


Research Article

Integral Inequalities Involving Strongly Convex Functions

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Received 25 February 2018; Accepted 6 May 2018; Published 11 June 2018

Academic Editor: Alberto Fiorenza

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We study the notions of strongly convex function as well as F -strongly convex function. We present here some new integral inequalities of Jensen's type for these classes of functions. A refinement of companion inequality to Jensen's inequality established by Matic and Pečarić is shown to be recaptured as a particular instance. Counterpart of the integral Jensen inequality for strongly convex functions is also presented. Furthermore, we present integral Jensen-Steffensen and Slater's inequality for strongly convex functions.

1. Introduction and Preliminaries

The word “convexity” is the most important, natural, and fundamental notations in mathematics. Convex functions were presented by Johan Jensen over 100 years ago. Over the past few years, multiple generalizations and extensions have been made for convexity. These extensions and generalizations in the theory of inequalities have made valuable contributions in many areas of mathematics. Some new generalized concepts in this point of view are quasiconvex [1], strongly convex [2], approximately convex [3], logarithmically convex [4], midconvex functions [5], pseudoconvex [6], φ -convex [7], λ -convex [8], h -convex [9], delta-convex [10], Schur convex [11–15], and others [16–19].

The main ingredient of our investigation is the strongly convex function [2]. Let Ψ be the real function defined on interval I and c be positive number, then we say that the function Ψ is strongly convex with modulus c on I if

$$\Psi(\eta z + (1 - \eta)y) \leq \eta\Psi(z) + (1 - \eta)\Psi(y) - c\eta(1 - \eta)(z - y)^2 \quad (1)$$

for all $y, z \in I$ and $\eta \in [0, 1]$.

Every strongly convex function is convex, but the converse is not true in general. Strongly convex functions have been utilized for proving the convergence of a gradient type

algorithm for minimizing a function. They play a significant role in mathematical economics, approximation theory, and optimization theory. Many applications and properties of them can be found in [2, 9, 20]. In 2016, Adamek [21] further generalized the notion of strongly convex function. They replaced the nonnegative term $c(z - y)^2$ by a nonnegative real valued function F and defined it as follows: function Ψ is said to be F -strongly convex function if

$$\Psi(\eta z + (1 - \eta)y) \leq \eta\Psi(z) + (1 - \eta)\Psi(y) - \eta(1 - \eta)F(z - y) \quad (2)$$

for all $y, z \in I$ and $\eta \in [0, 1]$. From [22], we also have

$$\Psi(y) + \Psi'_+(y)(z - y) + F(z - y) \leq \Psi(z), \quad (3)$$

where Ψ is F -strongly convex function.

In literature the following inequality is well-known as Jensen inequality.

Theorem 1 (see [4]). *Let (Λ, X, μ) be a measure space with $0 < \mu(\Lambda) < \infty$ and $\Psi : I \rightarrow \mathbb{R}$ be convex function. Suppose $\phi : \Lambda \rightarrow I$ is such that $\phi, \Psi(\phi) \in L^1(\mu)$, then one has*

$$\Psi\left(\frac{1}{\mu(\Lambda)} \int_{\Lambda} \phi d\mu\right) \leq \frac{1}{\mu(\Lambda)} \int_{\Lambda} \Psi(\phi) d\mu. \quad (4)$$

In 1981, Slater proved a companion inequality to the Jensen inequality [23].

Theorem 2. Let (Λ, X, μ) be a measure space with $0 < \mu(\Lambda) < \infty$ and $\Psi : I \rightarrow \mathbb{R}$ be increasing and convex function. Suppose $\phi : \Lambda \rightarrow I$ is such that $\Psi(\phi)$, $\Psi'_+(\phi)$, and $\phi\Psi'_+(\phi) \in L^1(\mu)$. If $\int_{\Lambda} \Psi'_+(\phi) d\mu \neq 0$, then one has

$$\frac{1}{\mu(\Lambda)} \int_{\Lambda} \Psi(\phi) d\mu \leq \Psi \left(\frac{\int_{\Lambda} \phi \Psi'_+(\phi) d\mu}{\int_{\Lambda} \Psi'_+(\phi) d\mu} \right). \quad (5)$$

In the case when Ψ is strictly convex, then equality holds in (5) if and only if ϕ is constant almost everywhere on Λ .

Remark 3. Some improvements and reversions of Slater's inequality are given in [24, 25].

The following inequality is the integral analogue of another companion inequality to the Jensen inequality.

Theorem 4 (see [26]). Let (Λ, X, μ) be a measure space with $0 < \mu(\Lambda) < \infty$ and $\Psi : I \rightarrow \mathbb{R}$ be convex function. Suppose $\phi : \Lambda \rightarrow I$ is such that ϕ , $\Psi(\phi)$, $\Psi'_+(\phi)$, and $\phi\Psi'_+(\phi) \in L^1(\mu)$ and

$$\begin{aligned} \bar{\phi} &= \frac{1}{\mu(\Lambda)} \int_{\Lambda} \phi d\mu \\ \bar{\Psi} &= \frac{1}{\mu(\Lambda)} \int_{\Lambda} \Psi(\phi) d\mu. \end{aligned} \quad (6)$$

Then the following inequalities hold:

$$0 \leq \bar{\Psi} - \Psi(\bar{\phi}) \leq \frac{1}{\mu(\Lambda)} \int_{\Lambda} (\phi - \bar{\phi}) \Psi'_+(\phi) d\mu. \quad (7)$$

If the function Ψ is strictly convex, then equality holds in (7) if and only if ϕ is constant almost everywhere on Λ .

Matić and Pečarić established a general inequality from which one can directly obtain inequalities (5) and (7).

Theorem 5 (see [27]). Let all the assumptions of Theorem 4 be fulfilled. If $d_1, d_2 \in I$, then one has

$$\begin{aligned} \Psi(d_1) + \Psi'_+(d_1)(\bar{\phi} - d_1) &\leq \bar{\Psi} \\ &\leq \Psi(d_2) + \frac{1}{\mu(\Lambda)} \int_{\Lambda} (\phi - d_2) \Psi'_+(\phi) d\mu. \end{aligned} \quad (8)$$

Also, when Ψ is strictly convex, then equality in the left side in (8) holds if and only if $\phi = d_1$ almost everywhere on Λ , while equality in the right side in (8) holds if and only if $\phi = d_2$ almost everywhere on Λ .

Remark 6. Under the assumptions of Theorem 4, let $\int_{\Lambda} \Psi'_+(\phi) d\mu \neq 0$ and $\bar{\Psi} = \int_{\Lambda} \phi \Psi'_+(\phi) d\mu / \int_{\Lambda} \Psi'_+(\phi) d\mu \in I$, then by setting $d_2 = \bar{\Psi}$ in (8), we get Slater's inequality (5), and similarly by setting $d_2 = \bar{\phi}$ in (8), we get (7).

Merentes and Nikodem improved the Jensen inequality for strongly convex functions as follows.

Theorem 7 (see [28]). Let (Λ, X, μ) be a probability measure space and $\Psi : I \rightarrow \mathbb{R}$ be strongly convex function with modulus c . Suppose $\phi : \Lambda \rightarrow I$ is a Lebesgue integrable function and $\bar{\theta} = \int_{\Lambda} \phi d\mu$. Then the following inequality holds:

$$0 \leq \int_{\Lambda} \Psi(\phi) d\mu - \Psi(\bar{\theta}) - c \int_{\Lambda} (\phi - \bar{\theta})^2 d\mu. \quad (9)$$

For more recent results related to strongly convex function and Jensen type inequalities we recommend [22, 29–34].

This paper is organized as follows. In Section 2, we establish general inequalities for F -strongly convex function as well as strongly convex functions. As a consequence, we obtain integral Jensen inequality and Slater's inequality for strongly convex functions. Also by the virtue of these general inequalities we deduce converse of Jensen inequality. In Section 3, we give some properties of strongly convex functions. By using these properties of strongly convex functions we prove Jensen-Steffensen and Slater's type inequalities.

2. Jensen's Type Inequalities

We start this section to give the following general theorem.

Theorem 8. Let (Λ, X, μ) be a measure space with $0 < \mu(\Lambda) < \infty$ and $\Psi : I \rightarrow \mathbb{R}$ be F -strongly convex function. Suppose $\phi : \Lambda \rightarrow I$ is such that $\Psi(\phi)$, $\Psi'_+(\phi)$, and $\phi\Psi'_+(\phi) \in L^1(\mu)$ and also $\bar{\phi} = (1/\mu(\Lambda)) \int_{\Lambda} \phi d\mu$, $\bar{\Psi} = (1/\mu(\Lambda)) \int_{\Lambda} \Psi(\phi) d\mu$. If $d_1, d_2 \in I$, then one has

$$\begin{aligned} \Psi(d_1) + \Psi'_+(d_1)(\bar{\phi} - d_1) + \frac{1}{\mu(\Lambda)} \int_{\Lambda} F(\phi - d_1) d\mu \\ \leq \bar{\Psi} \\ \leq \Psi(d_2) + \frac{1}{\mu(\Lambda)} \int_{\Lambda} (\phi - d_2) \Psi'_+(\phi) d\mu \\ - \frac{1}{\mu(\Lambda)} \int_{\Lambda} F(d_2 - \phi) d\mu. \end{aligned} \quad (10)$$

Proof. Since Ψ is strongly convex function, therefore

$$\Psi(y) - \Psi'_+(y)(y - z) + F(z - y) \leq \Psi(z). \quad (11)$$

Letting $y \rightarrow \phi$ and $z \rightarrow d_2$, in (11), we get

$$\Psi(\phi) - \Psi'_+(\phi)(\phi - d_2) + F(d_2 - \phi) \leq \Psi(d_2). \quad (12)$$

Taking integral of (12) and then dividing by $\mu(\Lambda)$, we obtain

$$\begin{aligned} \bar{\Psi} \leq \Psi(d_2) + \frac{1}{\mu(\Lambda)} \int_{\Lambda} (\phi - d_2) \Psi'_+(\phi) d\mu \\ - \frac{1}{\mu(\Lambda)} \int_{\Lambda} F(d_2 - \phi) d\mu. \end{aligned} \quad (13)$$

Similarly, rearranging (11), we get

$$\Psi(y) + \Psi'_+(y)(z - y) + F(z - y) \leq \Psi(z). \quad (14)$$

Letting $y \rightarrow d_1$ and $z \rightarrow \phi$, in (14), we have

$$\Psi(d_1) + \Psi'_+(d_1)(\phi - d_1) + F(\phi - d_1) \leq \Psi(\phi). \quad (15)$$

Taking integral of (15) and then dividing by $\mu(\Lambda)$, we get

$$\begin{aligned} \Psi(d_1) + \Psi'_+(d_1)(\bar{\phi} - d_1) + \frac{1}{\mu(\Lambda)} \int_{\Lambda} F(\phi - d_1) d\mu \\ \leq \bar{\Psi}. \end{aligned} \quad (16)$$

Combining (13) and (16), we obtain (10). \square

By virtue of Theorem 8, we can deduce some new and interesting consequences.

Proposition 9. *Suppose that all the assumptions of Theorem 8 are satisfied. Then*

$$\begin{aligned} \frac{1}{\mu(\Lambda)} \int_{\Lambda} \Psi(\phi) d\mu \\ \leq \frac{1}{\mu(\Lambda)} \int_{\Lambda} \phi \Psi'_+(\phi) d\mu \\ + \inf_{z \in I} \left\{ \Psi(z) - \frac{z}{\mu(\Lambda)} \int_{\Lambda} \Psi'_+(\phi) d\mu \right\} \\ - \inf_{z \in I} \left\{ \frac{1}{\mu(\Lambda)} \int_{\Lambda} F(z - \phi) d\mu \right\}. \end{aligned} \quad (17)$$

Proof. If we set $y = \phi$ in (11) and taking integral over Λ and then dividing by $\mu(\Lambda)$, we have

$$\begin{aligned} \frac{1}{\mu(\Lambda)} \int_{\Lambda} \Psi(\phi) d\mu - \frac{1}{\mu(\Lambda)} \int_{\Lambda} \phi \Psi'_+(\phi) d\mu \\ + \frac{z}{\mu(\Lambda)} \int_{\Lambda} \Psi'_+(\phi) d\mu + \frac{1}{\mu(\Lambda)} \int_{\Lambda} F(z - \phi) d\mu \\ \leq \Psi(z), \end{aligned} \quad (18)$$

or equivalently

$$\begin{aligned} \frac{1}{\mu(\Lambda)} \int_{\Lambda} \Psi(\phi) d\mu \leq \frac{1}{\mu(\Lambda)} \int_{\Lambda} \phi \Psi'_+(\phi) d\mu + \Psi(z) \\ - \frac{z}{\mu(\Lambda)} \int_{\Lambda} \Psi'_+(\phi) d\mu \\ - \frac{1}{\mu(\Lambda)} \int_{\Lambda} F(z - \phi) d\mu. \end{aligned} \quad (19)$$

Taking the infimum over $z \in I$, we obtain (17). \square

Proposition 10. *Suppose that all the assumptions of Theorem 8 are satisfied and $\bar{x} = (1/\mu(\Lambda)) \int_{\Lambda} \Psi'_+(\phi) d\mu$, then*

$$\begin{aligned} 0 \leq \bar{\Psi} - \Psi(\bar{\phi}) - \frac{1}{\mu(\Lambda)} \int_{\Lambda} F(\phi - \bar{\phi}) d\mu \\ \leq \inf_{z \in I} \{ \Psi(z) - z\bar{x} \} + \frac{1}{\mu(\Lambda)} \int_{\Lambda} \phi \Psi'_+(\phi) d\mu - \Psi(\bar{\phi}) \\ - \inf_{z \in I} \left\{ \frac{1}{\mu(\Lambda)} \int_{\Lambda} F(z - \phi) d\mu \right\} \\ - \frac{1}{\mu(\Lambda)} \int_{\Lambda} F(\phi - \bar{\phi}) d\mu \\ \leq \frac{1}{\mu(\Lambda)} \int_{\Lambda} \phi \Psi'_+(\phi) d\mu - \bar{\phi}\bar{x} \\ - \inf_{z \in I} \left\{ \frac{1}{\mu(\Lambda)} \int_{\Lambda} F(z - \phi) d\mu \right\} \\ - \frac{1}{\mu(\Lambda)} \int_{\Lambda} F(\phi - \bar{\phi}) d\mu. \end{aligned} \quad (20)$$

Proof. By setting $d_1 = \bar{\phi}$ and $d_2 = z \in I$ in (10), we have

$$\begin{aligned} \Psi(\bar{\phi}) + \frac{1}{\mu(\Lambda)} \int_{\Lambda} F(\phi - \bar{\phi}) d\mu \leq \bar{\Psi} \\ \leq \Psi(z) + \frac{1}{\mu(\Lambda)} \int_{\Lambda} (\phi - z) \Psi'_+(\phi) d\mu \\ - \frac{1}{\mu(\Lambda)} \int_{\Lambda} F(z - \phi) d\mu. \end{aligned} \quad (21)$$

Indeed, the following equivalent form of (21) is

$$\begin{aligned} 0 \leq \bar{\Psi} - \Psi(\bar{\phi}) - \frac{1}{\mu(\Lambda)} \int_{\Lambda} F(\phi - \bar{\phi}) d\mu \\ \leq \Psi(z) + \frac{1}{\mu(\Lambda)} \int_{\Lambda} (\phi - z) \Psi'_+(\phi) d\mu - \Psi(\bar{\phi}) \\ - \frac{1}{\mu(\Lambda)} \int_{\Lambda} F(z - \phi) d\mu \\ - \frac{1}{\mu(\Lambda)} \int_{\Lambda} F(\phi - \bar{\phi}) d\mu. \end{aligned} \quad (22)$$

Taking the infimum over $z \in I$, we can easily derive the first and the second inequality in (20). The remaining third inequality in (20) follows because

$$\inf_{z \in I} \{ \Psi(z) - z\bar{x} \} \leq \Psi(\bar{\phi}) - \bar{\phi}\bar{x}. \quad (23)$$

\square

Corollary 11. *Let (Λ, X, μ) be a measure space with $0 < \mu(\Lambda) < \infty$ and $\Psi : I \rightarrow \mathbb{R}$ be strongly convex function with modulus c . Suppose $\phi : \Lambda \rightarrow I$ is such that $\phi, \Psi(\phi)$,*

$\Psi'_+(\phi)$, and $\phi\Psi'_+(\phi) \in L^1(\mu)$ and also $\bar{\phi} = (1/\mu(\Lambda)) \int_{\Lambda} \phi d\mu$, $\bar{\Psi} = (1/\mu(\Lambda)) \int_{\Lambda} \Psi(\phi)d\mu$. If $d_1, d_2 \in I$, then one has

$$\begin{aligned} & \Psi(d_1) + \Psi'_+(d_1)(\bar{\phi} - d_1) + \frac{c}{\mu(\Lambda)} \int_{\Lambda} (\phi - d_1)^2 d\mu \\ & \leq \bar{\Psi} \\ & \leq \Psi(d_2) + \frac{1}{\mu(\Lambda)} \int_{\Lambda} (\phi - d_2) \Psi'_+(\phi) d\mu \\ & \quad - \frac{c}{\mu(\Lambda)} \int_{\Lambda} (\phi - d_2)^2 d\mu. \end{aligned} \quad (24)$$

Remark 12. If we put $d_1 = \bar{\phi}$ in (24) and take probability measure space, then we obtain integral Jensen inequality (9) for strongly convex function.

In the following corollary, we obtain integral Slater's inequality for strongly convex function.

Corollary 13. Suppose Ψ , ϕ , $\mu(\Lambda)$, $\bar{\phi}$, and $\bar{\Psi}$ are stated as in Corollary 11 and assume that $\int_{\Lambda} \Psi'_+(\phi)d\mu \neq 0$; also if $\bar{\phi} = \int_{\Lambda} \phi\Psi'_+(\phi)d\mu / \int_{\Lambda} \Psi'_+(\phi)d\mu \in I$, then

$$\bar{\Psi} \leq \Psi(\bar{\phi}) - \frac{c}{\mu(\Lambda)} \int_{\Lambda} (\phi - \bar{\phi})^2 d\mu. \quad (25)$$

Proof. By setting $d_2 = \bar{\phi}$ in (24), we deduced

$$\begin{aligned} \bar{\Psi} & \leq \Psi \left(\frac{\int_{\Lambda} \phi\Psi'_+(\phi) d\mu}{\int_{\Lambda} \Psi'_+(\phi) d\mu} \right) \\ & \quad + \frac{1}{\mu(\Lambda)} \int_{\Lambda} \left(\phi - \frac{\int_{\Lambda} \phi\Psi'_+(\phi) d\mu}{\int_{\Lambda} \Psi'_+(\phi) d\mu} \right) \Psi'_+(\phi) d\mu \\ & \quad - \frac{c}{\mu(\Lambda)} \int_{\Lambda} \left(\phi - \frac{\int_{\Lambda} \phi\Psi'_+(\phi) d\mu}{\int_{\Lambda} \Psi'_+(\phi) d\mu} \right)^2 d\mu. \end{aligned} \quad (26)$$

Since $(1/\mu(\Lambda)) \int_{\Lambda} (\phi - \int_{\Lambda} \phi\Psi'_+(\phi)d\mu / \int_{\Lambda} \Psi'_+(\phi)d\mu) \Psi'_+(\phi)d\mu = 0$, therefore (26) is equivalent to (25). \square

In the following corollary, we obtain a converse of the Jensen inequality for strongly convex function.

Corollary 14. Suppose Ψ , ϕ , $\mu(\Lambda)$, $\bar{\phi}$, and $\bar{\Psi}$ are stated as in Corollary 11, then one has

$$\begin{aligned} 0 & \leq \bar{\Psi} - \Psi(\bar{\phi}) - \frac{c}{\mu(\Lambda)} \int_{\Lambda} (\phi - \bar{\phi})^2 d\mu \\ & \leq \frac{1}{\mu(\Lambda)} \int_{\Lambda} (\phi - \bar{\phi}) \Psi'_+(\phi) d\mu \\ & \quad - \frac{2c}{\mu(\Lambda)} \int_{\Lambda} (\phi - \bar{\phi})^2 d\mu. \end{aligned} \quad (27)$$

Proof. By setting $d_1 = d_2 = \bar{\phi}$ in (24), we obtain (27). \square

3. Jensen-Steffensen Inequality for Riemann-Stieltjes Integrals

To prove the main results of this section, first we prove the following lemma which will play a key role in the proof of main results.

Lemma 15. Let $\Psi : (\gamma, \beta) \rightarrow \mathbb{R}$ be strongly convex function with modulus c . Suppose y is fixed element from interval (γ, β) , then

(a) the function $\Delta_y(\zeta) : (\gamma, \beta) \rightarrow \mathbb{R}$ defined by

$$\Delta_y(\zeta) = \Psi(\zeta) - \Psi(y) - \Psi'_+(y)(\zeta - y) - c(\zeta - y)^2 \quad (28)$$

(b) and the function $\bar{\Delta}_y(\zeta) : (\gamma, \beta) \rightarrow \mathbb{R}$ defined by

$$\bar{\Delta}_y(\zeta) = \Psi(y) - \Psi(\zeta) - \Psi'_+(\zeta)(y - \zeta) - c(y - \zeta)^2 \quad (29)$$

are nonnegative on (γ, β) , decreasing on $(\gamma, y]$ and increasing on $[y, \beta)$.

Proof. By definition of strongly convexity we have

$$\begin{aligned} \Delta_y(\zeta) & = \Psi(\zeta) - \Psi(y) - \Psi'_+(y)(\zeta - y) - c(\zeta - y)^2 \\ & \geq 0, \quad \forall \zeta, y \in (\gamma, \beta) \end{aligned} \quad (30)$$

It means that $\Delta_y(\zeta)$ is nonnegative on (γ, β) .

Let $\gamma < \zeta_1 < \zeta_2 \leq y$. Since Ψ is strongly convex with modulus c , therefore $\Psi'_+(\zeta) - 2c\zeta$ is increasing and hence

$$\Psi'_+(\zeta_2) - 2c\zeta_2 \leq \Psi'_+(y) - 2cy; \quad (31)$$

it follows

$$\begin{aligned} \Psi'_+(y)(\zeta_1 - \zeta_2) & \leq \Psi'_+(\zeta_2)(\zeta_1 - \zeta_2) - 2c\zeta_2(\zeta_1 - \zeta_2) \\ & \quad + 2cy(\zeta_1 - \zeta_2). \end{aligned} \quad (32)$$

Setting $\zeta = \zeta_1$ and $\zeta = \zeta_2$ in (28) and then taking the difference, we get

$$\begin{aligned} \Delta_y(\zeta_1) - \Delta_y(\zeta_2) & = \Psi(\zeta_1) - \Psi(\zeta_2) \\ & \quad - \Psi'_+(y)(\zeta_1 - \zeta_2) - c(\zeta_1 - y)^2 \\ & \quad + c(\zeta_2 - y)^2 \\ & \geq \Psi(\zeta_1) - \Psi(\zeta_2) \\ & \quad - \Psi'_+(\zeta_2)(\zeta_1 - \zeta_2) \\ & \quad + 2c\zeta_2(\zeta_1 - \zeta_2) - 2cy(\zeta_1 - \zeta_2) \\ & \quad - c(\zeta_1 - y)^2 + c(\zeta_2 - y)^2 \\ & \quad \text{(by using (32))} \\ & = \Psi(\zeta_1) - \Psi(\zeta_2) \\ & \quad - \Psi'_+(\zeta_2)(\zeta_1 - \zeta_2) - c(\zeta_1 - \zeta_2)^2 \\ & \geq 0. \end{aligned} \quad (33)$$

Hence, $\Delta_y(\zeta)$ is decreasing on $(\gamma, y]$.

Also, if $y \leq \zeta_1 < \zeta_2 < \beta$, then similarly as above by strongly convexity we have

$$\Psi'_+(y) - 2cy \leq \Psi'_+(\zeta_1) - 2c\zeta_1, \tag{34}$$

from which it follows that

$$\begin{aligned} \Psi'_+(y)(\zeta_1 - \zeta_2) &\geq \Psi'_+(\zeta_1)(\zeta_1 - \zeta_2) - 2c\zeta_1(\zeta_1 - \zeta_2) \\ &\quad + 2cy(\zeta_1 - \zeta_2). \end{aligned} \tag{35}$$

Setting $\zeta = \zeta_1$ and $\zeta = \zeta_2$ in (28), then taking the difference we get

$$\begin{aligned} \Delta_y(\zeta_1) - \Delta_y(\zeta_2) &= \Psi(\zeta_1) - \Psi(\zeta_2) \\ &\quad - \Psi'_+(y)(\zeta_1 - \zeta_2) - c(\zeta_1 - y)^2 \\ &\quad + c(\zeta_2 - y)^2 \\ &\leq \Psi(\zeta_1) - \Psi(\zeta_2) \\ &\quad - \Psi'_+(\zeta_1)(\zeta_1 - \zeta_2) \\ &\quad + 2c\zeta_1(\zeta_1 - \zeta_2) - 2cy(\zeta_1 - \zeta_2) \\ &\quad - c(\zeta_1 - y)^2 + c(\zeta_2 - y)^2 \\ &\quad \text{(by using (35))} \\ &= \Psi(\zeta_1) - \Psi(\zeta_2) \\ &\quad - \Psi'_+(\zeta_1)(\zeta_1 - \zeta_2) - c(\zeta_1 - \zeta_2)^2 \\ &\leq 0. \end{aligned} \tag{36}$$

Hence, $\Delta_y(\zeta)$ is increasing on $[y, \beta]$.

In the same manner as above, we also obtain $\bar{\Delta}_y(\zeta)$ is nonnegative on (y, β) .

Let $y < \zeta_1 < \zeta_2 \leq y$, then by strongly convexity we have

$$\Psi'_+(\zeta_1) - 2c\zeta_1 \leq \Psi'_+(\zeta_2) - 2c\zeta_2, \tag{37}$$

$$\begin{aligned} \text{that is, } \Psi'_+(\zeta_2)(y - \zeta_2) \\ \geq \Psi'_+(\zeta_1)(y - \zeta_2) - 2c\zeta_1(y - \zeta_2) + 2c\zeta_2(y - \zeta_2). \end{aligned} \tag{38}$$

Setting $\zeta = \zeta_1$ and $\zeta = \zeta_2$ in (29) and then taking the difference we get

$$\begin{aligned} \bar{\Delta}_y(\zeta_1) - \bar{\Delta}_y(\zeta_2) &= \Psi(\zeta_2) - \Psi(\zeta_1) \\ &\quad - \Psi'_+(\zeta_1)(y - \zeta_1) \\ &\quad + \Psi'_+(\zeta_2)(y - \zeta_2) - c(y - \zeta_1)^2 \\ &\quad + c(y - \zeta_2)^2 \\ &\geq \Psi(\zeta_2) - \Psi(\zeta_1) \end{aligned}$$

$$\begin{aligned} &- \Psi'_+(\zeta_1)(y - \zeta_1) \\ &+ \Psi'_+(\zeta_1)(y - \zeta_2) \\ &- 2c\zeta_1(y - \zeta_2) + 2c\zeta_2(y - \zeta_2) \\ &- c(y - \zeta_1)^2 + c(y - \zeta_2)^2 \\ &\quad \text{(by (38))} \\ &= \Psi(\zeta_2) - \Psi(\zeta_1) \\ &\quad - \Psi'_+(\zeta_1)(\zeta_2 - \zeta_1) - c(\zeta_2 - \zeta_1)^2 \\ &\geq 0. \end{aligned} \tag{39}$$

Hence, $\bar{\Delta}_y(\zeta)$ is decreasing on $(y, \beta]$.

Also, if $y \leq \zeta_1 < \zeta_2 < \beta$, then by strongly convexity we have

$$\Psi'_+(\zeta_1) - 2c\zeta_1 \leq \Psi'_+(\zeta_2) - 2c\zeta_2, \tag{40}$$

$$\begin{aligned} \text{i.e., } \Psi'_+(\zeta_1)(y - \zeta_1) &\geq \Psi'_+(\zeta_2)(y - \zeta_1) \\ &\quad - 2c\zeta_2(y - \zeta_1) \\ &\quad + 2c\zeta_1(y - \zeta_1). \end{aligned} \tag{41}$$

Setting $\zeta = \zeta_1$ and $\zeta = \zeta_2$ in (29) and then taking the difference we get

$$\begin{aligned} \bar{\Delta}_y(\zeta_1) - \bar{\Delta}_y(\zeta_2) &= \Psi(\zeta_2) - \Psi(\zeta_1) \\ &\quad - \Psi'_+(\zeta_1)(y - \zeta_1) \\ &\quad + \Psi'_+(\zeta_2)(y - \zeta_2) - c(y - \zeta_1)^2 \\ &\quad + c(y - \zeta_2)^2 \\ &\leq \Psi(\zeta_2) - \Psi(\zeta_1) \\ &\quad - \Psi'_+(\zeta_2)(y - \zeta_1) \\ &\quad + \Psi'_+(\zeta_2)(y - \zeta_2) \\ &\quad - 2c\zeta_1(y - \zeta_1) + 2c\zeta_2(y - \zeta_1) \\ &\quad - c(y - \zeta_1)^2 + c(y - \zeta_2)^2 \\ &\quad \text{(by (41))} \\ &= \Psi(\zeta_2) - \Psi(\zeta_1) \\ &\quad - \Psi'_+(\zeta_2)(\zeta_1 - \zeta_2) + c(\zeta_1 - \zeta_2)^2 \\ &\leq 0. \end{aligned} \tag{42}$$

Hence, $\bar{\Delta}_y(\zeta)$ is increasing on $[y, \beta]$. This completes the proof. \square

The following lemma is given in [35].

Lemma 16. Let $\phi : [y, \beta] \rightarrow \mathbb{R}$ be a nonnegative function and suppose $\xi : [y, \beta] \rightarrow \mathbb{R}$ is either a bounded variation or

continuous. Also assume that the functions ϕ and ξ have no common discontinuity points.

(a) If ϕ is increasing on $[\gamma, \beta]$, then

$$\begin{aligned} \phi(\beta) \inf_{\gamma \leq \tau \leq \beta} \int_{\tau}^{\beta} d\xi(x) &\leq \int_{\gamma}^{\beta} \phi(x) d\xi(x) \\ &\leq \phi(\beta) \sup_{\gamma \leq \tau \leq \beta} \int_{\tau}^{\beta} d\xi(x). \end{aligned} \quad (43)$$

(b) If ϕ is decreasing on $[\gamma, \beta]$, then

$$\begin{aligned} \phi(\gamma) \inf_{\gamma \leq \tau \leq \beta} \int_{\gamma}^{\tau} d\xi(x) &\leq \int_{\gamma}^{\beta} \phi(x) d\xi(x) \\ &\leq \phi(\gamma) \sup_{\gamma \leq \tau \leq \beta} \int_{\gamma}^{\tau} d\xi(x). \end{aligned} \quad (44)$$

In the next result, we prove some general integral inequalities for strongly convex functions.

Theorem 17. Suppose $\phi : [\gamma, \beta] \rightarrow (\kappa, \nu)$ is monotonic and continuous function and let $\Psi : (\kappa, \nu) \rightarrow \mathbb{R}$ be a strongly convex function with modulus c . If $\xi : [\gamma, \beta] \rightarrow \mathbb{R}$ is either a bounded variation or continuous and satisfying $\xi(\gamma) \leq \xi(x) \leq \xi(\beta)$ for all $x \in [\gamma, \beta]$, $\xi(\beta) - \xi(\gamma) > 0$, then $\bar{\phi}$ and $\bar{\Psi}$ given by

$$\begin{aligned} \bar{\phi} &= \frac{1}{\xi(\beta) - \xi(\gamma)} \int_{\gamma}^{\beta} \phi(x) d\xi(x), \\ \bar{\Psi} &= \frac{1}{\xi(\beta) - \xi(\gamma)} \int_{\gamma}^{\beta} \Psi(\phi(x)) d\xi(x) \end{aligned} \quad (45)$$

are well defined and $\bar{\phi} \in (\kappa, \nu)$. Also, if $\Psi'_+(\phi(x))$ and ξ have no common discontinuity points, then for $d_1, d_2 \in (\kappa, \nu)$, one has

$$\begin{aligned} &\Psi(d_1) + \Psi'_+(d_1)(\bar{\phi} - d_1) \\ &+ \frac{c}{\xi(\beta) - \xi(\gamma)} \int_{\gamma}^{\beta} (\phi(x) - d_1)^2 d\xi(x) \leq \bar{\Psi} \\ &\leq \Psi(d_2) \\ &+ \frac{1}{\xi(\beta) - \xi(\gamma)} \int_{\gamma}^{\beta} (\phi(x) - d_2) \Psi'_+(\phi(x)) d\xi(x) \\ &- \frac{c}{\xi(\beta) - \xi(\gamma)} \int_{\gamma}^{\beta} (\phi(x) - d_2)^2 d\xi(x). \end{aligned} \quad (46)$$

Proof. Under the given conditions in [35], it has been shown that

$$\phi([\gamma, \beta]) = [\phi(\gamma), \phi(\beta)] \subseteq (\kappa, \nu). \quad (47)$$

We define the function $\lambda : [\gamma, \beta] \rightarrow \mathbb{R}$ by $\lambda(x) = \Delta_{d_1}(\phi(x))$, where Δ_{d_1} is defined as in Lemma 15(a), i.e.,

$$\begin{aligned} \lambda(x) &= \Delta_{d_1}(\phi(x)) \\ &= \Psi(\phi(x)) - \Psi(d_1) - \Psi'_+(d_1)(\phi(x) - d_1) \\ &\quad - c(\phi(x) - d_1)^2. \end{aligned} \quad (48)$$

From Lemma 15(a) it follows that λ is nonnegative and since ϕ and $\Psi(\phi)$ are continuous therefore the integral $\int_{\gamma}^{\beta} \lambda(x) d\xi(x)$ exists. Thus we discuss the following three cases:

(i) If $\phi(\beta) \leq d_1$, since ϕ is increasing on $[\gamma, \beta]$ and by Lemma 15(a) Δ_{d_1} is decreasing on $(\kappa, d_1]$, therefore $\lambda = \Delta_{d_1}(\phi(x))$ is decreasing on $[\gamma, \beta]$. So using Lemma 16 (b), we obtain

$$\int_{\gamma}^{\beta} \lambda(x) d\xi(x) \geq \lambda(\gamma) \inf_{\gamma \leq \tau \leq \beta} \{\xi(\tau) - \xi(\gamma)\} = 0. \quad (49)$$

(ii) If $d_1 \leq \phi(\gamma)$, since Δ_{d_1} is increasing on $[d_1, \nu)$ by Lemma 15(a), therefore $\lambda = \Delta_{d_1}(\phi(x))$ is increasing on $[\gamma, \beta]$. So using Lemma 16 (a), we have

$$\int_{\gamma}^{\beta} \lambda(x) d\xi(x) \geq \lambda(\beta) \inf_{\gamma \leq \tau \leq \beta} \{\xi(\beta) - \xi(\tau)\} = 0. \quad (50)$$

(iii) If $\phi(\gamma) < d_1 < \phi(\beta)$, since ϕ is continuous on $[\gamma, \beta]$, there exists at least one point $\bar{x} \in (\gamma, \beta)$ such that $\phi(\bar{x}) = d_1$. Also by Lemma 15(a), λ is decreasing on $[\gamma, \bar{x}]$ and λ is increasing on $[\bar{x}, \beta]$. Using Lemma 16, we have

$$\begin{aligned} \int_{\gamma}^{\beta} \lambda(x) d\xi(x) &= \int_{\gamma}^{\bar{x}} \lambda(x) d\xi(x) + \int_{\bar{x}}^{\beta} \lambda(x) d\xi(x) \\ &\geq \lambda(\gamma) \inf_{\gamma \leq \tau \leq \bar{x}} \{\xi(\bar{x}) - \xi(\tau)\} \\ &\quad + \lambda(\beta) \inf_{\bar{x} \leq \tau \leq \beta} \{\xi(\beta) - \xi(\tau)\} = 0. \end{aligned} \quad (51)$$

From the above three subcases we conclude that

$$\begin{aligned} \int_{\gamma}^{\beta} \lambda(x) d\xi(x) &= \int_{\gamma}^{\beta} \{\Psi(\phi(x)) - \Psi(d_1) \\ &\quad - \Psi'_+(d_1)(\phi(x) - d_1) \\ &\quad - c(\phi(x) - d_1)^2\} d\xi(x) \geq 0. \end{aligned} \quad (52)$$

i.e., $\Psi(d_1) \{\xi(\beta) - \xi(\gamma)\} + \Psi'_+(d_1)$

$$\begin{aligned} &\cdot \left\{ \int_{\gamma}^{\beta} (\phi(x)) d\xi(x) - d_1 \{\xi(\beta) - \xi(\gamma)\} \right\} \\ &+ c \int_{\gamma}^{\beta} (\phi(x) - d_1)^2 d\xi(x) \\ &\leq \int_{\gamma}^{\beta} \Psi(\phi(x)) d\xi(x). \end{aligned} \quad (53)$$

Dividing (53) by $\xi(\beta) - \xi(\gamma) > 0$, we obtain the left side of the inequality (46). Similarly, if ϕ is decreasing we consider the cases $\phi(\gamma) \leq d_1$ (λ is increasing on $[\gamma, \beta]$), $d_1 \leq \phi(\beta)$ (λ is decreasing on $[\gamma, \beta]$), and $\phi(\gamma) < d_1 < \phi(\beta)$ (λ is decreasing on $[\gamma, \bar{x}]$ and increasing on $[\bar{x}, \beta]$). In all three cases we obtain $\int_{\gamma}^{\beta} \lambda(x) d\xi(x) \geq 0$ from the first inequality in (46) which directly follows.

Similarly, we can prove the right side of the inequality (46). We define the function $\bar{\lambda} : [\gamma, \beta] \rightarrow \mathbb{R}$ by $\bar{\lambda}(x) = \bar{\Delta}_{d_2}(\phi(x))$, where $\bar{\Delta}_{d_2}$ is defined as in Lemma 15(b), i.e.,

$$\begin{aligned} \bar{\lambda}(x) &= \bar{\Delta}_{d_2}(\phi(x)) \\ &= \Psi(d_2) - \Psi(\phi(x)) - \Psi'_+(\phi(x))(d_2 - \phi(x)) \\ &\quad - c(d_2 - \phi(x))^2. \end{aligned} \quad (54)$$

Since ϕ is monotonic and continuous, $\Psi(\phi)$ is continuous and $\Psi'_+(\phi)$ is monotonic which have no common discontinuity point with ξ . Therefore the integral $\int_{\gamma}^{\beta} \bar{\lambda}(x) d\xi(x)$ exists. Using the same process as above we have $\int_{\gamma}^{\beta} \bar{\lambda}(x) d\xi(x) \geq 0$, which means that

$$\begin{aligned} &\int_{\gamma}^{\beta} \left\{ \Psi(d_2) - \Psi(\phi(x)) - \Psi'_+(\phi(x))(d_2 - \phi(x)) \right. \\ &\quad \left. - c(d_2 - \phi(x))^2 \right\} d\xi(x) \geq 0. \\ \text{i.e., } &\{ \xi(\beta) - \xi(\gamma) \} \Psi(d_2) - \int_{\gamma}^{\beta} \Psi(\phi(x)) d\xi(x) \\ &+ \int_{\gamma}^{\beta} (\phi(x) - d_2) \Psi'_+(\phi(x)) d\xi(x) \\ &- c \int_{\gamma}^{\beta} (\phi(x) - d_2)^2 d\xi(x) \geq 0. \end{aligned} \quad (55)$$

If we divide by $\xi(\beta) - \xi(\gamma) > 0$, we obtain the second inequality of (46). So, both inequalities of (46) are proved. \square

Now, we are in a situation to obtain the following result.

Corollary 18. *Suppose all the assumptions of Theorem 17 are satisfied, then one has*

$$\begin{aligned} 0 &\leq \bar{\Psi} - \Psi(\bar{\phi}) - \frac{c}{\xi(\beta) - \xi(\gamma)} \int_{\gamma}^{\beta} (\phi(x) - \bar{\phi})^2 d\xi(x) \\ &\leq \frac{1}{\xi(\beta) - \xi(\gamma)} \int_{\gamma}^{\beta} (\phi(x) - \bar{\phi}) \Psi'_+(\phi(x)) d\xi(x) \\ &\quad - \frac{2c}{\xi(\beta) - \xi(\gamma)} \int_{\gamma}^{\beta} (\phi(x) - \bar{\phi})^2 d\xi(x). \end{aligned} \quad (56)$$

Proof. By setting $d_1 = d_2 = \bar{\phi}$ in (46), we obtain (56). \square

Remark 19. If we set $c = 0$ in (56), we obtain Theorem 3.2 in [35].

In the following corollary, we obtain integral Slater's inequality for strongly convex functions.

Corollary 20. *Suppose all the assumptions of Theorem 17 are satisfied and assume $\int_{\gamma}^{\beta} \Psi'_+(\phi(x)) d\xi(x) \neq 0$; also if $\bar{\phi} = \int_{\gamma}^{\beta} \phi(x) \Psi'_+(\phi(x)) d\xi(x) / \int_{\gamma}^{\beta} \Psi'_+(\phi(x)) d\xi(x) \in I$, then*

$$\bar{\Psi} \leq \Psi(\bar{\phi}) - \frac{c}{\xi(\beta) - \xi(\gamma)} \int_{\gamma}^{\beta} (\phi(x) - \bar{\phi})^2 d\xi(x). \quad (57)$$

Proof. Similar to the proof of Corollary 13, setting $d_2 = \bar{\phi}$ in the right hand side of (46), we get (57). \square

Remark 21. If we set $c = 0$ in (57), we obtain Slater's inequality for convex functions given in [35].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

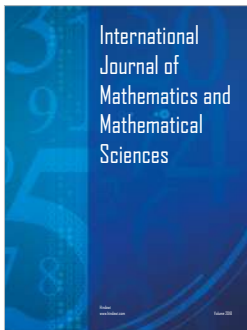
Acknowledgments

The research was supported by the Natural Science Foundation of China (Grants nos. 61673169 and 11601485).

References

- [1] B. de Finetti, "Sulle stratificazioni convesse," *Annali di Matematica Pura ed Applicata*, vol. 30, no. 4, pp. 173–183, 1949.
- [2] B. T. Polyak, "Existence theorems and convergence of minimizing sequences for extremal problems with constraints," *Soviet Doklady Mathematics*, vol. 7, pp. 72–75, 1966.
- [3] D. H. Hyers and S. M. Ulam, "Approximately convex functions," *Proceedings of the American Mathematical Society*, vol. 3, pp. 821–828, 1952.
- [4] J. E. Pečarić, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, vol. 187, Academic Press, Boston, MA, USA, 1992.
- [5] J. L. W. V. Jensen, "Om konvexe funktioner og uligheder mellem middlvaerdier," *Nyt Tidsskrift for Matematik. B*, vol. 16, pp. 49–69, 1905.
- [6] O. L. Mangasarian, "Pseudo-convex functions," *Journal of the Society for Industrial and Applied Mathematics Series A Control*, vol. 3, pp. 281–290, 1965.
- [7] S. S. Dragomir, "Inequalities of Hermite-Hadamard type for ϕ -convex functions on linear spaces," *Preprint RGMIA Res. Rep. Coll*, vol. 87, 2013, <http://rgmia.org/papers/v16/v16a87.pdf>.
- [8] S. S. Dragomir, "Inequalities of Hermite-Hadamard type for λ -convex functions on linear spaces," *Preprint RGMIA Res. Rep. Coll*, vol. 17, no. 18, 2014, <http://rgmia.org/papers/v17/v17a13.pdf>.
- [9] S. Varošanec, "On h-convexity," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 303–311, 2007.
- [10] T. Rajba, "On strong delta-convexity and Hermite-Hadamard type inequalities for delta-convex functions of higher order," *Mathematical Inequalities & Applications*, vol. 18, no. 1, pp. 267–293, 2015.

- [11] Y. Chu and X. Zhang, "Necessary and sufficient conditions such that extended mean values are Schur-convex or Schur-concave," *Kyoto Journal of Mathematics*, vol. 48, no. 1, pp. 229–238, 2008.
- [12] N. Elezović and J. Pečarić, "A note on Schur-convex functions," *Rocky Mountain Journal of Mathematics*, vol. 30, no. 3, pp. 853–856, 2000.
- [13] H.-N. Shi, "Schur-convex functions related to Hadamard-type inequalities," *Journal of Mathematical Inequalities*, vol. 1, no. 1, pp. 127–136, 2007.
- [14] C. Stępniaak, "An effective characterization of Schur-convex functions with applications," *Journal of Convex Analysis*, vol. 14, no. 1, pp. 103–108, 2007.
- [15] X.-M. Zhang, "Schur-convex functions and isoperimetric inequalities," *Proceedings of the American Mathematical Society*, vol. 126, no. 2, pp. 461–470, 1998.
- [16] G. A. Anastassiou, "Basic and s -convexity Ostrowski and Grüss type inequalities involving several functions," *Communications in Applied Analysis*, vol. 17, no. 2, pp. 189–212, 2013.
- [17] E. Set, M. E. Özdemir, and S. S. Dragomir, "On Hadamard-type inequalities involving several kinds of convexity," *Journal of Inequalities and Applications*, vol. 2010, Article ID 286845, 12 pages, 2010.
- [18] Z.-H. Yang, W.-M. Qian, Y.-M. Chu, and W. Zhang, "On approximating the error function," *Mathematical Inequalities & Applications*, vol. 21, no. 2, pp. 469–479, 2018.
- [19] Z.-H. Yang, W.-M. Qian, Y.-M. Chu, and W. Zhang, "On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind," *Journal of Mathematical Analysis and Applications*, vol. 462, no. 2, pp. 1714–1726, 2018.
- [20] A. W. Roberts and D. E. Varberg, *Convex Functions*, Academic Press, London, UK, 1973.
- [21] M. Adamek, "On a problem connected with strongly convex functions," *Mathematical Inequalities & Applications*, vol. 19, no. 4, pp. 1287–1293, 2016.
- [22] H. R. Moradi, M. E. Omidvar, M. Adil Khan, and K. Nikodem, "Around Jensen's inequality for strongly convex functions," *Aequationes Mathematicae*, vol. 92, no. 1, pp. 25–37, 2018.
- [23] M. L. Slater, "A companion inequality to Jensen's inequality," *Journal of Approximation Theory*, vol. 32, no. 2, pp. 160–166, 1981.
- [24] M. Adil Khan and J. Pečarić, "On Slater's integral inequality," *Journal of Mathematical Inequalities*, vol. 5, no. 2, pp. 231–241, 2011.
- [25] M. Adil Khan and J. E. Pečarić, "Improvement and reversion of Slater's inequality and related results," *Journal of Inequalities and Applications*, vol. 2010, Article ID 646034, 14 pages, 2010.
- [26] S. S. Dragomir and C. J. Goh, "A counterpart of Jensen's discrete inequality for differentiable convex mappings and applications in information theory," *Mathematical and Computer Modelling*, vol. 24, no. 2, pp. 1–11, 1996.
- [27] M. Matić and J. Pečarić, "Some companion inequalities to Jensen's inequality," *Mathematical Inequalities & Applications*, vol. 3, no. 3, pp. 355–368, 2000.
- [28] N. Merentes and K. Nikodem, "Remarks on strongly convex functions," *Aequationes Mathematicae*, vol. 80, no. 1-2, pp. 193–199, 2010.
- [29] M. Klaričić Bakula and K. Nikodem, "On the converse Jensen inequality for strongly convex functions," *Journal of Mathematical Analysis and Applications*, vol. 434, no. 1, pp. 516–522, 2016.
- [30] M. Adil Khan, J. Khan, and Pečarić, "Generalization of Jensen's and Jensen-Steffensen's inequalities by generalized majorization theorem," *Journal of Mathematical Inequalities*, vol. 11, no. 4, pp. 1049–1074, 2017.
- [31] S. S. Dragomir, M. Adil Khan, and A. Abathun, "Refinement of the Jensen integral inequality," *Open Mathematics*, vol. 14, pp. 221–228, 2016.
- [32] F.-C. Mitroi-Symeonidis and N. Minculete, "On the Jensen functional and strong convexity," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 41, no. 1, pp. 311–319, 2018.
- [33] K. Nikodem, *On Strongly Convex Functions and Related Classes of Functions, Handbook of Functional Equations*, Springer, New York, NY, USA, 2014.
- [34] K. Nikodem and Z. Páles, "Characterizations of inner product spaces by strongly convex functions," *Banach Journal of Mathematical Analysis*, vol. 5, no. 1, pp. 83–87, 2011.
- [35] M. Klaričić Bakula, M. Matić, and J. Pečarić, "Generalizations of the Jensen-Steffensen and related inequalities," *Central European Journal of Mathematics*, vol. 7, no. 4, pp. 787–803, 2009.




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