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Integral manifolds of differential equations with piecewise constant argument of generalized type

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Abstract

In this paper we introduce a general type of differential equations with piecewise constant argument (EPCAG). The existence of global integral manifolds of the quasilinear EPCAG is established when the associated linear homogeneous system has an exponential dichotomy. The smoothness of the manifolds is investigated. The existence of bounded and periodic solutions is considered. A new technique of investigation of equations with piecewise argument, based on an integral representation formula, is proposed. Appropriate illustrating examples are given.

MSC: 34A36; 34K12; 34K13; 34K19

Keywords: Integral manifolds; Piecewise constant argument of generalized type; Differentiability of integral manifolds; Periodic solutions

1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{R}$ be the sets of all integers, natural and real numbers, respectively. Denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^n$, $n \in \mathbb{N}$.

In this paper we are concerned with the quasilinear system

$$y' = A(t)y + f(t, y(t), y(\beta(t))), \tag{1}$$

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where $y \in \mathbb{R}^n$, $t \in \mathbb{R}$, $\beta(t) = \theta_i$ if $\theta_i \le t < \theta_{i+1}$, $i \in \mathbb{Z}$, is an identification function, θ_i, $i \in \mathbb{Z}$, is a strictly ordered sequence of real numbers, $|\theta_i| \to \infty$ as $|i| \to \infty$, and there exists a number $\theta > 0$ such that $\theta_{i+1} - \theta_i \le \theta$, $i \in \mathbb{Z}$. The theory of differential equations with piecewise constant argument (EPCA) of the type

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x(t), x([t])), \tag{2}$$

where $[\cdot]$ signifies the greatest integer function, was initiated in [9] and developed by many authors [1,5,9,15,20,21,27,29–33].

System (1) is a general case (EPCAG) of Eq. (2). Indeed, if we take $\theta_i = i$, $i \in \mathbb{Z}$, then (1) takes the form of (2).

The existing method of investigation of EPCA, as proposed by its founders, is based on the reduction of EPCA to discrete equations. We propose another approach to the problem. In fact, this approach consists of the construction of the equivalent integral equation. Consequently, for every result of our paper we prove a corresponding equivalence lemma. Thus, while investigating EPCAG, we need not impose any conditions on the reduced discrete equations, and, hence, we require more easily verifiable conditions, similar to those for ordinary differential equations. It become less cumbersome to solve the problems of EPCAG theory (as well as of EPCA theory).

The theory of integral manifolds was founded by Poincaré and Lyapunov [24,17], and it became a very powerful instrument for investigating various problems of the qualitative theory of differential equations. For example, one can talk about the exceptional role of manifolds in the reduction of the dimensions of equations. It is natural that the exploration of manifolds, of their properties and neighborhoods, is one of the most interesting problems [6–8,13,14,18,19,22,23,25]. One should not be surprised that manifolds are one of the major subjects of investigation for specific types of differential and difference equations [2,3,12,16,26,28]. In [21] the subject was discussed for EPCA. Obviously, it is not possible to mention all the results pertaining to integral sets in this paper.

The proof of main theorems of the paper is very similar to that of the classic assertions about integral manifolds [13,18,19,22,23].

The paper is organized in the following manner. In Section 2 we give the definitions of global solutions of the initial value problem, and formulate and prove the existence and uniqueness theorems for the solutions. The main results of our work are given in Section 3. We consider the existence of stable and unstable manifolds, and prove two lemmas about the equivalence of the EPCAG and some integral equations under certain conditions. Special properties of solutions which start outside of the manifolds are also discussed. In Section 4 the smoothness of the manifolds is investigated. The existence of bounded and periodic solutions is considered in the last section. Moreover, an appropriate example of the logistic EPCAG is given.

2. Preliminaries

In what follows, we use the uniform norm $\|T\| = \sup\{\|Tx\| \,|\, \|x\| = 1\}$ for matrices. The following assumptions will be needed throughout the paper.

$(C1)$ $A(t)$ is a continuous $n \times n$ matrix and $\sup_{\mathbb{R}} \|A(t)\| = \mu < \infty$;

$(C2)$ $f(t, x, z)$ is continuous in the first argument, $f(t, 0, 0) = 0$, $t \in \mathbb{R}$, and f is Lipschitzian in the second and the third arguments with a Lipschitz constant l such that

$$\|f(t, y_1, w_1) - f(t, y_2, w_2)\| \le l(\|y_1 - y_2\| + \|w_1 - w_2\|);$$

($C3$) the linear homogeneous system associated with (1)

$$x' = A(t)x \tag{3}$$

has an exponential dichotomy on $\mathbb{R}$. That is, there exists a projection P and positive constants K and σ such that

$$\begin{aligned} &\|X(t)PX^{-1}(s)\| \le K\exp(-\sigma(t-s)),\ t \ge s,\\ &\|X(t)(I-P)X^{-1}(s)\| \le K\exp(\sigma(s-t)),\ t \le s, \end{aligned} \tag{4}$$

where $X(t)$ is a fundamental matrix of (3).

2.1. *The solutions of the EPCAG*

In [29] the initial value problem (IVP) problem for EPCA is investigated in the case when the initial moment is an integer, and global solutions have been considered. The approach is reasonable since it is a consequence of the method of reduction to the discrete equations.

We combine the approach of EPCA [9,20,29] with a more careful analysis of the initial data. In particular, we prove that under the conditions of our paper a global solution of (1) with arbitrary initial data $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ is a restriction of a global solution of an IVP for (1) with the initial moment equal to one of the elements of the sequence θ_i, $i \in \mathbb{Z}$ (see Theorem 2.2). The analysis of the general problem of IVP for EPCAG is the subject of discussion for another paper.

In our article it is assumed that all solutions are continuous functions.

Solutions which start at points θ_i, $i \in \mathbb{Z}$, and exist to the right.

Definition 2.1. A solution $y(t) = y(t, \theta_i, y_0)$, $y(\theta_i) = y_0$, $i \in \mathbb{Z}$, of (1) on $[\theta_i, \infty)$ is a continuous function such that

(i) the derivative $y'(t)$ exists at each point $t \in [\theta_i, \infty)$, with the possible exception of the points θ_j, $j \ge i$, where one-sided derivatives exist;
(ii) Eq. (1) is satisfied by $y(t)$ on each interval $[\theta_j, \theta_{j+1})$, $j \ge i$.

Remark 2.1. Definition 2.1 is a version of Definition 1.1 from [9] adapted to our general case.

Theorem 2.1. *Suppose conditions ($C1$)–($C3$) are fulfilled. Then for every $y_0 \in \mathbb{R}^n$ and $i \in \mathbb{Z}$ there exists a unique solution $y(t)$ of (1) in the sense of Definition 2.1.*

Proof. If $\theta_i \le t \le \theta_{i+1}$ then $y(t)$ coincides with a solution of the following IVP:

$$\begin{aligned} &\frac{\mathrm{d}\xi}{\mathrm{d}t} = A(t)\xi + f(t, \xi, y_0)\\ &\xi(\theta_i) = y_0. \end{aligned} \tag{5}$$

Eq. (5) is an ordinary differential equation, where function $f(t, \xi, y_0)$ is Lipschitzian in ξ, and, consequently, $y(t)$ exists and unique on $[\theta_i, \theta_{i+1}]$.

Assume, now, that $y(t)$ is defined uniquely on an interval $[\theta_i, \theta_k]$, $k \ge i+1$. Then for $[\theta_k, \theta_{k+1}]$ it is a solution of the IVP

$$\begin{aligned} &\frac{\mathrm{d}\xi}{\mathrm{d}t} = A(t)\xi + f(t, \xi, y(\theta_k))\\ &\xi(\theta_k) = y(\theta_k). \end{aligned} \tag{6}$$

It is obvious that $y(t)$ exists and unique on the interval.

By induction we can conclude that for every $y_0 \in \mathbb{R}^n$ and $i \in \mathbb{Z}$ there exists a unique solution $y(t)$ of (1) on $[\theta_i, \infty)$, satisfying $y(\theta_i) = y_0$. The theorem is proved. □

The existence and the uniqueness of solutions on $\mathbb{R}$.

Lemma 2.1 ([11]). *Assume that condition* $(C1)$ *is fulfilled. Then*

$$\|X(t,s)\| \le \exp(\mu|t-s|),\ t, s \in \mathbb{R}.$$

Lemma 2.2. *Assume that condition* $(C1)$ *is fulfilled. Then*

$$\|X(t,s)\| \ge \exp(-\mu|t-s|),\ t, s \in \mathbb{R}.$$

Proof. The proof follows immediately from the equality $X(t,s)X(s,t) = I$, where I is an $n \times n$ identity matrix.

The last two lemmas imply the following, simple but useful in what follows, inequalities:

$$\|X(t,s)\| \le M, \quad \|X(t,s)\| \ge m, \tag{7}$$

which hold if $|t-s| \le \theta$, where $M = \exp(\mu\theta)$, $m = \exp(-\mu\theta)$.

Assume that $\theta_i < t_0 < \theta_{i+1}$ for some $i \in \mathbb{Z}$, and $y(t) = y(t, t_0, y_0)$, $y_0 \in \mathbb{R}^n$, is a solution of (1). We can easily see that even for the local existence of $y(t)$ around t_0 the solution should be defined also at the moment $t = \theta_i$, since $y(t)$ for all $t \in [\theta_i, \theta_{i+1}]$ is a solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = A(t)y + f(t, y, y(\theta_i)). \tag{8}$$

In other words the solution must exist globally to the left from $t = t_0$, on section $[\theta_i, t_0]$.

The following example shows that even for simple EPCA solutions do not always exist to the left, and moreover, difficulties with uniqueness may appear. □

Example 2.1. Consider the following equation:

$$x' = 2x - x^2([t]), \tag{9}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$. Fix $z \in \mathbb{R}$ and let $x(t) = x(t, 1, z)$, $x(1) = z$, be a solution of (9). Let us try to define the solution for $t < 1$. It is known that if $t \in [0, 1]$ then $x(t)$ satisfies the equation

$$x' = 2x - x^2(0). \tag{10}$$

Denote $x(0) = x_0$, and introduce an operator $T : \mathbb{R} \to \mathbb{R}$ such that

$$Tx_0 = \exp(2)x_0 + \int_0^1 \exp(2(1-s))x_0^2 \, \mathrm{d}s.$$

If $x(t)$ exists on $[0, 1]$ then the equation $Tx_0 = z$ must be solvable with respect to x_0. But the last equation has a form $[\exp(2) - 1]x_0^2 + 2\exp(2)x_0 - 2z = 0$. Since the latter can be solved not for all $z \in \mathbb{R}$, not all solutions $x(t, 1, z)$, $z \in \mathbb{R}$, can be proceeded to $t = 0$.

Further we shall consider the uniqueness of solutions continued to the left. Fix numbers $x_0, x_1 \in \mathbb{R}$, $x_1 \ne x_2$, such that $(x_0 + x_1)(1 - \exp(2)) = 2\exp(2)$. Denote $x_0(t) = x(t, 0, x_0)$ and $x_1(t) = x(t, 0, x_1)$, solutions of (9). For $t \in [0, 1]$, they are solutions of the equations $x' = 2x - x_0^2$ and $x' = 2x - x_1^2$, respectively. One can easily check that $Tx_0 = Tx_1$, that is $x_0(1) = x_1(1)$, and the solution $x(t, 1, x_1(1))$ of (9) cannot be continued to $t = 0$ uniquely.

From now on we make the assumptions:

$(C4)$ $2Ml\theta < 1$;
$(C5)$ $Ml\theta[1 + M(1 + l\theta)\exp(Ml\theta)] < m$.

Let us prove the following theorem which provides conditions of unique existence of solutions to the left when the initial moment t_0 is an arbitrary real number.

Theorem 2.2. *Assume that conditions $(C1)$–$(C5)$ are fulfilled. Then for every $y_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, $\theta_i < t_0 \le \theta_{i+1}$, $i \in \mathbb{Z}$, there exists a unique solution $\bar{y}(t) = y(t, \theta_i, \bar{y}_0)$ of (1) in the sense of Definition 2.1 such that $\bar{y}(t_0) = y_0$.*

Proof. *Existence.* Consider a solution $\xi(t) = y(t, t_0, y_0)$ of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = A(t)y + f(t, y, \eta) \tag{11}$$

on $[\theta_i, t_0]$.

We have that

$$\xi(t) = X(t, t_0)y_0 + \int_{t_0}^{t} X(t, s) f(s, \xi(s), \eta)\, \mathrm{d}s, \tag{12}$$

and should prove that there exists a vector $\eta \in \mathbb{R}$ and a solution $\xi(t)$ of (12) which is defined on $[\theta_i, t_0]$, and satisfies $\xi(\theta_i) = \eta$.

Take $\xi_0(t) = X(t, t_0)y_0$ and construct

$$\xi_{m+1}(t) = X(t, t_0)y_0 + \int_{t_0}^{t} X(t, s) f(s, \xi_m(s), \xi_m(\theta_i))\, \mathrm{d}s, \quad m \ge 0.$$

One can easily check that

$$\max_{[\theta_i, t_0]} \|\xi_{m+1}(t) - \xi_m(t)\|_0 \le [2Ml\theta]^m M\|y_0\|.$$

That is sequence $\xi_m(t)$ is convergent and its limit $\xi(t)$ satisfies (12) on $[\theta_i, t_0]$ with $\eta = \xi(\theta_i)$. The existence is proved.

Uniqueness. It is sufficient to check that for each $t \in (\theta_i, \theta_{i+1}]$, and $y_2, y_1 \in \mathbb{R}^n$, $y_2 \ne y_1$, the condition $y(t, \theta_i, y_1) \ne y(t, \theta_i, y_2)$ is valid.

Denote by $y_1(t) = y(t, \theta_i, y_1)$, $y_2(t) = y(t, \theta_i, y_2)$, $y_1 \ne y_2$, solutions of (1). Assume, on the contrary, that there exists $t^* \in (\theta_i, \theta_{i+1}]$ such that $y_1(t^*) = y_2(t^*)$. Then

$$X(t^*, \theta_i)(y_2 - y_1) = \int_{\theta_i}^{t^*} X(t^*, s)[f(s, y_1(s), y_1) - f(s, y_2(s), y_2)]\, \mathrm{d}s. \tag{13}$$

We have that

$$\|X(t^*, \theta_i)(y_2 - y_1)\| \ge m\|y_2 - y_1\|. \tag{14}$$

Moreover, for $t \in (\theta_i, \theta_{i+1}]$ the following inequality is valid:

$$\|y_1(t) - y_2(t)\| \le M\|y_2 - y_1\| + \int_{\theta_i}^{t} Ml[\|y_1(s) - y_2(s)\| + \|y_2 - y_1\|]\, \mathrm{d}s.$$

Hence, using Gronwall–Bellman inequality we can write that

$$\|y_1(t) - y_2(t)\| \le M(1 + l\theta)\exp(Ml\theta)\|y_2 - y_1\|.$$

Consequently,

$$\left\| \int_{\theta_i}^{t^*} X(t^*, s)[f(s, y_1(s), y_1) - f(s, y_2(s), y_2)]\,\mathrm{d}s \right\| \\ \le lM\theta[1 + M(1 + l\theta)\exp(Ml\theta)]\|y_2 - y_1\|. \tag{15}$$

Finally, one can see that $(C5)$, (14) and (15) contradict (13). The theorem is proved. □

Let us illustrate the last theorem by the following example.

Example 2.2. Consider the following EPCAG:

$$x'(t) = 3x(t) - x(t)x(\beta(t)), \tag{16}$$

where $\beta(t) = \theta_i$ if $\theta_i \le t < \theta_{i+1}, i \in \mathbb{Z}, \theta_{2j-1} = j - \frac{1}{5}, \theta_{2j} = j + \frac{1}{5}, j \in \mathbb{Z}$. The distance $\theta_{i+1} - \theta_i, i \in \mathbb{Z}$, is equal either to $\theta = \frac{3}{5}$ or to $\bar{\theta} = \frac{2}{5}$.

We shall find conditions of existence of solution $x(t) = x(t, t_0, x_0), \theta_i < t_0 < \theta_{i+1}, i \in \mathbb{Z}$, of (16) on $[\theta_i, \infty)$, applying two ways: (a) by using Theorem 2.2; (b) directly through specific properties of the equation. Then we compare the conditions obtained by these two methods.

(a). If $t \in [\theta_i, \theta_{i+1}]$ then $x(t)$ satisfies the following equation:

$$x'(t) = 3x(t) - x(t)x(\theta_i).$$

Hence,

$$x(t) = x(\theta_i)\mathrm{e}^{(3-x(\theta_i))(t-\theta_i)}. \tag{17}$$

The last equality implies that every nontrivial solution of (16) either is positive or negative. That is why, without loss of generality, we consider only positive solutions. For a fixed $H > 0$ denote $G_H = \{x : 0 < x < H\}$.

If $x_1, x_2, y_1, y_2 \in G_H$ then $|x_1y_1 - x_2y_2| \le H(|x_1 - x_2| + |y_1 - y_2|)$. Moreover, we have that constants $m,\ M$, which are defined in Theorem 2.2, are equal

$$m = \min_{|t-s|\le\theta} \mathrm{e}^{3(t-s)} = \mathrm{e}^{\frac{-9}{5}},\ M = \max_{|t-s|\le\theta} \mathrm{e}^{3(t-s)} = \mathrm{e}^{\frac{9}{5}}.$$

Hence, condition $(C4)$ and Theorem 2.2 imply that if $0 < x_0 \le H$ then solution $x(t, t_0, x_0)$ exists on $[\theta_i, \infty)$ if positive constant H satisfies the inequality

$$H < \frac{5}{6}\mathrm{e}^{\frac{-9}{5}}. \tag{18}$$

(b). Investigate again the problem of existence of $x(t, t_0, x_0)$ on $[\theta_i, \infty)$, but this time without applying Theorem 2.2.

Using (17) we find that

$$x_0 = x(\theta_i)\mathrm{e}^{(3-x(\theta_i))(t_0-\theta_i)}. \tag{19}$$

Consider (19) as an equation with respect to $x = x(\theta_i)$. Introduce the following functions: $F_1(x) = x\mathrm{e}^{(3-x)\bar{\theta}}$ and $F_2(x) = x\mathrm{e}^{(3-x)\theta}$. The critical values of x for the functions are $x_{\max}^{(1)} = \bar{\theta}^{-1} = \frac{5}{2} < 3$ and $x_{\max}^{(2)} = \theta^{-1} = \frac{5}{3} < 3$ respectively, and maximal values of these functions are

$$F_{\max}^{(1)} = F_1(x_{\max}^{(1)}) = \frac{5}{2}\mathrm{e}^{\frac{1}{5}}, \quad F_{\max}^{(2)} = F_2(x_{\max}^{(2)}) = \frac{5}{3}\mathrm{e}^{\frac{4}{5}}. \tag{20}$$

We have that $F_{\min} = \min(F_{\max}^{(1)},\ F_{\max}^{(2)}) = \frac{5}{2}\mathrm{e}^{\frac{1}{5}}$.

It is clear that if

$$0 < x_0 \le F_{\min} \tag{21}$$

then equation

$$x_0 = x\mathrm{e}^{(3-x)(t_0-\theta_i)}$$

is solvable, and, hence, solution $x(t, t_0, x_0)$ exists on $[\theta_i, \infty)$.

One can see that $H < F_{\min}$. That is, if x_0 satisfies conditions of Theorem 2.2 of existence of $x(t, t_0, x_0)$ on $[\theta_i, \infty)$, then the solution exists on $[\theta_i, \infty)$ by condition (21) on the basis of the discussion in (b). In other words, we have verified that in the particular case of equation (16), Theorem 2.2 does not contradict the evaluation in (b).

We shall use the following definition, which is a version of a definition from [21], adapted for our general case.

Definition 2.2. A function $y(t)$ is a solution of (1) on $\mathbb{R}$ if:

(i) $y(t)$ is continuous on $\mathbb{R}$;
(ii) the derivative $y'(t)$ exists at each point $t \in \mathbb{R}$ with the possible exception of the points $\theta_i, i \in \mathbb{Z}$, where one-sided derivatives exist;
(iii) Eq. (1) is satisfied on each interval $[\theta_i, \theta_{i+1}), i \in \mathbb{Z}$.

Theorem 2.3. *Suppose that conditions* $(C1)$–$(C5)$ *are fulfilled. Then for every* $(t_0, y_0) \in \mathbb{R}\times\mathbb{R}^n$, *there exists a unique solution* $y(t) = y(t, t_0, y_0)$ *of* (1) *in the sense of Definition* 2.2 *such that* $y(t_0) = y_0$.

Proof. Assume that $\theta_i < t_0 \le \theta_{i+1}$ for a fixed $i \in \mathbb{Z}$. By Theorem 2.2 there exists a unique solution $y(t, \theta_i, y^i)$ of (1) with some vector $y^i \in \mathbb{R}^n$ such that $y(t_0, \theta_i, y^i) = y_0$. Applying the theorem again we can find a unique solution $y(t, \theta_{i-1}, y^{i-1})$ of (1) such that $y(\theta_i, \theta_{i-1}, y^{i-1}) = y^i$, and, hence, $y(t_0, \theta_{i-1}, y^{i-1}) = y_0$. We can complete the proof using induction. □

Remark 2.2. The last theorem is of major importance for our paper. It arranges the correspondence between points $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}^n$ and all solutions of (1), and there is not a solution of the equation out of the correspondence. Using the assertion we can say that the definition of the IVP for the EPCAG is similar to the problem for an ordinary differential equation, although the EPCAG is an equation with delay argument. In the rest of the paper we shall use the correspondence to prove the main theorems.

2.2. Reduction to a system with a box-diagonal matrix of coefficients

Using Gram–Schmidt orthogonalization of the columns of $X(t)$ one can obtain that by the transformation $y = U(t)z$, where $U(t)$ is a Lyapunov matrix, (1) can be reduced to the following system [10]:

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= B_+(t)u + g_+(t, z(t), z(\beta(t))),\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= B_-(t)v + g_-(t, z(t), z(\beta(t))),\end{aligned} \tag{22}$$

where

$$z = (u, v), u \in R^k, v \in R^{n-k}, \operatorname{diag}\{B_+(t), B_-(t)\} = U^{-1}(t)A(t)U(t),$$
$$(g_+(t, z(t), z(\beta(t))), g_-(t, z(t), z(\beta(t)))) = f(t, U(t)z(t)), U(\beta(t))z(\beta(t)).$$

One can check that the Lipschitz condition is valid

$$\|g_+(t, z_1, w_1) - g_+(t, z_2, w_2)\| + \|g_-(t, z_1, w_1) - g_-(t, z_2, w_2)\|$$
$$\leq L(\|z_1 - z_2\| + \|w_1 - w_2\|)$$

for all $t \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^k, w_1, w_2 \in \mathbb{R}^{(n-k)}$, and $L = 2\sup_{\mathbb{R}} \|U(t)\|l$.

The normed fundamental matrices $U(t, s), V(t, s)$ of the systems

$$\frac{du}{dt} = B_+(t)u, \quad \frac{dv}{dt} = B_-(t)v, \tag{23}$$

respectively, satisfy the following inequalities:

$$\|U(t, s)\| \leq K \exp(-\sigma(t - s)),\ t \geq s, \quad \|V(t, s)\| \leq K \exp(\sigma(s - t)),\ t \leq s.$$

The boundedness and continuity of matrices $U, U^{-1}, \frac{dU}{dt}, \frac{dU^{-1}}{dt}$, imply that the solutions and integral sets of (1) and (22) have the same asymptotic properties and smoothness. Moreover, the linear part of system (22) has box-diagonal form with boxes corresponding to the stable and unstable subspaces. For this reason it is easier to investigate Eq. (22) in what follows.

3. The existence of manifolds

Theorem 3.1. *Suppose that conditions $(C1)$–$(C5)$ are satisfied. Then for arbitrary $\epsilon > 0, \alpha \in (0, \sigma)$ and a sufficiently small Lipschitz constant L, there exists a continuous function $F(t, u)$ satisfying*

$$F(t, 0) = 0, \tag{24}$$

$$\|F(t, u_1) - F(t, u_2)\| \leq \frac{2K^2L(1 + \exp(\sigma\theta))}{\sigma + \alpha}\|u_1 - u_2\|, \tag{25}$$

for all t, u_1, u_2, such that $v_0 = F(t_0, u_0)$ determines a solution $z(t)$ of (22) on $\mathbb{R}$ and

$$\|z(t)\| \leq (K + \epsilon)\|u_0\| \exp(-\alpha(t - t_0)),\ t \geq t_0. \tag{26}$$

Let us denote $S^+ = \{(t, u, v) \in \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{n-k} \mid v = F(t, u)\}$.

To prove the last theorem we need the following assertion which is of major importance for our paper.

Lemma 3.1. *Fix $N \in \mathbb{R}, N > 0, \alpha \in (0, \sigma)$ and assume that conditions $(C1)$–$(C3)$ are valid. A function $z(t) = (u, v), \|z(t)\| \leq N \exp(-\alpha(t - t_0)), t \geq t_0$, where t_0 is a real fixed number, is a solution of (22) on $\mathbb{R}$ if and only if it is a solution on $\mathbb{R}$ of the following system of integral equations:*

$$u(t) = U(t, t_0)u(t_0) + \int_{t_0}^{t} U(t, s)g_+(s, z(s), z(\beta(s)))\,ds,$$
$$v(t) = -\int_{t}^{\infty} V(t, s)g_-(s, z(s), z(\beta(s)))\,ds. \tag{27}$$

Proof. *Necessity*. Assume that $z(t) = (u, v)$, $\|z(t)\| \le N \exp(-\alpha(t - t_0))$, $t \ge t_0$, is a solution of (22). Denote

$$\begin{aligned} \phi(t) &= U(t, t_0)u(t_0) + \int_{t_0}^{t} U(t, s)g_+(s, z(s), z(\beta(s)))\, \mathrm{d}s, \\ \psi(t) &= -\int_{t}^{\infty} V(t, s)g_-(s, z(s), z(\beta(s)))\, \mathrm{d}s. \end{aligned} \tag{28}$$

By straightforward evaluation we can see that the integrals converge and are bounded if $t \in [t_0, \infty)$. Assume that $t \ne \theta_i$, $i \in \mathbb{Z}$. Then

$$\begin{aligned} \phi'(t) &= B_+(t)\phi(t) + g_+(t, z(t), z(\beta(t))), \\ \psi'(t) &= B_-(t)\psi(t) + g_-(t, z(t), z(\beta(t))), \end{aligned}$$

and

$$\begin{aligned} u'(t) &= B_+(t)u(t) + g_+(t, z(t), z(\beta(t))), \\ v'(t) &= B_-(t)v(t) + g_-(t, z(t), z(\beta(t))). \end{aligned}$$

Hence,

$$\begin{aligned} [\phi(t) - u(t)]' &= B_+(t)[\phi(t) - u(t)], \\ [\psi(t) - v(t)]' &= B_-(t)[\psi(t) - v(t)]. \end{aligned}$$

Calculating the limit values at $\theta_j \in \mathbb{Z}$ we can find that

$$\begin{aligned} \phi'(\theta_j \pm 0) &= B_+(\theta_j \pm 0)\phi(\theta_j \pm 0) + g_+(\theta_j \pm 0, z(\theta_j \pm 0), z(\beta(\theta_j \pm 0))), \\ u'(\theta_j \pm 0) &= B_+(\theta_j \pm 0)u(\theta_j \pm 0) + g_+(\theta_j \pm 0, z(\theta_j \pm 0), z(\beta(\theta_j \pm 0))), \\ \psi'(\theta_j \pm 0) &= B_+(\theta_j \pm 0)\psi(\theta_j \pm 0) + g_-(\theta_j \pm 0, z(\theta_j \pm 0), z(\beta(\theta_j \pm 0))), \\ v'(\theta_j \pm 0) &= B_+(\theta_j \pm 0)v(\theta_j \pm 0) + g_-(\theta_j \pm 0, z(\theta_j \pm 0), z(\beta(\theta_j \pm 0))). \end{aligned}$$

Consequently,

$$\begin{aligned} &[\phi(t) - u(t)]' \,|_{t=\theta_j+0} = [\phi(t) - u(t)]' \,|_{t=\theta_j-0}, \\ &[\psi(t) - v(t)]' \,|_{t=\theta_j+0} = [\psi(t) - v(t)]' \,|_{t=\theta_j-0}\,. \end{aligned}$$

Thus, $(\phi(t) - u(t), \psi(t) - v(t))$ is a continuously differentiable on $\mathbb{R}$ function satisfying (23) with the initial condition $\phi(t_0) - u(t_0) = 0$. Assume that $\psi(t_0) - v(t_0) \ne 0$. Then $\psi(t) - v(t)$ is unbounded on $[t_0, \infty)$. This contradiction proves that $\phi(t) - u(t) = 0$ and $\psi(t) - v(t) = 0$ on $\mathbb{R}$.

Sufficiency. Suppose that (27) is valid. Fix $i \in \mathbb{Z}$ and consider the interval $[\theta_i, \theta_{i+1})$. If $t \in (\theta_i, \theta_{i+1})$, then by differentiating one can see that $z(t)$ satisfies (4). Moreover, considering $t \to \theta_i+$, and taking into account that $z(\beta(t))$ is a right-continuous function, we find that $z(t)$ satisfies (22) on $[\theta_i, \theta_{i+1})$. The lemma is proved. □

Similarly to the last lemma, one can prove that the following assertion is valid.

Lemma 3.2. *Fix $N \in \mathbb{R}$, $N > 0$, $\alpha \in (0, \sigma)$, and assume that conditions $(C1)$–$(C3)$ are valid. A function $z(t) = (u, v)$, $\|z(t)\| \le N \exp(\alpha(t - t_0))$, $t \le t_0$, where t_0 is a real fixed number, is a solution of (22) on $\mathbb{R}$ if and only if it is a solution of the following system of integral* *equations:*

$$u(t) = \int_{-\infty}^{t} U(t,s)g_+(s, z(s), z(\beta(s)))\,\mathrm{d}s,$$
$$v(t) = V(t,t_0)v(t_0) + \int_{t_0}^{t} V(t,s)g_-(s, z(s), z(\beta(s)))\,\mathrm{d}s. \tag{29}$$

Proof of Theorem 3.1. Let us consider system (27) and apply the method of successive approximations to it. Denote $z_0 = (0,0)^T$, $z_m = (u_m, v_m)^T$, $m \in \mathbb{N}$, where for $m \ge 0$

$$u_{m+1}(t) = U(t,t_0)u(t_0) + \int_{t_0}^{t} U(t,s)g_+(s, z_m(s), z_m(\beta(s)))\,\mathrm{d}s,$$
$$v_{m+1}(t) = -\int_{t}^{\infty} V(t,s)g_-(s, z_m(s), z_m(\beta(s)))\,\mathrm{d}s. \tag{30}$$

One can show by induction that

$$\|z_m(t,c)\| \le (K+\epsilon)\|c\| \exp(-\alpha(t-t_0)),\ t \ge t_0, \tag{31}$$

provided that

$$K(K+\epsilon)\frac{2\sigma}{\sigma^2-\alpha^2}(1+\exp(\sigma\theta))L < \epsilon. \tag{32}$$

Similarly, one can establish the following inequalities:

$$\|v_m(t,c_1) - v_m(t,c_2)\| \le \frac{2K^2L(1+\exp(\sigma\theta))}{\sigma+\alpha}\|c_1-c_2\| \exp(-\alpha(t-t_0)),$$
$$\|z_m(t,c_1) - z_m(t,c_2)\| \le 2K\|c_1-c_2\| \exp(-\alpha(t-t_0)),$$

if $4\sigma KL(1+\exp(\sigma\theta)) < \sigma^2 - \alpha^2$.

And

$$\|z_m(t,c) - z_{m-1}(t,c)\| \le K\|c\| \left(\frac{2KL(1+\exp(\sigma\theta))}{\sigma-\alpha}\right)^{m-1} \exp(-\alpha(t-t_0)). \tag{33}$$

The last inequality and the assumption

$$L < \frac{\sigma-\alpha}{2K(1+\exp(\sigma\theta))} \tag{34}$$

imply that the sequence z_m converges uniformly for all c and $t \ge t_0$. Define the limit function $z(t,t_0,c) = (u(t,t_0,c), v(t,t_0,c))$. It can be easily seen that this function is a solution of (27). By Lemma 3.1, $z(t,t_0,c)$ is also a solution of (22). Taking $t = t_0$ in (27) we have that

$$u(t_0,t_0,c) = c,\ v(t_0,t_0,c) = -\int_{t_0}^{\infty} V(t,s)g_-(s, z(s,t_0,c), z(\beta(s),t_0,c))\,\mathrm{d}s.$$

Denote $F(t_0,c) = v(t_0,t_0,c)$. One can see that it satisfies all the conditions which should be verified. The theorem is proved. □

Definition 3.1. The set Σ in the $(t,y)-$ space is said to be an integral set of system (1) if any solution $y(t) = y(t,t_0,y_0)$, $y(t_0) = y_0$, with $(t_0,y_0) \in \Sigma$, has the property that $(t, y(t)) \in \Sigma$, $t \in \mathbb{R}$.

Theorem 3.2. *The set S^+ is an integral set.*

Proof. Assume that $(t_0, u_0, v_0) \in S^+, z_0 = (u_0, v_0)$. We must show that if $z(t) = z(t, t_0, z_0)$, then $(t^*, z(t^*)) \in S^+$ for all $t^* \in \mathbb{R}$. Indeed, if $t^* > t_0$, then $\|z(t)\| \le (K + \epsilon)\|u_0\| \exp(-\alpha(t^* - t_0)) \exp(-\alpha(t - t^*)), t^* > t_0$. Lemma 3.1 implies that the point $(t^*, z(t^*))$ satisfies the equation $v = F(t, u)$. If $t^* < t_0$, then $\|z(t)\| \le \bar{K}\|u_0\| \exp(-\alpha(t^* - t_0)) \exp(-\alpha(t - t^*))$, where $\bar{K} = \max\{\max_{[t^*, t_0]} \|z(t)\|, (K + \epsilon)\|u_0\|\}$. Using Lemma 3.1 again one can see that $(t^*, z(t^*)) \in S^+$. The theorem is proved. □

Theorem 3.3. *For every fixed $(t_0, u(t_0))$ system (27) admits only one solution bounded on $[t_0, \infty)$.*

Proof. By Theorem 3.1 there exists a solution of (27) bounded on $[t_0, \infty)$. Assume that z_1, z_2 are two bounded solutions of (27). By straightforward evaluation it can be shown that

$$\sup_{[t_0,\infty)} \|z_1 - z_2\| \le \frac{2KL(1 + \exp(\sigma))}{\sigma - \alpha} \sup_{[t_0,\infty)} \|z_1 - z_2\|.$$

Hence, in view of (34) the theorem is proved. □

Theorem 3.4. *If $(t_0, z_0) \notin S^+$ then the solution $z(t, t_0, z_0)$ of (22) is unbounded on $[t_0, \infty)$.*

Proof. Assume, on the contrary, that $z(t) = z(t, t_0, z_0) = (u, v)$ is a bounded solution of (22) and $(t_0, z_0) \notin S^+$. It is obvious that

$$\begin{aligned} u(t) &= U(t, t_0)u(t_0) + \int_{t_0}^{t} U(t, s)g_+(s, z(s), z(\beta(s)))\, ds, \\ v(t) &= V(t, t_0)\kappa - \int_{t}^{\infty} V(t, s)g_-(s, z(s), z(\beta(s)))\, ds, \end{aligned} \tag{35}$$

where

$$\kappa = v(t_0) + \int_{t_0}^{\infty} V(t, s)g_-(s, z(s), z(\beta(s)))\, ds,$$

and the improper integral converges and is bounded on $[t_0, \infty)$. But the inequality $\|V(t, s)\| \ge K^{-1} \exp(\alpha(t-s)), t \ge s$ implies that $z(t)$ is bounded only if $\kappa = 0$. By Theorem 3.3 $z(t)$ satisfies (27) with $c = u(t_0)$. Hence $u(t_0), v(t_0), t_0$ satisfy (27) and $(t_0, z_0) \notin S^+$. The contradiction proves our theorem. □

Using Lemma 3.2 one can prove the existence of an unstable manifold S^- for system (22), which has the same asymptotic properties as the stable manifold S^+, but for decreasing t.

On the basis of Theorems 3.1–3.4 and their analogues for $t \to -\infty$ one can conclude that there exist two integral manifolds Σ^+, Σ^- of Eq. (1) such that every solution which starts at Σ^+ tends to zero as $t \to \infty$, and every solution which starts at Σ^- tends to zero as $t \to -\infty$. All solutions on Σ^+, Σ^- exist on $\mathbb{R}$. If a solution starts outside of Σ^+ then it is unbounded on $[t_0, \infty)$, and if a solution starts outside of Σ^- then it is unbounded on $(-\infty, t_0]$.

The following assertions of this section describe the structure of the set of solutions which do not belong to the integral sets. Consider a solution $z(t) = \{u(t), v(t)\}$ of (22) with the initial data $t_0, u_0, v_0, z_0 \ne 0$.

Lemma 3.3. *Assume* $(t_0, u_0, v_0) \notin S^+$, $K(K^2+1)(1+\exp(\alpha\theta))L < \sigma$, *and* $\|u_0\| \le \|v_0\|$. *Then* $\|u(t)\| \le K^2\|v(t)\|, t \ge t_0$.

Proof. Assume that $K > 1$ (otherwise the proof is more simple). Suppose on the contrary that there exist moments $\bar{t}, \tilde{t}$ such that

$$\|u(\bar{t})\| = \|v(\bar{t})\|, \|u(\tilde{t})\| = K^2\|v(\tilde{t})\|, \tag{36}$$

and

$$\|v(t)\| \le \|u(t)\| \le K^2\|v(t)\|, \bar{t} < t < \tilde{t}. \tag{37}$$

Since

$$u(t) = U(t,\bar{t})u(\bar{t}) + \int_{\bar{t}}^{t} U(t,s)g_+(s, z(s), z(\beta(s)))\,\mathrm{d}s,$$

we have that

$$\|u(t)\| \le K\mathrm{e}^{-\sigma(t-\bar{t})}\|u(\bar{t})\| + 2KL(1+\mathrm{e}^{\alpha\theta})\int_{\bar{t}}^{t} \mathrm{e}^{-\sigma(t-s)}\|u(s)\|\,\mathrm{d}s.$$

Applying the Gronwall–Bellman Lemma one can obtain that

$$\|u(\tilde{t})\| \le K\|u(\bar{t})\|\mathrm{e}^{-(\sigma-2KL(1+\mathrm{e}^{\alpha\theta}))(\tilde{t}-\bar{t})}. \tag{38}$$

Similarly we can find that

$$\|v(\bar{t})\| \le K\|v(\tilde{t})\|\mathrm{e}^{-(\sigma-KL(1+K^2)(1+\mathrm{e}^{\alpha\theta}))(\tilde{t}-\bar{t})}. \tag{39}$$

Hence, using (38) and the equality $\|u(\bar{t})\| = \|v(\bar{t})\|$, one can obtain the inequality $\|u(\tilde{t})\| < K^2\|v(\tilde{t})\|$, which contradicts the assumption. The lemma is proved. □

Lemma 3.4. *Assume that all conditions of Lemma 3.3 are valid. Then*

$$\|v(t)\| \ge \frac{\|v_0\|}{K}\mathrm{e}^{(\sigma-KL(1+K^2)(1+\mathrm{e}^{\alpha\theta}))(t-t_0)}. \tag{40}$$

Proof. Similarly to (39), applying Lemma 3.3 we may write that

$$\|v(t_0)\| \le K\|v(t)\|\mathrm{e}^{(\sigma-KL(1+K^2)(1+\mathrm{e}^{\alpha\theta}))(t_0-t)}.$$

The last inequality is equivalent to (40). The lemma is proved. □

Theorem 3.5. *Assume that* $(t_0, u_0, v_0) \notin S^+$ *and* $K(K^2+1)(1+\mathrm{e}^{\alpha\theta})L < \sigma$. *Then* $\|v(t)\| \to \infty$ *as* $t \to \infty$.

Proof. If $\|u_0\| \le \|v_0\|$, then the proof is similar to that of Lemma 3.4. Assume that $\|u_0\| > \|v_0\|$. In the same way as we obtained (38), we show that

$$\|u(t)\| \le K\|u(t_0)\|\mathrm{e}^{-(\sigma-2KL(1+\mathrm{e}^{\alpha\theta}))(t-t_0)}. \tag{41}$$

Now, Theorem 3.4 and the last inequality imply that there exists $\bar{t}$ such that $\|u(\bar{t})\| = \|v(\bar{t})\|$. The theorem is proved. □

Similarly to Theorem 3.5, one can prove that the following theorem is valid.

Theorem 3.6. *Assume* $(t_0, u_0, v_0) \notin S^-$ *and* $K(K^2+1)(1+\mathrm{e}^{\alpha\theta})L < \sigma$. *Then* $\|u(t)\| \to \infty$ *as* $t \to -\infty$.

4. The smoothness of the manifolds

The following condition is needed in this part of the paper.

$(C6)$ The function $f(t, x, w)$ is uniformly continuously differentiable in x, w for all t, x, w.

Theorem 4.1. *Suppose conditions $(C1)$–$(C6)$ are fulfilled. Then the function $F(t, u)$ is continuously differentiable in u for all t, u.*

Proof. By Theorem 3.1 there exists a solution $z(t) = z(t, t_0, c)$ of (22). Let us show that it is continuously differentiable in c. Denote $e_j = (0, \ldots, 0, 1, 0, \ldots 0)^T$, where the j-th coordinate is one, and $h_j = he_j$, where $h \in \mathbb{R}$ is fixed. Denote $\Delta z(t) = z(t, t_0, c + h_j) - z(t, t_0, c)$, $\Delta z = (\Delta u, \Delta v)$. We have that

$$\begin{aligned}
\Delta u(t) &= U(t, t_0)h_j + \int_{t_0}^{t} U(t, s)\left[\frac{\partial g_+(s, \Delta z(s), \Delta z(\beta(s)))}{\partial x}\Delta z(s)\right. \\
&\quad \left. + \frac{\partial g_+(s, \Delta z(s), \Delta z(\beta(s)))}{\partial w}\Delta z(\beta(s)) + o_+(\Delta z)\right] \mathrm{d}s, \\
\Delta v(t) &= -\int_{t}^{\infty} V(t, s)\left[\frac{\partial g_-(s, \Delta z(s), \Delta z(\beta(s)))}{\partial x}\Delta z(s)\right. \\
&\quad \left. + \frac{\partial g_-(s, \Delta z(s), \Delta z(\beta(s)))}{\partial w}\Delta z(\beta(s)) + o_-(\Delta z)\right] \mathrm{d}s.
\end{aligned} \tag{42}$$

Consider the following system of integral equations:

$$\begin{aligned}
\phi(t) &= U(t, t_0)e_j + \int_{t_0}^{t} U(t, s)\left[\frac{\partial g_+(s, \Delta z(s), \Delta z(\beta(s)))}{\partial x}\omega(s)\right. \\
&\quad \left. + \frac{\partial g_+(s, \Delta z(s), \Delta z(\beta(s)))}{\partial w}\omega(\beta(s))\right] \mathrm{d}s, \\
v(t) &= -\int_{t}^{\infty} V(t, s)\left[\frac{\partial g_-(s, \Delta z(s), \Delta z(\beta(s)))}{\partial x}\omega(s)\right. \\
&\quad \left. + \frac{\partial g_-(s, \Delta z(s), \Delta z(\beta(s)))}{\partial w}\omega(\beta(s))\right] \mathrm{d}s,
\end{aligned} \tag{43}$$

where $\omega = (\phi, \psi)$.

Condition $(C6)$ implies that there exists a constant $L > 0$ such that

$$\left\|\frac{\partial g_+(s, \Delta z(s), \Delta z(\beta(s)))}{\partial x}\right\| \le L, \quad \left\|\frac{\partial g_+(s, \Delta z(s), \Delta z(\beta(s)))}{\partial w}\right\| \le L,$$
$$\left\|\frac{\partial g_-(s, \Delta z(s), \Delta z(\beta(s)))}{\partial x}\right\| \le L, \quad \left\|\frac{\partial g_-(s, \Delta z(s), \Delta z(\beta(s)))}{\partial w}\right\| \le L.$$

Consequently, from Theorem 3.1 one can obtain that there exists a unique bounded solution $\omega(t) = \omega(t, t_0, e_j) = (\phi, \psi)$ of system (43). Equalities (42) and (43) imply that

$$\begin{aligned}
\frac{\Delta u(t)}{h} - \phi &= \int_{t_0}^{t} U(t, s)\left[\frac{\partial g_+(s, \Delta z(s), \Delta z(\beta(s)))}{\partial x}\left(\frac{\Delta z(s)}{h} - \omega(s)\right)\right. \\
&\quad \left. + \frac{\partial g_+(s, \Delta z(s), \Delta z(\beta(s)))}{\partial w}\left(\frac{\Delta z(\beta(s))}{h} - \omega(\beta(s))\right) + \frac{o_+(\Delta z)}{h}\right] \mathrm{d}s,
\end{aligned}$$

$$\frac{\Delta v(t)}{h} - \psi = -\int_t^{\infty} V(t,s)\left[\frac{\partial g_-(s,\Delta z(s),\Delta z(\beta(s)))}{\partial x}\left(\frac{\Delta z(s)}{h} - \omega(s)\right)\right.$$
$$\left. + \frac{\partial g_-(s,\Delta z(s),\Delta z(\beta(s)))}{\partial w}\left(\frac{\Delta z(s)}{h} - \omega(s)\right) + \frac{o_-(\Delta z)}{h}\right] \mathrm{d}s. \tag{44}$$

Since solutions of (27) satisfy the Lipschitz condition, $\|\Delta z(t)\| \le 2K|h|\exp(-\alpha(t-t_0))$ and $(C6)$ is valid, $\|o_\pm\| \le O(h)|h|\exp(-\alpha(t-t_0)),\ i = 1, 2$. Using the last relation and (44) we can check that

$$\sup_{[t_0,\infty)} \left\|\frac{\Delta z}{h} - \omega\right\| \le \frac{2KL\exp(\sigma)}{\alpha} \sup_{[t_0,\infty)} \left\|\frac{\Delta z}{h} - \omega\right\| + \frac{2KL\exp(\sigma)}{\alpha} O(h),$$

where $O(h) \to 0$ as $h \to 0$. By the assumption we can take L arbitrarily small. Then

$$\frac{\partial z(t,t_0,c)}{\partial c_j} = \omega(t).$$

The theorem is proved. □

The last theorem implies the smoothness of the surface Σ^+. Similarly, one can verify the smoothness of the surface Σ^-.

5. The existence of bounded and periodic solutions

In the same way as we did in Lemma 3.1, we can prove that the following assertion is valid.

Lemma 5.1. *Assume that conditions $(C1)$–$(C3)$ are valid. A function $z(t) = (u, v)$ bounded on $\mathbb{R}$ is a solution of (22) if and only if it is a solution of the following system of integral equations:*

$$u(t) = \int_{-\infty}^{t} U(t,s)g_+(s,z(s),z(\beta(s)))\,\mathrm{d}s,$$
$$v(t) = -\int_t^{\infty} V(t,s)g_-(s,z(s),z(\beta(s)))\,\mathrm{d}s. \tag{45}$$

In what follows, we shall need the following two conditions:

$(C7)$ $\|f(t,0,0)\| < h,\ t \in \mathbb{R}$, for some $h \in \mathbb{R}, h > 0$;
$(C8)$ $\frac{2KL}{\sigma} < 1$.

Denote $H = \sup_{\mathbb{R}} \|U(t)\|h$.

Theorem 5.1. *Suppose conditions $(C1)$–$(C3)$, $(C7)$, $(C8)$ are valid. Then there exists a unique solution $z(t)$ of (22), bounded on $\mathbb{R}$, and*

$$\|z(t)\| \le \frac{2KH}{\sigma - 2KL}. \tag{46}$$

Proof. Let us solve system (45), applying the method of successive approximations. Assume that $z_0(t) = 0,\ t \in \mathbb{R}$, and

$$u_m(t) = \int_{-\infty}^{t} U(t,s)g_+(s,z_{m-1}(s),z_{m-1}(\beta(s)))\,\mathrm{d}s,$$
$$v_m(t) = -\int_t^{\infty} V(t,s)g_-(s,z_{m-1}(s),z_{m-1}(\beta(s)))\,\mathrm{d}s, \tag{47}$$

where $z_m(t) = (u_m, v_m)^T$, $m \geq 1$. We have that

$$\|u_1(t)\| \leq \int_{-\infty}^{t} \|U(t,s)\| \|g_+(s,0,0)\| \,\mathrm{d}s,$$

$$\|v_1(t)\| \leq \int_{t}^{\infty} \|V(t,s)\| \|g_-(s,0,0)\| \,\mathrm{d}s.$$

Then

$$\|u_1(t)\| \leq \frac{KH}{\sigma}, \|v_1(t)\| \leq \frac{KH}{\sigma},$$

and hence

$$\|z_1(t)\| \leq \frac{2KH}{\sigma}.$$

We can obtain by induction that the following inequality is valid:

$$\|z_{m+1}(t) - z_m(t)\| \leq \left(\frac{2KL}{\sigma}\right)^m \frac{2KH}{\sigma}, 0 \leq m.$$

Condition $(C8)$ implies that the sequence z_m converges uniformly to a function $\kappa(t)$, $t \in \mathbb{R}$. It is easy to check that

$$\|\kappa(t)\| \leq \frac{2KH}{\sigma - 2KL}$$

and $\kappa(t)$ is a solution of system (45). Assume that there are two different bounded solutions z^1, z^2 of (45). Then one can see that

$$\sup_{\mathbb{R}} \|z^1(t) - z^2(t)\| \leq \frac{2KL}{\sigma} \sup_{\mathbb{R}} \|z^1(t) - z^2(t)\|.$$

The last formula contradicts assumption $(C6)$. The theorem is proved. □

Let us make the following additional assumptions:

$(C9)$ the matrix $A(t)$ and the function $f(t,x,y)$ are periodic in t with a period ω;
$(C10)$ there exist a real number $\bar{\omega}$ and a positive integer p, such that $\theta_{i+p} = \theta_i + \bar{\omega}$, $i \in \mathbb{Z}$.

Theorem 5.2. *Suppose conditions* $(C1)$–$(C3)$*, and* $(C7)$–$(C10)$ *are valid. Moreover,*

$$\frac{\omega}{\bar{\omega}} = \frac{k}{m}, \quad k, m \in \mathbb{N}. \tag{48}$$

Then there exists an $m\omega$*-periodic solution of* (22)*.*

Indeed, one can verify that all approximations $z_m(t)$ are $m\omega$-periodic functions. The definition of $\kappa(t)$ implies that it is $m\omega$-periodic, too. The theorem is proved.

Example 5.1. Consider the following logistic equation:

$$N'(t) = N(t)[a(t) - f(N(\beta(t)))], \tag{49}$$

where $N, t \in \mathbb{R}$, $a(t)-$ is an ω-periodic, positive-valued, continuous function, $f(0) = 0$, $f(N) > 0$ if $N > 0$, and there exists a Lipschitz constant $l > 0$ such that $|f(N_1) - f(N_2)| \leq l|N_1 - N_2|$ for all N_1, N_2. Moreover, we assume that sequence θ_i satisfies condition $(C10)$.

Let m be the number from (48). We shall find conditions which guarantee the existence of an $m\omega$-periodic, nonnegative solution of (49).

If we denote $\sigma = \frac{1}{\omega}\int_0^\omega a(u)\mathrm{d}u > 0$ then there exists a constant $K \geq 1$ such that

$$\mathrm{e}^{\int_s^t a(u)\,\mathrm{d}u} \leq K\mathrm{e}^{\sigma(t-s)}, t \leq s. \tag{50}$$

Fix $H > 0$ and denote $G_H = \{N \in \mathbb{R} \mid 0 < N < H\}$. One can easily check that if $N_1, N_2, M_1, M_2 \in G_H$, then

$$|M_1 f(N_1) - M_2 f(N_2)| \leq lH(|N_1 - N_2| + |M_1 - M_2|).$$

Fix $\phi_0(t)$, an $m\omega$-periodic, positive and continuous function. The last theorem implies that if $\frac{2KHl}{\sigma} < 1, lH < 1$ and $\sigma > 2(1 - lH)$ then a sequence $\phi_n(t)$ of $m\omega$-periodic and positive functions

$$\phi_{n+1}(t) = \int_t^\infty \mathrm{e}^{\int_s^t a(u)\,\mathrm{d}u} f(\phi_n(\beta(s)))\,\mathrm{d}s,\ n \geq 0,$$

converges to the nonnegative $m\omega$-periodic solution of (49).

Remark 5.1. Almost periodic solutions of logistic EPCA were considered using the reduction to discrete equations in [27], and by applying an equivalent integral equation in [4]. Another interesting possibility is to consider the existence of quasiperiodic solutions [33] using the method of equivalent integral equations.

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