}(f \circ \psi)\left(\psi^{-1}(v_2)\right)\right]$$
$$+ \left[\frac{\eta_1}{B(\eta_1)}{}^{RL}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\psi^{-1}(v_2)-}(f \circ \psi)\left(\psi^{-1}(v_1)\right) + \frac{1 - \eta_1}{B(\eta_1)}(f \circ \psi)\left(\psi^{-1}(v_1)\right)\right]$$
$$= \frac{\eta_1}{B(\eta_1)}\left[{}^{RL}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\psi^{-1}(v_1)+}(f \circ \psi)\left(\psi^{-1}(v_2)\right) + {}^{RL}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\psi^{-1}(v_2)-}(f \circ \psi)\left(\psi^{-1}(v_1)\right)\right]$$
$$+ \frac{1 - \eta_1}{B(\eta_1)}\left[f(v_1) + f(v_2)\right].$$

Since the quantities $\Gamma(\eta_1 + 1)$ and $2(v_2 - v_1)^{\eta_1}$ are positive, then by using inequality (18), we get

$$\frac{2(v_2 - v_1)^{\eta_1}}{\Gamma(\eta_1 + 1)}f\left(\frac{v_1 + v_2}{2}\right)$$
$$\leq {}^{RL}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\psi^{-1}(v_1)+}(f \circ \psi)\left(\psi^{-1}(v_2)\right) + {}^{RL}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\psi^{-1}(v_2)-}(f \circ \psi)\left(\psi^{-1}(v_1)\right)$$
$$\leq \frac{2(v_2 - v_1)^{\eta_1}}{\Gamma(\eta_1 + 1)} \cdot \frac{f(v_1) + f(v_2)}{2}.$$

By multiplying by $\frac{\eta_1}{B(\eta_1)}$ and adding $\frac{1-\eta_1}{B(\eta_1)}[f(v_1) + f(v_2)]$, we get AB integrals in the middle:

$$\frac{2(v_2 - v_1)^{\eta_1}}{\Gamma(\eta_1)B(\eta_1)}f\left(\frac{v_1 + v_2}{2}\right) + \frac{1 - \eta_1}{B(\eta_1)}\left[f(v_1) + f(v_2)\right]$$
$$\leq {}^{AB}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\psi^{-1}(v_1)+}(f \circ \psi)\left(\psi^{-1}(v_2)\right) + {}^{AB}_{\psi(x)}\mathfrak{J}^{\eta_1}_{\psi^{-1}(v_2)-}(f \circ \psi)\left(\psi^{-1}(v_1)\right)$$
$$\leq \frac{2(v_2 - v_1)^{\eta_1}}{\Gamma(\eta_1)B(\eta_1)} \cdot \frac{f(v_1) + f(v_2)}{2} + \frac{1 - \eta_1}{B(\eta_1)}\left[f(v_1) + f(v_2)\right]. \tag{20}$$

By convexity of f, we have $f(\frac{v_1+v_2}{2}) \leq \frac{f(v_1)+f(v_2)}{2}$, so we can deduce

$$\left[\frac{2(v_2-v_1)^{\eta_1}}{\Gamma(\eta_1)B(\eta_1)} + \frac{2(1-\eta_1)}{B(\eta_1)}\right]f\left(\frac{v_1+v_2}{2}\right)$$

$$\leq {}^{AB}_{\psi(x)}\mathsf{J}^{\eta_1}_{\psi^{-1}(v_1)+}(f\circ\psi)\left(\psi^{-1}(v_2)\right) + {}^{AB}_{\psi(x)}\mathsf{J}^{\eta_1}_{\psi^{-1}(v_2)-}(f\circ\psi)\left(\psi^{-1}(v_1)\right)$$

$$\leq \left[\frac{2(v_2-v_1)^{\eta_1}}{\Gamma(\eta_1)B(\eta_1)} + \frac{2(1-\eta_1)}{B(\eta_1)}\right]\frac{f(v_1)+f(v_2)}{2},$$

which rearranges to the required result. □

Corollary 3.1 *Theorem* 3.1 *with* $\psi(x)=x$ *becomes*

$$f\left(\frac{v_1+v_2}{2}\right) \leq \frac{B(\eta_1)\Gamma(\eta_1)}{2[(v_2-v_1)^{\eta_1}+(1-\eta_1)\Gamma(\eta_1)]}\left[{}^{AB}\mathsf{J}^{\eta_1}_{v_1}f(v_2) + {}^{AB}\mathsf{J}^{\eta_1}_{v_2}f(v_1)\right]$$

$$\leq \frac{f(v_1)+f(v_2)}{2},$$

which is proved by Fernandez and Mohammed in [19, *Proposition* 2.1].

Theorem 3.2 *If* $f:[v_1,v_2]\to\mathfrak{R}$ *is an* L^1 *convex function,* ψ *is an increasing positive function on* $(v_1,v_2]$ *with* $\psi'(x)\in L^1(v_1,v_2)$, *and* $\eta_1\in(0,1)$, *then we have*

$$\mathcal{Q}_1(\eta_1,v_2-v_1)f\left(\frac{v_1+v_2}{2}\right) + \mathcal{Q}_2(\eta_1,v_2-v_1)\frac{f(v_1)+f(v_2)}{2}$$

$$\leq \frac{1-\eta_1}{2B(\eta_1)E_{\eta_1}(\frac{-\eta_1}{1-\eta_1}(v_2-v_1)^{\eta_1})}$$

$$\times \left[{}^{ABR}_{\psi(x)}\mathsf{D}^{\eta_1}_{\psi^{-1}(v_1)+}(f\circ\psi)\left(\psi^{-1}(v_2)\right) + {}^{ABR}_{\psi(x)}\mathsf{D}^{\eta_1}_{\psi^{-1}(v_2)-}(f\circ\psi)\left(\psi^{-1}(v_1)\right)\right]$$

$$\leq \mathcal{Q}_2(\eta_1,v_2-v_1)f\left(\frac{v_1+v_2}{2}\right) + \mathcal{Q}_1(\eta_1,v_2-v_1)\frac{f(v_1)+f(v_2)}{2}, \tag{21}$$

where the multipliers $\mathcal{Q}_1(\eta_1,v_2-v_1)$ *and* $\mathcal{Q}_2(\eta_1,v_2-v_1)$ *sum to* 1, *therefore organizing weighted averages on both ends of the inequality, and these are defined as follows*:

$$\mathcal{Q}_1(\eta_1,v_2-v_1) = \frac{E_{2\eta_1}((\frac{-\eta_1}{1-\eta_1})^2(v_2-v_1)^{2\eta_1})}{E_{\eta_1}(\frac{-\eta_1}{1-\eta_1}(v_2-v_1)^{\eta_1})};$$

$$\mathcal{Q}_2(\eta_1,v_2-v_1) = \frac{\frac{-\eta_1}{1-\eta_1}(v_2-v_1)^{\eta_1}E_{2\eta_1,\eta_1+1}((\frac{-\eta_1}{1-\eta_1})^2(v_2-v_1)^{2\eta_1})}{E_{\eta_1}(\frac{-\eta_1}{1-\eta_1}(v_2-v_1)^{\eta_1})}.$$

Proof Following [17], we have the following key result:

$$^{ABR}_{\psi(x)}\mathsf{D}^{\eta_1}_{\psi^{-1}(v_1)+}f(x) = \frac{B(\eta_1)}{1-\eta_1}\sum_{k=0}^{\infty}\left(\frac{-\eta_1}{1-\eta_1}\right)^k {}^{RL}_{\psi(x)}\mathsf{J}^{k\eta_1}_{\psi^{-1}(v_1)+}f(x). \tag{22}$$

The series being locally uniformly convergent (see [17]). From this formula, we see that the AB derivatives can be written purely in terms of RL integrals. In view of this result, many results can be proved in the AB model directly from the corresponding results in the RL model.

Here, we apply the HH-inequality for fractional ψ-RL integrals (18) with η_1 replaced by $k\eta_1$:

$$
f\left(\frac{v_1 + v_2}{2}\right) \le \frac{\Gamma(k\eta_1 + 1)}{2(v_2 - v_1)^{k\eta_1}} \Big[{}^{RL}_{\psi(x)}\mathrm{J}^{k\eta_1}_{\psi^{-1}(v_1)+}(f \circ \psi)\big(\psi^{-1}(v_2)\big)
$$
$$
+ {}^{RL}_{\psi(x)}\mathrm{J}^{k\eta_1}_{\psi^{-1}(v_2)-}(f \circ \psi)\big(\psi^{-1}(v_1)\big) \Big]
$$
$$
\le \frac{f(v_1) + f(v_2)}{2}.
$$

Multiplying on all sides of the inequality by positive constants, we get

$$
\frac{2(v_2 - v_1)^{k\eta_1}}{\Gamma(k\eta_1 + 1)} f\left(\frac{v_1 + v_2}{2}\right)
$$
$$
\le {}^{RL}_{\psi(x)}\mathrm{J}^{k\eta_1}_{\psi^{-1}(v_1)+}(f \circ \psi)\big(\psi^{-1}(v_2)\big) + {}^{RL}_{\psi(x)}\mathrm{J}^{k\eta_1}_{\psi^{-1}(v_2)-}(f \circ \psi)\big(\psi^{-1}(v_1)\big)
$$
$$
\le \frac{2(v_2 - v_1)^{k\eta_1}}{\Gamma(k\eta_1 + 1)} \cdot \frac{f(v_1) + f(v_2)}{2}.
$$

The constant $\frac{-\eta_1}{1-\eta_1}$ is negative since $0 < \eta_1 < 1$. So, we need to split the above inequality into two cases according to the parity of k.

If k is even, then we have

$$
\left(\frac{-\eta_1}{1-\eta_1}\right)^k \frac{2(v_2 - v_1)^{k\eta_1}}{\Gamma(k\eta_1 + 1)} f\left(\frac{v_1 + v_2}{2}\right)
$$
$$
\le \left(\frac{-\eta_1}{1-\eta_1}\right)^k \Big[{}^{RL}_{\psi(x)}\mathrm{J}^{k\eta_1}_{\psi^{-1}(v_1)+}(f \circ \psi)\big(\psi^{-1}(v_2)\big) + {}^{RL}_{\psi(x)}\mathrm{J}^{k\eta_1}_{\psi^{-1}(v_2)-}(f \circ \psi)\big(\psi^{-1}(v_1)\big) \Big]
$$
$$
\le \left(\frac{-\eta_1}{1-\eta_1}\right)^k \frac{2(v_2 - v_1)^{k\eta_1}}{\Gamma(k\eta_1 + 1)} \cdot \frac{f(v_1) + f(v_2)}{2}.
$$

If k is odd, then we have

$$
\left(\frac{-\eta_1}{1-\eta_1}\right)^k \frac{2(v_2 - v_1)^{k\eta_1}}{\Gamma(k\eta_1 + 1)} \cdot \frac{f(v_1) + f(v_2)}{2}
$$
$$
\le \left(\frac{-\eta_1}{1-\eta_1}\right)^k \Big[{}^{RL}_{\psi(x)}\mathrm{J}^{k\eta_1}_{\psi^{-1}(v_1)+}(f \circ \psi)\big(\psi^{-1}(v_2)\big) + {}^{RL}_{\psi(x)}\mathrm{J}^{k\eta_1}_{\psi^{-1}(v_2)-}(f \circ \psi)\big(\psi^{-1}(v_1)\big) \Big]
$$
$$
\le \left(\frac{-\eta_1}{1-\eta_1}\right)^k \frac{2(v_2 - v_1)^{k\eta_1}}{\Gamma(k\eta_1 + 1)} f\left(\frac{v_1 + v_2}{2}\right).
$$

Summing over all k and making use of the series formula (22), we deduce

$$
\sum_{k \text{ even}} \left(\frac{-\eta_1}{1-\eta_1}\right)^k \frac{2(v_2 - v_1)^{k\eta_1}}{\Gamma(k\eta_1 + 1)} f\left(\frac{v_1 + v_2}{2}\right)
$$
$$
+ \sum_{k \text{ odd}} \left(\frac{-\eta_1}{1-\eta_1}\right)^k \frac{2(v_2 - v_1)^{k\eta_1}}{\Gamma(k\eta_1 + 1)} \cdot \frac{f(v_1) + f(v_2)}{2}
$$
$$
\le \frac{1 - \eta_1}{B(\eta_1)} \Big[{}^{ABR}_{\psi(x)}\mathrm{J}^{\eta_1}_{\psi^{-1}(v_1)+}(f \circ \psi)\big(\psi^{-1}(v_2)\big) + {}^{ABR}_{\psi(x)}\mathrm{J}^{\eta_1}_{\psi^{-1}(v_2)-}(f \circ \psi)\big(\psi^{-1}(v_1)\big) \Big]
$$

$$\leq \sum_{k \text{ even}} \left(\frac{-\eta_1}{1-\eta_1} \right)^k \frac{2(\nu_2-\nu_1)^{k\eta_1}}{\Gamma(k\eta_1+1)} \cdot \frac{f(\nu_1)+f(\nu_2)}{2}$$

$$+ \sum_{k \text{ odd}} \left(\frac{-\eta_1}{1-\eta_1} \right)^k \frac{2(\nu_2-\nu_1)^{k\eta_1}}{\Gamma(k\eta_1+1)} f\left(\frac{\nu_1+\nu_2}{2} \right).$$

The sums over even and odd k can be rewritten more precisely as follows:

$$\sum_{n=0}^{\infty} \left(\frac{-\eta_1}{1-\eta_1} \right)^{2n} \frac{2(\nu_2-\nu_1)^{2n\eta_1}}{\Gamma(2n\eta_1+1)} f\left(\frac{\nu_1+\nu_2}{2} \right)$$

$$+ \sum_{n=0}^{\infty} \left(\frac{-\eta_1}{1-\eta_1} \right)^{2n+1} \frac{2(\nu_2-\nu_1)^{(2n+1)\eta_1}}{\Gamma((2n+1)\eta_1+1)} \cdot \frac{f(\nu_1)+f(\nu_2)}{2}$$

$$\leq \frac{1-\eta_1}{B(\eta_1)} \left[{}^{ABR}_{\psi(x)} I^{\eta_1}_{\psi^{-1}(\nu_1)+} (f \circ \psi)(\psi^{-1}(\nu_2)) + {}^{ABR}_{\psi(x)} \mathbb{J}^{\eta_1}_{\psi^{-1}(\nu_2)-} (f \circ \psi)(\psi^{-1}(\nu_1)) \right]$$

$$\leq \sum_{n=0}^{\infty} \left(\frac{-\eta_1}{1-\eta_1} \right)^{2n} \frac{2(\nu_2-\nu_1)^{2n\eta_1}}{\Gamma(2n\eta_1+1)} \cdot \frac{f(\nu_1)+f(\nu_2)}{2}$$

$$+ \sum_{n=0}^{\infty} \left(\frac{-\eta_1}{1-\eta_1} \right)^{2n+1} \frac{2(\nu_2-\nu_1)^{(2n+1)\eta_1}}{\Gamma((2n+1)\eta_1+1)} f\left(\frac{\nu_1+\nu_2}{2} \right).$$

The infinite series appearing here can be re-explained as the power series for ML functions:

$$2 E_{2\eta_1} \left(\left(\frac{-\eta_1}{1-\eta_1} \right)^2 (\nu_2-\nu_1)^{2\eta_1} \right) f\left(\frac{\nu_1+\nu_2}{2} \right)$$

$$+ 2 \left(\frac{-\eta_1}{1-\eta_1} \right) (\nu_2-\nu_1)^{\eta_1} E_{2\eta_1,\eta_1+1} \left(\left(\frac{-\eta_1}{1-\eta_1} \right)^2 (\nu_2-\nu_1)^{2\eta_1} \right) \frac{f(\nu_1)+f(\nu_2)}{2}$$

$$\leq \frac{1-\eta_1}{B(\eta_1)} \left[{}^{ABR}_{\psi(x)} \mathbb{J}^{\eta_1}_{\psi^{-1}(\nu_1)+} (f \circ \psi)(\psi^{-1}(\nu_2)) + {}^{ABR}_{\psi(x)} \mathbb{J}^{\eta_1}_{\psi^{-1}(\nu_2)-} (f \circ \psi)(\psi^{-1}(\nu_1)) \right]$$

$$\leq 2 E_{2\eta_1} \left(\left(\frac{-\eta_1}{1-\eta_1} \right)^2 (\nu_2-\nu_1)^{2\eta_1} \right) \frac{f(\nu_1)+f(\nu_2)}{2}$$

$$+ 2 \left(\frac{-\eta_1}{1-\eta_1} \right) (\nu_2-\nu_1)^{\eta_1} E_{2\eta_1,\eta_1+1} \left(\left(\frac{-\eta_1}{1-\eta_1} \right)^2 (\nu_2-\nu_1)^{2\eta_1} \right) f\left(\frac{\nu_1+\nu_2}{2} \right). \qquad (23)$$

We observe that the two functions $E_{2\eta_1}((\frac{-\eta_1}{1-\eta_1})^2 (\nu_2 - \nu_1)^{2\eta_1})$ and $(\frac{-\eta_1}{1-\eta_1})(\nu_2 - \nu_1)^{\eta_1} \times E_{2\eta_1,\eta_1+1}((\frac{-\eta_1}{1-\eta_1})^2 (\nu_2 - \nu_1)^{2\eta_1})$ that appear as multipliers, since they arise from the even and odd parts of a single infinite series, become a simpler ML function when they are added together:

$$E_{2\eta_1} \left(\left(\frac{-\eta_1}{1-\eta_1} \right)^2 (\nu_2-\nu_1)^{2\eta_1} \right) + \left(\frac{-\eta_1}{1-\eta_1} \right) (\nu_2-\nu_1)^{\eta_1} E_{2\eta_1,\eta_1+1} \left(\left(\frac{-\eta_1}{1-\eta_1} \right)^2 (\nu_2-\nu_1)^{2\eta_1} \right)$$

$$= \sum_{k \text{ even}} \left(\frac{-\eta_1}{1-\eta_1} \right)^k \frac{(\nu_2-\nu_1)^{k\eta_1}}{\Gamma(k\eta_1+1)} + \sum_{k \text{ odd}} \left(\frac{-\eta_1}{1-\eta_1} \right)^k \frac{(\nu_2-\nu_1)^{k\eta_1}}{\Gamma(k\eta_1+1)}$$

$$= \sum_{k=0}^{\infty} \left(\frac{-\eta_1}{1-\eta_1} \right)^k \frac{(v_2 - v_1)^{k\eta_1}}{\Gamma(k\eta_1 + 1)}$$

$$= E_{\eta_1} \left(\frac{-\eta_1}{1-\eta_1} (v_2 - v_1)^{\eta_1} \right).$$

Also, the sum function $E_{\eta_1}(\frac{-\eta_1}{1-\eta_1}(v_2 - v_1)^{\eta_1})$ is strictly positive [45], so it allows us to divide by it in inequality (23) to get

$$\frac{E_{2\eta_1}((\frac{-\eta_1}{1-\eta_1})^2 (v_2 - v_1)^{2\eta_1})}{E_{\eta_1}(\frac{-\eta_1}{1-\eta_1}(v_2 - v_1)^{\eta_1})} f\left(\frac{v_1 + v_2}{2} \right)$$

$$+ \frac{\frac{-\eta_1}{1-\eta_1}(v_2 - v_1)^{\eta_1} E_{2\eta_1, \eta_1 + 1}((\frac{-\eta_1}{1-\eta_1})^2 (v_2 - v_1)^{2\eta_1})}{E_{\eta_1}(\frac{-\eta_1}{1-\eta_1}(v_2 - v_1)^{\eta_1})} \cdot \frac{f(v_1) + f(v_2)}{2}$$

$$\leq \frac{1 - \eta_1}{2B(\eta_1) E_{\eta_1}(\frac{-\eta_1}{1-\eta_1}(v_2 - v_1)^{\eta_1})}$$

$$\times \left[{}^{ABR}_{\psi(x)} \mathcal{J}^{\eta_1}_{\psi^{-1}(v_1)+} (f \circ \psi)(\psi^{-1}(v_2)) + {}^{ABR}_{\psi(x)} \mathcal{J}^{\eta_1}_{\psi^{-1}(v_2)-} (f \circ \psi)(\psi^{-1}(v_1)) \right]$$

$$\leq \frac{E_{2\eta_1}((\frac{-\eta_1}{1-\eta_1})^2 (v_2 - v_1)^{2\eta_1})}{E_{\eta_1}(\frac{-\eta_1}{1-\eta_1}(v_2 - v_1)^{\eta_1})} \cdot \frac{f(v_1) + f(v_2)}{2}$$

$$+ \frac{\frac{-\eta_1}{1-\eta_1}(v_2 - v_1)^{\eta_1} E_{2\eta_1, \eta_1 + 1}((\frac{-\eta_1}{1-\eta_1})^2 (v_2 - v_1)^{2\eta_1})}{E_{\eta_1}(\frac{-\eta_1}{1-\eta_1}(v_2 - v_1)^{\eta_1})} f\left(\frac{v_1 + v_2}{2} \right),$$

which is the required result. □

Corollary 3.2 *Theorem* 3.2 *with* $\psi(x) = x$ *becomes*

$$\mathcal{Q}_1(\eta_1, v_2 - v_1) f\left(\frac{v_1 + v_2}{2} \right) + \mathcal{Q}_2(\eta_1, v_2 - v_1) \frac{f(v_1) + f(v_2)}{2}$$

$$\leq \frac{1 - \eta_1}{2B(\eta_1) E_{\eta_1}(\frac{-\eta_1}{1-\eta_1}(v_2 - v_1)^{\eta_1})} \left[{}^{ABR}D^{\eta_1}_{v_1+} f(v_2) + {}^{ABR}D^{\eta_1}_{v_2-} f(v_1) \right]$$

$$\leq \mathcal{Q}_2(\eta_1, v_2 - v_1) f\left(\frac{v_1 + v_2}{2} \right) + \mathcal{Q}_1(\eta_1, v_2 - v_1) \frac{f(v_1) + f(v_2)}{2},$$

where the multipliers $\mathcal{Q}_1(\eta_1, v_2 - v_1)$ *and* $\mathcal{Q}_2(\eta_1, v_2 - v_1)$ *are defined as follows*:

$$\mathcal{Q}_1(\eta_1, v_2 - v_1) = \frac{E_{2\eta_1}((\frac{-\eta_1}{1-\eta_1})^2 (v_2 - v_1)^{2\eta_1})}{E_{\eta_1}(\frac{-\eta_1}{1-\eta_1}(v_2 - v_1)^{\eta_1})};$$

$$\mathcal{Q}_2(\eta_1, v_2 - v_1) = \frac{\frac{-\eta_1}{1-\eta_1}(v_2 - v_1)^{\eta_1} E_{2\eta_1, \eta_1 + 1}((\frac{-\eta_1}{1-\eta_1})^2 (v_2 - v_1)^{2\eta_1})}{E_{\eta_1}(\frac{-\eta_1}{1-\eta_1}(v_2 - v_1)^{\eta_1})},$$

which is proved by Fernandez and Mohammed in [19, *Theorem* 2.1].

Example 3.1 We verify the result of Theorem 3.2 for $\psi(x) = x$ and the convex function $f(x) = e^{3x}$. Due to [32], we have

$$^{RL}\mathcal{J}^{\eta_1}_{c+} e^{3x} = e^{3x} 3^{-\eta_1} \frac{\gamma(\eta_1, 3(x-c))}{\Gamma(\eta_1)}, \qquad ^{RL}\mathcal{J}^{\eta_1}_{c-} e^{3x} = e^{3x} (-3)^{-\eta_1} \frac{\gamma(\eta_1, 3(x-c))}{\Gamma(\eta_1)}. \tag{24}$$

By making use of the series formula (22), we have

$$
{}^{ABR}\mathrm{D}_{c+}^{\eta_1}e^{3x} = \frac{B(\eta_1)}{1-\eta_1}e^{3x}\sum_{k=0}^{\infty}\left(\frac{-\eta_1}{1-\eta_1}\right)^k 3^{-k\eta_1}\frac{\gamma(k\eta_1,3(x-c))}{\Gamma(k\eta_1)},
$$

$$
{}^{ABR}\mathrm{D}_{c-}^{\eta_1}e^{3x} = \frac{B(\eta_1)}{1-\eta_1}e^{3x}\sum_{k=0}^{\infty}\left(\frac{-\eta_1}{1-\eta_1}\right)^k (-3)^{-k\eta_1}\frac{\gamma(k\eta_1,3(x-c))}{\Gamma(k\eta_1)}.
$$

Substituting the above expressions into HH-inequality (21), we get

$$
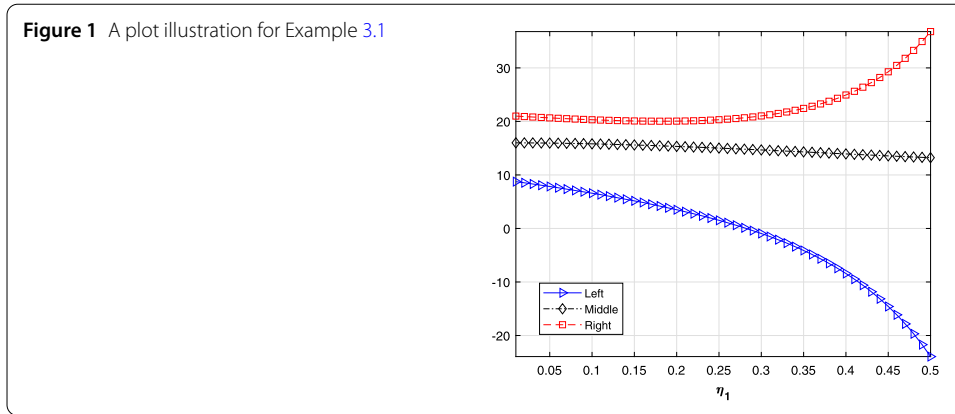\mathcal{Q}_1(\eta_1,\nu_2-\nu_1)e^{3(\nu_1+\nu_2)/2} + \mathcal{Q}_2(\eta_1,\nu_2-\nu_1)\frac{e^{3\nu_1}+e^{3\nu_2}}{2}
$$

$$
\leq \frac{1-\eta_1}{2B(\eta_1)E_{\eta_1}\left(\frac{-\eta_1}{1-\eta_1}(\nu_2-\nu_1)^{\eta_1}\right)}\left[\frac{B(\eta_1)}{1-\eta_1}e^{3\nu_2}\sum_{k=0}^{\infty}\left(\frac{-\eta_1}{1-\eta_1}\right)^k 3^{-k\eta_1}\frac{\gamma(k\eta_1,3(\nu_2-\nu_1))}{\Gamma(k\eta_1)}\right.
$$

$$
\left.+ \frac{B(\eta_1)}{1-\eta_1}e^{3\nu_1}\sum_{k=0}^{\infty}\left(\frac{-\eta_1}{1-\eta_1}\right)^k (-3)^{-k\eta_1}\frac{\gamma(k\eta_1,3(\nu_1-b))}{\Gamma(k\eta_1)}\right]
$$

$$
\leq \mathcal{Q}_2(\eta_1,\nu_2-\nu_1)e^{3(\nu_1+\nu_2)/2} + \mathcal{Q}_1(\eta_1,\nu_2-\nu_1)\frac{e^{3\nu_1}+e^{3\nu_2}}{2}.
$$

Additional simplification process gives

$$
\mathcal{Q}_1(\eta_1,\nu_2-\nu_1)e^{3(\nu_1+\nu_2)/2} + \mathcal{Q}_2(\eta_1,\nu_2-\nu_1)\frac{e^{3\nu_1}+e^{3\nu_2}}{2}
$$

$$
\leq \frac{1}{2E_{\eta_1}\left(\frac{-\eta_1}{1-\eta_1}(\nu_2-\nu_1)^{\eta_1}\right)}
$$

$$
\times \sum_{k=0}^{\infty}\frac{(\frac{-\eta_1}{1-\eta_1})^k}{\Gamma(k\eta_1)}\left[3^{-k\eta_1}e^{3\nu_2}\gamma\left(k\eta_1,3(\nu_2-\nu_1)\right)+(-3)^{-k\eta_1}e^{3\nu_1}\gamma\left(k\eta_1,3(\nu_1-b)\right)\right]
$$

$$
\leq \mathcal{Q}_2(\eta_1,\nu_2-\nu_1)e^{3(\nu_1+\nu_2)/2} + \mathcal{Q}_1(\eta_1,\nu_2-\nu_1)\frac{e^{3\nu_1}+e^{3\nu_2}}{2},
$$

or dropping the multiplier $\mathcal{Q}$ leads to

$$
2E_{2\eta_1}\left(\left(\frac{-\eta_1}{1-\eta_1}\right)^2(\nu_2-\nu_1)^{2\eta_1}\right)e^{3(\nu_1+\nu_2)/2}
$$

$$
+ \frac{-\eta_1}{1-\eta_1}(\nu_2-\nu_1)^{\eta_1}E_{2\eta_1,\eta_1+1}\left(\left(\frac{-\eta_1}{1-\eta_1}\right)^2(\nu_2-\nu_1)^{2\eta_1}\right)\left(e^{3\nu_1}+e^{3\nu_2}\right)
$$

$$
\leq \sum_{k=0}^{\infty}\frac{(\frac{-\eta_1}{1-\eta_1})^k}{\Gamma(k\eta_1)}\left[3^{-k\eta_1}e^{3\nu_2}\gamma\left(k\eta_1,3(\nu_2-\nu_1)\right)+(-3)^{-k\eta_1}e^{3\nu_1}\gamma\left(k\eta_1,3(\nu_1-b)\right)\right]
$$

$$
\leq 2\frac{-\eta_1}{1-\eta_1}(\nu_2-\nu_1)^{\eta_1}E_{2\eta_1,\eta_1+1}\left(\left(\frac{-\eta_1}{1-\eta_1}\right)^2(\nu_2-\nu_1)^{2\eta_1}\right)e^{3(\nu_1+\nu_2)/2}
$$

$$
+ E_{2\eta_1}\left(\left(\frac{-\eta_1}{1-\eta_1}\right)^2(\nu_2-\nu_1)^{2\eta_1}\right)\left(e^{3\nu_1}+e^{3\nu_2}\right).
$$

Figure 1 A plot illustration for Example 3.1

For a specific value of $\eta_1 \in (0, \frac{1}{2})$ and $v_1 = 0$, $v_2 = 1$, this inequality becomes

$$
2E_{2\eta_1}\left(\left(\frac{-\eta_1}{1-\eta_1}\right)^2\right)e^{3/2} + \frac{-\eta_1}{1-\eta_1}E_{2\eta_1,\eta_1+1}\left(\left(\frac{-\eta_1}{1-\eta_1}\right)^2\right)\left(1+e^3\right)
$$

$$
\leq \sum_{k=0}^{\infty}\frac{(\frac{-\eta_1}{1-\eta_1})^k}{\Gamma(k\eta_1)}\left[3^{-k\eta_1}e^3\gamma(k\eta_1,3) + (-3)^{-k\eta_1}\gamma(k\eta_1,-3)\right]
$$

$$
\leq 2\frac{-\eta_1}{1-\eta_1}E_{2\eta_1,\eta_1+1}\left(\left(\frac{-\eta_1}{1-\eta_1}\right)^2\right)e^{3/2} + E_{2\eta_1}\left(\left(\frac{-\eta_1}{1-\eta_1}\right)^2\right)\left(1+e^3\right). \tag{25}
$$

The three functions given by the left-, middle-, and right-hand sides of the double inequality (25) are plotted in Fig. 1 against $\eta_1 \in (0, \frac{1}{2})$ to demonstrate clearly that both inequalities are valid.

Theorem 3.3 *If $f : [v_1, v_2] \to \Re$ is an L^1 convex function, ψ is an increasing positive function on $(v_1, v_2]$ with $\psi'(x) \in L^1(v_1, v_2)$, and $\eta_1, \eta_2, \gamma, \omega > 0$, then we have*

$$
f\left(\frac{v_1+v_2}{2}\right) \leq \frac{{}^{P}_{\psi(x)}\mathrm{J}^{\eta_1,\eta_2,\gamma,\omega}_{\psi^{-1}(v_1)+}(f\circ\psi)(\psi^{-1}(v_2)) + {}^{P}_{\psi(x)}\mathrm{J}^{\eta_1,\eta_2,\gamma,\omega}_{\psi^{-1}(v_2)-}(f\circ\psi)(\psi^{-1}(v_1))}{2(v_2-v_1)^{\eta_2}E^{\gamma}_{\eta_1,\eta_2+1}(\omega(v_2-v_1)^{\eta_1})}
$$

$$
\leq \frac{f(v_1)+f(v_2)}{2}. \tag{26}
$$

Proof As in the theorem proof, analogous to (22), we have the following key result due to [17]:

$$
{}^{P}_{\psi(x)}\mathrm{J}^{\eta_1,\eta_2,\gamma,\omega}_{\psi^{-1}(v_1)+}f(x) = \sum_{k=0}^{\infty}\frac{\Gamma(\gamma+k)\omega^k}{\Gamma(\gamma)k!}{}^{RL}_{\psi(x)}\mathrm{J}^{k\eta_1+\eta_2}_{\psi^{-1}(v_1)+}f(x). \tag{27}
$$

Again, this series is locally uniformly convergent (see [17]).

By making use of HH-inequality (18) and replacing η_1 by $k\eta_1 + \eta_2$, we get

$$
f\left(\frac{v_1+v_2}{2}\right) \leq \frac{\Gamma(k\eta_1+\eta_2+1)}{2(v_2-v_1)^{k\eta_1+\eta_2}}\left[{}^{RL}_{\psi(x)}\mathrm{J}^{k\eta_1+\eta_2}_{\psi^{-1}(v_1)+}(f\circ\psi)(\psi^{-1}(v_2))\right.
$$

$$
\left. + {}^{RL}_{\psi(x)}\mathrm{J}^{k\eta_1+\eta_2}_{\psi^{-1}(v_2)-}(f\circ\psi)(\psi^{-1}(v_1))\right] \leq \frac{f(v_1)+f(v_2)}{2}.
$$

Since the multiplier in the central term is positive and $\gamma, \omega > 0$ in series (27), so we can proceed as follows:

$$
\frac{\Gamma(\gamma + k)\omega+}{\Gamma(\gamma)k!} \cdot \frac{2(\nu_2 - \nu_1)^{k\eta_1 + \eta_2}}{\Gamma(k\eta_1 + \eta_2 + 1)} f\left(\frac{\nu_1 + \nu_2}{2}\right)
$$

$$
\leq \frac{\Gamma(\gamma + k)\omega+}{\Gamma(\gamma)k!} \left[{}^{RL}_{\psi(x)}\mathbb{J}^{k\eta_1 + \eta_2}_{\psi^{-1}(\nu_1)+}(f \circ \psi)\left(\psi^{-1}(\nu_2)\right) + {}^{RL}_{\psi(x)}\mathbb{J}^{k\eta_1 + \eta_2}_{\psi^{-1}(\nu_2)-}(f \circ \psi)\left(\psi^{-1}(\nu_1)\right) \right]
$$

$$
\leq \frac{\Gamma(\gamma + k)\omega+}{\Gamma(\gamma)k!} \cdot \frac{2(\nu_2 - \nu_1)^{k\eta_1 + \eta_2}}{\Gamma(k\eta_1 + \eta_2 + 1)} \cdot \frac{f(\nu_1) + f(\nu_2)}{2}.
$$

Summing over all k to get

$$
\sum_{k=0}^{\infty} \frac{\Gamma(\gamma + k)\omega+}{\Gamma(\gamma)k!} \cdot \frac{2(\nu_2 - \nu_1)^{k\eta_1 + \eta_2}}{\Gamma(k\eta_1 + \eta_2 + 1)} f\left(\frac{\nu_1 + \nu_2}{2}\right)
$$

$$
\leq {}^{P}_{\psi(x)}\mathbb{J}^{\eta_1,\eta_2,\gamma,\omega}_{\psi^{-1}(\nu_1)+}(f \circ \psi)\left(\psi^{-1}(\nu_2)\right) + {}^{P}_{\psi(x)}\mathbb{J}^{\eta_1,\eta_2,\gamma,\omega}_{\psi^{-1}(\nu_2)-}(f \circ \psi)\left(\psi^{-1}(\nu_1)\right)
$$

$$
\leq \sum_{k=0}^{\infty} \frac{\Gamma(\gamma + k)\omega+}{\Gamma(\gamma)k!} \cdot \frac{2(\nu_2 - \nu_1)^{k\eta_1 + \eta_2}}{\Gamma(k\eta_1 + \eta_2 + 1)} \cdot \frac{f(\nu_1) + f(\nu_2)}{2}.
$$

The left- and right-hand sides series in this inequality can be rewritten as the power series for the three-parameter ML function:

$$
2(\nu_2 - \nu_1)^{\eta_2} E^{\gamma}_{\eta_1, \eta_2 + 1}\left(\omega(\nu_2 - \nu_1)^{\eta_1}\right) f\left(\frac{\nu_1 + \nu_2}{2}\right)
$$

$$
\leq {}^{P}_{\psi(x)}\mathbb{J}^{\eta_1,\eta_2,\gamma,\omega}_{\psi^{-1}(\nu_1)+}(f \circ \psi)\left(\psi^{-1}(\nu_2)\right) + {}^{P}_{\psi(x)}\mathbb{J}^{\eta_1,\eta_2,\gamma,\omega}_{\psi^{-1}(\nu_2)-}(f \circ \psi)\left(\psi^{-1}(\nu_1)\right)
$$

$$
\leq 2(\nu_2 - \nu_1)^{\eta_2} E^{\gamma}_{\eta_1, \eta_2 + 1}\left(\omega(\nu_2 - \nu_1)^{\eta_1}\right) \frac{f(\nu_1) + f(\nu_2)}{2}.
$$

The result follows by a simple rearrangement. □

Corollary 3.3 *Theorem* 3.3 *with* $\psi(x) = x$ *becomes*

$$
f\left(\frac{\nu_1 + \nu_2}{2}\right) \leq \frac{{}^{P}\mathbb{J}^{\eta_1,\eta_2,\gamma,\omega}_{\nu_1+}f(\nu_2) + {}^{P}\mathbb{J}^{\eta_1,\eta_2,\gamma,\omega}_{\nu_2-}f(\nu_1)}{2(\nu_2 - \nu_1)^{\eta_2} E^{\gamma}_{\eta_1, \eta_2 + 1}(\omega(\nu_2 - \nu_1)^{\eta_1})} \leq \frac{f(\nu_1) + f(\nu_2)}{2},
$$

which is proved by Fernandez and Mohammed in [19, *Theorem* 2.2].

Example 3.2 We verify our result in Theorem 3.3 for $\psi(x) = x$ and the convex function $f(x) = e^{2x}$ on the interval $[0, 1]$. By making use of (24) and (27), it follows that

$$
{}^{P}\mathbb{J}^{\eta_1,\eta_2,\gamma,\omega}_{d+}e^{2x} = e^{2x}\sum_{k=0}^{\infty} \frac{\Gamma(\gamma + k)\omega+}{\Gamma(\gamma)k!} 2^{-k\eta_1 - \eta_2} \frac{\gamma(k\eta_1 + \eta_2, 2(x - d))}{\Gamma(k\eta_1 + \eta_2)},
$$

$$
{}^{P}\mathbb{J}^{\eta_1,\eta_2,\gamma,\omega}_{d-}e^{2x} = e^{2x}\sum_{k=0}^{\infty} \frac{\Gamma(\gamma + k)\omega+}{\Gamma(\gamma)k!} (-2)^{-k\eta_1 - \eta_2} \frac{\gamma(k\eta_1 + \eta_2, 2(x - d))}{\Gamma(k\eta_1 + \eta_2)}.
$$

(a) For $\eta_1 = \frac{1}{2}, \gamma = 4$ (b) For $\eta_1 = \frac{1}{4}, \gamma = 2$

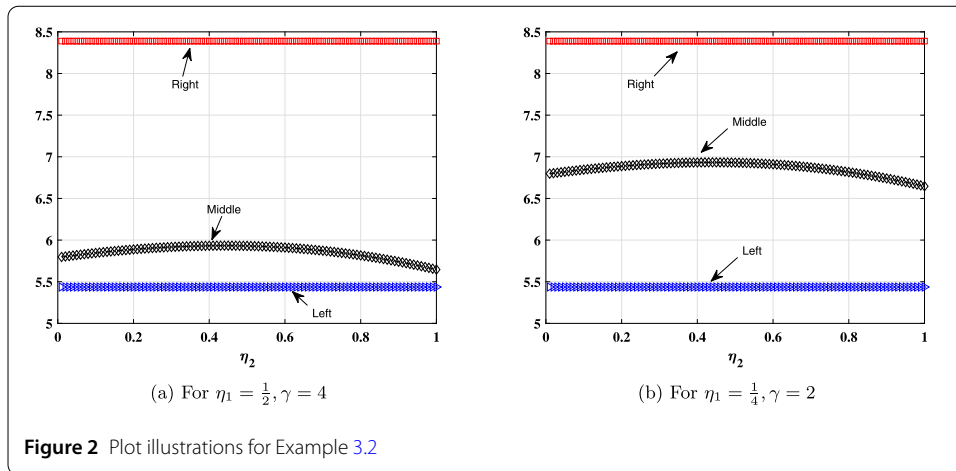**Figure 2** Plot illustrations for Example 3.2

Substituting the above expressions into HH-inequality (26), we get

$$
e \leq \frac{e^2 \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)\omega+}{\Gamma(\gamma)k!} 2^{-k\eta_1 - \eta_2} \frac{\gamma(k\eta_1 + \eta_2, 2)}{\Gamma(k\eta_1 + \eta_2)} + \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)\omega+}{\Gamma(\gamma)k!} (-2)^{-k\eta_1 - \eta_2} \frac{\gamma(k\eta_1 + \eta_2, -2)}{\Gamma(k\eta_1 + \eta_2)}}{2 E_{\eta_1, \eta_2 + 1}^{\gamma}(\omega)}
$$

$$
\leq \frac{1 + e^2}{2},
$$

which may be simplified to

$$
2e \leq \frac{1}{\Gamma(\gamma) E_{\eta_1, \eta_2 + 1}^{\gamma}(\omega)} \sum_{k=0}^{\infty} \frac{\Gamma(\gamma + k)\omega+}{k! \Gamma(k\eta_1 + \eta_2)} \big[e^2 2^{-k\eta_1 - \eta_2} \gamma(k\eta_1 + \eta_2, 2)
$$

$$
+ (-2)^{-k\eta_1 - \eta_2} \gamma(k\eta_1 + \eta_2, -2) \big] \leq 1 + e^2. \tag{28}
$$

For further illustration of this result, we set $\omega = 1$ and allow η_2 to vary between 0 and 1. Also, for several choices of η_1 and γ, the results are shown in Fig. 2. In each situation, the middle inequality of (28) is between the constant values of the left- and right-hand sides.

4 A related integral equality

Hermite–Hadamard (HH) inequalities are strongly connected with further well-known integral inequalities which are said to be of trapezoidal type in the literature, see e.g. [11, 36–42, 47, 53, 54]. In the coming theorem, we prepare an equality of trapezoidal type for the new integral operator. Also, we believe this result will be vital information for the future studies of these models of fractional calculus.

Theorem 4.1 *Suppose that $f : [v_1, v_2] \to \Re$ is an L^1 function with $\eta_1 \in (0, 1)$. Also, suppose that $f' \in \mathrm{L}^1([v_1, v_2])$, and ψ is an increasing positive function on $(v_1, v_2]$ with $\psi'(x) \in \mathrm{L}^1(v_1, v_2)$, then we have*

$$
\frac{f(v_1) + f(v_2)}{2} - \frac{B(\eta_1) \Gamma(\eta_1)}{2[(v_2 - v_1)^{\eta_1} + (1 - \eta_1)\Gamma(\eta_1)]} \Big[{}_{\psi(x)}^{AB}\mathfrak{J}_{\psi^{-1}(v_1)+}^{\eta_1} (f \circ \psi)(\psi^{-1}(v_2))
$$

$$
+ {}_{\psi(x)}^{AB}\mathfrak{J}_{\psi^{-1}(v_2)-}^{\eta_1} (f \circ \psi)(\psi^{-1}(v_1)) \Big]
$$

$$= \frac{(v_2 - v_1)^{\eta_1 - 1}}{2[(v_2 - v_1)^{\eta_1} + (1 - \eta_1)\Gamma(\eta_1)]}$$

$$\times \int_{\psi^{-1}(v_1)}^{\psi^{-1}(v_2)} \left[(\psi(\xi) - v_1)^{\eta_1} - (v_2 - \psi(\xi))^{\eta_1} \right] (f' \circ \psi)(\xi)\psi'(\xi)\, d\xi. \tag{29}$$

Proof By making use of Lemma 3.1, we get

$$_{\psi(x)}^{RL}\mathcal{J}_{\psi^{-1}(v_1)+}^{\eta_1}(f \circ \psi)(\psi^{-1}(v_2)) = \frac{B(\eta_1)}{\eta_1}\, _{\psi(x)}^{AB}\mathcal{J}_{\psi^{-1}(v_1)+}^{\eta_1}(f \circ \psi)(\psi^{-1}(v_2)) - \frac{1 - \eta_1}{\eta_1}f(v_2);$$

$$_{\psi(x)}^{RL}\mathcal{J}_{\psi^{-1}(v_2)-}^{\eta_1}(f \circ \psi)(\psi^{-1}(v_1)) = \frac{B(\eta_1)}{\eta_1}\, _{\psi(x)}^{AB}\mathcal{J}_{\psi^{-1}(v_2)-}^{\eta_1}(f \circ \psi)(\psi^{-1}(v_1)) - \frac{1 - \eta_1}{\eta_1}f(v_1).$$

By adding the last two inequalities together and then multiplying by $\frac{\Gamma(\eta_1+1)}{2(v_2-v_1)^{\eta_1}} > 0$, we obtain

$$\frac{\Gamma(\eta_1 + 1)}{2(v_2 - v_1)^{\eta_1}} \left[_{\psi(x)}^{RL}\mathcal{J}_{\psi^{-1}(v_1)+}^{\eta_1}f(v_2) + _{\psi(x)}^{RL}\mathcal{J}_{\psi^{-1}(v_2)-}^{\eta_1}f(v_1) \right]$$

$$= \frac{B(\eta_1)\Gamma(\eta_1)}{2(v_2 - v_1)^{\eta_1}} \left[_{\psi(x)}^{AB}\mathcal{J}_{\psi^{-1}(v_1)+}^{\eta_1}(f \circ \psi)(\psi^{-1}(v_2)) + _{\psi(x)}^{AB}\mathcal{J}_{\psi^{-1}(v_2)-}^{\eta_1}(f \circ \psi)(\psi^{-1}(v_1)) \right]$$

$$- \frac{(1 - \eta_1)\Gamma(\eta_1)}{(v_2 - v_1)^{\eta_1}} \cdot \frac{f(v_1) + f(v_2)}{2}.$$

This gives

$$\frac{f(v_1) + f(v_2)}{2} - \frac{\Gamma(\eta_1 + 1)}{2(v_2 - v_1)^{\eta_1}} \left[_{\psi(x)}^{RL}\mathcal{J}_{\psi^{-1}(v_1)+}^{\eta_1}f(v_2) + _{\psi(x)}^{RL}\mathcal{J}_{\psi^{-1}(v_2)-}^{\eta_1}f(v_1) \right]$$

$$= \frac{(v_2 - v_1)^{\eta_1} + (1 - \eta_1)\Gamma(\eta_1)}{(v_2 - v_1)^{\eta_1}} \cdot \frac{f(v_1) + f(v_2)}{2} - \frac{B(\eta_1)\Gamma(\eta_1)}{2(v_2 - v_1)^{\eta_1}}$$

$$\times \left[_{\psi(x)}^{AB}\mathcal{J}_{\psi^{-1}(v_1)+}^{\eta_1}(f \circ \psi)(\psi^{-1}(v_2)) + _{\psi(x)}^{AB}\mathcal{J}_{\psi^{-1}(v_2)-}^{\eta_1}(f \circ \psi)(\psi^{-1}(v_1)) \right]. \tag{30}$$

Following [30, Lemma 3.1], it follows that

$$\frac{f(v_1) + f(v_2)}{2} - \frac{\Gamma(\eta_1 + 1)}{2(v_2 - v_1)^{\eta_1}} \left[_{\psi(x)}^{RL}\mathcal{J}_{\psi^{-1}(v_1)+}^{\eta_1}(f \circ \psi)(\psi^{-1}(v_2)) \right.$$

$$\left. + _{\psi(x)}^{RL}\mathcal{J}_{\psi^{-1}(v_2)-}^{\eta_1}(f \circ \psi)(\psi^{-1}(v_1)) \right]$$

$$= \frac{1}{2(v_2 - v_1)} \int_{\psi^{-1}(v_1)}^{\psi^{-1}(v_2)} \left[(\psi(\xi) - v_1)^{\eta_1} - (v_2 - \psi(\xi))^{\eta_1} \right] (f' \circ \psi)(\xi)\psi'(\xi)\, d\xi. \tag{31}$$

Making use of equations (30) and (31) completes the proof. □

Corollary 4.1 *Theorem 4.1 with $\psi(x) = x$ becomes*

$$\frac{f(v_1) + f(v_2)}{2} - \frac{B(\eta_1)\Gamma(\eta_1)}{2[(v_2 - v_1)^{\eta_1} + (1 - \eta_1)\Gamma(\eta_1)]} \left[^{AB}\mathcal{J}_{v_1+}^{\eta_1}f(v_2) + ^{AB}\mathcal{J}_{v_2-}^{\eta_1}f(v_1) \right]$$

$$= \frac{(v_2 - v_1)^{\eta_1+1}}{2[(v_2 - v_1)^{\eta_1} + (1 - \eta_1)\Gamma(\eta_1)]} \int_0^1 \left[(1 - \xi)^{\eta_1} - \xi^{\eta_1} \right] f'(tv_1 + (1 - t)v_2)\, d\xi,$$

which is proved by Fernandez and Mohammed in [19, Theorem 3.1].

5 Conclusions

In this article, we have performed a connection between the Atangana–Baleanu and Riemann–Liouville fractional integrals of a function with respect to an increasing function with nonsingular kernel. Also, we have recalled the iterated formula of the Prabhakar fractional operator implemented by Fernandez and Baleanu [17]. In view of these indices, we have presented new efficient inequalities of Hermite–Hadamard type (HH-type) in the context of fractional calculus with respect to functions involving nonsingular kernels. In fact, our findings generalize those in [19]. Due to the extensive recent studies and applications of AB and Prabhakar fractional operators, we believe that our results will be important for the future studies of those models of fractional calculus and integral inequality.

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