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INTEGRAL INEQUALITIES FOR LIPSCHITZIAN MAPPINGS BETWEEN TWO BANACH SPACES AND APPLICATIONS

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Abstract

In this paper we obtain some inequalities of Ostrowski and Hermite-Hadamard type for Lipschitzian mappings between two Banach spaces. Applications for functions of norms in Banach spaces and functions defined by power series in Banach algebras are provided as well.

1. Introduction

Let $\mathscr{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^*A)^{1/2}$.

One of the central problems in perturbation theory is to find bounds for

$$\|f(A) - f(B)\|$$

in terms of $\|A - B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [5], [32] and the references therein.

It is known that [4] in the infinite-dimensional case the map $f(A) := |A|$ is not *Lipschitz continuous* on $\mathscr{B}(H)$ with the usual operator norm, i.e. there is no constant $L > 0$ such that

$$\| \, |A| - |B| \, \| \le L\|A - B\|$$

for any $A, B \in \mathscr{B}(H)$.

However, as shown by Farforovskaya in [30], [31] and Kato in [37], the following inequality holds

$$\| \, |A| - |B| \, \| \le \frac{2}{\pi}\|A - B\|\left(2 + \log\left(\frac{\|A\| + \|B\|}{\|A - B\|}\right)\right) \tag{1.1}$$

for any $A, B \in \mathscr{B}(H)$ with $A \ne B$.

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If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS} := (\operatorname{tr} C^*C)^{1/2}$ of an operator C, then the following inequality is true [2]

$$\| |A| - |B| \|_{HS} \le \sqrt{2}\|A - B\|_{HS} \tag{1.2}$$

for any $A, B \in \mathscr{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [4] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\| |A| - |B| \| \le a_1\|A - B\| + a_2\|A - B\|^2 + O(\|A - B\|^3) \tag{1.3}$$

where

$$a_1 = \|A^{-1}\| \, \|A\| \quad \text{and} \quad a_2 = \|A^{-1}\| + \|A^{-1}\|^3\|A\|^2.$$

In [3] the author also obtained the following *Lipschitz type inequality*

$$\|f(A) - f(B)\| \le f'(a)\|A - B\| \tag{1.4}$$

where f is an *operator monotone function* on $(0, \infty)$ and $A, B \ge aI_H > 0$.

If we use, for instance, the inequality (1.4), then we can state that

$$\|f((1-t)A + tB) - f((1-s)A + sB)\| \le f'(a)\|A - B\| \, |t - s| \tag{1.5}$$

for any $t, s \in [0, 1]$, where f is an operator monotone function on $(0, \infty)$ and $A, B \ge aI_H > 0$.

Further on, if we integrate the inequality (1.5) over $s \in [0, 1]$ and if we use the integral triangle inequality then we have successively

$$\begin{aligned} &\left\| f((1-t)A + tB) - \int_0^1 f((1-s)A + sB)\, ds \right\| \\ &\le \int_0^1 \|f((1-t)A + tB) - f((1-s)A + sB)\|\, ds \\ &\le f'(a)\|A - B\| \int_0^1 |t - s|\, ds = \frac{1}{2} f'(a)\|A - B\|[t^2 + (1-t)^2] \\ &= \left[\frac{1}{4} + \left(t - \frac{1}{2}\right)^2\right] f'(a)\|A - B\| \end{aligned} \tag{1.6}$$

for any $t \in [0, 1]$.

This is an Ostrowski type inequality for operator monotone functions.

In particular, we get the following Hermite-Hadamard type inequality or mid-point inequality

$$\left\| f\left(\frac{A+B}{2}\right) - \int_0^1 f((1-s)A + sB)\, ds \right\| \le \frac{1}{4} f'(a)\|A - B\|$$

where f is an operator monotone function on $(0, \infty)$ and $A, B \ge aI_H > 0$.

We recall the original Ostrowski inequality [42]

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] M(b-a) \tag{1.7}$$

provided that $f : [a,b] \to \mathbf{R}$ is differentiable on (a,b) and $|f'(u)| \le M$ for all $u \in (a,b)$. The best inequality one can obtain from (1.7) is the mid-point inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \le \frac{1}{4} M(b-a) \tag{1.8}$$

with $\frac{1}{4}$ the best possible constant.

For Hermite-Hadamard's type inequalities, namely

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(u)\,du \le \frac{f(a)+f(b)}{2}, \tag{1.9}$$

where $f : [a,b] \to \mathbf{R}$ is convex on $[a,b]$, see for instance [10], [11], [12], [33], [35], [36], [38], [40], [41], [44], [45], [46], [47], [48] and the references therein.

For Ostrowski's type inequalities for the Lebesgue integral, see [1], [8]–[9] and [14]–[27]. Inequalities for the Riemann-Stieltjes integral may be found in [16], [18] while the generalization for isotonic functionals was provided in [19]. For the case of functions of self-adjoint operators on complex Hilbert spaces, see the recent monograph [22].

Motivated by the above results we investigate in this paper some norm inequalities of Ostrowski and Hermite-Hadamard type to approximate the integral $\int_0^1 F[uy + (1-u)x]\,du$ for the mapping $F : C \subset X \to Y$ that is a *Lipschitzian* mapping with the constant $L > 0$ on the convex subset C of the Banach space X and with values in another Banach space Y. Applications for functions of norms in Banach spaces and functions defined by power series in Banach algebras are provided as well.

2. Integral inequalities

Let $(X; \|\cdot\|_X)$ and $(Y; \|\cdot\|_Y)$ be two Banach spaces over the complex number field $\mathbf{C}$. Let C be a convex set in X. For any mapping $F : C \subset X \to Y$ we can consider the associated function $\Phi_{F,x,y,\lambda} : [0,1] \to Y$, where $x, y \in C$, $\lambda \in [0,1]$, defined by

$$\begin{aligned} \Phi_{F,x,y,\lambda}(t) := {} & (1-\lambda)F[(1-t)((1-\lambda)x + \lambda y) + ty] \\ & + \lambda F[(1-t)x + t((1-\lambda)x + \lambda y)]. \end{aligned} \tag{2.1}$$

We observe that for $\lambda = 0$ and $\lambda = 1$ we have

$$\Phi_{F,x,y,0}(t) = \Phi_{F,x,y,1}(t) = F[(1-t)x + ty]$$

and

$$\Phi_{F,x,y,1/2}(t) = \frac{1}{2}\left(F\left[(1-t)\frac{x+y}{2} + ty\right] + F\left[(1-t)x + t\frac{x+y}{2}\right]\right)$$

where $x, y \in B$.

There are some particular values of interest, namely

$$\Phi_{F,x,y,0}(0) = \Phi_{F,x,y,1}(0) = F(x), \quad \Phi_{F,x,y,0}(1) = \Phi_{F,x,y,1}(1) = F(y),$$

$$\Phi_{F,x,y,0}\left(\frac{1}{2}\right) = \Phi_{F,x,y,1}\left(\frac{1}{2}\right) = F\left(\frac{x+y}{2}\right),$$

$$\Phi_{F,x,y,1/2}(0) = \frac{1}{2}\left[F\left(\frac{x+y}{2}\right) + F(x)\right], \quad \Phi_{F,x,y,1/2}(1) = \frac{1}{2}\left[F(y) + F\left(\frac{x+y}{2}\right)\right]$$

and

$$\Phi_{F,x,y,1/2}\left(\frac{1}{2}\right) = \frac{1}{2}\left[F\left(\frac{3x+y}{4}\right) + F\left(\frac{x+3y}{4}\right)\right]$$

where $x, y \in B$.

The following result holds.

Lemma 1. *Let $F : C \subset X \to Y$ be a continuous mapping on the convex subset C of X. If $x, y \in C$, then*

$$\int_0^1 \Phi_{F,x,y,\lambda}(t)\, dt = \int_0^1 F[uy + (1-u)x]\, du \tag{2.2}$$

for any $\lambda \in [0,1]$.

Proof. Since F is continuous on C the integrals involved in (2.2) exist.

For $\lambda = 0$ and $\lambda = 1$ the equality (2.2) is obvious.

Let $\lambda \in (0,1)$. Observe that

$$\int_0^1 F[(1-t)(\lambda y + (1-\lambda)x) + ty]\, dt = \int_0^1 F[((1-t)\lambda + t)y + (1-t)(1-\lambda)x]\, dt$$

and

$$\int_0^1 F[t(\lambda y + (1-\lambda)x) + (1-t)x]\, dt = \int_0^1 F[t\lambda y + (1-\lambda t)x]\, dt.$$

If we make the change of variable $u := (1-t)\lambda + t$ then we have $1-u = (1-t)(1-\lambda)$ and $du = (1-\lambda)\, du$. Then

$$\int_0^1 F[((1-t)\lambda + t)y + (1-t)(1-\lambda)x]\, dt = \frac{1}{1-\lambda}\int_\lambda^1 F[uy + (1-u)x]\, du.$$

If we make the change of variable $u := \lambda t$ then we have $du = \lambda\, dt$ and

$$\int_0^1 F[t\lambda y + (1 - \lambda t)x]\, dt = \frac{1}{\lambda}\int_0^\lambda F[uy + (1 - u)x]\, du.$$

Therefore

$$\begin{aligned}(1-\lambda)&\int_0^1 F[(1-t)(\lambda y + (1-\lambda)x) + ty]\, dt \\ &+ \lambda\int_0^1 F[t(\lambda y + (1-\lambda)x) + (1-t)x]\, dt \\ &= \int_\lambda^1 F[uy + (1-u)x]\, du + \int_0^\lambda F[uy + (1-u)x]\, du \\ &= \int_0^1 F[uy + (1-u)x]\, du\end{aligned}$$

and the identity (2.2) is proved. □

We say that the mapping $F : B \subset X \to Y$ is *Lipschitzian* with the constant $L > 0$ on the subset B of X if

$$\|F(x) - F(y)\|_Y \le L\|x - y\|_X \quad \text{for any } x, y \in B. \tag{2.3}$$

The following lemma is of interest in itself.

Lemma 2. *Let $F : C \subset X \to Y$ be a Lipschitzian mapping with the constant $L > 0$ on the convex subset C of X. If $x, y \in C$ and $\lambda \in [0, 1]$ then for any $t_1, t_2 \in [0, 1]$ we have*

$$\|\Phi_{F,x,y,\lambda}(t_2) - \Phi_{F,x,y,\lambda}(t_1)\|_Y \le L|t_2 - t_1|\, \|x - y\|_X[(1-\lambda)^2 + \lambda^2]. \tag{2.4}$$

In particular we have

$$\begin{aligned}\|\Phi_{F,x,y,0}(t_2) - \Phi_{F,x,y,0}(t_1)\|_Y &\le L|t_2 - t_1|\, \|x - y\|_X, \\ \|\Phi_{F,x,y,1}(t_2) - \Phi_{F,x,y,1}(t_1)\|_Y &\le L|t_2 - t_1|\, \|x - y\|_X\end{aligned} \tag{2.5}$$

and

$$\|\Phi_{F,x,y,1/2}(t_2) - \Phi_{F,x,y,1/2}(t_1)\|_Y \le \frac{1}{2}L|t_2 - t_1|\, \|x - y\|_X \tag{2.6}$$

for any $x, y \in C$.

Proof. For any $t_1, t_2 \in [0,1]$ we have, by the triangle inequality, that

$$\begin{aligned} (2.7)\quad \|\Phi_{F,x,y,\lambda}(t_2) - \Phi_{F,x,y,\lambda}(t_1)\|_Y &= \|(1-\lambda)F[(1-t_2)((1-\lambda)x+\lambda y)+t_2 y] \\ &\quad + \lambda F[(1-t_2)x + t_2((1-\lambda)x+\lambda y)] \\ &\quad - (1-\lambda)F[(1-t_1)((1-\lambda)x+\lambda y)+t_1 y] \\ &\quad - \lambda F[(1-t_1)x + t_1((1-\lambda)x+\lambda y)]\|_Y \\ &\le (1-\lambda)\|F[(1-t_2)((1-\lambda)x+\lambda y)+t_2 y] \\ &\quad - F[(1-t_1)((1-\lambda)x+\lambda y)+t_1 y]\|_Y \\ &\quad + \lambda\|F[(1-t_2)x + t_2((1-\lambda)x+\lambda y)] \\ &\quad - F[(1-t_1)x + t_1((1-\lambda)x+\lambda y)]\|_Y \end{aligned}$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

Since $F : C \subset X \to Y$ is a *Lipschitzian* mapping with the constant $L > 0$, then

$$\begin{aligned} (2.8)\quad &\|F[(1-t_2)((1-\lambda)x+\lambda y)+t_2 y] - F[(1-t_1)((1-\lambda)x+\lambda y)+t_1 y]\|_Y \\ &\quad \le L\|(1-t_2)((1-\lambda)x+\lambda y)+t_2 y - (1-t_1)((1-\lambda)x+\lambda y) - t_1 y\|_Y \\ &\quad = L(1-\lambda)|t_2 - t_1|\,\|x-y\|_X \end{aligned}$$

and

$$\begin{aligned} (2.9)\quad &\|F[(1-t_2)x + t_2((1-\lambda)x+\lambda y)] - F[(1-t_1)x + t_1((1-\lambda)x+\lambda y)]\|_Y \\ &\quad \le L\|(1-t_2)x + t_2((1-\lambda)x+\lambda y) - (1-t_1)x - t_1((1-\lambda)x+\lambda y)\|_Y \\ &\quad = L\lambda|t_2 - t_1|\,\|x-y\|_X \end{aligned}$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

From (2.6)–(2.9) we get the inequality (2.4).

The other inequalities are obvious. □

The following result holds:

THEOREM 1. *Let $F : C \subset X \to Y$ be a Lipschitzian mapping with the constant $L > 0$ on the convex subset C of X. If $x, y \in C$, then we have*

$$\begin{aligned} (2.10)\quad &\left\| \Phi_{F,x,y,\lambda}(t) - \int_0^1 F[sy + (1-s)x]\,ds \right\|_Y \\ &\qquad \le 2L\left[\frac{1}{4} + \left(t - \frac{1}{2}\right)^2\right]\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right]\|x-y\|_X \end{aligned}$$

for any $t \in [0,1]$ and $\lambda \in [0,1]$.

Proof. By Lemma 2 we have

$$\|\Phi_{F,x,y,\lambda}(t) - \Phi_{F,x,y,\lambda}(s)\|_Y \le L|t-s|\,\|x-y\|_X[(1-\lambda)^2 + \lambda^2] \tag{2.11}$$

for any $t, s \in [0,1]$ and $\lambda \in [0,1]$.

Integrating (2.11) over s on $[0,1]$ we get

$$\begin{aligned}\left\|\Phi_{F,x,y,\lambda}(t) - \int_0^1 \Phi_{F,x,y,\lambda}(s)\,ds\right\|_Y &\le \int_0^1 \|\Phi_{F,x,y,\lambda}(t) - \Phi_{F,x,y,\lambda}(s)\|_Y\,ds \\ &\le L\|x-y\|_X[(1-\lambda)^2+\lambda^2]\int_0^1 |t-s|\,ds,\end{aligned} \tag{2.12}$$

for any $t \in [0,1]$ and $\lambda \in [0,1]$.

By Lemma 1 we have

$$\int_0^1 \Phi_{F,x,y,\lambda}(s)\,ds = \int_0^1 F[sy+(1-s)x]\,ds$$

and since

$$\int_0^1 |t-s|\,ds = \frac{1}{2}[t^2 + (t-1)^2] = \left[\frac{1}{4} + \left(t - \frac{1}{2}\right)^2\right]$$

and

$$(1-\lambda)^2 + \lambda^2 = 2\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right],$$

then by (2.12) we get the desired result (2.10). □

The best inequality we can get from (2.10) is as follows:

Corollary 1. *Let $F : C \subset X \to Y$ be a Lipschitzian mapping with the constant $L > 0$ on the convex subset C of X. If $x, y \in C$, then we have*

$$\left\|\frac{1}{2}\left[F\left(\frac{3x+y}{4}\right) + F\left(\frac{x+3y}{4}\right)\right] - \int_0^1 F[sy+(1-s)x]\,ds\right\|_Y \le \frac{1}{8}L\|x-y\|_X. \tag{2.13}$$

The constant $\frac{1}{8}$ is best possible in (2.13).

Proof. The inequality (2.13) follows by (2.10) by taking $t = \lambda = \frac{1}{2}$.

Consider the function $F : [0,1] \to \mathbf{R}$,

$$F(t) = \begin{cases} \left|t - \frac{1}{4}\right|, & t \in \left[0, \frac{1}{2}\right] \\ \left|t - \frac{3}{4}\right|, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

We observe that F is absolutely continuos with $\|F'\|_{[0,1],\infty} = 1$ and therefore Lipschitzian with the constant $L = 1$ and if we take $x = 0$, $y = 1$ then we get

$$\frac{1}{2}\left[F\left(\frac{1}{4}\right) + F\left(\frac{3}{4}\right)\right] - \int_0^{1/2}\left|s - \frac{1}{4}\right| ds - \int_{1/2}^1 \left|s - \frac{3}{4}\right| ds = -\frac{1}{8},$$

showing that the equality case is realized in (2.13). □

Remark 1. If we take in (2.10) $\lambda = 0$ then we get the Ostrowski type inequality

$$\left\|F[(1-t)x + ty] - \int_0^1 F[sy + (1-s)x]\, ds\right\|_Y \le L\left[\frac{1}{4} + \left(t - \frac{1}{2}\right)^2\right]\|x - y\|_X \tag{2.14}$$

for any $x, y \in C$ and $t \in [0,1]$.

The best inequality we can get from (2.14) is for $t = \frac{1}{2}$, namely, the mid-point inequality

$$\left\|F\left(\frac{x+y}{2}\right) - \int_0^1 F[sy + (1-s)x]\, ds\right\|_Y \le \frac{1}{4}L\|x - y\|_X. \tag{2.15}$$

The constant $\frac{1}{4}$ is best possible in (2.15). The scalar case was obtained in [11].

If we take in (2.10) $\lambda = \frac{1}{2}$ then we get

$$\begin{aligned} &\left\|\frac{1}{2}\left(F\left[(1-t)\frac{x+y}{2} + ty\right] + F\left[(1-t)x + t\frac{x+y}{2}\right]\right)\right. \\ &\qquad \left. - \int_0^1 F[sy + (1-s)x]\, ds\right\| \\ &\qquad \le \frac{1}{2}L\left[\frac{1}{4} + \left(t - \frac{1}{2}\right)^2\right]\|x - y\|_X \end{aligned} \tag{2.16}$$

for any $x, y \in C$ and $t \in [0,1]$.

The best inequality we can get from (2.16) is for $t = \frac{1}{2}$, namely, the inequality (2.13).

Let C be a convex set in X and a mapping $F : C \subset X \to Y$. We can also consider another associated function $\Psi_{F,x,y,\lambda} : [0,1] \to Y$, where $x, y \in C$, $\lambda \in [0,1]$, defined by

$$\begin{aligned} \Psi_{F,x,y,\lambda}(t) := {} & (1-\lambda)F[(1-t)((1-\lambda)x + \lambda y) + ty] \\ & + \lambda F[tx + (1-t)((1-\lambda)x + \lambda y)]. \end{aligned} \tag{2.17}$$

We observe that for $\lambda = 0$ and $\lambda = 1$ we have

$$\Psi_{F,x,y,0}(t) = F[(1-t)x + ty], \quad \Psi_{F,x,y,1}(t) = F[tx + (1-t)y]$$

and

$$\Psi_{F,x,y,1/2}(t) = \frac{1}{2}\left(F\left[(1-t)\frac{x+y}{2} + ty\right] + F\left[tx + (1-t)\frac{x+y}{2}\right]\right),$$

where $x, y \in B$.

There are some particular values of interest, namely

$$\Psi_{F,x,y,0}(0) = F(x), \quad \Psi_{F,x,y,1}(0) = F(y), \quad \Psi_{F,x,y,0}(1) = F(y), \quad \Psi_{F,x,y,1}(1) = F(x)$$

$$\Psi_{F,x,y,0}\left(\frac{1}{2}\right) = \Psi_{F,x,y,1}\left(\frac{1}{2}\right) = F\left(\frac{x+y}{2}\right),$$

$$\Psi_{F,x,y,1/2}(0) = F\left(\frac{x+y}{2}\right), \quad \Psi_{F,x,y,1/2}(1) = \frac{1}{2}[F(x) + F(y)]$$

and

$$\Psi_{F,x,y,1/2}\left(\frac{1}{2}\right) = \frac{1}{2}\left[F\left(\frac{3x+y}{4}\right) + F\left(\frac{x+3y}{4}\right)\right]$$

where $x, y \in B$.

Lemma 3. *Let $F : C \subset X \to Y$ be a continuous mapping on the convex subset C of X. If $x, y \in C$, then*

$$\int_0^1 \Psi_{F,x,y,\lambda}(t)\, dt = \int_0^1 F[uy + (1-u)x]\, du. \tag{2.18}$$

for any $\lambda \in [0, 1]$.

Proof. Since F is continuous on C the integrals involved in (2.2) exist. For $\lambda = 0$ and $\lambda = 1$ the equality (2.18) is obvious. Let $\lambda \in (0, 1)$. Observe that

$$\int_0^1 \Psi_{F,x,y,\lambda}(t)\, dt = (1-\lambda)\int_0^1 F[(1-t)((1-\lambda)x + \lambda y) + ty]\, dt$$
$$+ \lambda \int_0^1 F[tx + (1-t)((1-\lambda)x + \lambda y)]\, dt.$$

If we change the variable $1 - t = s$, then we have

$$\int_0^1 F[tx + (1-t)((1-\lambda)x + \lambda y)]\, dt = \int_0^1 F[(1-s)x + s((1-\lambda)x + \lambda y)]\, ds.$$

Making use of the equalities provided by the change of variables in the proof of Lemma 1 we get the desired result (2.18). We omit the details. □

We have the following properties for the function $\Psi_{F,x,y,\lambda}$:

LEMMA 4. *Let $F : C \subset X \to Y$ be a Lipschitzian mapping with the constant $L > 0$ on the convex subset C of X. If $x, y \in C$ and $\lambda \in [0,1]$, then for any $t_1, t_2 \in [0,1]$ the inequalities* (2.4), (2.5) *and* (2.6) *hold for the function $\Psi_{F,x,y,\lambda}$ as well.*

Using Lemma 3 and 4 we can also state the following inequality:

THEOREM 2. *Let $F : C \subset X \to Y$ be a Lipschitzian mapping with the constant $L > 0$ on the convex subset C of X. If $x, y \in C$, then we have*

$$\left\| \Psi_{F,x,y,\lambda}(t) - \int_0^1 F[sy + (1-s)x]\, ds \right\|_Y \leq 2L\left[\frac{1}{4} + \left(t - \frac{1}{2}\right)^2\right]\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right]\|x - y\|_X \tag{2.19}$$

for any $t \in [0,1]$ and $\lambda \in [0,1]$.

Remark 2. If we take in (2.19) $\lambda = \frac{1}{2}$, then we get

$$\left\| \frac{1}{2}\left(F\left[(1-t)\frac{x+y}{2} + ty\right] + F\left[tx + (1-t)\frac{x+y}{2}\right]\right) - \int_0^1 F[sy + (1-s)x]\, ds \right\|_Y \leq \frac{1}{2}L\left[\frac{1}{4} + \left(t - \frac{1}{2}\right)^2\right]\|x - y\|_X \tag{2.20}$$

for any $t \in [0,1]$ and $x, y \in C$.

In particular, if we take in (2.20) $t = 1$, then we get the trapezoid type inequality

$$\left\| \frac{1}{2}[F(x) + F(y)] - \int_0^1 F[sy + (1-s)x]\, ds \right\|_Y \leq \frac{1}{4}L\|x - y\|_X \tag{2.21}$$

for any $x, y \in C$. The constant $\frac{1}{4}$ is best possible in (2.21). The scalar case was obtained in [41].

If we take in (2.20) $t = 0$, then we get the mid-point inequality (2.15). If we take in (2.20) $t = \frac{1}{2}$, then we get the inequality (2.13).

3. Applications for norm inequalities

Let X be a real linear space, $a, b \in X$, $a \neq b$ and let $[a,b] := \{(1-\lambda)a + \lambda b, \lambda \in [0,1]\}$ be the *segment* generated by a and b. We consider the function $f : [a,b] \to \mathbf{R}$ and the attached function $g(a,b) : [0,1] \to \mathbf{R}$, $g(a,b)(t) := f[(1-t)a + tb]$, $t \in [0,1]$.

It is well known that f is convex on $[a,b]$ iff $g(a,b)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $g'_{\pm}(a,b)(s) = (\nabla_{\pm} f[(1-s)a + sb])(b-a)$, $s \in [0,1)$

(ii) $g'_{+}(a,b)(0) = (\nabla_{+} f(a))(b-a)$

(iii) $g'_{-}(a,b)(1) = (\nabla_{-} f(b))(b-a)$

where $(\nabla_{\pm} f(x))(y)$ are the *Gâteaux lateral derivatives*, we recall that

$$(\nabla_{+} f(x))(y) := \lim_{h \to 0+} \frac{f(x+hy) - f(x)}{h},$$

$$(\nabla_{-} f(x))(y) := \lim_{k \to 0-} \frac{f(x+ky) - f(x)}{k}, \quad x, y \in X.$$

Now, assume that $(X, \|\cdot\|)$ is a normed linear space. The function $f(x) = \frac{1}{2}\|x\|^2$, $x \in X$ is convex and thus the following limits exist

(iv) $\langle x, y \rangle_s := (\nabla_{+} f_0(y))(x) = \lim_{t \to 0+} \dfrac{\|y + tx\|^2 - \|y\|^2}{2t}$;

(v) $\langle x, y \rangle_i := (\nabla_{-} f_0(y))(x) = \lim_{s \to 0-} \dfrac{\|y + sx\|^2 - \|y\|^2}{2s}$;

for any $x, y \in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [13]), assuming that $p, q \in \{s, i\}$ and $p \neq q$:

(a) $\langle x, x \rangle_p = \|x\|^2$ for all $x \in X$;

(aa) $\langle \alpha x, \beta y \rangle_p = \alpha\beta \langle x, y \rangle_p$ if $\alpha, \beta \geq 0$ and $x, y \in X$;

(aaa) $|\langle x, y \rangle_p| \leq \|x\| \|y\|$ for all $x, y \in X$;

(av) $\langle \alpha x + y, x \rangle_p = \alpha \langle x, x \rangle_p + \langle y, x \rangle_p$ if $x, y \in X$ and $\alpha \in \mathbf{R}$;

(v) $\langle -x, y \rangle_p = -\langle x, y \rangle_q$ for all $x, y \in X$;

(va) $\langle x + y, z \rangle_p \leq \|x\| \|z\| + \langle y, z \rangle_p$ for all $x, y, z \in X$;

(vaa) The mapping $\langle \cdot, \cdot \rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p = s$ (or $p = i$);

(vaaa) The normed linear space $(X, \|\cdot\|)$ is smooth at the point $x_0 \in X \backslash \{0\}$ if and only if $\langle y, x_0 \rangle_s = \langle y, x_0 \rangle_i$ for all $y \in X$; in general $\langle y, x \rangle_i \leq \langle y, x \rangle_s$ for all $x, y \in X$;

(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle \cdot, \cdot \rangle$, then $\langle y, x \rangle_i = \langle y, x \rangle = \langle y, x \rangle_s$ for all $x, y \in X$.

Now, consider the functions $f(x) = \|x\|^r$, $r \geq 1$, $x \in X$ where $(X, \|\cdot\|)$ is a Banach space.

The following result holds:

LEMMA 5. *Let* $r \geq 1$ *and* $x, y \in X \backslash \{0\}$, *then*

$$| \, \|y\|^r - \|x\|^r | \leq r \|y - x\| \int_0^1 \|(1-t)x + ty\|^{r-1} \, dt. \tag{3.1}$$

The inequality (3.1) *is sharp.*

Proof. Let $r > 1$ and $x, y \in X \backslash \{0\}$.

Assume that x and y are in X and such that $(1-t)x + ty \neq 0$ for any $t \in [0,1]$. The function $g(x,y)(t) := \|(1-t)x + ty\|^r$ is convex on $[0,1]$ and absolutely continuous on $[0,1]$. The derivative $g'(x,y)$ exists almost everywhere on $[0,1]$ and

$$g'(x,y)(t) = r\|(1-t)x + ty\|^{r-2} \langle y - x, (1-t)x + ty \rangle_p$$

for almost every $t \in [0,1]$, where p is either s or i.

Therefore

$$\begin{aligned} \|y\|^r - \|x\|^r &= g(x,y)(1) - g(x,y)(0) \\ &= \int_0^1 g'(x,y)(t) \, dt \\ &= r \int_0^1 \|(1-t)x + ty\|^{r-2} \langle y - x, (1-t)x + ty \rangle_p \, dt, \end{aligned} \tag{3.2}$$

which is an equality of interest in itself.

Taking the modulus in (3.2) and utilizing the Schwarz inequality (aaa) we get

$$\begin{aligned} | \, \|y\|^r - \|x\|^r | &\leq r \int_0^1 \|(1-t)x + ty\|^{r-2} |\langle y - x, (1-t)x + ty \rangle_p| \, dt \\ &\leq r\|y - x\| \int_0^1 \|(1-t)x + ty\|^{r-1} \, dt \end{aligned}$$

and the inequality (3.1) is proved.

Now, if there exists $\alpha \in (0,1)$ such that $(1-\alpha)x + \alpha y = 0$, then $y = \frac{\alpha - 1}{\alpha} x$.

We then have

$$| \, \|y\|^r - \|x\|^r | = |\alpha^r - (1-\alpha)^r| \frac{\|x\|^r}{\alpha^r}$$

and

$$\begin{aligned} r\|y - x\| \int_0^1 \|(1-t)x + ty\|^{r-1} \, dt &= r \frac{\|x\|^r}{\alpha^r} \int_0^1 |t - \alpha|^{r-1} \, dt \\ &= r \frac{\|x\|^r}{\alpha^r} \frac{\alpha^r + (1-\alpha)^r}{r} \\ &= [\alpha^r + (1-\alpha)^r] \frac{\|x\|^r}{\alpha^r} \end{aligned}$$

and the inequality (3.1) becomes

$$|\alpha^r - (1-\alpha)^r| \le \alpha^r + (1-\alpha)^r,$$

which is obvious.

If $r = 1$, then (3.1) becomes the continuity inequality for the norm.

If we take $r = 2$, $y = b$, $x = a$ where $b > a > 0$, then we get in both sides of (3.1) the same quantity $b^2 - a^2 > 0$. □

Corollary 2. *Let $r \ge 1$ and $x, y \in X \backslash \{0\}$ with $\|x\|, \|y\| \le M$ for some $M > 0$, then*

$$| \, \|y\|^r - \|x\|^r| \le rM^{r-1} \|y - x\|. \tag{3.3}$$

Proof. It follows by (3.1) on observing that, if $\|x\|, \|y\| \le M$, then $\|(1-t)x + ty\| \le M$ for any $t \in [0,1]$, which implies that $\int_0^1 \|(1-t)x + ty\|^{r-1}\, dt \le M^{r-1}$. □

Remark 3. Let $r \ge 1$. Utilising the inequalities (2.10) and (2.19), we have for $x, y \in X \backslash \{0\}$ with $\|x\|, \|y\| \le M$ that

$$\begin{aligned} &\left| \Lambda_{r,x,y,\lambda}(t) - \int_0^1 \|sy + (1-s)x\|^r \, ds \right| \\ &\qquad \le 2rM^{r-1} \left[\frac{1}{4} + \left(t - \frac{1}{2}\right)^2 \right] \left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2 \right] \|x - y\| \end{aligned} \tag{3.4}$$

where either

$$\begin{aligned} \Lambda_{r,x,y,\lambda}(t) := &(1-\lambda)\|(1-t)((1-\lambda)x + \lambda y) + ty\|^r \\ &+ \lambda \|(1-t)x + t((1-\lambda)x + \lambda y)\|^r \end{aligned}$$

or

$$\begin{aligned} \Lambda_{r,x,y,\lambda}(t) := &(1-\lambda)\|(1-t)((1-\lambda)x + \lambda y) + ty\|^r \\ &+ \lambda \|tx + (1-t)((1-\lambda)x + \lambda y)\|^r \end{aligned}$$

and $t \in [0,1]$, $\lambda \in [0,1]$.

We also have the norm inequalities

$$\begin{aligned} &\left| \frac{1}{2}\left(\left\| (1-t)\frac{x+y}{2} + ty \right\|^r + \left\| (1-t)x + t\frac{x+y}{2} \right\|^r \right) - \int_0^1 \|sy + (1-s)x\|^r \, ds \right| \\ &\quad \le \frac{1}{2} rM^{r-1} \left[\frac{1}{4} + \left(t - \frac{1}{2}\right)^2 \right] \|x - y\|, \end{aligned} \tag{3.5}$$

$$\left|\frac{1}{2}\left(\left\|(1-t)\frac{x+y}{2}+ty\right\|^r+\left\|tx+(1-t)\frac{x+y}{2}\right\|^r\right)-\int_0^1\|sy+(1-s)x\|^r\,ds\right| \le \frac{1}{2}rM^{r-1}\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\|x-y\| \tag{3.6}$$

and

$$\left|\|(1-t)x+ty\|^r-\int_0^1\|sy+(1-s)x\|^r\,ds\right| \le rM^{r-1}\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\|x-y\| \tag{3.7}$$

for any $t\in[0,1]$ and $x,y\in X\backslash\{0\}$ with $\|x\|,\|y\|\le M$.

In particular, we have

$$0\le\frac{1}{2}\left[\left\|\frac{3x+y}{4}\right\|^r+\left\|\frac{x+3y}{4}\right\|^r\right]-\int_0^1\|sy+(1-s)x\|^r\,ds \le\frac{1}{8}rM^{r-1}\|x-y\|, \tag{3.8}$$

$$0\le\int_0^1\|sy+(1-s)x\|^r\,ds-\left\|\frac{x+y}{2}\right\|^r\le\frac{1}{4}rM^{r-1}\|x-y\| \tag{3.9}$$

and

$$0\le\frac{\|x\|^r+\|y\|^r}{2}-\int_0^1\|sy+(1-s)x\|^r\,ds\le\frac{1}{4}rM^{r-1}\|x-y\| \tag{3.10}$$

for any $x,y\in X\backslash\{0\}$ with $\|x\|,\|y\|\le M$ and $r\ge 1$.

The positivity of the terms in (3.8)–(3.10) follows by the Hermite-Hadamard inequality for the convex function $[0,1]\ni s\mapsto\|sy+(1-s)x\|^r$ for $r\ge 1$.

Now, by the help of power series $f(\lambda)=\sum_{n=0}^{\infty}\alpha_n\lambda^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficients of the original series, namely, $f_a(\lambda):=\sum_{n=0}^{\infty}|\alpha_n|\lambda^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $\alpha_n\ge 0$, then $f_a=f$.

In general, we have the following result:

Theorem 3. *Let $f(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbf{C}$, $R>0$. For any $x,y\in X$ with $\|x\|,\|y\|<R$ we have*

$$|f(\|y\|)-f(\|x\|)|\le\|y-x\|\int_0^1 f_a'(\|(1-t)x+ty\|)\,dt. \tag{3.11}$$

Proof. Now, for any $m \geq 1$, by making use of the inequality (3.1) we have

$$\left|\sum_{n=0}^{m} \alpha_n \|y\|^n - \sum_{n=0}^{m} \alpha_n \|x\|^n\right| = \left|\sum_{n=1}^{m} \alpha_n (\|y\|^n - \|x\|^n)\right| \tag{3.12}$$
$$\leq \sum_{n=1}^{m} |\alpha_n| \, | \, \|y\|^n - \|x\|^n|$$
$$\leq \|y - x\| \sum_{n=1}^{m} n|\alpha_n| \int_0^1 \|(1-t)x + ty\|^{n-1} \, dt$$
$$= \|y - x\| \int_0^1 \left(\sum_{n=1}^{m} n|\alpha_n| \, \|(1-t)x + ty\|^{n-1}\right) dt.$$

Moreover, since $\|x\|, \|y\| < R$, then the series $\sum_{n=0}^{\infty} \alpha_n y^n$, $\sum_{n=0}^{\infty} \alpha_n x^n$ and

$$\sum_{n=1}^{\infty} n|\alpha_n| \, \|(1-t)x + ty\|^{n-1}$$

are convergent and

$$\sum_{n=0}^{\infty} \alpha_n \|y\|^n = f(\|y\|), \quad \sum_{n=0}^{\infty} \alpha_n \|x\|^n = f(\|x\|)$$

while

$$\sum_{n=1}^{\infty} n|\alpha_n| \, \|(1-t)x + ty\|^{n-1} = f_a'(\|(1-t)x + ty\|).$$

Therefore, by taking the limit over $m \to \infty$ in the inequality (3.12) we deduce the desired result (3.11). □

Corollary 3. *With the assumptions of Theorem* 3, *for any* $x, y \in X$ *with* $\|x\|, \|y\| \leq M < R$ *and* $M > 0$, *we have*

$$|f(\|y\|) - f(\|x\|)| \leq f_a'(M)\|y - x\|. \tag{3.13}$$

Now, if we consider the associated function $\lambda_{f,x,y,\lambda} : [0,1] \to [0,\infty)$ that is given either by

$$\lambda_{f,x,y,\lambda}(t) := (1-\lambda) f(\|(1-t)((1-\lambda)x + \lambda y) + ty\|)$$
$$+ \lambda f(\|(1-t)x + t((1-\lambda)x + \lambda y)\|),$$

or by

$$\lambda_{f,x,y,\lambda}(t) := (1-\lambda)f(\|(1-t)((1-\lambda)x+\lambda y)+ty\|) + \lambda f(\|tx+(1-t)((1-\lambda)x+\lambda y)\|),$$

then we get the inequality

$$\left|\lambda_{f,x,y,\lambda}(t) - \int_0^1 f(\|sy+(1-s)x\|)\,ds\right| \le 2f_a'(M)\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|x-y\|, \tag{3.14}$$

where $f(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ is a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset \mathbf{C}$, $R>0$ and $x,y\in X$ with $\|x\|,\|y\|\le M<R$ and $M>0$.

In particular, we have from (3.14) the following inequalities

$$\left|\frac{1}{2}\left(f\left(\left\|(1-t)\frac{x+y}{2}+ty\right\|\right)+f\left(\left\|(1-t)x+t\frac{x+y}{2}\right\|\right)\right) - \int_0^1 f(\|sy+(1-s)x\|)\,ds\right| \le \frac{1}{2}f_a'(M)\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\|x-y\|, \tag{3.15}$$

$$\left|\frac{1}{2}\left(f\left(\left\|(1-t)\frac{x+y}{2}+ty\right\|\right)+f\left(\left\|tx+(1-t)\frac{x+y}{2}\right\|\right)\right) - \int_0^1 f(\|sy+(1-s)x\|)\,ds\right| \le \frac{1}{2}f_a'(M)\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\|x-y\| \tag{3.16}$$

and

$$\left|f(\|(1-t)x+ty\|) - \int_0^1 f(\|sy+(1-s)x\|)\,ds\right| \le f_a'(M)\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\|x-y\| \tag{3.17}$$

for any $t\in[0,1]$ and $x,y\in X\backslash\{0\}$ with $\|x\|,\|y\|\le M$.

Moreover, we have

$$(3.18)\qquad \left|\frac{1}{2}\left[f\left(\left\|\frac{3x+y}{4}\right\|\right)+f\left(\left\|\frac{x+3y}{4}\right\|\right)\right]-\int_0^1 f(\|sy+(1-s)x\|)\,ds\right| \le \frac{1}{8}f_a'(M)\|x-y\|,$$

$$(3.19)\qquad \left|f\left(\left\|\frac{x+y}{2}\right\|\right)-\int_0^1 f(\|sy+(1-s)x\|)\,ds\right| \le \frac{1}{4}f_a'(M)\|x-y\|$$

and

$$(3.20)\qquad \left|\frac{f(\|x\|)+f(\|y\|)}{2}-\int_0^1 f(\|sy+(1-s)x\|)\,ds\right| \le \frac{1}{4}f_a'(M)\|x-y\|$$

for any $x, y \in X\backslash\{0\}$ with $\|x\|, \|y\| \le M$.

As some natural examples that are useful for applications, we can point out that, if

$$(3.21)\qquad f(\lambda)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\lambda^n=\ln\frac{1}{1+\lambda},\quad \lambda\in D(0,1);$$

$$g(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\lambda^{2n}=\cos\lambda,\quad \lambda\in \mathbf{C};$$

$$h(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\lambda^{2n+1}=\sin\lambda,\quad \lambda\in \mathbf{C};$$

$$l(\lambda)=\sum_{n=0}^{\infty}(-1)^n\lambda^n=\frac{1}{1+\lambda},\quad \lambda\in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$(3.22)\qquad f_a(\lambda)=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^n=\ln\frac{1}{1-\lambda},\quad \lambda\in D(0,1);$$

$$g_a(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n}=\cosh\lambda,\quad \lambda\in \mathbf{C};$$

$$h_a(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1}=\sinh\lambda,\quad \lambda\in \mathbf{C};$$

$$l_a(\lambda)=\sum_{n=0}^{\infty}\lambda^n=\frac{1}{1-\lambda},\quad \lambda\in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n \quad \lambda \in \mathbf{C}, \tag{3.23}$$

$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \quad \lambda \in D(0,1);$$

$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1}, \quad \lambda \in D(0,1);$$

$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \quad \lambda \in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n, \quad \alpha,\beta,\gamma > 0,$$

$$\lambda \in D(0,1);$$

where Γ is *Gamma function*.

4. Applications for banach algebras

Let $\mathscr{B}$ be an algebra. An *algebra norm* on $\mathscr{B}$ is a map $\|\cdot\| : \mathscr{B} \to [0,\infty)$ such that $(\mathscr{B}, \|\cdot\|)$ is a normed space, and, further:

$$\|ab\| \le \|a\|\,\|b\|$$

for any $a, b \in \mathscr{B}$. The normed algebra $(\mathscr{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm*.

We assume that the Banach algebra is *unital*, this means that $\mathscr{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathscr{B}$ be a unital algebra. An element $a \in \mathscr{B}$ is *invertible* if there exists an element $b \in \mathscr{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\dfrac{1}{a}$. The set of invertible elements of $\mathscr{B}$ is denoted by $\operatorname{Inv}\mathscr{B}$. If $a, b \in \operatorname{Inv}\mathscr{B}$ then $ab \in \operatorname{Inv}\mathscr{B}$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathscr{B}$ and $\lim_{n\to\infty}\|a^n\|^{1/n} < 1$, then $1 - a \in \operatorname{Inv}\mathscr{B}$;

(ii) $\{a \in \mathscr{B} : \|1-b\| < 1\} \subset \operatorname{Inv}\mathscr{B}$;

(iii) $\operatorname{Inv}\mathscr{B}$ is an *open subset* of $\mathscr{B}$;

(iv) The map $\operatorname{Inv}\mathscr{B} \ni a \mapsto a^{-1} \in \operatorname{Inv}\mathscr{B}$ is continuous.

For simplicity, we denote $z1$, where $z \in \mathbf{C}$ and 1 is the identity of $\mathscr{B}$, by z. The *resolvent set* of $a \in \mathscr{B}$ is defined by

$$\rho(a) := \{z \in \mathbf{C} : z - a \in \operatorname{Inv}\mathscr{B}\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbf{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to \operatorname{Inv} \mathscr{B}$,

$$R_a(z) := (z-a)^{-1}.$$

For each $z, w \in \rho(a)$ we have the identity

$$R_a(w) - R_a(z) = (z-w)R_a(z)R_a(w).$$

We also have that

$$\sigma(a) \subset \{z \in \mathbf{C} : |z| \le \|a\|\}.$$

The *spectral radius* of a is defined as

$$v(a) = \sup\{|z| : z \in \sigma(a)\}.$$

If a, b are *commuting* elements in $\mathscr{B}$, i.e. $ab = ba$, then

$$v(ab) \le v(a)v(b) \quad \text{and} \quad v(a+b) \le v(a) + v(b).$$

Let $\mathscr{B}$ a unital Banach algebra and $a \in \mathscr{B}$. Then
(i) The resolvent set $\rho(a)$ is open in $\mathbf{C}$;
(ii) For any *bounded linear functionals* $\lambda : \mathscr{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbf{C}$;
(iv) We have

$$v(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

Let f be an analytic functions on the open disk $D(0,R)$ given by the *power series*

$$f(z) := \sum_{j=0}^{\infty} \alpha_j z^j \quad (|z| < R).$$

If $v(a) < R$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathscr{B}$ because $\sum_{j=0}^{\infty} |\alpha_j| \, \|a^j\| < \infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathscr{B}$ that can be constructed in this way. For instance, the *exponential map* on $\mathscr{B}$ denoted exp and defined as

$$\exp a := \sum_{j=0}^{\infty} \frac{1}{j!} a^j \quad \text{for each } a \in \mathscr{B}.$$

If $\mathscr{B}$ is not commutative, then many of the familiar properties of the exponential function from the scalar case do not hold. The following key formula is valid, however with the additional hypothesis of commutativity for a and b from $\mathcal{B}$

$$\exp(a+b) = \exp(a)\exp(b).$$

In a general Banach algebra $\mathcal{B}$ it is difficult to determine the elements in the range of the exponential map $\exp(\mathcal{B})$, i.e. the element which have a "*logarithm*". However, it is easy to see that if a is an element in B such that $\|1-a\| < 1$, then a is in $\exp(\mathcal{B})$. That follows from the fact that if we set

$$b = -\sum_{n=1}^{\infty} \frac{1}{n}(1-a)^n,$$

then the series converges absolutely and, as in the scalar case, substituting this series into the series expansion for $\exp(b)$ yields $\exp(b) = a$.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [29] and [43].

The following result is valid, see also [28]. For the sake of completeness, we give here a simple proof.

LEMMA 6. *For any $x, y \in \mathcal{B}$ and $n \geq 1$ we have*

$$\|y^n - x^n\| \leq n\|y-x\| \int_0^1 \|(1-t)x+ty\|^{n-1}\,dt. \tag{4.1}$$

Proof. We use the identity (see for instance [6, p. 254])

$$a^n - b^n = \sum_{j=0}^{n-1} a^{n-1-j}(a-b)b^j \tag{4.2}$$

that holds for any $a, b \in \mathcal{B}$ and $n \geq 1$.

For $x, y \in \mathcal{B}$ we consider the function $\varphi : [0,1] \to \mathcal{B}$ defined by $\varphi(t) = [(1-t)x+ty]^n$. For $t \in (0,1)$ and $\varepsilon \neq 0$ with $t+\varepsilon \in (0,1)$ we have from (4.2) that

$$\begin{aligned}\varphi(t+\varepsilon) - \varphi(t) &= [(1-t-\varepsilon)x + (t+\varepsilon)y]^n - [(1-t)x+ty]^n \\ &= \varepsilon \sum_{j=0}^{n-1} [(1-t-\varepsilon)x + (t+\varepsilon)y]^{n-1-j}(y-x)[(1-t)x+ty]^j.\end{aligned}$$

Dividing with $\varepsilon \neq 0$ and taking the limit over $\varepsilon \to 0$ we have in the norm topology of $\mathcal{B}$ that

$$\begin{aligned}\varphi'(t) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\varphi(t+\varepsilon) - \varphi(t)] \\ &= \sum_{j=0}^{n-1} [(1-t)x+ty]^{n-1-j}(y-x)[(1-t)x+ty]^j.\end{aligned} \tag{4.3}$$

Integrating on $[0,1]$ we get from (4.3) that

$$\int_0^1 \varphi'(t)\,dt = \sum_{j=0}^{n-1} \int_0^1 [(1-t)x + ty]^{n-1-j}(y-x)[(1-t)x+ty]^j\,dt$$

and since

$$\int_0^1 \varphi'(t)\,dt = \varphi(1) - \varphi(0) = y^n - x^n$$

then we get the following *equality of interest*

$$y^n - x^n = \sum_{j=0}^{n-1} \int_0^1 [(1-t)x + ty]^{n-1-j}(y-x)[(1-t)x+ty]^j\,dt$$

for any $x, y \in \mathscr{B}$ and $n \geq 1$.

Taking the norm and utilizing the properties of Bochner integral for vector valued functions (see for instance [39, p. 21]) we have

$$\begin{aligned}
\|y^n - x^n\| &\leq \sum_{j=0}^{n-1} \left\| \int_0^1 [(1-t)x + ty]^{n-1-j}(y-x)[(1-t)x+ty]^j\,dt \right\| \\
&\leq \sum_{j=0}^{n-1} \int_0^1 \|[(1-t)x + ty]^{n-1-j}(y-x)[(1-t)x+ty]^j\|\,dt \\
&\leq \sum_{j=0}^{n-1} \int_0^1 \|[(1-t)x + ty]^{n-1-j}\|\,\|y-x\|\,\|[(1-t)x+ty]^j\|\,dt \\
&\leq \sum_{j=0}^{n-1} \int_0^1 \|(1-t)x + ty\|^{n-1-j}\|y-x\|\,\|(1-t)x+ty\|^j\,dt \\
&= n\|y-x\| \int_0^1 \|(1-t)x + ty\|^{n-1}\,dt
\end{aligned}$$

for any $x, y \in \mathscr{B}$ and $n \geq 1$. □

The following result is valid.

THEOREM 4. *Let* $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ *be a function defined by power series with complex coefficients and convergent on the open disk* $D(0,R) \subset \mathbf{C}$, $R > 0$. *For any* $x, y \in \mathscr{B}$ *with* $\|x\|, \|y\| < R$ *we have*

$$\|f(y) - f(x)\| \leq \|y - x\| \int_0^1 f_a'(\|(1-t)x + ty\|)\,dt. \tag{4.4}$$

The proof is similar to the one from Theorem 3 by making use of the Lemma 6. We omit the details.

COROLLARY 4. *With the assumptions of Theorem* 4, *for any* $x, y \in \mathcal{B}$ *with* $\|x\|, \|y\| \le M < R$ *and* $M > 0$, *we have*

$$\|f(y) - f(x)\| \le f_a'(M)\|y - x\|. \tag{4.5}$$

Now, if we consider the associated function $\delta_{f,x,y,\lambda} : [0,1] \to \mathcal{B}$ that is given either by

$$\delta_{f,x,y,\lambda}(t) := (1-\lambda)f((1-t)((1-\lambda)x + \lambda y) + ty) + \lambda f((1-t)x + t((1-\lambda)x + \lambda y)),$$

or by

$$\delta_{f,x,y,\lambda}(t) := (1-\lambda)f((1-t)((1-\lambda)x + \lambda y) + ty) + \lambda f(tx + (1-t)((1-\lambda)x + \lambda y)),$$

then we get the inequality

$$\left\|\delta_{f,x,y,\lambda}(t) - \int_0^1 f(sy + (1-s)x)\, ds\right\| \le 2f_a'(M)\left[\frac{1}{4} + \left(t - \frac{1}{2}\right)^2\right]\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right]\|x - y\|, \tag{4.6}$$

where $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ is a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbf{C}$, $R > 0$ and $x, y \in X$ with $\|x\|, \|y\| \le M < R$ and $M > 0$.

In particular, we have from (4.6) the following inequalities

$$\left\|\frac{1}{2}\left(f\left((1-t)\frac{x+y}{2} + ty\right) + f\left((1-t)x + t\frac{x+y}{2}\right)\right) - \int_0^1 f(sy + (1-s)x)\, ds\right\| \le \frac{1}{2}f_a'(M)\left[\frac{1}{4} + \left(t - \frac{1}{2}\right)^2\right]\|x - y\|, \tag{4.7}$$

$$\left\|\frac{1}{2}\left(f\left((1-t)\frac{x+y}{2} + ty\right) + f\left(tx + (1-t)\frac{x+y}{2}\right)\right) - \int_0^1 f(sy + (1-s)x)\, ds\right\| \le \frac{1}{2}f_a'(M)\left[\frac{1}{4} + \left(t - \frac{1}{2}\right)^2\right]\|x - y\| \tag{4.8}$$

and

$$\left\| f((1-t)x+ty) - \int_0^1 f(sy+(1-s)x)\,ds \right\| \leq f_a'(M)\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\|x-y\| \tag{4.9}$$

for any $t \in [0,1]$ and $x, y \in \mathscr{B}\backslash\{0\}$ with $\|x\|, \|y\| \leq M$.

Moreover, we have

$$\left\| \frac{1}{2}\left[f\left(\frac{3x+y}{4}\right)+f\left(\frac{x+3y}{4}\right)\right] - \int_0^1 f(sy+(1-s)x)\,ds \right\| \leq \frac{1}{8} f_a'(M)\|x-y\|, \tag{4.10}$$

$$\left\| f\left(\frac{x+y}{2}\right) - \int_0^1 f(sy+(1-s)x)\,ds \right\| \leq \frac{1}{4} f_a'(M)\|x-y\| \tag{4.11}$$

and

$$\left\| \frac{f(x)+f(y)}{2} - \int_0^1 f(sy+(1-s)x)\,ds \right\| \leq \frac{1}{4} f_a'(M)\|x-y\| \tag{4.12}$$

for any $x, y \in \mathscr{B}\backslash\{0\}$ with $\|x\|, \|y\| \leq M$.

One can obtain various examples by taking the functions mentioned in the previous section. The details are left to the interested reader.

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