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Integral Operator of Analytic Functions with Positive Real Part

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Abstract. In this paper, we introduce the integral operator $I_\beta(p_1,...,p_n;\alpha_1,...,\alpha_n)(z)$ of analytic functions with positive real part. The radius of convexity of this integral operator when $\beta = 1$ is determined. In particular, we get the radius of starlikeness and convexity of the analytic functions with $Re\{f(z)/z\} > 0$ and $Re\{f'(z)\} > 0$. Furthermore, we derive sufficient condition for the integral operator $I_\beta(p_1,...,p_n;\alpha_1,...,\alpha_n)(z)$ to be analytic and univalent in the open unit disc, which leads to univalency of the operators $\int_0^z (f(t)/t)^\alpha \, dt$ and $\int_0^z (f'(t))^\alpha \, dt$.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. A function $f(z)$ belonging to $\mathcal{S}$ is said to be starlike if it satisfies

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathcal{U}) \tag{1.1}$$

We denote by $\mathcal{S}^*$ the subclass of $\mathcal{A}$ consisting of functions which are starlike in $\mathcal{U}$. Also, a function $f(z)$ belonging to $\mathcal{S}$ is said to be convex if it satisfies

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \qquad (z \in \mathcal{U}) \tag{1.2}$$

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We denote by $\mathcal{K}$ the subclass of $\mathcal{A}$ consisting of functions which are convex in $\mathcal{U}$.

Let $\alpha_i \in \mathbb{C}$ for all $i = 1, \ldots, n$, $n \in \mathbb{N}$, $\beta \in \mathbb{C}$ with $Re(\beta) > 0$. We let $I_\beta : A^n \to A$ be the integral operator defined by

$$I_\beta(p_1, ..., p_n; \alpha_1, ..., \alpha_n)(z) = \left\{ \int_0^z \beta t^{\beta-1} (p_1(t))^{\alpha_1} ... (p_n(t))^{\alpha_n} \, dt \right\}^{\frac{1}{\beta}}, \tag{1.3}$$

where $p_i(z)$ are analytic in $\mathcal{U}$ and satisfy $p_i(0) = 1$ for all $i = 1, \ldots, n$. Here and throughout in the sequel every many-valued function is taken with the principal branch.

Remark 1.1. Note that the integral operator $I_\beta(p_1, ..., p_n; \alpha_1, ..., \alpha_n)(z)$ generalizes many operators introduced and studied by several authors, for example:

(2) For $p_i(t) = \frac{D_\lambda^{m,\gamma} f_i(t)}{t}$; $1 \le i \le n$, we obtain the following integral operator introduced and studied by Bulut [7]

$$I_\beta^{m,\gamma}(f_1, ..., f_n)(z) = \left\{ \int_0^z \beta t^{\beta-1} \left(\frac{D_\lambda^{m,\gamma} f_1(t)}{t} \right)^{\alpha_1} \cdots \left(\frac{D_\lambda^{m,\gamma} f_n(t)}{t} \right)^{\alpha_n} dt \right\}^{\frac{1}{\beta}} \tag{1.4}$$

where $D_\lambda^{m,\gamma} f(\ z) = z + \sum_{n=2}^{\infty} \left(\frac{\Gamma(n+1)\Gamma(2-\gamma)}{\Gamma(n+1-\gamma)} (1 + (n-1))\lambda \right)^m a_n z^n$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the generalized Al-Oboudi operator [2].

(3) For $p_i(t) = \frac{f_i(t)}{t}$; $1 \le i \le n$, we obtain the integral operator

$$I_\beta(f_1, ..., f_n)(z) = \left\{ \int_0^z \beta t^{\beta-1} \left(\frac{f_1(t)}{t} \right)^{\alpha_1} \cdots \left(\frac{f_n(t)}{t} \right)^{\alpha_n} dt \right\}^{\frac{1}{\beta}} \tag{1.5}$$

introduced and studied by Breaz and Breaz [3].

(1) For $\beta = 1$ and $p_i(t) = \frac{(f_i * g_i)(t)}{t}$; $1 \le i \le n$, we obtain the integral operator

$$I(f_1, ..., f_n; g_1, ..., g_n)(z) = \int_0^z \left(\frac{(f_1 * g_1)(t)}{t} \right)^{\alpha_1} \cdots \left(\frac{(f_n * g_n)(t)}{t} \right)^{\alpha_n} dt \tag{1.6}$$

introduced and studied by Frasin [9].

(4) For $\beta = 1$ and $p_i(t) = \frac{f_i(t)}{t}$; $1 \le i \le n$, we obtain the integral operator

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t} \right)^{\alpha_1} \cdots \left(\frac{f_n(t)}{t} \right)^{\alpha_n} dt \tag{1.7}$$

introduced and studied by Breaz and Breaz [3].

(5) For $\beta = 1$ and $p_i(t) = f_i'(t)$; $1 \le i \le n$, we obtain the integral operator

$$F_{\alpha_1,\dots,\alpha_n}(z) = \int_0^z (f_1'(t))^{\alpha_1} \dots (f_n'(t))^{\alpha_n}\, dt \tag{1.8}$$

introduced and studied by Breaz *et al.* [5].

(6) For $\beta = 1$ and $p_i(t) = \frac{R^k f_i(t)}{t}$; $1 \le i \le n$, we obtain the integral operator introduced in [11]

$$I(f_1, ..., f_n)(z) = \int_0^z \left(\frac{R^k f_1(t)}{t}\right)^{\alpha_1} \dots \left(\frac{R^k f_n(t)}{t}\right)^{\alpha_n} dt, \tag{1.9}$$

where $R^k f(z) = z + \sum_{n=2}^{\infty} C_{k+n-1}^{k} a_n z^n$, $k \in \mathbb{N}_0$ is Ruscheweyh differential operator [17].

(7) For $\beta = 1$ and $p_i(t) = \frac{D^k f_i(t)}{t}$; $1 \le i \le n$, we obtain the integral operator introduced and studied by Breaz *et al.* [4]

$$D^k F(z) = \int_0^z \left(\frac{D^k f_1(t)}{t}\right)^{\alpha_1} \dots \left(\frac{D^k f_n(t)}{t}\right)^{\alpha_n} dt, \tag{1.10}$$

where $D^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n$, $k \in \mathbb{N}_0$ is Sălăgean differential operator [18].

(8) For $\beta = 1$ and $p_i(t) = \frac{D_\lambda^k f_i(t)}{t}$; $1 \le i \le n$, we obtain the following integral operator introduced and studied by Bulut [6]

$$I_n(f_1, ..., f_n)(z) = \int_0^z \left(\frac{D_\lambda^k f_1(t)}{t}\right)^{\alpha_1} \dots \left(\frac{D_\lambda^k f_n(t)}{t}\right)^{\alpha_n} dt, \tag{1.11}$$

where $D_\lambda^k f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)\lambda]^k a_n z^n$, $0 \le \lambda \le 1$, is Al-Oboudi differential operator [2] .

(9) For $\beta = 1$ and $p_i(t) = \frac{L(a,c) f_i(t)}{t}$; $1 \le i \le n$, we obtain the integral operator introduced and studied by Selvaraj and Karthikeyan [19]

$$F_\alpha(a, c; z) = \int_0^z \left(\frac{L(a,c) f_1(t)}{t}\right)^{\alpha_1} \dots \left(\frac{L(a,c) f_n(t)}{t}\right)^{\alpha_n} dt, \tag{1.12}$$

where $L(a,c)f(z) := z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} a_n z^n$ is the Carlson-Shaffer linear operator [8].

(5) For $\beta = 1$, $n = 1$, $\alpha_1 = \alpha$ and $p_1(t) = \frac{f(t)}{t}$, we obtain the integral operator

$$F_\alpha(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt \tag{1.13}$$

studied in [13]. In particular, for $\alpha = 1$, we obtain Alexander integral operator introduced in [1]

$$I(z) = \int_0^z \frac{f(t)}{t} dt \tag{1.14}$$

(6) For $\beta = 1$, $n = 1$, $\alpha_1 = \alpha$ and $p_1(t) = f'(t)$, we obtain the integral operator

$$G_\alpha(z) = \int_0^z (f'(t))^\alpha \, dt \tag{1.15}$$

studied in [15] (see also [16]) .

In the present paper, the radius of convexity of the integral operator defined by (1.3) when $\beta = 1$ are determined. In particular, we get the radius of starlikeness and convexity of the analytic functions with $Re\{f(z)/z\} > 0$ and $Re\{f'(z)\} > 0$ obtained by MacGregor [12]. Furthermore, we derive sufficient condition for the integral operator $I_\beta(p_1, ..., p_n; \alpha_1, ..., \alpha_n)(z)$ to be analytic and univalent in $\mathcal{U}$, which leads to univalency of the operators $\int_0^z (f(t)/t)^\alpha \, dt$ and $\int_0^z (f'(t))^\alpha \, dt$ obtained by Kim and Merkes [10] and Pfaltzgraff [16].

In the proofs of our main results we need the following lemmas

Lemma 1.2([12]). *Let $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be analytic in $\mathcal{U}$ and satisfy $p(0) = 1$ with $Re\{p(z)\} > 0$, then we have*

$$\left|\frac{zp'(z)}{p(z)}\right| < \frac{2|z|}{1 - |z|^2}, \qquad (z \in \mathcal{U}). \tag{1.16}$$

Lemma 1.3([14]). *Let $\beta \in \mathbb{C}$ with $Re(\beta) > 0$. If $f \in \mathcal{A}$ satisfies*

$$(1 - |z|^{2\mathrm{Re}(\beta)}) \left|\frac{zf''(z)}{f'(z)}\right| \leq \mathrm{Re}(\beta),$$

for all $z \in \mathcal{U}$, then the integral operator

$$F_\beta(z) = \left\{\beta \int_0^z t^{\beta-1} f'(t) dt\right\}^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

2. Convexity of $I_1(p_1, ..., p_n; \alpha_1, ..., \alpha_n)(z)$

In this section, we obtain the radius of convexity of the integral operator $I_\beta(p_1, ..., p_n; \alpha_1, ..., \alpha_n)(z)$ defined by (1.3) when $\beta = 1$.

Theorem 2.1. *Let* $p_i(0) = 1$ *and* $Re\{p_i(z)\} > 0$ *for all* $i = 1, \ldots, n$. *Then the integral operator defined by*

$$I_1(z) = \int_0^z \prod_{i=1}^{n} (p_i(t))^{\alpha_i}\, dt \tag{2.1}$$

is convex in $|z| = r < \sqrt{\left(\sum_{i=1}^{n} \alpha_i\right)^2 + 1} - \sum_{i=1}^{n} \alpha_i$; *where* $\alpha_i > 0$ *for all* $i = 1, \ldots, n$.

Proof. From (2.1), it is easy to see that $I_1(0) = I_1'(0) - 1 = 0$ and

$$I_1'(z) = \prod_{i=1}^{n} (p_i(z))^{\alpha_i}. \tag{2.2}$$

Differentiate (2.2) logarithmically with respect with z, we obtain

$$\frac{zI_1''(z)}{I_1'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zp_i'(z)}{p_i(z)}\right)$$

or, equivalently,

$$1 + \frac{zI_1''(z)}{I_1'(z)} = 1 + \sum_{i=1}^{n} \alpha_i \left(\frac{zp_i'(z)}{p_i(z)}\right). \tag{2.3}$$

Now, using the estimate (1.16) in Lemma 1.2, from (2.3) it follows that

$$\begin{aligned} Re\left\{1 + \frac{zI_1''(z)}{I_1'(z)}\right\} &= 1 + Re \sum_{i=1}^{n} \alpha_i \left(\frac{zp_i'(z)}{p_i(z)}\right) \\ &\geq 1 - \left|\sum_{i=1}^{n} \alpha_i \left(\frac{zp_i'(z)}{p_i(z)}\right)\right| \\ &\geq 1 - \left(\frac{2|z|}{1 - |z|^2}\right) \sum_{i=1}^{n} \alpha_i, \qquad (|z| < 1). \end{aligned}$$

If $|z| < \sqrt{\left(\sum_{i=1}^{n} \alpha_i\right)^2 + 1} - \sum_{i=1}^{n} \alpha_i$, then $Re\left\{1 + \frac{zI_1''(z)}{I_1'(z)}\right\} > 0$. Therefore, $I_1(z)$ is convex in $|z| < \sqrt{\left(\sum_{i=1}^{n} \alpha_i\right)^2 + 1} - \sum_{i=1}^{n} \alpha_i$. This proves the theorem. □

Letting $n = 1$, $\alpha_1 = 1$ and $p_1 = p$ in Theorem 2.1, we have

Corollary 2.2. *Let* $p(0) = 1$ *and* $Re\{p(z)\} > 0$. *Then the integral operator* $\int_0^z p(t)dt$ *is convex in* $|z| < \sqrt{2} - 1$.

Letting $p(z) = f(z)/z$ and $p(z) = f'(z)$ in Corollary 2.2, we get the following interesting results due to MacGregor [12].

Corollary 2.3. *Let* $f(z) \in \mathcal{A}$. *If* $Re\{f(z)/z\} > 0$ *for* $|z| < 1$, *then* $f(z)$ *is starlike in* $|z| < \sqrt{2}-1$. *The result is sharp for the extremal function* $f(z) = (z+z^2)/(1-z^2)$.

Corollary 2.4. *Let* $f(z) \in \mathcal{A}$. *If* $Re\{f'(z)\} > 0$ *for* $|z| < 1$, *then* $f(z)$ *is convex in* $|z| < \sqrt{2}-1$. *The result is sharp for the extremal function* $f(z) = \int_0^z (1+t)/(1-t)dt$.

Remark 2.5. Taking different choices of $p_i(z)$ as stated in Section 1, Theorems 2.1 leads to new radius of convexity for the integral operators defined in Section 1 when $\beta = 1$.

3. Univalency of $I_\beta(p_1, ..., p_n; \alpha_1, ..., \alpha_n)(z)$

Next, we obtain the following sufficient condition for the integral operator $I_\beta(p_1, ..., p_n; \alpha_1, ..., \alpha_n)(z)$ to be analytic and univalent in $\mathcal{U}$.

Theorem 3.1. *Let* $\alpha_i \in \mathbb{C}$ *for all* $i = 1, \ldots, n$ *and* $\beta \in \mathbb{C}$ *with* $Re(\beta) = a$. *If*

$$\sum_{i=1}^{n} |\alpha_i| \leq \begin{cases} a/2 & \text{if } 0 < a \leq 1/2 \\ 1/4 & \text{if } a \geq 1/2. \end{cases} \tag{3.1}$$

Then the integral operator $I_\beta(p_1, ..., p_n; \alpha_1, ..., \alpha_n)(z)$ *defined by (1.3) is analytic and univalent in* $\mathcal{U}$, *where* $p_i(z)$ *are analytic in* $\mathcal{U}$ *and satisfy* $p_i(0) = 1$*with* $Re\{p_i(z)\} > 0$ *for all* $i = 1, \ldots, n$.

Proof. Define

$$h(z) = \int_0^z \prod_{i=1}^{n} (p_i(t))^{\alpha_i} dt,$$

so that, obviously,

$$h'(z) = \prod_{i=1}^{n} (p_i(z))^{\alpha_i}. \tag{3.2}$$

Making use of the logarithmic differentiation on both sides of (3.2) and multiplying by z, we have

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zp_i'(z)}{p_i(z)} \right). \tag{3.3}$$

From (1.16) and (3.3), we obtain

$$\begin{aligned}(1-|z|^{2a}) \left| \frac{zh''(z)}{h'(z)} \right| &\leq (1-|z|^{2a}) \sum_{i=1}^{n} |\alpha_i| \left| \frac{zp_i'(z)}{p_i(z)} \right| \\ &\leq (1-|z|^{2a}) \left(\frac{2\,|z|}{1-|z|^2} \right) \sum_{i=1}^{n} |\alpha_i| .\end{aligned}$$

Since $\frac{2|z|}{1-|z|^2} \leq \frac{2}{1-|z|}$ for $z \in \mathcal{U}$, we have

$$(1-|z|^{2a}) \left| \frac{zh''(z)}{h'(z)} \right| \leq \left(\frac{2(1-|z|^{2a})}{1-|z|} \right) \sum_{i=1}^{n} |\alpha_i| .$$

Define the function $\Psi : (0,1) \to \mathbb{R}$ by

$$\Psi(x) = \frac{2(1-x^{2a})}{1-x}, \quad (a > 0, x = |z|).$$

It is easy to show that

$$\Psi(x) \leq \begin{cases} 2 & \text{if } \ 0 < a \leq 1/2 \\ 4a & \text{if } a \geq 1/2. \end{cases} \tag{3.4}$$

We thus find from (3.4) and the hypothesis (3.1) that

$$(1-|z|^{2a}) \left| \frac{zh''(z)}{h'(z)} \right| \leq a, \qquad (z \in \mathcal{U}).$$

Applying Lemma 1.3 for the function $h(z)$ with $\text{Re}(\beta) = a$, we prove that $I_\beta(p_1, ..., p_n; \alpha_1, ..., \alpha_n)(z) \in \mathcal{S}$. This evidently completes the proof of Theorem 3.1. □

Letting $n = 1$, $\alpha_1 = \alpha$, $p_1 = p$ in Theorem 3.1, we have

Corollary 3.2. *Let $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$ with Re$(\beta) = a$. If*

$$|\alpha| \leq \begin{cases} a/2 & \text{if } \ 0 < a \leq 1/2 \\ 1/4 & \text{if } a \geq 1/2. \end{cases} \tag{3.5}$$

Then the integral operator $I_\beta(p;\alpha)(z) = \left\{ \int_0^z \beta t^{\beta-1} \left(p(t)\right)^\alpha dt \right\}^{\frac{1}{\beta}}$ *is analytic and univalent in $\mathcal{U}$, where $p(z)$ is analytic in $\mathcal{U}$ and satisfy $p(0) = 1$, Re$\{p(z)\} > 0$.*

Letting $\beta = 1$ in Corollary 3.2, we have

Corollary 3.3. *Let $p(z)$ be analytic in $\mathcal{U}$ and satisfy $p(0) = 1$, $Re\{p(z)\} > 0$. Then the integral operator $\int_0^z (p(t))^\alpha \, dt$ is analytic and univalent in $\mathcal{U}$, where $|\alpha| \le 1/4$; $\alpha \in \mathbb{C}$.*

Letting $p(z) = f(z)/z$ in Corollary 3.3, we get the following result obtained by Kim and Merkes [10].

Corollary 3.4. *Let $f \in \mathcal{S}$, and $\alpha \in \mathbb{C}$; $|\alpha| \le 1/4$. Then the integral operator $\int_0^z (f(t)/t)^\alpha \, dt$ is analytic and univalent in $\mathcal{U}$.*

Letting $p(z) = f'(z)$ in Corollary 3.3, we get the following result obtained by Pfaltzgraff [16].

Corollary 3.5. *Let $f \in \mathcal{S}$, and $\alpha \in \mathbb{C}$; $|\alpha| \le 1/4$. Then the integral operator $\int_0^z (f'(t))^\alpha \, dt$ is analytic and univalent in $\mathcal{U}$.*

Remark 3.6. Taking different choices of $p_i(z)$, where $p_i(z)$ as stated in Section 1, Theorem 3.1 leads to new sufficient conditions for the integral operators defined in Section 1 to be analytic and univalent in $\mathcal{U}$.

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