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Integral Inequalities of Hadamard Type for r-Convex Functions

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Abstract

The main aim of the present note is to establish new Hadamard like integral inequalities involving r-convex functions.

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1 Introduction

The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

which holds for all convex functions $f : [a, b] \to \mathbf{R}$ is also known as Hadamard's inequality.

C.E.M. Pearce, J.Pecaric and V. Simic generalized this inequality to a r-convex positive function f which is defined on an interval [a,b], for all $x, y \in [a, b]$ and $t \in [0, 1]$

$$f(tx+(1-t)y) \leq \begin{cases} (t[f(x)]^r + (1-t)[f(y)]^r)^{1/r}, \text{ if } r \neq 0, \\ [f(x)]^t[f(y)]^{(1-t)}, \text{ if } r = 0. \end{cases}$$

We have that 0-convex functions are simply log-convex functions and 1-convex functions are ordinary convex functions.

2 Main Results

We start with the following theorem.

Theorem 2.1. *Let* $f : [a,b] \to (0,\infty)$ *be r-convex function on* $[a,b]$ *with* $a < b$ *. Then the following inequality holds for* $0 < r \leq 1$*:*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \left(\frac{r}{r+1}\right)^{1/r} \left([f(a)]^r + [f(b)]^r\right)^{1/r}.$$

Proof. Since f is r-convex function and $r > 0$, we have

$$f(ta + (1-t)b) \leq \left(t[f(a)]^r + (1-t)[f(b)]^r\right)^{1/r}$$

for all $t \in [0,1]$. It is easy to observe that

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)dx &= \int_0^1 f(ta+(1-t)b)dt \\ &\leq \int_0^1 \left(t[f(a)]^r + (1-t)[f(b)]^r\right)^{1/r} dt\end{aligned}$$

Using Minkowski's inequality, we have

$$\begin{aligned}\int_0^1 \left(t[f(a)]^r + (1-t)[f(b)]^r\right)^{1/r} dt &\leq \left[\left(\int_0^1 t^{1/r} f(a)dt\right)^r + \left(\int_0^1 (1-t)^{1/r} f(b)dt\right)^r\right]^{1/r} \\ &= \left(\frac{r}{r+1}[f(a)]^r + \frac{r}{r+1}[f(b)]^r\right)^{1/r} \\ &= \left(\frac{r}{r+1}\right)^{1/r} \left([f(a)]^r + [f(b)]^r\right)^{1/r}.\end{aligned}$$

Thus

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \left(\frac{r}{r+1}\right)^{1/r} \left([f(a)]^r + [f(b)]^r\right)^{1/r}.$$

This proof is complete. □

Corollary 2.2. *Let* $f : [a,b] \to (0,\infty)$ *be 1-convex function on* $[a,b]$ *with* $a < b$*. Then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Theorem 2.3. *Let $f, g : [a, b] \to (0, \infty)$ be r-convex and s-convex functions respectively on $[a, b]$ with $a < b$. Then the following inequality holds for $0 < r, s \le 2$:*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{1}{2}\left(\frac{r}{r+2}\right)^{2/r}\left([f(a)]^r + [f(b)]^r\right)^{2/r} + \frac{1}{2}\left(\frac{s}{s+2}\right)^{2/s}\left([g(a)]^s + [g(b)]^s\right)^{2/s}$$

Proof. Since f is r-convex function and g is s-convex function $(r > 0, s > 0)$, we have

$$f(ta + (1-t)b) \le (t[f(a)]^r + (1-t)[f(b)]^r)^{1/r}$$

$$g(ta + (1-t)b) \le (t[g(a)]^s + (1-t)[g(b)]^s)^{1/s}$$

for all $t \in [0, 1]$

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx = \int_0^1 f(ta + (1-t)b)g(ta + (1-t)b)dt$$

$$\le \int_0^1 (t[f(a)]^r + (1-t)[f(b)]^r)^{1/r}(t[g(a)]^s + (1-t)[g(b)]^s)^{1/s}dt$$

Using Cauchy's inequality, we have

$$\int_0^1 (t[f(a)]^r + (1-t)[f(b)]^r)^{1/r}(t[g(a)]^s + (1-t)[g(b)]^s)^{1/s}dt$$

$$\le \frac{1}{2}\int_0^1 (t[f(a)]^r + (1-t)[f(b)]^r)^{2/r}dt$$

$$+ \frac{1}{2}\int_0^1 (t[g(a)]^s + (1-t)[g(b)]^s)^{2/s}dt$$

Using Minkowski's inequality, we have

$$\begin{aligned}
&\int_0^1 (t[f(a)]^r + (1-t)[f(b)]^r)^{2/r} dt \\
&\leq \left[\left(\int_0^1 t^{2/r}[f(a)]^2 dt \right)^{r/2} + \left(\int_0^1 (1-t)^{2/r}[f(b)]^2 dt \right)^{r/2} \right]^{2/r} \\
&= \left([f(a)]^r \int_0^1 t^{2/r} dt + [f(b)]^r \int_0^1 (1-t)^{2/r} dt \right)^{2/r} \\
&= \left(\frac{r}{r+2}[f(a)]^r + \frac{r}{r+2}[f(b)]^r \right)^{2/r} \\
&= \left(\frac{r}{r+2} \right)^{2/r} ([f(a)]^r + [f(b)]^r)^{2/r}
\end{aligned}$$

Similiary we have:

$$\int_0^1 (t[g(a)]^s + (1-t)[f(b)]^s)^{2/s} dt \leq \left(\frac{s}{s+2} \right)^{2/s} ([g(a)]^s + [g(b)]^s)^{2/s}$$

Thus

$$\begin{aligned}
\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq & \frac{1}{2} \left(\frac{r}{r+2} \right)^{2/r} ([f(a)]^r + [f(b)]^r)^{2/r} \\
& + \frac{1}{2} \left(\frac{s}{s+2} \right)^{2/s} ([g(a)]^s + [g(b)]^s)^{2/s}
\end{aligned}$$

□

Corollary 2.4. *In Theorem 2.3, if $s = r = 2$, we have*

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{4} \left([f(a)]^2 + [f(b)]^2 + [g(a)]^2 + [g(b)]^2 \right)$$

Corollary 2.5. *In Theorem 2.3, if $s = r = 2$ and $f(x) = g(x)$, we have*

$$\frac{1}{b-a} \int_a^b [f(x)]^2 dx \leq \frac{1}{2} \left([f(a)]^2 + [f(b)]^2 \right)$$

Theorem 2.6. *Let $f, g : [a,b] \to (0,\infty)$ be r-convex and s-convex functions respectively on $[a,b]$ with $a < b$. Then the following inequality holds if $r > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$:*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \left(\frac{[f(a)]^r + [f(b)]^r}{2}\right)^{1/r}\left(\frac{[g(a)]^s + [g(b)]^s}{2}\right)^{1/s}$$

Proof. Since f is r-convex function and g is s-convex function ($r > 0, s > 0$), we have

$$f(ta + (1-t)b) \leq (t[f(a)]^r + (1-t)[f(b)]^r)^{1/r}$$
$$g(ta + (1-t)b) \leq (t[g(a)]^s + (1-t)[g(b)]^s)^{1/s}$$

for all $t \in [0,1]$.

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(x)dx &= \int_0^1 f(ta + (1-t)b)g(ta + (1-t)b)dt \\ &\leq \int_0^1 (t[f(a)]^r + (1-t)[f(b)]^r)^{1/r}(t[g(a)]^s + (1-t)[g(b)]^s)^{1/s}dt\end{aligned}$$

Using Holder's inequality, we have

$$\begin{aligned}&\int_0^1 (t[f(a)]^r + (1-t)[f(b)]^r)^{1/r}(t[g(a)]^s + (1-t)[g(b)]^s)^{1/s}dt \\ &\leq \left(\int_0^1 (t[f(a)]^r + (1-t)[f(b)]^r)dt\right)^{1/r}\left(\int_0^1 (t[g(a)]^s + (1-t)[g(b)]^s)dt\right)^{1/s} \\ &= \left([f(a)]^r\int_0^1 tdt + [f(b)]^r\int_0^1 (1-t)dt\right)^{1/r}\left([g(a)]^s\int_0^1 tdt + [g(b)]^s\int_0^1 (1-t)dt\right)^{1/s} \\ &= \left(\frac{[f(a)]^r+[f(b)]^r}{2}\right)^{1/r}\left(\frac{[g(a)]^s+[g(b)]^s}{2}\right)^{1/s}\end{aligned}$$

Thus

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \left(\frac{[f(a)]^r + [f(b)]^r}{2}\right)^{1/r}\left(\frac{[g(a)]^s + [g(b)]^s}{2}\right)^{1/s}.$$

□

Corollary 2.7. *In Theorem 2.6, if $r = s = 2$, we have*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \sqrt{\frac{[f(a)]^2 + [f(b)]^2}{2}}\sqrt{\frac{[g(a)]^2 + [g(b)]^2}{2}}.$$

Corollary 2.8. *In Theorem 2.6, if $r = s = 2$ and $f(x) = g(x)$, we have*

$$\frac{1}{b-a}\int_a^b [f(x)]^2 dx \leq \frac{[f(a)]^2 + [f(b)]^2}{2}.$$

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