Hindawi
Journal of Function Spaces
Volume 2019, Article ID 9487823, 11 pages
https://doi.org/10.1155/2019/9487823

Research Article

Integral Majorization Type Inequalities for the Functions in the Sense of Strong Convexity

Syed Zaheer Ullah,[1] Muhammad Adil Khan,[1] Zareen Abdulhameed Khan,[2] and Yu-Ming Chu[3]

[1] *Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan*
[2] *Department of Mathematics, Princess Nora Bint Abdulrahman University, Riyadh 11538, Saudi Arabia*
[3] *Department of Mathematics, Huzhou University, Huzhou 313000, China*

Correspondence should be addressed to Yu-Ming Chu; chuyuming@zjhu.edu.cn

Received 24 January 2019; Accepted 27 March 2019; Published 2 May 2019

Guest Editor: Vita Leonessa

In this article, we establish several integral majorization type and generalized Favard's inequalities for the class of strongly convex functions. Our results generalize and improve the previous known results.

1. Introduction

It is well known that convex functions are a class of important functions in the fields of mathematics and other natural sciences; they have been studied for more than one hundred years. In recent years there is a growing interest in generalized convex functions (such as quasi-convex function [1], strongly convex function [2–4], s-convex function [5], approximately convex function [6], logarithmically convex function [7, 8], midconvex function [9], pseudo-convex function [10], φ-convex function [11], λ-convex function [12], h-convex function [13], delta-convex function [14], Schur convex function [15–21], and other convex functions [22–29]) among the researchers of applied mathematics due to the fact that mathematical models with these functions are more suitable to describe problems of the real world than models using conventional convex functions. Recently, a large number of remarkable results and applications for the generalized convex functions can be found in the literature [30–49].

In the article, our focus is on the integral majorization type inequalities for the strongly convex functions.

Definition 1. Let ϕ be a real-valued function defined on the interval $[\lambda_1, \xi_1]$ and c a positive real number. Then ϕ is said to be strongly convex with modulus c if the inequality

$$\phi(\eta u_1 + (1-\eta)v_1) \le \eta\phi(u_1) + (1-\eta)\phi(v_1) - c\eta(1-\eta)(u_1-v_1)^2 \tag{1}$$

holds for all $u_1, v_1 \in [\lambda_1, \xi_1]$ and $\eta \in [0,1]$. From (1) we clearly see that

$$\phi(u_1) - \phi(v_1) \ge \phi'_+(v_1)(u_1-v_1) + c(u_1-v_1)^2. \tag{2}$$

The following Lemma 2 for strongly convex function is given in [2] (see also [50, Proposition 1.1.2]).

Lemma 2. *A real-valued function $\phi : [\lambda_1, \xi_1] \longrightarrow \mathbb{R}$ is a strongly convex function with modulus c if and only if the function $\varphi : [\lambda_1, \xi_1] \longrightarrow \mathbb{R}$ defined by $\varphi(r) = \phi(r) - cr^2$ is a convex function.*

Every strongly convex function is convex, but the converse is not true in general. Strongly convex functions have been utilized for showing the convergence of a gradient type algorithm for minimizing a function. They play a significant role in mathematical economics, approximation theory, and optimization theory; many applications and properties for strongly functions can be found in [2–4, 13, 30].

Next we are going to present some basic theories of majorization.

There is a natural description of the indefinite notion that the entries of n-tuple $\boldsymbol{\delta}$ are more nearly equal, or less spread out than, to the entries of n-tuple $\boldsymbol{\beta}$. The applicable assertion is that $\boldsymbol{\delta}$ majorizes $\boldsymbol{\beta}$; it means that the sum of ℓ largest entries of $\boldsymbol{\beta}$ does not exceed the sum of ℓ largest entries of $\boldsymbol{\delta}$ for all $\ell = 1, 2, \ldots, n-1$ with equality for $\ell = n$. That is, let $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_n)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$ be two real n−tuples and let

$$\delta_1^{\downarrow} \geq \delta_2^{\downarrow} \geq \cdots \geq \delta_n^{\downarrow}, \\ \beta_1^{\downarrow} \geq \beta_2^{\downarrow} \geq \cdots \geq \beta_n^{\downarrow} \tag{3}$$

be their ordered entries. Then the n−tuple $\boldsymbol{\delta}$ is said to majorize $\boldsymbol{\beta}$ (or $\boldsymbol{\beta}$ is said to be majorized by $\boldsymbol{\delta}$), in symbol $\boldsymbol{\delta} \succ \boldsymbol{\beta}$, if

$$\sum_{j=1}^{\ell} \delta_j^{\downarrow} \geq \sum_{j=1}^{\ell} \beta_j^{\downarrow} \tag{4}$$

holds for $\ell = 1, 2, \ldots, n-1$ and

$$\sum_{j=1}^{n} \delta_j = \sum_{j=1}^{n} \beta_j. \tag{5}$$

The theory of majorization is a very significant topic in mathematics; a remarkable and complete reference on the majorization subject is the book by Olkin and Marshall [51]. For example, the theory of majorization is an essential tool that permits us to transform nonconvex complicated constrained optimization problems that involve matrix valued variables into simple problems with scalar variables that can be easily solved [52–55].

The definition of majorization for integrable functions can be stated as follows (see [7]).

Definition 3. Let $\mathfrak{f}$ and $\mathfrak{g}$ be two decreasing real-valued integrable functions on the interval $[\lambda_1, \xi_1]$. Then $\mathfrak{f}$ is said to majorize $\mathfrak{g}$ (or $\mathfrak{g}$ is said to be majorized by $\mathfrak{f}$), in symbol, $\mathfrak{f} \succ \mathfrak{g}$, if the inequality

$$\int_{\lambda_1}^{x} \mathfrak{g}(r)\, dr \leq \int_{\lambda_1}^{x} \mathfrak{f}(r)\, dr \tag{6}$$

holds for all $x \in [\lambda_1, \xi_1)$ and

$$\int_{\lambda_1}^{\xi_1} \mathfrak{g}(r)\, dr = \int_{\lambda_1}^{\xi_1} \mathfrak{f}(r)\, dr. \tag{7}$$

Theorem 4 (See [56]). *Let $\mathfrak{f}$ and $\mathfrak{g}$ be two continuous and increasing real-valued functions defined on $[\lambda_1, \xi_1]$, and let $\Omega : [\lambda_1, \xi_1] \longrightarrow \mathbb{R}$ be a bounded variation function. Then the following statements are true.*

(a) *If*

$$\int_{\lambda_1}^{x} \mathfrak{f}(r)\, d\Omega(r) \leq \int_{\lambda_1}^{x} \mathfrak{g}(r)\, d\Omega(r) \tag{8}$$

for all $x \in [\lambda_1, \xi_1]$ and

$$\int_{\lambda_1}^{\xi_1} \mathfrak{f}(r)\, d\Omega(r) = \int_{\lambda_1}^{\xi_1} \mathfrak{g}(r)\, d\Omega(r), \tag{9}$$

then

$$\int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{f}(r)\}\, d\Omega(r) \leq \int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{g}(r)\}\, d\Omega(r) \tag{10}$$

holds for every continuous convex function Ψ.

(b) *If (8) and (9) hold, then (10) holds for every continuous increasing convex function Ψ.*

Theorem 5 (See [57]). *Let $\Psi : [0, \infty) \longrightarrow \mathbb{R}$ be a convex function, $\mathfrak{f}$, $\mathfrak{g}$ and Ω be three positive and integrable functions defined on $[\lambda_1, \xi_1]$ such that*

$$\int_{\lambda_1}^{x} \mathfrak{f}(r)\,\Omega(r)\, dr \leq \int_{\lambda_1}^{x} \mathfrak{g}(r)\,\Omega(r)\, dr \tag{11}$$

for all $x \in [\lambda_1, \xi_1]$ and

$$\int_{\lambda_1}^{\xi_1} \mathfrak{f}(r)\,\Omega(r)\, dr = \int_{\lambda_1}^{\xi_1} \mathfrak{g}(r)\,\Omega(r)\, dr. \tag{12}$$

Then the following statements are true:

(a) *If $\mathfrak{f}$ is decreasing on $[\lambda_1, \xi_1]$, then*

$$\int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{f}(r)\}\,\Omega(r)\, dr \leq \int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{g}(r)\}\,\Omega(r)\, dr. \tag{13}$$

(b) *If $\mathfrak{g}$ is increasing on $[\lambda_1, \xi_1]$, then*

$$\int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{g}(r)\}\,\Omega(r)\, dr \leq \int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{f}(r)\}\,\Omega(r)\, dr. \tag{14}$$

Let $p > 1$, $\mathfrak{f}$ be a positive and continuous concave function defined on $[\lambda_1, \xi_1]$, and let Ψ be a convex function defined on $[0, 2\bar{\mathfrak{f}}_1]$ with

$$\bar{\mathfrak{f}}_1 = \frac{1}{\xi_1 - \lambda_1} \int_{\lambda_1}^{\xi_1} \mathfrak{f}(r)\, dr. \tag{15}$$

Then Favard [58] proved that the inequalities

$$\begin{aligned} \int_0^1 \Psi\left(2s\bar{\mathfrak{f}}_1\right) &= \frac{1}{2\bar{\mathfrak{f}}_1} \int_0^{2\bar{\mathfrak{f}}_1} \Psi(y)\, dy \\ &\geq \frac{1}{\xi_1 - \lambda_1} \int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{f}(r)\}\, dr \end{aligned} \tag{16}$$

and

$$\frac{1}{\xi_1 - \lambda_1} \int_{\lambda_1}^{\xi_1} \mathfrak{f}^{p}(r)\, dr \leq \frac{2^p}{p+1} \left(\frac{1}{\xi_1 - \lambda_1} \int_{\lambda_1}^{\xi_1} \mathfrak{f}(r)\, dr \right)^{p} \tag{17}$$

hold.

The main purpose of the article is to establish several integral majorization type and generalized Favard's inequalities for strongly convex functions.

2. Main Results

Theorem 6. *Let $c > 0$, $\Psi : [0,\infty) \longrightarrow \mathbb{R}$ be a continuous strongly convex function with modulus c, and let $\mathfrak{f}$, $\mathfrak{g}$, and Ω be three positive and integrable functions defined on $[\lambda_1, \xi_1]$ such that*

$$\int_{\lambda_1}^{x} \mathfrak{f}(r)\Omega(r)\,dr \le \int_{\lambda_1}^{x} \mathfrak{g}(r)\Omega(r)\,dr \tag{18}$$

for all $x \in [\lambda_1, \xi_1]$ and

$$\int_{\lambda_1}^{\xi_1} \mathfrak{f}(r)\Omega(r)\,dr = \int_{\lambda_1}^{\xi_1} \mathfrak{g}(r)\Omega(r)\,dr. \tag{19}$$

Then the following statements are true.

(a) *If $\mathfrak{f}$ is decreasing on $[\lambda_1, \xi_1]$, then we have*

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{g}(r)\}\Omega(r)\,dr &\ge \int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{f}(r)\}\Omega(r)\,dr \\ &\quad + c\int_{\lambda_1}^{\xi_1} \{\mathfrak{f}(r) - \mathfrak{g}(r)\}^2\Omega(r)\,dr.\end{aligned} \tag{20}$$

(b) *If $\mathfrak{g}$ is increasing on $[\lambda_1, \xi_1]$, then one has*

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{f}(r)\}\Omega(r)\,dr &\ge \int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{g}(r)\}\Omega(r)\,dr \\ &\quad + c\int_{\lambda_1}^{\xi_1} \{\mathfrak{f}(r) - \mathfrak{g}(r)\}^2\Omega(r)\,dr.\end{aligned} \tag{21}$$

Proof. (a) Let $v_1 = \mathfrak{f}$ and $u_1 = \mathfrak{g}$. Then it follows from (2) and the proof of Lemma 2 given in [57] that

$$\begin{aligned}&\Psi(\mathfrak{f}(r))\Omega(r) + \Psi'_+(\mathfrak{f}(r))(\mathfrak{g}(r) - \mathfrak{f}(r))\Omega(r) \\ &\quad + c(\mathfrak{g}(r) - \mathfrak{f}(r))^2\Omega(r) \le \Psi(\mathfrak{g}(r))\Omega(r).\end{aligned} \tag{22}$$

Let $\mathscr{F}(x) = \int_{\lambda_1}^{x}\{\mathfrak{f}(r) - \mathfrak{g}(r)\}\Omega(r)dr$. Then (18) and (19) lead to $\mathscr{F}(x) \le 0$ for all $x \in [\lambda_1, \xi_1]$, $\mathscr{F}(\lambda_1) = \mathscr{F}(\xi_1) = 0$, and

$$\begin{aligned}&\int_{\lambda_1}^{\xi_1} [\Psi\{\mathfrak{f}(r)\} - \Psi\{\mathfrak{g}(r)\}]\Omega(r)\,dr \\ &\quad + c\int_{\lambda_1}^{\xi_1} \{\mathfrak{f}(r) - \mathfrak{g}(r)\}^2\Omega(r)\,dr \\ &\le \int_{\lambda_1}^{\xi_1} \Psi'_+\{\mathfrak{f}(r)\}\{\mathfrak{f}(r) - \mathfrak{g}(r)\}\Omega(r)\,dr \\ &= \int_{\lambda_1}^{\xi_1} \Psi'_+\{\mathfrak{f}(r)\}\,d\mathscr{F}(r) = \Psi'_+\{\mathfrak{f}(r)\}\mathscr{F}(r)\Big|_{\lambda_1}^{\xi_1} \\ &\quad - \int_{\lambda_1}^{\xi_1} \mathscr{F}(r)\,d\left\{\Psi'_+\{\mathfrak{f}(r)\}\right\} \\ &= -\int_{\lambda_1}^{\xi_1} \mathscr{F}(r)\,d\left\{\Psi'_+\{\mathfrak{f}(r)\}\right\} \le 0.\end{aligned} \tag{23}$$

Since $\mathfrak{f}$ is decreasing on $[\lambda_1, \xi_1]$, therefore inequality (20) can be deduced easily from the above inequality. Similarly, we can prove part (b) for increasing function $\mathfrak{g}$ defined on $[\lambda_1, \xi_1]$. □

Theorem 7. *Suppose that all the assumptions of Theorem 6 hold. Then the following statements are true.*

(a) *If $\mathfrak{f}$ is decreasing on $[\lambda_1, \xi_1]$, then*

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{g}(r)\}\Omega(r)\,dr &\ge \int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{f}(r)\}\Omega(r)\,dr \\ &\quad + c\int_{\lambda_1}^{\xi_1} \left\{\mathfrak{g}^2(r) - \mathfrak{f}^2(r)\right\}\Omega(r)\,dr.\end{aligned} \tag{24}$$

(b) *If $\mathfrak{g}$ is increasing on $[\lambda_1, \xi_1]$, then*

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{g}(r)\}\Omega(r)\,dr &\le \int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{f}(r)\}\Omega(r)\,dr \\ &\quad + c\int_{\lambda_1}^{\xi_1} \left\{\mathfrak{g}^2(r) - \mathfrak{f}^2(r)\right\}\Omega(r)\,dr.\end{aligned} \tag{25}$$

Proof. Since Ψ is a strongly convex function with modulus c, therefore $\Psi(r) - cr^2$ is a convex function, and inequalities (24) and (25) follow easily from the convexity of the function $\Psi(r) - cr^2$ and Lemma 2 given in [57]. □

Remark 8. Inequalities (13) and (14) can be obtained by (24) and (25) immediately.

Remark 9. Generally, the assumptions of both the functions $\mathfrak{f}$ and $\mathfrak{g}$ are monotonic in majorization theorem, but in Theorems 6 and 7 we only need one of the functions $\mathfrak{f}$ and $\mathfrak{g}$ to be monotonic.

Theorem 10. *Suppose that $\Psi : [\lambda_1, \xi_1] \longrightarrow \mathbb{R}$ is a continuous strongly convex function with modulus c, and $\mathfrak{f}$, $\mathfrak{g}$, and Ω are three integrable functions on $[\lambda_1, \xi_1]$. If $\mathfrak{g}$ and $\mathfrak{f} - \mathfrak{g}$*

are nondecreasing (nonincreasing) functions on $[\lambda_1, \xi_1]$ *and* $\int_{\lambda_1}^{\xi_1} \mathfrak{f}(r)\Omega(r)dr = \int_{\lambda_1}^{\xi_1} \mathfrak{g}(r)\Omega(r)dr$, *then*

$$\int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{f}(r)\}\Omega(r)\,dr \geq \int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{g}(r)\}\Omega(r)\,dr + c\int_{\lambda_1}^{\xi_1} \{\mathfrak{f}(r)-\mathfrak{g}(r)\}^2\Omega(r)\,dr. \quad (26)$$

Proof. Since Ψ is a strongly convex function, therefore using (2) for $u_1 = \mathfrak{f}$ and $v_1 = \mathfrak{g}$, we have

$$\int_{\lambda_1}^{\xi_1} [\Psi(\mathfrak{f}(r)) - \Psi(\mathfrak{g}(r))]\Omega(r)\,dr - c\int_{\lambda_1}^{\xi_1} (\mathfrak{f}(r)-\mathfrak{g}(r))^2\Omega(r)\,dr \geq \int_{\lambda_1}^{\xi_1} \Psi'_+(\mathfrak{g}(r))(\mathfrak{f}(r)-\mathfrak{g}(r))\Omega(r)\,dr. \quad (27)$$

It follows from the Čebyšev inequality [59] that

$$\frac{1}{\int_{\lambda_1}^{\xi_1}\Omega(r)\,dr}\int_{\lambda_1}^{\xi_1} \Psi'_+(\mathfrak{g}(r))(\mathfrak{f}(r)-\mathfrak{g}(r))\Omega(r)\,dr \geq \frac{1}{\int_{\lambda_1}^{\xi_1}\Omega(r)\,dr}\int_{\lambda_1}^{\xi_1} \Psi'_+(\mathfrak{g}(r))\Omega(r)\,dr\frac{1}{\int_{\lambda_1}^{\xi_1}\Omega(r)\,dr} \cdot \int_{\lambda_1}^{\xi_1} (\mathfrak{f}(r)-\mathfrak{g}(r))\Omega(r)\,dr \geq 0. \quad (28)$$

Therefore, inequality (26) follows from (27) and (28). □

Making use of the similar idea as in the proof of Theorem 10, we can obtain the following Theorem 11 immediately.

Theorem 11. *Suppose that* $\Psi : [\lambda_1, \xi_1] \longrightarrow \mathbb{R}$ *is a continuous strongly convex function with modulus c, and* $\mathfrak{f}$, $\mathfrak{g}$, *and* Ω *are three integrable functions on* $[\lambda_1, \xi_1]$. *If* $\mathfrak{g}$ *and* $\mathfrak{f} - \mathfrak{g}$ *are nondecreasing (nonincreasing) functions on* $[\lambda_1, \xi_1]$, *and* $\int_{\lambda_1}^{\xi_1} \mathfrak{f}(r)\Omega(r)dr \geq \int_{\lambda_1}^{\xi_1} \mathfrak{g}(r)\Omega(r)dr$, *then inequality (26) holds.*

Theorem 12. *The inequality*

$$\int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{f}(r)\}\Omega(r)\,dr \geq \int_{\lambda_1}^{\xi_1} \Psi\{\mathfrak{g}(r)\}\Omega(r)\,dr + c\int_{\lambda_1}^{\xi_1} \left(\mathfrak{f}^2(r)-\mathfrak{g}^2(r)\right)\Omega(r)\,dr \quad (29)$$

holds if all the assumptions of Theorem 10 are satisfied.

Proof. Since Ψ is a strongly convex function with modulus c, therefore $\Psi(r) - cr^2$ is a convex function and inequality (29) can be deduced by applying this convex function in Theorem 6 of [59]. □

Using strongly convex function we can give an extension of [60, Theorem 2] in the following form.

Theorem 13. *Let* $\varphi, \Psi : [0,\infty) \longrightarrow \mathbb{R}$ *be two functions such that* φ *is a strictly increasing and* $\Psi^\circ\varphi^{-1}$ *is strongly convex with modulus c,* $\mathfrak{f}$, $\mathfrak{g}$, *and* Ω *being three positive and integrable functions on* $[\lambda_1, \xi_1]$ *such that*

$$\int_{\lambda_1}^{x} \varphi(\mathfrak{f}(r))\Omega(r)\,dr \leq \int_{\lambda_1}^{x} \varphi(\mathfrak{g}(r))\Omega(r)\,dr \quad (30)$$

for all $x \in [\lambda_1, \xi_1]$ *and*

$$\int_{\lambda_1}^{\xi_1} \varphi(\mathfrak{f}(r))\Omega(r)\,dr = \int_{\lambda_1}^{\xi_1} \varphi(\mathfrak{g}(r))\Omega(r)\,dr. \quad (31)$$

Then the following statements are true.

(a) *If* $\mathfrak{f}$ *is decreasing on* $[\lambda_1, \xi_1]$, *then inequality (20) holds.*

(b) *If* $\mathfrak{g}$ *is increasing on* $[\lambda_1, \xi_1]$, *then inequality (21) holds.*

Proof. We clearly see that it is sufficient to prove the case of $\varphi(r) = r$, but this case is already proved in Theorem 6. □

Similarly, we have Theorem 14 as follows.

Theorem 14. *Suppose that all the assumptions of Theorem 13 are satisfied. Then the following statements are true.*

(a) *If* $\mathfrak{f}$ *is decreasing on* $[\lambda_1, \xi_1]$, *then inequality (24) holds.*

(b) *If* $\mathfrak{g}$ *is increasing on* $[\lambda_1, \xi_1]$, *then inequality (25) holds.*

The following Lemma 15 was given in [57].

Lemma 15. *Let* χ *be a positive integrable function and let* $\hbar$ *be an increasing function on* (λ_1, ξ_1); *then*

$$\int_{\lambda_1}^{x} \hbar(r)\chi(r)\,dr\int_{\lambda_1}^{\xi_1}\chi(r)\,dr \leq \int_{\lambda_1}^{\xi_1} \hbar(r)\chi(r)\,dr\int_{\lambda_1}^{x}\chi(r)\,dr. \quad (32)$$

If $\hbar$ *is a decreasing function on* (λ_1, ξ_1), *then inequality (32) holds in reverse directions.*

Lemma 16. *Let* $\mathfrak{f}$ *be a real-valued function defined on* $[\lambda_1, \xi_1]$. *Then the following statements are true.*

(i) *If* $\mathfrak{f}$ *is a strongly concave function with modulus c, then*

(a) *the function* $\hbar_1(r) = \mathfrak{f}(r)/(r-\lambda_1) - cr$ *is decreasing on* $(\lambda_1, \xi_1]$;

(b) *the function* $\hbar_2(r) = \mathfrak{f}(r)/(\xi_1 - r) + cr$ *is increasing on* $[\lambda_1, \xi_1)$.

(ii) *If $\mathfrak{f}$ is a strongly convex function with modulus c, then*

(c) *the function $\hbar_1(r) = \mathfrak{f}(r)/(r-\lambda_1)-cr$ is increasing on $(\lambda_1, \xi_1]$ if $\mathfrak{f}(\lambda_1) = 0$;*

(d) *the function $\hbar_2(r) = \mathfrak{f}(r)/(\xi_1-r)+cr$ is decreasing on $[\lambda_1, \xi_1)$ if $\mathfrak{f}(\xi_1) = 0$.*

Proof. (i) Suppose that $\mathfrak{f}$ is a strongly concave function with modulus c.

(a) To show that the function $\hbar_1(r) = \mathfrak{f}(r)/(r-\lambda_1) - cr$ is decreasing on $(\lambda_1, \xi_1]$, in fact, for $\lambda_1 < r_1 \le r_2 \le \xi_1$ we have

$$\begin{aligned}\mathfrak{f}(r_1) &= \mathfrak{f}\left(\frac{r_1-\lambda_1}{r_2-\lambda_1}r_2 + \frac{r_2-\lambda_1-(r_1-\lambda_1)}{r_2-\lambda_1}\lambda_1\right) \\ &\ge \frac{r_1-\lambda_1}{r_2-\lambda_1}\mathfrak{f}(r_2) + \left(1-\frac{r_1-\lambda_1}{r_2-\lambda_1}\right)\mathfrak{f}(\lambda_1) \\ &\quad - c\left(\frac{r_1-\lambda_1}{r_2-\lambda_1}\right)\left(1-\frac{r_1-\lambda_1}{r_2-\lambda_1}\right)(r_2-\lambda_1)^2 \\ &\ge \frac{r_1-\lambda_1}{r_2-\lambda_1}\mathfrak{f}(r_2) - c(r_1-\lambda_1)(r_2-r_1),\end{aligned} \tag{33}$$

which shows that the function $\hbar_1(r) = \mathfrak{f}(r)/(r-\lambda_1) - cr$ is decreasing on $(\lambda_1, \xi_1]$.

(b) To show that the function $\hbar_2(r) = \mathfrak{f}(r)/(\xi_1 - r) + cr$ is increasing on $[\lambda_1, \xi_1)$, in fact, for $\lambda_1 \le r_1 \le r_2 < \xi_1$ we have

$$\begin{aligned}\mathfrak{f}(r_2) &= \mathfrak{f}\left(\frac{\xi_1-r_2}{\xi_1-r_1}r_1 + \frac{\xi_1-r_1-(\xi_1-r_2)}{\xi_1-r_1}\xi_1\right) \\ &\ge \frac{\xi_1-r_2}{\xi_1-r_1}\mathfrak{f}(r_1) + \left(1-\frac{\xi_1-r_2}{\xi_1-r_1}\right)\mathfrak{f}(\xi_1) \\ &\quad - c_1\left(\frac{\xi_1-r_2}{\xi_1-r_1}\right)\left(1-\frac{\xi_1-r_2}{\xi_1-r_1}\right)(r_1-\xi_1)^2 \\ &\ge \frac{\xi_1-r_2}{\xi_1-r_1}\mathfrak{f}(r_1) - c_1(\xi_1-r_2)(r_2-r_1),\end{aligned} \tag{34}$$

which shows that the function $\hbar_2(r) = \mathfrak{f}(r)/(\xi_1 - r) + cr$ is increasing on $[\lambda_1, \xi_1)$.

(ii) Suppose that $\mathfrak{f}$ is a strongly convex function with modulus c.

(c) Since $\mathfrak{f}(\lambda_1) = 0$, by similar method of (a) we can easily prove that the function $\hbar_1(r) = \mathfrak{f}(r)/(r-\lambda_1) - cr$ is increasing on $(\lambda_1, \xi_1]$.

(d) Since $\mathfrak{f}(\xi_1) = 0$, by similar method of (b) we can prove that the function $\hbar_2(r) = \mathfrak{f}(r)/(\xi_1 - r) + cr$ is decreasing on $(\lambda_1, \xi_1]$. □

Next, we establish several Favard type inequalities for strongly convex functions.

Theorem 17. (a) *Let $\mathfrak{f}$ be a strongly concave function with modulus c_1 on $[\lambda_1, \xi_1]$ such that $\mathfrak{g}(r) = \mathfrak{f}(r) - c_1 r(r-\lambda_1)$ is a positive increasing function, let Ψ be a strongly convex function with modulus c_2 on $[0, 2\bar{\mathfrak{f}}_1]$, $\bar{z}_1 = \lambda_1(1-s) + \xi_1 s$, and*

$$\bar{\mathfrak{f}}_1 = \frac{(\xi_1-\lambda_1)\int_{\lambda_1}^{\xi_1}\mathfrak{g}(r)\,\Omega(r)\,dr}{2\int_{\lambda_1}^{\xi_1}(r-\lambda_1)\,\Omega(r)\,dr}. \tag{35}$$

Then

$$\begin{aligned}&\frac{1}{\xi_1-\lambda_1}\int_{\lambda_1}^{\xi_1}\Psi\{\mathfrak{g}(r)\}\,\Omega(r)\,dr \le \int_0^1 \Psi\left(2s\bar{\mathfrak{f}}_1\right) \\ &\quad\cdot\Omega(\bar{z}_1)\,ds \\ &\quad - c_2\int_0^1\left\{\left[\mathfrak{f}(\bar{z}_1) - c_1\bar{z}_1(s\xi_1 - s\lambda_1)\right] - 2s\bar{\mathfrak{f}}_1\right\}^2 \\ &\quad\cdot\Omega(\bar{z}_1)\,ds.\end{aligned} \tag{36}$$

If $\mathfrak{f}$ is a strongly convex function with modulus c_1 on $[\lambda_1, \xi_1]$ such that $\mathfrak{g}(r) = \mathfrak{f}(r) - c_1 r(r-\lambda_1)$ is a positive increasing function and $\mathfrak{f}(\lambda_1) = 0$, then the reverse inequality in (36) holds.

(b) *Let $\mathfrak{f}$ be a strongly concave function with modulus c_1 on $[\lambda_1, \xi_1]$ such that $\mathfrak{g}(r) = \mathfrak{f}(r) + c_1 r(\xi_1 - r)$ is a positive decreasing function, let Ψ be a strongly convex function with modulus c_2 on $[0, 2\bar{\mathfrak{f}}_2]$, $\bar{z}_2 = \lambda_1 s + \xi_1(1-s)$, and*

$$\bar{\mathfrak{f}}_2 = \frac{(\xi_1-\lambda_1)\int_{\lambda_1}^{\xi_1}\mathfrak{g}(r)\,\Omega(r)\,dr}{2\int_{\lambda_1}^{\xi_1}(\xi_1-r)\,\Omega(r)\,dr}. \tag{37}$$

Then

$$\begin{aligned}&\frac{1}{\xi_1-\lambda_1}\int_{\lambda_1}^{\xi_1}\Psi\{\mathfrak{g}(r)\}\,\Omega(r)\,dr \le \int_0^1 \Psi\left(2s\bar{\mathfrak{f}}_2\right) \\ &\quad\cdot\Omega(\bar{z}_2)\,ds \\ &\quad - c_2\int_0^1\left\{\left[\mathfrak{f}(\bar{z}_2) + c_1\bar{z}_2(s\xi_1 - s\lambda_1)\right] - 2s\bar{\mathfrak{f}}_2\right\}^2 \\ &\quad\cdot\Omega(\bar{z}_2)\,ds.\end{aligned} \tag{38}$$

If $\mathfrak{f}$ is a strongly convex function with modulus c_1 on $[\lambda_1, \xi_1]$ such that $\mathfrak{g}(r) = \mathfrak{f}(r) + c_1 r(\xi_1 - r)$ is a positive decreasing function and $\mathfrak{f}(\xi_1) = 0$, then the reverse inequality in (38) holds.

Proof. (a) From Lemma 16(a) we know that the function $\hbar_1(r) = \mathfrak{f}(r)/(r-\lambda_1) - c_1 r$ is decreasing; then using Lemma 15 to the functions $\chi(r) = (r-\lambda_1)\Omega(r)$ and $\hbar_1(r)$, we obtain

$$\begin{aligned}&\int_{\lambda_1}^{\xi_1}\mathfrak{g}(r)\,\Omega(r)\,dr\int_{\lambda_1}^{x}(r-\lambda_1)\,\Omega(r)\,dr \\ &\quad\le \int_{\lambda_1}^{x}\mathfrak{g}(r)\,\Omega(r)\,dr\int_{\lambda_1}^{\xi_1}(r-\lambda_1)\,\Omega(r)\,dr.\end{aligned} \tag{39}$$

It follows from (35) that inequality (39) can be rewritten as

$$\int_{\lambda_1}^{x}\frac{(r-\lambda_1)}{\xi_1-\lambda_1}2\bar{\mathfrak{f}}_1\Omega(r)\,dr \le \int_{\lambda_1}^{x}\mathfrak{g}(r)\,\Omega(r)\,dr \tag{40}$$

for all $x \in [\lambda_1, \xi_1]$.

As $\mathfrak{g}$ is an increasing function, and by use of Theorem 6(b), we have

$$\int_{\lambda_1}^{\xi_1}\Psi\{\mathfrak{g}(r)\}\Omega(r)\,dr \le \int_{\lambda_1}^{\xi_1}\Psi\left\{\frac{(r-\lambda_1)}{\xi_1-\lambda_1}2\bar{\mathfrak{f}}_1\right\}\Omega(r)\,dr - c_2\int_{\lambda_1}^{\xi_1}\left\{\mathfrak{g}(r)-\frac{(r-\lambda_1)}{\xi_1-\lambda_1}2\bar{\mathfrak{f}}_1\right\}^2\Omega(r)\,dr. \quad (41)$$

Note that

$$\begin{aligned}&\frac{1}{\xi_1-\lambda_1}\int_{\lambda_1}^{\xi_1}\Psi\left\{\frac{(r-\lambda_1)}{\xi_1-\lambda_1}2\bar{\mathfrak{f}}_1\right\}\Omega(r)\,dr-\frac{c_2}{\xi_1-\lambda_1}\cdot\int_{\lambda_1}^{\xi_1}\left\{\mathfrak{g}(r)-\frac{(r-\lambda_1)}{\xi_1-\lambda_1}2\bar{\mathfrak{f}}_1\right\}^2\Omega(r)\,dr=\frac{1}{2\bar{\mathfrak{f}}_1}\\&\cdot\int_0^{2\bar{\mathfrak{f}}_1}\Psi(y)\,\Omega\left(\lambda_1+y\frac{\xi_1-\lambda_1}{2\bar{\mathfrak{f}}_1}\right)dy-\frac{c_2}{2\bar{\mathfrak{f}}_1}\cdot\int_0^{2\bar{\mathfrak{f}}_1}\left\{\left[\mathfrak{f}\left(\lambda_1+y\frac{\xi_1-\lambda_1}{2\bar{\mathfrak{f}}_1}\right)\right.\right.\\&\left.\left.-c_1\left(\lambda_1+y\frac{\xi_1-\lambda_1}{2\bar{\mathfrak{f}}_1}\right)\left(y\frac{\xi_1-\lambda_1}{2\bar{\mathfrak{f}}_1}\right)\right]-y\right\}^2\\&\times\Omega\left(\lambda_1+y\frac{\xi_1-\lambda_1}{2\bar{\mathfrak{f}}_1}\right)dy=\int_0^1\Psi\left(2s\bar{\mathfrak{f}}_1\right)\cdot\Omega\left[\lambda_1(1-s)+\xi_1 s\right]ds\\&-c_2\int_0^1\left\{\left[\mathfrak{f}\left(\lambda_1(1-s)+\xi_1 s\right)-c_1\left(\lambda_1(1-s)+\xi_1 s\right)\left(s\xi_1-s\lambda_1\right)\right]-2s\bar{\mathfrak{f}}_1\right\}^2\\&\times\Omega\left[\lambda_1(1-s)+\xi_1 s\right]ds.\end{aligned} \quad (42)$$

Therefore, we get

$$\frac{1}{\xi_1-\lambda_1}\int_{\lambda_1}^{\xi_1}\Psi\{\mathfrak{g}(r)\}\Omega(r)\,dr\le\int_0^1\Psi\left(2s\bar{\mathfrak{f}}_1\right)\cdot\Omega\left(\overline{z}_1\right)ds-c_2\int_0^1\left\{\left[\mathfrak{f}\left(\overline{z}_1\right)-c_1\left(\overline{z}_1\right)\left(s\xi_1-s\lambda_1\right)\right]-2s\bar{\mathfrak{f}}_1\right\}^2\cdot\Omega\left(\overline{z}_1\right)ds. \quad (43)$$

If $\mathfrak{f}$ is a strongly convex function with modulus c_1 on $[\lambda_1,\xi_1]$ such that $\mathfrak{g}(r)=\mathfrak{f}(r)-c_1 r(r-\lambda_1)$ is a positive increasing function and $\mathfrak{f}(\lambda_1)=0$, then the reverse inequality in (36) can be proved by using a similar method as in the proof of part (a) and Lemma 16(c).

(b) From Lemma 16(b) we know that the function $\hbar_2(r)=\mathfrak{f}(r)/(\xi_1-r)+c_1 r$ is increasing; then using Lemma 15 to the functions $\chi(r)=(\xi_1-r)\Omega(r)$ and $\hbar_2(r)$, we obtain

$$\int_{\lambda_1}^{x}\mathfrak{g}(r)\Omega(r)\,dr\int_{\lambda_1}^{\xi_1}(\xi_1-r)\Omega(r)\,dr\le\int_{\lambda_1}^{\xi_1}\mathfrak{g}(r)\Omega(r)\,dr\int_{\lambda_1}^{x}(\xi_1-r)\Omega(r)\,dr. \quad (44)$$

From (37) we clearly see that inequality (44) can be rewritten as

$$\int_{\lambda_1}^{x}\mathfrak{g}(r)\Omega(r)\,dr\le\int_{\lambda_1}^{x}\frac{(\xi_1-r)}{\xi_1-\lambda_1}2\bar{\mathfrak{f}}_2\Omega(r)\,dr \quad (45)$$

for all $x\in[\lambda_1,\xi_1]$.

As $\mathfrak{g}$ is decreasing function, and by using Theorem 6(a) we have

$$\int_{\lambda_1}^{\xi_1}\Psi\{\mathfrak{g}(r)\}\Omega(r)\,dr\le\int_{\lambda_1}^{\xi_1}\Psi\left\{\frac{(\xi_1-r)}{\xi_1-\lambda_1}2\bar{\mathfrak{f}}_2\right\}\Omega(r)\,dr-c_2\int_{\lambda_1}^{\xi_1}\left\{\mathfrak{g}(r)-\frac{(\xi_1-r)}{\xi_1-\lambda_1}2\bar{\mathfrak{f}}_2\right\}^2\Omega(r)\,dr. \quad (46)$$

Note that

$$\begin{aligned}&\frac{1}{\xi_1-\lambda_1}\int_{\lambda_1}^{\xi_1}\Psi\left\{\frac{(\xi_1-r)}{\xi_1-\lambda_1}2\bar{\mathfrak{f}}_2\right\}\Omega(r)\,dr-\frac{c_2}{\xi_1-\lambda_1}\cdot\int_{\lambda_1}^{\xi_1}\left\{\mathfrak{g}(r)-\frac{(\xi_1-r)}{\xi_1-\lambda_1}2\bar{\mathfrak{f}}_2\right\}^2\Omega(r)\,dr=\frac{1}{2\bar{\mathfrak{f}}_2}\\&\cdot\int_0^{2\bar{\mathfrak{f}}_2}\Psi(y)\,\Omega\left(\xi_1-y\frac{\xi_1-\lambda_1}{2\bar{\mathfrak{f}}_2}\right)dy-\frac{c_2}{2\bar{\mathfrak{f}}_2}\cdot\int_0^{2\bar{\mathfrak{f}}_2}\left\{\left[\mathfrak{f}\left(\xi_1-y\frac{\xi_1-\lambda_1}{2\bar{\mathfrak{f}}_2}\right)\right.\right.\\&\left.\left.+c_1\left(\xi_1-y\frac{\xi_1-\lambda_1}{2\bar{\mathfrak{f}}_2}\right)\left(y\frac{\xi_1-\lambda_1}{2\bar{\mathfrak{f}}_2}\right)\right]-y\right\}^2\\&\times\Omega\left(\xi_1-y\frac{\xi_1-\lambda_1}{2\bar{\mathfrak{f}}_2}\right)dy=\int_0^1\Psi\left(2s\bar{\mathfrak{f}}_2\right)\Omega\left[\lambda_1 s+\xi_1(1-s)\right]ds-c_2\int_0^1\left\{\left[\mathfrak{f}\left(\lambda_1 s+\xi_1(1-s)\right)\right.\right.\\&\left.\left.+c_1\left(\lambda_1 s+\xi_1(1-s)\right)\left(s\xi_1-s\lambda_1\right)\right]-2s\bar{\mathfrak{f}}_2\right\}^2\\&\times\Omega\left[\lambda_1 s+\xi_1(1-s)\right]ds.\end{aligned} \quad (47)$$

Therefore,

$$\frac{1}{\xi_1-\lambda_1}\int_{\lambda_1}^{\xi_1}\Psi\{\mathfrak{g}(r)\}\Omega(r)\,dr \le \int_0^1 \Psi\left(2s\bar{\mathfrak{f}}_2\right)\cdot\Omega(\bar{z}_2)\,ds - c_2\int_0^1\left\{[\mathfrak{f}(\bar{z}_2)+c_1\bar{z}_2(s\xi_1-s\lambda_1)]-2s\bar{\mathfrak{f}}_2\right\}^2\cdot\Omega(\bar{z}_2)\,ds. \quad (48)$$

If $\mathfrak{f}$ is a strongly convex function with modulus c_1 on $[\lambda_1,\xi_1]$ such that $\mathfrak{g}(r)=\mathfrak{f}(r)+c_1r(\xi_1-r)$ is a positive decreasing function and $\mathfrak{f}(\xi_1)=0$, then the reverse inequality in (38) can be proved by using a similar method as in the proof of part (b) and Lemma 16(d). □

Theorem 18. *The following statements are true under the assumptions of Theorem 17.*

(a) *If $\mathfrak{g}$ is a positive increasing function, then*

$$\frac{1}{\xi_1-\lambda_1}\int_{\lambda_1}^{\xi_1}\Psi\{\mathfrak{g}(r)\}\Omega(r)\,dr \le \int_0^1 \Psi\left(2s\bar{\mathfrak{f}}_1\right)\cdot\Omega(\bar{z}_1)\,ds + c_2\int_0^1\left\{[\mathfrak{f}(\bar{z}_1)-c_1\bar{z}_1(s\xi_1-s\lambda_1)]^2-\left(2s\bar{\mathfrak{f}}_1\right)^2\right\}\cdot\Omega(\bar{z}_1)\,ds. \quad (49)$$

(b) *If $\mathfrak{g}$ is a positive decreasing function, then*

$$\frac{1}{\xi_1-\lambda_1}\int_{\lambda_1}^{\xi_1}\Psi\{\mathfrak{g}(r)\}\Omega(r)\,dr \le \int_0^1 \Psi\left(2s\bar{\mathfrak{f}}_2\right)\cdot\Omega(\bar{z}_2)\,ds + c_2\int_0^1\left\{[\mathfrak{f}(\bar{z}_2)+c_1\bar{z}_2(s\xi_1-s\lambda_1)]^2-\left(2s\bar{\mathfrak{f}}_2\right)^2\right\}\cdot\Omega(\bar{z}_2)\,ds. \quad (50)$$

Proof. (a) We clearly see that $\Psi(r)-c_2r^2$ is convex function due to Ψ being a strongly convex function with modulus c_2. Therefore, inequality (49) follows easily from [57, Theorem 1(i)] and the strong convexity of $\Psi(r)-c_2r^2$ together with the fact that $\mathfrak{f}$ is a strongly concave function with modulus c_1 on $[\lambda_1,\xi_1]$ such that $\mathfrak{g}(r)=\mathfrak{f}(r)-c_1r(r-\lambda_1)$ is positive increasing function.

(b) Similarly, inequality (50) follows easily from [57, Theorem 1(ii)] and the strong convexity of $\Psi(r)-c_2r^2$ together with the fact that $\mathfrak{f}$ is a strongly concave function with modulus c_1 on $[\lambda_1,\xi_1]$ such that $\mathfrak{g}(r)=\mathfrak{f}(r)+c_1r(\xi_1-r)$ is positive decreasing function. □

Theorem 19. *Let $\mathfrak{g}(r)=\mathfrak{f}(r)-c_1r(r-\lambda_1)$ be an increasing function on $(0,1)$, let $\mathfrak{g}/h$ be a decreasing function on $(0,1)$, let $\mathfrak{g}$, h, and Ω be three positive functions on $(0,1)$, and let $\mathfrak{g}\Omega$ and $h\Omega$ be integrable on $(0,1)$ such that*

$$\phi=\frac{\int_0^1\mathfrak{g}(r)\Omega(r)\,dr}{\int_0^1 h(r)\Omega(r)\,dr}\ge 0. \quad (51)$$

And let Ψ be a strongly convex function with modulus c_2. Then the inequality holds

$$\int_0^1\Psi\{k\phi h(r)\}\Omega(r)\,dr \ge \int_0^1\Psi\{k\mathfrak{g}(r)\}\Omega(r)\,dr + c_2k^2\int_0^1\{\mathfrak{g}(r)-\phi h(r)\}^2\Omega(r)\,dr \quad (52)$$

for all $k>0$.

Proof. From $h>0$ and (51), applying Lemma 15 to the function $\chi(r)=h(r)\Omega(r)$ and the decreasing function $\hbar(r)=\mathfrak{g}(r)/h(r)$ we get

$$\int_0^x k\phi h(r)\Omega(r)\,dr \le \int_0^x k\mathfrak{g}(r)\Omega(r)\,dr. \quad (53)$$

Since $\mathfrak{g}$ is increasing, therefore by using Theorem 6 we have

$$\int_0^1\Psi\{k\phi h(r)\}\Omega(r)\,dr \ge \int_0^1\Psi\{k\mathfrak{g}(r)\}\Omega(r)\,dr + c_2k^2\int_0^1\{\mathfrak{g}(r)-\phi h(r)\}^2\Omega(r)\,dr. \quad (54)$$

□

Theorem 20. *Let Ψ be a strongly convex function with modulus c_2. Then the inequality*

$$\int_0^1\Psi\{k\phi h(r)\}\Omega(r)\,dr \ge \int_0^1\Psi\{k\mathfrak{g}(r)\}\Omega(r)\,dr + c_2k^2\int_0^1\left[(\phi h(r))^2-(\mathfrak{g}(r))^2\right]\Omega(r)\,dr. \quad (55)$$

holds for all $k>0$ if all the assumptions of Theorem 19 are satisfied.

Proof. We clearly see that $\Psi(r)-c_2r^2$ is a convex function due to Ψ being a strongly convex function with modulus c_2. Therefore, inequality (55) follows easily from [60, Theorem 3] and the convexity of the function $\Psi(r)-c_2r^2$. □

Remark 21. Clearly, [60, Theorem 3] can be deduced from (52) due to

$$\int_0^1 \{\mathfrak{g}(r) - \phi h(r)\}^2 \Omega(r)\, dr \geq 0 \tag{56}$$

or from (55) due to

$$\int_0^1 \left[(\phi h(r))^2 - (\mathfrak{g}(r))^2\right] \Omega(r)\, dr \geq 0 \tag{57}$$

for convex function $\Psi(r) = r^2$.

The following Theorem 22 is an extension of Theorem 19.

Theorem 22. *Let $\Psi : [0,\infty) \longrightarrow \mathbb{R}$ be a continuous strongly convex function with modulus c_2, $\mathfrak{g}(r) = \mathfrak{f}(r) - c_1 r(r-\lambda_1)$ (c_1 is a nonnegative real number), h and Ω two positive integrable functions on $[\lambda_1, \xi_1]$, $z_1(r) = \mathfrak{g}(r)/\int_{\lambda_1}^{\xi_1}\mathfrak{g}(r)\Omega(r)dr$, and $z_2(r) = h(r)/\int_{\lambda_1}^{\xi_1} h(r)\Omega(r)dr$. Then the following statements are true.*

(a) *If $\mathfrak{g}$ is increasing on $[\lambda_1, \xi_1]$ and $\mathfrak{g}/h$ is decreasing on $[\lambda_1, \xi_1]$, then*

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} &\Psi(z_2(r))\Omega(r)\,dr \\ &\geq \int_{\lambda_1}^{\xi_1} \Psi(z_1(r))\Omega(r)\,dr \\ &\quad + c_2\int_{\lambda_1}^{\xi_1} (z_1(r) - z_2(r))^2\Omega(r)\,dr.\end{aligned} \tag{58}$$

(b) *If h is increasing on $[\lambda_1, \xi_1]$ and $\mathfrak{g}/h$ is increasing on $[\lambda_1, \xi_1]$, then*

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} &\Psi(z_1(r))\Omega(r)\,dr \\ &\geq \int_{\lambda_1}^{\xi_1} \Psi(z_2(r))\Omega(r)\,dr \\ &\quad + c_2\int_{\lambda_1}^{\xi_1} (z_1(r) - z_2(r))^2\Omega(r)\,dr.\end{aligned} \tag{59}$$

Proof. (a) Let $h > 0$; then applying Lemma 15 to the function $\chi(r) = h(r)\Omega(r)$ and the decreasing function $\hbar(r) = \mathfrak{g}(r)/h(r)$, we have

$$\int_{\lambda_1}^{x} z_2(r)\Omega(r)\,dr \leq \int_{\lambda_1}^{x} z_1(r)\Omega(r)\,dr. \tag{60}$$

Therefore, inequality (58) follows from Theorem 6 and the fact that $\mathfrak{g}$ is an increasing function on $[\lambda_1, \xi_1]$.

(b) Let $h > 0$; then applying Lemma 15 to the function $\chi(r) = h(r)\Omega(r)$ and the increasing function $\hbar(r) = \mathfrak{g}(r)/h(r)$, we get

$$\int_{\lambda_1}^{x} z_1(r)\Omega(r)\,dr \leq \int_{\lambda_1}^{x} z_2(r)\Omega(r)\,dr. \tag{61}$$

Therefore, inequality (59) follows from Theorem 6 and the fact that h is an increasing function on $[\lambda_1, \xi_1]$. □

Theorem 23. *The following statements are true under the assumptions of Theorem 22.*

(a) *If $\mathfrak{g}$ is an increasing function on $[\lambda_1, \xi_1]$ and $\mathfrak{g}/h$ is a decreasing function on $[\lambda_1, \xi_1]$, then*

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} &\Psi(z_1(r))\Omega(r)\,dr \\ &\leq \int_{\lambda_1}^{\xi_1} \Psi(z_2(r))\Omega(r)\,dr \\ &\quad + c_2\int_{\lambda_1}^{\xi_1} \left(z_1^2(r) - z_2^2(r)\right)\Omega(r)\,dr.\end{aligned} \tag{62}$$

(b) *If h is increasing on $[\lambda_1, \xi_1]$ and $\mathfrak{g}/h$ is an increasing function on $[\lambda_1, \xi_1]$, then*

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} &\Psi(z_2(r))\Omega(r)\,dr \\ &\leq \int_{\lambda_1}^{\xi_1} \Psi(z_1(r))\Omega(r)\,dr \\ &\quad + c_2\int_{\lambda_1}^{\xi_1} \left(z_2^2(r) - z_1^2(r)\right)\Omega(r)\,dr.\end{aligned} \tag{63}$$

Proof. We clearly see that $\Psi(r) - c_2 r^2$ is a convex function due to Ψ being a strongly convex function with modulus c_2. Therefore, inequalities (62) and (63) follow from [61, Theorem 2.3] and the convexity of the function $\Psi(r) - c_2 r^2$. □

Remark 24. Clearly, Theorem 2.3(1) and Theorem 2.3(2) given in [57] can be deduced by (62) and (63), respectively.

Remark 25. Theorem 22 is an extension of Favard's inequality given in Theorem 17. Indeed, let $\Psi(r)$ be a strongly convex function with modulus c_2, then $\Psi(kr)$ is also a strongly convex function with modulus $k^2 c_2$ for any $k \in \mathbb{R}$. Substituting $h(r) = r - \lambda_1$ in (58), one has

$$\begin{aligned}\int_{\lambda_1}^{\xi_1} &\Psi\left[\mathfrak{f}(r) - c_1 r(r-\lambda_1)\right]\Omega(r)\,dr \\ &\leq \int_{\lambda_1}^{\xi_1} \Psi\left(\frac{\int_{\lambda_1}^{\xi_1}\left[\mathfrak{f}(r) - c_1 r(r-\lambda_1)\right]\Omega(r)\,dr}{\int_{\lambda_1}^{\xi_1}(r-\lambda_1)\Omega(r)\,dr}(r \right. \\ &\quad \left. - \lambda_1)\right)\Omega(r)\,dr - k^2 c_2 \int_{\lambda_1}^{\xi_1}\left(\left[\mathfrak{f}(r)\right.\right.\end{aligned}$$

$$- c_1 r (r - \lambda_1)] - \frac{\int_{\lambda_1}^{\xi_1} [\mathfrak{f}(r) - c_1 r (r - \lambda_1)] \Omega(r) dr}{\int_{\lambda_1}^{\xi_1} (r - \lambda_1) \Omega(r) dr} (r - \lambda_1) \Bigg)^2 \cdot \Omega(r) dr. \tag{64}$$

Since $\mathfrak{f}$ is a strongly concave function with modulus c_1 on $[\lambda_1, \xi_1]$ such that $\mathfrak{g}(r) = \mathfrak{f}(r) - c_1 r(r - \lambda_1)$ is a positive increasing function, taking $k = 1$ and using (35) and $\overline{z}_1$ in (64), we obtain the Favard's inequalities given in Theorem 17.

Remark 26. From (64) we can easily obtain Remark 2.4 given in [61] due to

$$\int_{\lambda_1}^{\xi_1} \Bigg([\mathfrak{f}(r) - c_1 r (r - \lambda_1)] - \frac{\int_{\lambda_1}^{\xi_1} [\mathfrak{f}(r) - c_1 r (r - \lambda_1)] \Omega(r) dr}{\int_{\lambda_1}^{\xi_1} (r - \lambda_1) \Omega(r) dr} (r - \lambda_1) \Bigg)^2 \cdot \Omega(r) dr \geq 0. \tag{65}$$

For an application of Theorem 22, we get Corollary 27 as follows.

Corollary 27. *Let $r > 1$, $p \in (-\infty, 0) \cup (1, \infty)$, $\Psi(r) = r^p$, and Ω, $\mathfrak{g}$, h, z_1, and z_2 be stated as in Theorem 22. Then the following statements are true.*

(a) *If $\mathfrak{g}$ is increasing on $[\lambda_1, \xi_1]$ and $\mathfrak{g}/h$ is decreasing on $[\lambda_1, \xi_1]$, then*

$$\left(\frac{\int_{\lambda_1}^{\xi_1} \mathfrak{g}(r) \Omega(r) dr}{\int_{\lambda_1}^{\xi_1} h(r) \Omega(r) dr} \right)^p \geq \frac{\int_{\lambda_1}^{\xi_1} \mathfrak{g}^p(r) \Omega(r) dr}{\int_{\lambda_1}^{\xi_1} h^p(r) \Omega(r) dr} + c_2 \cdot \frac{\left(\int_{\lambda_1}^{\xi_1} \mathfrak{g}(r) \Omega(r) dr \right)^p}{\int_{\lambda_1}^{\xi_1} h^p(r) \Omega(r) dr} \cdot \int_{\lambda_1}^{\xi_1} (z_1(r) - z_2(r))^2 \Omega(r) dr. \tag{66}$$

(b) *If h is increasing on $[\lambda_1, \xi_1]$ and $\mathfrak{g}/h$ is an increasing function on $[\lambda_1, \xi_1]$, then*

$$\frac{\int_{\lambda_1}^{\xi_1} \mathfrak{g}^p(r) \Omega(r) dr}{\int_{\lambda_1}^{\xi_1} h^p(r) \Omega(r) dr} \geq \left(\frac{\int_{\lambda_1}^{\xi_1} \mathfrak{g}(r) \Omega(r) dr}{\int_{\lambda_1}^{\xi_1} h(r) \Omega(r) dr} \right)^p + c_2 \cdot \frac{\left(\int_{\lambda_1}^{\xi_1} \mathfrak{g}(r) \Omega(r) dr \right)^p}{\int_{\lambda_1}^{\xi_1} h^p(r) \Omega(r) dr} \cdot \int_{\lambda_1}^{\xi_1} (z_1(r) - z_2(r))^2 \Omega(r) dr. \tag{67}$$

Proof. (a) Since $\mathfrak{g}$ is increasing on $[\lambda_1, \xi_1]$ and $\mathfrak{g}/h$ is a decreasing function on $[\lambda_1, \xi_1]$, using (58) given in Theorem 22 and substituting $\Psi(r) = r^p$, we get

$$\begin{aligned} \int_{\lambda_1}^{\xi_1} z_2^p(r) \Omega(r) dr &\geq \int_{\lambda_1}^{\xi_1} z_1^p(r) \Omega(r) dr + c_2 \int_{\lambda_1}^{\xi_1} (z_1(r) - z_2(r))^2 \Omega(r) dr, \\ \int_{\lambda_1}^{\xi_1} \left(\frac{h(r)}{\int_{\lambda_1}^{\xi_1} h(r) \Omega(r) dr} \right)^p \Omega(r) dr &\geq \int_{\lambda_1}^{\xi_1} \left(\frac{\mathfrak{g}(r)}{\int_{\lambda_1}^{\xi_1} \mathfrak{g}(r) \Omega(r) dr} \right)^p \Omega(r) dr + c_2 \int_{\lambda_1}^{\xi_1} (z_1(r) - z_2(r))^2 \Omega(r) dr. \end{aligned} \tag{68}$$

Therefore, inequality (66) follows from (68).

(b) Since h is increasing on $[\lambda_1, \xi_1]$ and $\mathfrak{g}/h$ is an increasing function on $[\lambda_1, \xi_1]$, using (59) given in Theorem 22 and substituting $\Psi(r) = r^p$ we get inequality (67). □

Remark 28. Let $h(r) = r - \lambda_1$, $\Omega(r) = 1$, and $\mathfrak{f}$ be a strongly concave function with modulus c_1 on $[\lambda_1, \xi_1]$ such that $\mathfrak{g}(r) = \mathfrak{f}(r) - c_1 r(r - \lambda_1)$ is a positive increasing function. Then inequality (66) leads to the classical Favard's inequality for strongly convex functions with modulus c_2:

$$\frac{2^p}{p+1} \left(\frac{1}{\xi_1 - \lambda_1} \int_{\lambda_1}^{\xi_1} \mathfrak{g}(r) dr \right)^p \geq \frac{1}{\xi_1 - \lambda_1} \cdot \int_{\lambda_1}^{\xi_1} \mathfrak{g}^p(r) dr + \frac{c_2}{\xi_1 - \lambda_1} \left(\int_{\lambda_1}^{\xi_1} \mathfrak{g}(r) dr \right)^p \cdot \int_{\lambda_1}^{\xi_1} \left(\frac{\mathfrak{g}(r)}{\int_{\lambda_1}^{\xi_1} \mathfrak{g}(r) dr} - \frac{2(r - \lambda_1)}{(\xi_1 - \lambda_1)^2} \right)^2 dr. \tag{69}$$

Remark 29. From (69) we get the classical Favard's inequality given in [58] due to

$$\left(\int_{\lambda_1}^{\xi_1} \mathfrak{g}(r) dr \right)^p \cdot \int_{\lambda_1}^{\xi_1} \left(\frac{\mathfrak{g}(r)}{\int_{\lambda_1}^{\xi_1} \mathfrak{g}(r) dr} - \frac{2(r - \lambda_1)}{(\xi_1 - \lambda_1)^2} \right)^2 dr \geq 0. \tag{70}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by the Natural Science Foundation of China (Grant nos. 61673169, 11301127, 11701176, 11626101, 11601485).

References

[1] B. de Finetti, "Sulle stratificazioni convesse," *Annali di Matematica Pura ed Applicata*, vol. 30, no. 4, pp. 173–183, 1949.

[2] N. Merentes and K. Nikodem, "Remarks on strongly convex functions," *Aequationes Mathematicae*, vol. 80, no. 1-2, pp. 193–199, 2010.

[3] Y.-Q. Song, M. Adil Khan, S. Zaheer Ullah, and Y.-M. Chu, "Integral inequalities involving strongly convex functions," *Journal of Function Spaces*, vol. 2018, Article ID 6595921, 8 pages, 2018.

[4] W. Zhou and X. Chen, "On the convergence of a modified regularized Newton method for convex optimization with singular solutions," *Journal of Computational and Applied Mathematics*, vol. 239, pp. 179–188, 2013.

[5] M. Adil Khan, Y.-M. Chu, T. U. Khan, and J. Khan, "Some new inequalities of Hermite-Hadamard type for s-convex functions with applications," *Open Mathematics*, vol. 15, pp. 1414–1430, 2017.

[6] D. H. Hyers and S. M. Ulam, "Approximately convex functions," *Proceedings of the American Mathematical Society*, vol. 3, pp. 821–828, 1952.

[7] J. E. Pečarič, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, vol. 187 of *Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1992.

[8] T.-H. Zhao, Y.-M. Chu, and Y.-P. Jiang, "Monotonic and logarithmically convex properties of a function involving gamma functions," *Journal of Inequalities and Applications*, vol. 2009, Article ID 728612, 13 pages, 2009.

[9] J. L. W. V. Jensen, "Om konvekse Funktioner og Uligheder mellem Middelværdier," *Nyt tidsskrift for matematik. B*, vol. 16, pp. 49–68, 1905.

[10] O. L. Mangasarian, "Pseudo-convex functions," *SIAM Journal on Control and Optimization*, vol. 3, pp. 281–290, 1965.

[11] S. S. Dragomir, "Inequalities of Hermite-Hadamard type for φ-convex functions," http://rgmia.org/papers/v16/v16a87.pdf.

[12] S. S. Dragomir, "Inequalities of Hermite-Hadamard type for λ-convex functions on linear spaces," http://rgmia.org/papers/v17/v17a13.pdf.

[13] S. Varošanec, "On h-convexity," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 303–311, 2007.

[14] T. Rajba, "On strong delta-convexity and Hermite-Hadamard type inequalities for delta-convex functions of higher order," *Mathematical Inequalities & Applications*, vol. 18, no. 1, pp. 267–293, 2015.

[15] H. Hu and L. Liu, "Weighted inequalities for a general commutator associated to a singular integral operator satisfying a variant of Hörmander's condition," *Mathematical Notes*, vol. 101, no. 5-6, pp. 830–840, 2017.

[16] X. Zhou, "Weighted sharp function estimate and boundedness for commutator associated with singular integral operator satisfying a variant of Hörmander's condition," *Journal of Mathematical Inequalities*, vol. 9, no. 2, pp. 587–596, 2015.

[17] C. Huang, Z. Yang, T. Yi, and X. Zou, "On the basins of attraction for a class of delay differential equations with non-monotone bistable nonlinearities," *Journal of Differential Equations*, vol. 256, no. 7, pp. 2101–2114, 2014.

[18] C. Huang, S. Guo, and L. Liu, "Boundedness on Morrey space for Toeplitz type operator associated to singular integral operator with variable Calderón-Zygmund kernel," *Journal of Mathematical Inequalities*, vol. 8, no. 3, pp. 453–463, 2014.

[19] D. Lu, "Some sharp inequalities for multilinear integral operators," *Journal of Inequalities and Applications*, vol. 2013, article 445, 14 pages, 2013.

[20] X. Zhou, "Some sharp function estimates for vector-valued multilinear integral operator," *Miskolc Mathematical Notes*, vol. 14, no. 1, pp. 373–387, 2013.

[21] Z.-G. Wang, G.-W. Zhang, and F.-H. Wen, "Properties and characteristics of the Srivastava-Khairnar-More integral operator," *Applied Mathematics and Computation*, vol. 218, no. 15, pp. 7747–7758, 2012.

[22] H. Wang, W.-M. Qian, and Y.-M. Chu, "Optimal bounds for Gaussian arithmetic-geometric mean with applications to complete elliptic integral," *Journal of Function Spaces*, vol. 2016, Article ID 3698463, 6 pages, 2016.

[23] H.-H. Chu, Z.-H. Yang, Y.-M. Chu, and W. Zhang, "Generalized Wilker-type inequalities with two parameters," *Journal of Inequalities and Applications*, vol. 2016, article 187, 13 pages, 2016.

[24] X. Zhou, "Sharp maximal function inequalities and boundedness for Toeplitz type operator associated to singular integral operator with non-smooth kernel," *Journal of Inequalities and Applications*, vol. 2014, article 141, 19 pages, 2014.

[25] Q. Xiao and Q. Li, "Generalized lower and upper approximations in quantales," *Journal of Applied Mathematics*, vol. 2012, Article ID 648983, 11 pages, 2012.

[26] C. Huang, C. Peng, X. Chen, and F. Wen, "Dynamics analysis of a class of delayed economic model," *Abstract and Applied Analysis*, vol. 2013, Article ID 962738, 12 pages, 2013.

[27] M. Adil Khan, A. Iqbal, M. Suleman, and Y.-M. Chu, "Hermite-Hadamard type inequalities for fractional integrals via Green's function," *Journal of Inequalities and Applications*, vol. 2018, article 161, 15 pages, 2018.

[28] Y. Khurshid, M. Adil Khan, and Y.-M. Chu, "Conformable integral inequaliites of the Hermite-Hadamard thpe in terms of GG- and GA-convexities," *Journal of Function Spaces*, vol. 2019, Article ID 6926107, 8 pages, 2019.

[29] M. Adil Khan, S.-H. Wu, H. Ullah, and Y.-M. Chu, "Discrete majorization type inequalities for convex functions on rectangles," *Journal of Inequalities and Applications*, vol. 2019, article 16, 18 pages, 2019.

[30] M.-K. Wang, Y.-M. Chu, and W. Zhang, "Monotonicity and inequalities involving zero-balanced hypergeometric function," *Mathematical Inequalities & Applications*, vol. 22, no. 2, pp. 601–617, 2019.

[31] J. Wu and Y. Liu, "Uniqueness results and convergence of successive approximations for fractional differential equations," *Hacettepe Journal of Mathematics and Statistics*, vol. 42, no. 2, pp. 149–158, 2013.

[32] R.-s. Yang and Y.-h. Ou, "Inverse coefficient problems for nonlinear elliptic variational inequalities," *Acta Mathematicae Applicatae Sinica*, vol. 27, no. 1, pp. 85–92, 2011.

[33] Z. Liu, Y. Zhang, J. Santos, and R. Ralha, "On computing complex square roots of real matrices," *Applied Mathematics Letters*, vol. 25, no. 10, pp. 1565–1568, 2012.

[34] L. Lin and Z.-Y. Liu, "An alternating projected gradient algorithm for nonnegative matrix factorization," *Applied Mathematics and Computation*, vol. 217, no. 24, pp. 9997–10002, 2011.

[35] W. Wang, "High order stable Runge-Kutta methods for nonlinear generalized pantograph equations on the geometric mesh," *Applied Mathematical Modelling: Simulation and Computation for Engineering and Environmental Systems*, vol. 39, no. 1, pp. 270–283, 2015.

[36] J. Wang, X. Chen, and L. Huang, "The number and stability of limit cycles for planar piecewise linear systems of node-saddle type," *Journal of Mathematical Analysis and Applications*, vol. 469, no. 1, pp. 405–427, 2019.

[37] F. Wang and Z. Liu, "Anti-periodic fractional boundary value problems for nonlinear differential equations of fractional order," *Advances in Difference Equations*, vol. 2012, article 116, 12 pages, 2012.

[38] Z.-H. Yang, W.-M. Qian, Y.-M. Chu, and W. Zhang, "Monotonicity rule for the quotient of two functions and its application," *Journal of Inequalities and Applications*, vol. 2017, article 106, 13 pages, 2017.

[39] S.-L. Qiu, X.-Y. Ma, and Y.-M. Chu, "Sharp Landen transformation inequalities for hypergeometric functions, with applications," *Journal of Mathematical Analysis and Applications*, vol. 474, no. 2, pp. 1306–1337, 2019.

[40] W. Wang, "A generalized Halanay inequality for stability of nonlinear neutral functional differential equations," *Journal of Inequalities and Applications*, vol. 2010, Article ID 475019, 16 pages, 2010.

[41] Z.-G. Wang, H. M. Srivastava, and S.-M. Yuan, "Some basic properties of certain subclasses of meromorphically starlike functions," *Journal of Inequalities and Applications*, vol. 2014, article 29, 13 pages, 2014.

[42] J. Li, F. Liu, L. Feng, and I. Turner, "A novel finite volume method for the Riesz space distributed-order advection-diffusion equation," *Applied Mathematical Modelling: Simulation and Computation for Engineering and Environmental Systems*, vol. 46, pp. 536–553, 2017.

[43] C. Huang and L. Liu, "Sharp function inequalities and boundness for Toeplitz type operator related to general fractional singular integral operator," *Publications de l'Institut Mathematique*, vol. 92, no. 106, pp. 165–176, 2012.

[44] Z.-H. Yang, Y.-M. Chu, and W. Zhang, "High accuracy asymptotic bounds for the complete elliptic integral of the second kind," *Applied Mathematics and Computation*, vol. 348, pp. 552–564, 2019.

[45] T.-H. Zhao, B.-C. Zhou, M.-K. Wang, and Y.-M. Chu, "On approximating the quasi-arithmetic mean," *Journal of Inequalities and Applications*, vol. 2019, article 42, 12 pages, 2019.

[46] E. Zhu and Y. Xu, "Pathwise estimation of stochastic functional Kolmogorov-type systems with infinite delay," *Journal of Inequalities and Applications*, vol. 2012, article 171, 15 pages, 2012.

[47] H.-H. Chu, W.-M. Qian, Y.-M. Chu, and Y.-Q. Song, "Optimal bounds for a Toader-type mean in terms of one-parameter quadratic and contraharmonic means," *Journal of Nonlinear Sciences and Applications. JNSA*, vol. 9, no. 5, pp. 3424–3432, 2016.

[48] S. Zaheer Ullah, M. Adil Khan, and Y.-M. Chu, "Majorization theorems for strongly convex functions," *Journal of Inequalities and Applications*, vol. 2019, article 58, 13 pages, 2019.

[49] S.-H. Wu and Y.-M. Chu, "Schur m-power convexity of generalized geometric Bonferroni mean involving three parameters," *Journal of Inequalities and Applications*, vol. 2019, article 57, 11 pages, 2019.

[50] J.-B. Hiriart-Urruty and C. Lemaréchal, *Fundamentals of Convex Analysis*, Springer, Berlin, Germany, 2001.

[51] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and its Applications*, vol. 143 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1979.

[52] Y.-M. Chu, H. Wang, and T.-H. Zhao, "Sharp bounds for the Neuman mean in terms of the quadratic and second Seiffert means," *Journal of Inequalities and Applications*, vol. 2014, article 299, 14 pages, 2014.

[53] R. Bhatia, *Matrix Analysis*, Springer, New York, NY, USA, 1997.

[54] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, Mass, USA, 1990.

[55] L. Huang, Y. Li, and K. Zhao, "Geometry of alternate matrices over Bezout domains," *Algebra Colloquium*, vol. 20, no. 2, pp. 197–214, 2013.

[56] J. E. Pečarić, "On some inequalities for functions with nondecreasing increments," *Journal of Mathematical Analysis and Applications*, vol. 98, no. 1, pp. 188–197, 1984.

[57] L. Maligranda, J. E. Pečarič, and L. E. Persson, "Weighted Favard and Berwald inequalities," *Journal of Mathematical Analysis and Applications*, vol. 190, no. 1, pp. 248–262, 1995.

[58] J. Favard, "Sur les valeurs moyennes," *Bulletin des Sciences Mathématiques*, vol. 57, pp. 54–64, 1933.

[59] N. S. Barnett, P. Cerone, and S. S. Dragomir, "Majorisation inequalities for Stieltjes integrals," *Applied Mathematics Letters*, vol. 22, no. 3, pp. 416–421, 2009.

[60] J. Pečarič and S. Abramovich, "On new majorization theorems," *Rocky Mountain Journal of Mathematics*, vol. 27, no. 3, pp. 903–911, 1997.

[61] N. Latif, J. Pečarič, and I. Perič, "On majorization, Favard and Berwald inequalities," *Annals of Functional Analysis*, vol. 2, no. 1, pp. 31–50, 2011.