Integral operators related to symplectic matrices

TECHNISCHE HOGESCHOOL EINDHOVEN

NEDERLAND

ONDERAFDELING DER WISKUNDE

TECHNOLOGICAL UNIVERSITY EINDHOVEN

THE NETHERLANDS

DEPARTMENT OF MATHEMATICS

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by

G.H.M. Frederix

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A. Abstract

In de Bruijn's theory of generalized functions and Wigner distributions there are a few points where generalization to the case of more than one variable presents an essential difficulty:

i) The integral transforms (possibly degenerated) corresponding to fractional linear transforms (see [B], section 27.3) and the induced transformations of the phase plane (see [B], section 27.12.2).
ii) The description of the classical limit of quantum mechanics by means of the Wigner distribution.

In both cases the 2×2 matrices with determinant 1 have to be replaced by $2n \times 2n$ symplectic matrices. The various difficulties that arise are solved in this report. As to point ii), this enables us to deal with the canonical transformations of linearized Hamilton systems.

AMS Subject Classifications: 15A21, 15A57, 46F05, 46F10.

B. Notation

We use Church's lambda calculus notation, but instead of his λ we have Ψ, as suggested by Freudenthal: if S is a set, then putting $\Psi_{x \in S}$ in front of an expression (usually containing x) means to indicate the function with domain S and with the function values given by the expression. We write Ψ_x instead of $\Psi_{x \in S}$, if it is clear from the context which set S is meant.

In this paper the symbol $\mathbb{R}$ is used for the set of all real numbers, and we use the symbol $\mathbb{C}$ for the set of all complex numbers. We use the symbol $M_{n\times m}(\mathbb{R})$ ($M_{n\times m}(\mathbb{C})$) to indicate the set of all matrices with n rows and m columns (n and m are positive integers) which have real (complex) entries. If $z \in \mathbb{C}$ then Re z (Im z) denotes the real (imaginary) part of z. If $T \in M_{n\times m}(\mathbb{C})$, we define Re(T) and Im(T) as the real matrices with

$$T = \mathrm{Re}(T) + i\ \mathrm{Im}(T) .$$

The overhead bar is used for complex conjugates for elements of $\mathbb{C}$ as well for elements of $M_{n\times m}(\mathbb{C})$. We shall write $\mathbb{N}$ for the set of all positive integers.

If $n \in \mathbb{N}$, and if x and y are elements of $\mathbb{C}^n$ with components $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$, we define

$$(x,y) := \sum_{i=1}^{n} x_i y_i .$$

Note that this is a bilinear form in x and y.

In general, we shall denote matrices by capitals. For instance, if $n \in \mathbb{N}$, we shall denote the unit matrix in $M_{n\times n}(\mathbb{C})$ by I_n. When dealing however with elements of $M_{n\times 1}(\mathbb{R})$ (or $M_{n\times 1}(\mathbb{C})$) we use lower case characters, and we shall identify elements of $\mathbb{R}^n$ (or $\mathbb{C}^n$) with the corresponding elements of $M_{n\times 1}(\mathbb{R})$ (or $M_{n\times 1}(\mathbb{C})$).

We shall indicate the transpose of a matrix $A \in M_{n\times m}(\mathbb{C})$ ($n \in \mathbb{N}$, $m \in \mathbb{N}$) by A^T and the complex conjugate of this transpose matrix by A^H. If A satisfies $A = A^T$, then A will be called symmetric, and if A satisfies $A = A^H$, it will be called hermitian.

If $n \in \mathbb{N}$ and $A \in \mathcal{M}_{n\times n}(\mathbb{C})$, then the formula $A > 0$ indicates that $A = A^H$ and that A is a matrix of a positive definite form, i.e. $x^H Ax > 0$ for every nonzero $x \in \mathcal{M}_{n\times 1}(\mathbb{C})$. (If A is real, it suffices to require that $A = A^T$ and $x^T Ax > 0$ for every nonzero $x \in \mathcal{M}_{n\times 1}(\mathbb{R})$). We express this by saying that A is positive definite. If B belongs to $\mathcal{M}_{n\times n}(\mathbb{C})$ too, then the formula $A > B$ indicates that $A = A^H$, $B = B^H$ and $A - B > 0$. The formulas $A \geq 0$ (positive semidefinite) and $A \geq B$ indicate the obvious analogy of the formulas $A > 0$ and $A > B$.

If $n \in \mathbb{N}$, $m \in \mathbb{N}$ and $Z \in \mathcal{M}_{n\times m}(\mathbb{C})$, we shall say that Z has block components $A \in \mathcal{M}_{n_1\times m_1}(\mathbb{C})$, $B \in \mathcal{M}_{n_1\times m_2}(\mathbb{C})$, $C \in \mathcal{M}_{n_2\times m_1}(\mathbb{C})$ and $D \in \mathcal{M}_{n_2\times n_2}(\mathbb{C})$, if $n_1 + n_2 = n$, $m_1 + m_2 = m$, and if

$$Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} .$$

If $n_1 = n_2$, then (A,B) and (C,D) will be called the first and the second matrix row of Z respectively. In case that $B = 0$ and $C = 0$, and $n_1 = m_1$, $n_2 = m_2$, we shall call Z block diagonal with blocks A and D, and this will be denoted by $Z = \operatorname{diag}(A,D)$. These notations (with the proper modifications) will be used in case of more than two blocks.

0. Introduction

0.1. We give a short survey of the fundamental notions and theorems of de Bruijn's theory of generalized functions as far as relevant for this report. A detailed treatment can be found in [B].

0.2. If A and B are positive numbers and if n is a positive integer we denote by $S^n_{A,B}$ the class of everywhere defined analytic functions f of n complex variables $z_1,\dots,z_n$ for which there exists a positive number M such that

$$|f(z)| \leq M \exp(-\pi A(\mathrm{Re}\, z,\mathrm{Re}\, z) + \pi B(\mathrm{Im}\, z,\mathrm{Im}\, z))$$

for every $z \in \mathbb{C}^n$. The set S^n of <u>smooth functions</u> of n complex variables is defined by

$$S^n := \bigcup_{A>0} \bigcup_{B>0} S^n_{A,B}$$

(see [B], 2.1).

0.3. In S^n we take the usual inner product and norm:

$$[f,g]_n := \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx \quad (f \in S^n,\ g \in S^n) ,$$

$$\|f\|_n := ([f,f]_n)^{\frac{1}{2}} \quad (f \in S^n) .$$

0.4. Let $n \in \mathbb{N}$. We consider a semigroup $(N_{\alpha,n})_{\alpha>0}$ of linear operators of S^n (the <u>smoothing operators</u>). The $N_{\alpha,n}$'s satisfy

$$N_{\alpha+\beta,n} = N_{\alpha,n}N_{\beta,n} \quad (\alpha > 0,\ \beta > 0) ,$$

where the product is the usual composition of mappings. These operators are defined as integral operators:

$$N_{\alpha,n} := \psi_{f\in S^n} \psi_{z\in\mathbb{C}^n} \int_{\mathbb{R}^n} K_{\alpha,n}(z,t)f(t)dt ,$$

where the kernel $K_{\alpha,n}$ is given by

$$K_{\alpha,n} := \Psi_{(z,t)\in\mathbb{C}^n\times\mathbb{C}^n}\ \sinh\alpha^{-\frac{n}{2}}\exp\left(-\frac{\pi}{\sinh\alpha}\{((z,z)+(t,t))\cosh\alpha - 2(z,t)\}\right).$$

(See [B], section 3,4,5 and 6.)

0.5. We summarize a number of properties of the $(N_{\alpha,n})_{\alpha>0}$.

i) $N_{\alpha,n}f = N_{\alpha,1}^{(1)}\, N_{\alpha,1}^{(2)}\, N_{\alpha,1}^{(3)} \dots N_{\alpha,1}^{(n)}f$ ($\alpha > 0$, $f \in S^n$), where $N_{\alpha,1}^{(i)}g$ denotes

$$\Psi_{(z_1,\dots,z_n)\in\mathbb{C}^n}\ N_{\alpha,1}(\Psi_{z_i}\, g(z_1,\dots,z_{i-1},z_i,z_{i+1},\dots,z_n))$$

$(1 \le i \le n,\ \alpha > 0,\ g \in S^n)$.

ii) $N_{\alpha,n}$ is symmetric, i.e.

$$[N_{\alpha,n}f,g]_n = [f,N_{\alpha,n}g]_n \qquad (\alpha > 0,\ f \in S^n,\ g \in S^n) .$$

See [B], 6.5.

iii) If $f \in S^n$ and $\alpha > 0$, then there is atmost one $g \in S^n$ such that $f = N_{\alpha,n}g$. In addition, if $f \in S^n$, then there exists an $\alpha > 0$ and a $g \in S^n$ such that $f = N_{\alpha,n}g$. This is a modification for the more dimensional case of [B], 10.1.

0.6. We give a number of examples of linear operators of S^n.

i) The smoothing operators $N_{\alpha,n}$ $(\alpha > 0)$.

ii) The Fourier transform F_n defined by

$$F_n := \Psi_{f\in S^n}\ \Psi_{z\in\mathbb{C}^n} \int_{\mathbb{R}^n} \exp(-2\pi i(z,t))f(t)dt .$$

iii) The shift operators T_a and R_b $(a \in \mathbb{C}^n,\ b \in \mathbb{C}^n)$ defined by

$$T_a := \Psi_{f\in S^n}\ \Psi_{z\in\mathbb{C}^n}\ f(z+a)$$

and

$$R_b := \Psi_{f\in S^n}\ \Psi_{z\in\mathbb{C}^n}\ \exp(-2\pi i(z,b))f(z) .$$

iv) The operators P_i and Q_i $(1 \le i \le n)$ defined by

$$P_i := \Psi_{f\in S^n} \Psi_{(z_1,z_2,\ldots,z_n)\in\mathbb{C}^n} \frac{1}{2\pi i}\left(\frac{\partial}{\partial z_i} f\right)(z_1,\ldots,z_n)$$

and

$$Q_i := \Psi_{f\in S^n} \Psi_{(z_1,\ldots,z_n)\in\mathbb{C}^n} z_i f(z_1,\ldots,z_n) \ .$$

For properties of these operators, see [B], section 8, 9 and 11, as far as the case $n = 1$ is concerned (in the general case we have similar properties).

0.7. A generalized function F of n variables ($n \in \mathbb{N}$) is a mapping of the set of positive numbers into S^n such that

$$N_{\alpha,n}F(\beta) = F(\alpha + \beta) \qquad (\alpha > 0,\ \beta > 0) \ .$$

(We also write $N_{\alpha,n}F$ instead of $F(\alpha)$ for $F \in S^{n*}$ and $\alpha > 0$).
The set of all generalized functions of n variables is denoted by S^{n*}. If $f \in S^n$, then its standard embedding in S^{n*} is defined by

$$\mathrm{emb}(f) := \Psi_{\alpha>0} N_{\alpha,n}f \ .$$

(See [B], section 17).

If $F \in S^{n*}$ and $g \in S^n$, then we can define the inner product $[F,g]_n$: write $g = N_{\alpha,n}h$ with some $\alpha > 0$ and $h \in S^n$ (see 0.5 iii)). Now $[F,g]_n := [F(\alpha),h]_n$ (this depends only on F and g: see [B], section 18).

0.8. Let T be a linear operator on S^n and assume that there exists a family $(Y_\alpha)_{\alpha>0}$ of linear mappings of $N_{\alpha,n}(S^{n*})$ into S^n such that

i) $\qquad Y_{\alpha+\beta}N_{\alpha+\beta,n}F = N_{\beta,n}Y_\alpha N_{\alpha,n}F$,

ii) $\qquad Y_\alpha N_{\alpha,n}f = N_{\alpha,n}Tf$

for every $\alpha > 0$, $\beta > 0$, $F \in S^{n*}$, $f \in S^n$.

We call T extendable by means of $(Y_\alpha)_{\alpha>0}$ (see [B], 19.3 and 19.4): It is possible to define a linear operator $\tilde{T}$ on S^{n*} such that $\tilde{T}\,\mathrm{emb}(f) = \mathrm{emb}(Tf)$ ($f \in S^n$). This $\tilde{T}$ is defined by

$$\tilde{T} := \Psi_{F\in S^{n*}} \Psi_{\alpha>0}\, Y_\alpha N_{\alpha,n} F$$

(see [B], 19.2).

There is another way to describe the extension of linear operators of S^n to linear operators of S^{n*}, as was investigated in [J] (see [J], appendix 1, section 3). The main result there is that a linear operator T of S^n is extendable to a linear operator T of S^{n*} such that T(emb(f)) = emb(Tf) for $f \in S^n$, if T has an adjoint. (In many cases this extension coincides with the one given above.) Although it may be proved that all operators investigated in the second chapter have an adjoint, we shall not use the theorem mentioned, since it seems to be more convenient in this report to have explicit formulas (given by the Y_α's) for the extended operators.

0.9. Convergence in S^n ($n \in \mathbb{N}$).
If $(f_m)_{m\in\mathbb{N}}$ is a sequence in S^n, then we write $f_m \overset{S^n}{\to} 0$ $(m \to \infty)$ if there are positive numbers A and B such that

$$f_m(z)\exp(\pi A(\mathrm{Re}\, z,\mathrm{Re}\, z) - \pi B(\mathrm{Im}\, z,\mathrm{Im}\, z)) \to 0$$

uniformly in $z \in \mathbb{C}^n$. If $f \in S^n$, then $f_m \to f$ $(m \to \infty)$ indicates that $f_m - f \to 0$ $(m \to \infty)$. For more details see [B], section 23.

0.10. Consider the set G_0 of matrices $Z \in M_{3\times 3}(\mathbb{C})$ which have the form

$$Z = \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix},$$

where a,b,c,d,e and f are complex numbers such that ad - cb = 1, and such that the mapping

0.10.1. $$\Psi_{t\in\mathbb{C},\mathrm{Re}(t)>0}\ \frac{at+b}{ct+d}$$

is well defined and maps the right-half plane into itself (see [B], 27.3.1). It is not hard to see that G_0 is a semigroup under matrix multiplication.

If $Z \in G_0$, we define an operator Γ of S related to Z as follows:

i) if $c \neq 0$

$$\Gamma = \Psi_{g\in S}\ \Psi_{z\in\mathbb{C}} \int_{-\infty}^{+\infty} c^{-\frac{1}{2}} \exp\left(-\pi\, \frac{az^2-2zt+dt^2+2(ce-af)z+2ft+af^2-cef}{c}\right) g(t)\,dt,$$

where $c^{-\frac{1}{2}}$ is a (properly chosen) complex number the square of which equals c^{-1},

ii) if $c = 0$

$$\Gamma = \Psi_{g\in S}\, \Psi_z\, d^{-\frac{1}{2}} g\left(\frac{z-f}{d}\right) \exp\left(-\pi\, \frac{bz^2 + 2(ed - bf)z + bf^2 - def}{d}\right) ,$$

where $d^{-\frac{1}{2}}$ denotes a complex number the square of which equals d. (There is a slight inconvenience caused by multivaluedness of square roots.) (See [B], 27.3.8 and 27.3.9.)

An important property of these operators is multiplicativity: if we compose two operators related to $Z_1 \in G_0$ and $Z_2 \in G_0$ respectively, we obtain an operator related to the product of Z_1 and Z_2.

Note that the operators $N_{\alpha,1}$, T_a, R_b and F_1 are obtained by suitable choices of the matrix $Z \in G_0$ ($\alpha > 0$, $a \in \mathbb{C}$, $b \in \mathbb{C}$). See [B], 27.3.

Several notions and theorems which are discussed in [B], 27.3 for the one dimensional case are generalized in this report to the more dimensional case. In chapter one we shall generalize the notion of fractional linear transform of 0.10.1, and in chapter two we shall be concerned with the generalization of the operators mentioned above.

1. Higher dimensional fractional linear transforms

1.0. Introduction

1.0.1. In this chapter we intend to develop a theory of higher dimensional fractional linear transforms of the right-half plane (higher dimensional means here that we consider square matrices instead of complex numbers). For the one dimensional case (i.e. if we take one-by-one matrices), such a theory is available (we refer to [B], 27.3). We wish to generalize the results of this theory to the higher dimensional case.

1.0.2. A non-trivial fractional linear transform is usually described by a matrix

$$Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a,b,c and d are complex numbers such that $ad - bc \neq 0$. If $ct + d \neq 0$ for every $t \in \mathbb{C}$ with $\mathrm{Re}(t) > 0$, we can relate a fractional linear transform to Z, viz. the mapping

$$\Psi_{t,\mathrm{Re}(t)>0} \frac{at + b}{ct + d}.$$

It is not hard to see that it suffices to consider matrices $Z \in M_{2\times 2}(\mathbb{C})$ with $\det(Z) = 1$.

1.0.3. One of our aims is to generalize the notion of right half-plane as well as the notion of fractional linear transform to the case that a,b,c,d and t are square matrices A,B,C,D and T instead of complex numbers. It will become clear that in this case we have to take symplectic matrices

$$Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A,B,C and D are elements of $M_{n\times n}(\mathbb{C})$ and that T is a symmetric matrix with positive definite real part. (See [M], § 4 and [S], chapter II).

1.0.4. We introduce the sets $G_0(n)$, $G_1(n)$ and $G_2(n)$ $(n \in \mathbb{N})$ in a way similar to the definitions of the sets G_0, G_1 and G_2 in [B] (see [B], 27.3). Many results obtained in [B] can be generalized to the case of more dimensions. We mention in particular that higher dimensional fractional linear transform related to elements of $G_1(n)$ map the generalized right half-plane onto itself, and that higher dimensional fractional linear transforms related to elements of $G_2(n)$ have a unique fixed point.

1.0.5. In the one dimensional case our theory coincides for a great deal with that of [B], 27.3, but in this case our theory gives a more unified approach for the cases that where treated seperatedly in [B].

1.0.6. The set $G_1(n)$ $(n \in \mathbb{N})$ was discussed before (although in a different form) in [M] an [S]. The authors used the generalized upper half-plane and the set of all symplectic matrices with real entries.

1.1. Preliminaries

In this section we shall give some introductory remarks and definitions that will be used throughout the whole report.

1.1.1. Definition. A matrix A is called symplectic of order n $(n \in \mathbb{N})$, if $Z \in M_{2n\times 2n}(\mathbb{C})$ and if Z satisfies

1.1.1.1. $$Z^T J_n Z = J_n ,$$

where

$$J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} .$$

(See [M], p. 31).

1.1.2. Let Z be symplectic of order n $(n \in \mathbb{N})$ and let $A,B,C,D \in M_{n\times n}(\mathbb{C})$ denote the block components of Z, i.e.

$$Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} .$$

Then it follows from 1.1.1.1 that

1.1.2.1. $$A^T D - C^T B = I_n,\ A^T C = C^T A,\ B^T D = D^T B .$$

Since J_n is non-singular, we obtain from 1.1.1.1 that Z is non-singular and that

1.1.2.2. $$Z^{-T}J_nZ^{-1} = J_n\ ,$$

hence Z^{-1} is symplectic. Furthermore, from $J_n^{-1} = -J_n$ we see by using 1.1.2.2 that

$$ZJ_nZ^T = J_n\ ,$$

so Z^T is also symplectic. If we apply 1.1.1.1 to Z^T we obtain

1.1.2.3. $$AB^T = BA^T,\ CD^T = DC^T,\ AD^T - BC^T = I_n\ .$$

Using 1.1.2.1 and 1.1.2.3 we easily see that

1.1.2.4. $$Z^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}.$$

1.1.2.5. For later references we mention the following property. If C is non-singular, then we conclude from the fact that A^TC and CD^T are symmetric that AC^{-1} and $C^{-1}D$ are also symmetric.

We shall denote the set of symplectic matrices of order n ($n \in \mathbb{N}$) by Sp(n). Clearly Sp(n) is a group under matrix multiplication. In case n = 1, Sp(n) equals SL(2) (the special linear group of order two, i.e. the set of all matrices in $M_{2\times 2}(\mathbb{C})$ with determinant one).

Let C_n denote the set of all symmetric matrices in $M_{n\times n}(\mathbb{C})$ ($n \in \mathbb{N}$) and let H_n denote the subset of C_n consisting of all $T \in C_n$ such that

$$\mathrm{Re}(T) > 0\ .$$

Note that we consider symmetric (instead of hermitian) matrices with complex entries.

We can consider C_n as a generalization of the complex plane to dimension n. From this point of view, H_n is the generalization of the right half-plane. C_n can be mapped one-to-one onto $\mathbb{C}^{\frac{1}{2}n(n+1)}$ in an obvious way and so C_n is endowed with a topology which is equivalent with the ordinary topology in $\mathbb{C}^{\frac{1}{2}n(n+1)}$.

1.1.3. For every $T \in \mathcal{H}_n$ $(n \in \mathbb{N})$ we have

$$T = \mathrm{Re}(T) + i\ \mathrm{Im}(T) ,$$

where Re(T) is positive definite and Im(T) is symmetric. Let $T \in \mathcal{H}_n$. If we abbreviate $X = \mathrm{Re}(T)$, $Y = \mathrm{Im}(T)$, then

$$X + YX^{-1}Y$$

is positive definite, hence

1.1.3.1. $$W := (I_n - iX^{-1}Y)(X + YX^{-1}Y)^{-1}$$

is well defined and satisfies $TW = I_n$. So every $T \in \mathcal{H}_n$ is non-singular and since $\mathrm{Re}(W) > 0$ we have $T^{-1} \in \mathcal{H}_n$. (Compare 1.1.3.1 with the expression for the inverse of a number $x + iy \in \mathbb{C}$ $(x \in \mathbb{R},\ y \in \mathbb{R})$ satisfying $x > 0$).

1.1.4. Let $Z \in Sp(n)$ $(n \in \mathbb{N})$ and let A,B,C,D denote the block components of Z (according to 1.1.2). If $CT + D$ is non-singular for every $T \in \mathcal{H}_n$, we can relate a mapping $\mathcal{L}_Z$ to Z defined by

$$\mathcal{L}_Z := \forall_{T \in \mathcal{H}_n} (AT + B)(CT + D)^{-1} .$$

We shall call this mapping the <u>(n-dimensional) fractional linear transform related to Z</u>. In the following, the phrase "$\mathcal{L}_Z$ is well defined" will mean that $CT + D$ is non-singular on $\mathcal{H}_n$.

If $S \in \mathcal{H}_n$, then we have $\mathcal{L}_Z(S) = (\mathcal{L}_Z(S))^T$, for we have by 1.1.2.1

$$(SC^T + D^T)(AS + B) = (SA^T + B^T)(CS + D) .$$

This means that $\mathcal{L}_Z(S) \in \mathcal{C}_n$.

1.1.4.1. Let $Z \in Sp(n)$ and assume that $\mathcal{L}_Z$ is well defined. If $T_1 \in \mathcal{H}_n$ and $T_2 \in \mathcal{H}_n$ satisfy

$$\mathcal{L}_Z(T_1) = \mathcal{L}_Z(T_2) ,$$

then, from

$$\mathcal{L}_Z(T_1) = (\mathcal{L}_Z(T_1))^T ,$$

we have

$$(T_1C^T + D^T)(AT_2 + B) = (T_1A^T + B^T)(CT_2 + D)$$

and, using the relations of 1.1.2.1, we infer that $T_1 = T_2$, hence $\mathcal{L}_Z$ is one-to-one.

1.1.4.2. Let $Z \in Sp(n)$ $(n \in \mathbb{N})$ and assume that $\mathcal{L}_Z$ is well defined. Since $\mathcal{L}_Z$ is one-to-one (see 1.1.4.1), $\mathcal{L}_Z$ has an inverse defined on the set $\mathcal{L}_Z(H_n)$. Assume that A,B,C and $D \in M_{n\times n}(\mathbb{C})$ denote the block components of Z, then

$$\forall_{S\in\mathcal{L}_Z(H_n)} \quad (D^TS - B^T)(-C^TS + A^T)^{-1}$$

is well defined and this mapping is the inverse of $\mathcal{L}_Z$ on $\mathcal{L}_Z(H_n)$. This can easily be seen as follows: Let $S \in \mathcal{L}_Z(H_n)$, then there exists exactly one $T \in H_n$ (see 1.4.1.1) such that

$$S = (AT + B)(CT + D)^{-1} \ .$$

Using 1.1.2.1, we have

$$-C^TS + A^T = (A^TD - C^TB)(CT + D)^{-1} = (CT + D)^{-1} \ ,$$

so $-C^TS + A^T$ is non-singular for every $S \in \mathcal{L}_Z(H_n)$. Using 1.1.2.1 and 1.1.2.3 it is easily seen that the mapping

$$\forall_{S\in\mathcal{L}_Z(H_n)} \quad (D^TS - B^T)(-C^TS + A^T)^{-1}$$

is the inverse of $\mathcal{L}_Z$ on $\mathcal{L}_Z(H_n)$.

1.1.4.3. If $Z_1 \in Sp(n)$ and $Z_2 \in Sp(n)$ $(n \in \mathbb{N})$ are such that $\mathcal{L}_{Z_1}$ and $\mathcal{L}_{Z_2}$ are well defined, and if furthermore $\mathcal{L}_{Z_1}(T) \in H_n$ for every $T \in H_n$, then $\mathcal{L}_{Z_2Z_1}$ is well defined and

$$\mathcal{L}_{Z_2}(\mathcal{L}_{Z_1}(T)) = \mathcal{L}_{Z_2Z_1}(T) \qquad (T \in H_n) \ .$$

This is easily seen by a matrix computation and by using the general properties of symplectic matrices (see 1.1.1 and 1.1.2).

1.1.4.4. We are now able to motivate why we consider symplectic matrices to define more dimensional fractional linear transforms.

Since we intend to define a mapping of the kind of $\mathcal{L}_Z$ that maps $\mathcal{H}_n$ into $\mathcal{C}_n$, the image of every $S \in \mathcal{H}_n$ under this mapping has to be symmetric, so

$$(SC^T + D^T)(AS + B) = (SA^T + B^T)(CS + D)$$

for every $S \in \mathcal{H}_n$. Therefore we have

$$S(C^TA - A^TC)S + D^TB - B^TD = S(A^TD - C^TB) + (D^TA - B^TC)S$$

for every $S \in \mathcal{H}_n$, and it is easily seen that this implies $C^TA = A^TC$, $B^TD = D^TB$, and that $A^TD - C^TB$ is a diagonal matrix. Considering a one dimensional fractional linear transform $\Psi_t(at+b)(ct+d)$, we see that we can restrict ourselves to complex numbers a,b,c and d with $ad - bc = 1$. The relation $A^TD - C^TB = I_n$ can be seen as a generalization of the relation $ad - bc = 1$.

There is another reason why we consider symplectic matrices. A linear transformation of $\mathbb{R}^{2n}$ ($n \in \mathbb{N}$) given by

$$\binom{q'}{p'} = Z\binom{q}{p} \qquad (q,q',p,p' \in \mathcal{M}_{n\times 1}(\mathbb{R})) ,$$

where Z denotes an element of $\mathcal{M}_{2n\times 2n}(\mathbb{R})$, is a canonical transformation (see [TH], p. 98) if and only if Z is symplectic. More details are given in section 2.6.

1.1.5. Definition. A pair (C,D) of matrices $C \in \mathcal{M}_{n\times n}(\mathbb{C})$ and $D \in \mathcal{M}_{n\times n}(\mathbb{C})$ $(n \in \mathbb{N})$ will be called admissible if it occurs as the second matrix row of a symplectic matrix of order n, and if furthermore $CT + D$ is non-singular for every $T \in \mathcal{H}_n$.

We conclude this section with an example that will be used extensively in the next sections.

1.1.6. Let $M_n(\alpha)$ $(n \in \mathbb{N}, \alpha > 0)$ be defined by

$$M_n(\alpha) := \begin{pmatrix} \cosh\alpha\ I_n & \sinh\alpha\ I_n \\ \sinh\alpha\ I_n & \cosh\alpha\ I_n \end{pmatrix} .$$

Obviously $M_n(\alpha)$ is symplectic of order n. Furthermore,

$$T + \coth\alpha\ I_n \in \mathcal{H}_n$$

for every $T \in H_n$, so, by using 1.1.3, we see that

$$\sinh\alpha\, T + \cosh\alpha\, I_n$$

is non-singular for every $T \in H_n$. This implies that $\mathcal{L}_{M_n(\alpha)}$ is well defined ($\alpha > 0$). It is not hard to see that

$$\mathcal{L}_{M_n(\alpha)}(T) = \coth\alpha I_n - \sinh^{-2}\alpha(T + \coth\alpha)^{-1} \quad (T \in H_n) \ ,$$

so application of 1.1.3 leads to

$$\begin{aligned} \mathrm{Re}(\mathcal{L}_{M_n(\alpha)}(T)) &= \coth\alpha\, I_n - \sinh^{-2}\alpha[\mathrm{Re}(T) + \coth\alpha\, I_n + \\ &\quad + \mathrm{Im}(T)(\mathrm{Re}(T) + \coth\alpha\, I_n)^{-1}\mathrm{Im}(T)]^{-1} \\ &\geq \coth\alpha\, I_n - \sinh^{-2}\alpha(\coth\alpha\, I_n)^{-1} \\ &= \tanh\alpha\, I_n \quad (T \in H_n) \ . \end{aligned}$$

Therefore $\mathcal{L}_{M_n(\alpha)}$ maps H_n into H_n for every $\alpha > 0$. Evidently we have

$$\mathcal{L}_{M_n(\alpha)}(\mathcal{L}_{M_n(\beta)}(T)) = \mathcal{L}_{M_n(\alpha+\beta)}(T) \quad (\alpha > 0,\ \beta > 0,\ T \in H_n) \ .$$

(See 1.1.4.2.)

1.2. Decomposition of a symplectic matrix

We shall discuss a result on the decomposition of symplectic matrices which plays an important rôle in the remainder of this report. The notations introduced in this section will be used in many of the subsequent sections.

Let $Z \in Sp(n)$ $(n \in \mathbb{N})$ and let A,B,C,D denote the block components of Z (according to 1.1.2). Furthermore, let r denote the rank of C and assume $0 < r < n$. A general theorem in matrix theory states that there exist permutation matrices $P,Q \in M_{n\times n}(\mathbb{R})$, matrices $X \in M_{r\times(n-r)}(\mathbb{C})$ and $Y \in M_{(n-r)\times r}(\mathbb{C})$, and a non-singular matrix $C_{11} \in M_{r\times r}(\mathbb{C})$ such that

1.2.1. $$C = P\begin{pmatrix} C_{11} & C_{11}X \\ YC_{11} & YC_{11}X \end{pmatrix}Q^T \ .$$

So

$$C = V_1 \begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix} U_2^T ,$$

where

$$V_1 := P \begin{pmatrix} I_r & 0 \\ Y & I_{n-r} \end{pmatrix}, \; U_2 := Q \begin{pmatrix} I_r & 0 \\ X^T & I_{n-r} \end{pmatrix} .$$

If we write

$$V_1^{-1} D U_2 = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} ,$$

where $D_{11} \in M_{r\times r}(\mathbb{C})$, $D_{12} \in M_{r\times(n-r)}(\mathbb{C})$, $D_{21} \in M_{(n-r)\times r}(\mathbb{C})$ and $D_{22} \in M_{(n-r)\times(n-r)}(\mathbb{C})$, then we deduce from $CD^T = DC^T$ (see 1.1.2.3) that

$$C_{11}D_{11}^T = D_{11}C_{11}^T, \; C_{11}D_{21}^T = 0 .$$

Since C_{11} is non-singular we have $D_{21} = 0$.
The rank of the matrix $(C,D) \in M_{n\times 2n}(\mathbb{C})$ is n (it occurs as the second matrix row of a non-singular matrix), hence

$$\operatorname{rank} V_1 \begin{pmatrix} C_{11} & 0 & D_{11} & D_{12} \\ 0 & 0 & 0 & D_{22} \end{pmatrix} \begin{pmatrix} U_2^T & 0 \\ 0 & U_2^{-1} \end{pmatrix} = n ,$$

and this implies that D_{22} is non-singular.
If we define U_1 by

$$U_1 := V_1 \begin{pmatrix} I_r & D_{12} \\ 0 & D_{22} \end{pmatrix} = P \begin{pmatrix} I_r & D_{12} \\ Y & YD_{12}+D_{22} \end{pmatrix} ,$$

then U_1 is non-singular and we have

1.2.2. $$C = U_1 C_1 U_2^T, \; D = U_1 D_1 U_2^{-1} ,$$

where

$$C_1 = \begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix}, \; D_1 = \begin{pmatrix} D_{11} & 0 \\ 0 & I_{n-r} \end{pmatrix} .$$

In addition, if we write

$$A_1 := U_1^T A U_2^{-T} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} ,$$

$$B_1 := U_1^T B U_2 = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} ,$$

where $A_{11}, B_{11} \in \mathcal{M}_{r\times r}(\mathbb{C})$, $A_{12}, B_{12} \in \mathcal{M}_{r\times(n-r)}(\mathbb{C})$, $A_{21}, B_{21} \in \mathcal{M}_{(n-r)\times r}(\mathbb{C})$, and $A_{22}, B_{22} \in \mathcal{M}_{(n-r)\times(n-r)}(\mathbb{C})$, then we have

1.2.3. $$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} U_1^T & 0 \\ 0 & U_1^{-1} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} U_2^{-T} & 0 \\ 0 & U_2 \end{pmatrix} ,$$

and since all matrices on the right-hand side are symplectic, we have

$$Z_1 := \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in Sp(n) .$$

Application of 1.1.2 to Z_1 and using the fact that C_{11} is non-singular gives

1.2.4. $$A_{12} = 0,\ A_{22} = I_{n-r},\ B_{22} = B_{22}^T,\ B_{21} = B_{12}^T D_{11},\ A_{21} = B_{12}^T C_{11} ,$$

and furthermore that the matrix $Z_{11} \in \mathcal{M}_{2r\times 2r}(\mathbb{C})$ defined by

1.2.5. $$Z_{11} := \begin{pmatrix} A_{11} & B_{11} \\ C_{11} & D_{11} \end{pmatrix}$$

is symplectic.

1.2.6. If we assume that $\mathcal{L}_Z$ is well defined (see 1.1.4) and if furthermore

$$T_1 := \mathcal{L}_Z(T) \qquad (T \in \mathcal{H}_n) ,$$

then we easily infer from 1.2.3 that

$$U_1^T T_1 U_1 = \mathcal{L}_{Z_1}(U_2^T T U_2) .$$

1.3. Admissibility

The aim of this section is to derive necessary and sufficient conditions for a pair of matrices $C, D \in \mathcal{M}_{n\times n}(\mathbb{C})$ $(n \in \mathbb{N})$ to be admissible.

First of all we want to characterize the pairs of matrices $C \in M_{n\times n}(\mathbb{C})$, $D \in M_{n\times n}(\mathbb{C})$ $(n \in \mathbb{N})$ that occur as the second matrix row of a symplectic matrix. We have the following lemma (see [M], p. 155).

1.3.1. Lemma. Let C and D be elements of $M_{n\times n}(\mathbb{C})$ $(n \in \mathbb{N})$. Then (C,D) occurs as the second matrix row of a symplectic matrix of order n if and only if

$$CD^T = DC^T, \ \mathrm{rank}[C,D] = n \ .$$

Proof. The "only if" part of our assertion is a trivial consequence of 1.1.2. Assume now CD^T is symmetric and $\mathrm{rank}[C,D] = n$. By elementary devisor theory (see [G], chapter II) we can choose non-singular matrices $U_1 \in M_{n\times n}(\mathbb{C})$ and $U_2 \in M_{2n\times 2n}(\mathbb{C})$ such that

$$U_1[C,D]U_2 = [I_n,0] \ ,$$

hence

$$[C,D]U_2\begin{pmatrix}U_1\\0\end{pmatrix} = I_n \ .$$

If we define X and $Y \in M_{n\times n}(\mathbb{C})$ by

$$\begin{pmatrix}X\\Y\end{pmatrix} := U_2\begin{pmatrix}U_1\\0\end{pmatrix} \ ,$$

then

$$CX + DY = I_n \ .$$

Now let

$$A := Y^T + X^TYC, \ B := -X^T + X^TYD \ ,$$

then

$$AB^T - BA^T = -Y^TX + (DY + CX)^TY^TX + X^TY - X^TY(CX + DY) = 0 \ ,$$

and

$$AD^T - BC^T = Y^TD^T + X^TC^T + X^TY(CD^T - DC^T) = I_n \ ,$$

hence

$$\begin{pmatrix}A & B\\C & D\end{pmatrix}^T$$

is symplectic (see 1.1.2.3), and thus (C,D) occurs as the second matrix of a symplectic matrix. Note that A and B are by no means uniquely determined: for every symmetric matrix $S \in M_{n\times n}(\mathbb{C})$ the matrix

$$\begin{pmatrix} A - SC & B - SD \\ C & D \end{pmatrix}$$

is symplectic. □

Let $C \in M_{n\times n}(\mathbb{C})$, $D \in M_{n\times n}(\mathbb{C})$ $(n \in \mathbb{N})$ be a pair of matrices such that (C,D) is the second matric row of a symplectic matrix of order n. We now want to investigate under which circumstances $CT + D$ is non-singular for every $T \in H_n$.

1.3.2. In case C is non-singular, we have that $CT + D$ is non-singular for every $T \in H_n$ if and only if $\mathrm{Re}(C^{-1}D) \geq 0$. This is an easy consequence of the symmetry of $C^{-1}D$ (see 1.3.1 and 1.1.2.5) and of the following lemma.

1.3.2.1. Lemma. Let $P \in M_{n\times n}(\mathbb{C})$ $(n \in \mathbb{N})$ be a symmetric matrix. Then $T + P$ is non-singular for every $T \in H_n$ if and only if $\mathrm{Re}(P) \geq 0$.

Proof. If $\mathrm{Re}(P) \geq 0$, then $T + P \in H_n$ for every $T \in H_n$, hence $T + P$ is non-singular (see 1.1.3). Now suppose that $\mathrm{Re}(P)$ has a negative eigenvalue $-\lambda$ $(\lambda > 0)$. If we define

$$T := \lambda I_n - i\,\mathrm{Im}(P) ,$$

then $T \in H_n$ and $T + P$ is singular, since 0 is an eigenvalue of $T + P$. □

The case discussed in 1.3.2 may seem a very special one: if (C,D) denotes the second matrix-row of a symplectic matrix, then, in general, C is singular. We shall see however that 1.3.2 in fact covers all cases. Before we deal with the general case we need an auxiliary result.

1.3.3. Lemma. If $T \in H_n$ $(n \in \mathbb{N})$, then there exists an $\alpha > 0$ and a $T' \in H_n$ such that

$$\mathcal{L}_{M_n(\alpha)}(T') = T .$$

Proof. We have to determine $\alpha > 0$ and $T' \in H_n$ such that

1.3.3.1. $$T = \coth\alpha\ I_n - \sinh^{-2}\alpha(T' + \coth\alpha\ I_n)^{-1}$$

(see 1.1.6). Let $\alpha_0 > 0$ be such that

$$\mathrm{Re}(\coth\alpha\ I_n - T) > 0 \quad (0 < \alpha < \alpha_0) .$$

Then for every $0 < \alpha < \alpha_0$ we can define

$$T'_\alpha := \sinh^{-2}\alpha[(\cosh\alpha\ I_n - \sinh\alpha\ T)^{-1} - \cosh\alpha\ I_n] ,$$

and obviously

$$\mathcal{L}_{M_n(\alpha)}(T'_\alpha) = T \quad (0 < \alpha < \alpha_0) .$$

From an easy matrix computation we obtain that $\mathrm{Re}(T'_\alpha) > 0$ if

$$\tanh\alpha\ I_n + \sinh\alpha\ \mathrm{Im}(T)(\cosh\alpha\ I_n - \sinh\alpha\ \mathrm{Re}(T))^{-1}\mathrm{Im}(T) < \mathrm{Re}(T) ,$$

and this is obviously true for α sufficiently small since the matrix on the left-hand side tends to the all zero matrix if α tends to zero. □

1.3.4. Let $Z \in Sp(n)$ $(n \in \mathbb{N})$ with second matrix row (C,D) $(C \in M_{n\times n}(\mathbb{C}),$ $D \in M_{n\times n}(\mathbb{C}))$. We intend to express the admissibility of (C,D) in terms of the admissibility of the second matrix row of the matrices $ZM_n(\alpha)$ $(\alpha > 0)$.

1.3.4.1. Let C_α and D_α be defined by

$$\left.\begin{aligned} C_\alpha &:= C\cosh\alpha + D\sinh\alpha\ , \\ D_\alpha &:= D\sinh\alpha + C\cosh\alpha\ . \end{aligned}\right\} \quad (\alpha > 0)$$

We have the following lemma.

1.3.4.2. Lemma. If the conditions of 1.3.4. are satisfied, then (C,D) is admissible if and only if (C_α,D_α) is admissible for every $\alpha > 0$.

Proof. From an easy matrix computation we infer that for every $\alpha > 0$ and every $T \in H_n$

1.3.4.3. $$C_\alpha T + D_\alpha = (C\mathcal{L}_{M_n(\alpha)}(T) + D)(\sinh\alpha\ T + \cosh\alpha\ I_n) .$$

Clearly $\sinh\alpha\, T + \cosh\alpha\, I_n$ is non-singular for every $T \in H_n$ and every $\alpha > 0$. Assume that (C,D) is admissible and that $\alpha > 0$. Then $C\mathcal{L}_{M_n(\alpha)}(T) + D$ is non-singular for every $T \in H_n$, and since (C_α, D_α) is the second matrix row of the symplectic matrix $ZM_n(\alpha)$, we see that (C_α, D_α) is admissible.
Conversely, assume that (C_α, D_α) is admissible for every $\alpha > 0$. If $T \in H_n$, then (according to lemma 1.3.3) there exists an $\alpha > 0$ and a $T' \in H_n$ such that

$$\mathcal{L}_{M_n(\alpha)}(T') = T ,$$

so, from 1.3.4.3, we obtain that

$$CT + D = (C_\alpha T' + D_\alpha)(\sinh\alpha\, T' + \cosh\alpha\, I_n)^{-1} ,$$

hence $CT + D$ is non-singular. Since by assumption (C,D) is the second matrix row of a symplectic matrix, we see that (C,D) is admissible. □

1.3.5. We shall now explain why we consider the matrices C_α and D_α (defined in 1.3.4) for studying the admissibility of (C,D).

1.3.5.1. Let $C \in M_{n\times n}(\mathbb{C})$, $D \in M_{n\times n}(\mathbb{C})$ $(n \in \mathbb{N})$ and assume that (C,D) is admissible. Since $\coth\alpha\, I_n \in H_n$, we easily see that C_α (defined in 1.3.4) is non-singular $(\alpha > 0)$. Using lemma 1.3.2.1 and lemma 1.3.4.2 we obtain that

$$\mathrm{Re}(C_\alpha^{-1} D_\alpha) \geq 0$$

for every $\alpha > 0$.
Conversely, if (C,D) denotes the second matrix row of a symplectic matrix of order n and if C_α and D_α (defined in 1.3.4) are such that C_α is non-singular and $\mathrm{Re}(C_\alpha^{-1} D_\alpha) \geq 0$ for every $\alpha > 0$, then (C_α, D_α) is admissible for every $\alpha > 0$ (see lemma 1.3.2.1), so, according to lemma 1.3.4.2 (C,D) is admissible. We have thus proved the following lemma.

1.3.5.2. <u>Lemma</u>. If (C,D) denotes the second matrix row of a symplectic matrix of order n $(n \in \mathbb{N})$, then (C,D) is admissible if and only if C_α is non-singular and $\mathrm{Re}(C_\alpha^{-1} D_\alpha) \geq 0$ for every $\alpha > 0$, where C_α and D_α are defined in 1.3.4.

1.3.5.3. Let $C \in M_{n\times n}(\mathbb{C})$, $D \in M_{n\times n}(\mathbb{C})$ $(n \in \mathbb{N})$ and assume that (C,D) is admissible and that C is non-singular. If C_α and D_α are defined as in 1.3.4, then C_α is non-singular and

$$\begin{aligned} C_\alpha^{-1}D_\alpha &= (I_n + \tanh\alpha\ C^{-1}D)^{-1}(\tanh\alpha\ I_n + C^{-1}D) \\ &= (I_n + \tanh\alpha\ C^{-1}D)^{-1}(\tanh^{-1}\alpha(\tanh\alpha\ C^{-1}D + I_n) + \\ &+ (\tanh\alpha - \tanh^{-1}\alpha)I_n) \\ &= \coth\alpha\ I_n - (\tanh\alpha - \coth\alpha)(I_n + \tanh\alpha\ C^{-1}D)^{-1} \qquad (\alpha > 0)\ . \end{aligned}$$

By using 1.1.3 and $\mathrm{Re}(C^{-1}D) \geq 0$ (see lemma 1.3.2.1), we have

$$\begin{aligned} \tanh\alpha\ \mathrm{Re}(C_\alpha^{-1}D_\alpha) &= I_n - \cosh^{-2}\alpha\mathrm{Re}(I_n + \tanh\alpha\ C^{-1}D)^{-1} \\ &\geq I_n - \cosh^{-2}\alpha(I_n + \tanh\alpha\ \mathrm{Re}(C^{-1}D))^{-1} \\ &\geq I_n(1 - \cosh^{-2}\alpha) > 0\ , \end{aligned}$$

so $\mathrm{Re}(C_\alpha^{-1}D_\alpha) > 0$ $(\alpha > 0)$.

1.3.5.4. Let (C,D) be admissible, then (C_β, D_β) is admissible and C_β is non-singular for every $\beta > 0$ (see 1.3.5.2). Since

$$C_{2\beta} = C_\beta\cosh\beta + D_\beta\sinh\beta$$

and

$$D_{2\beta} = C_\beta\sinh\beta + D_\beta\cosh\beta\ ,$$

we infer from 1.3.5.3 that $\mathrm{Re}(C_{2\beta}^{-1}D_{2\beta}) > 0$ for every $\beta > 0$, hence

$$\mathrm{Re}(C_\alpha^{-1}D_\alpha) > 0$$

for every $\alpha > 0$.

Summarizing now the results of 1.3.1, 1.3.4 and 1.3.5 we have the following theorem.

1.3.6. <u>Theorem</u>. Let $C \in M_{n\times n}(\mathbb{C})$, $D \in M_{n\times n}(\mathbb{C})$ $(n \in \mathbb{N})$ and let C_α and D_α be defined as in 1.3.4. Then (C,D) is admissible if and only if i), ii) and iii) hold, where

i) $CD^T = DC^T$,

ii) $\mathrm{rank}(C,D) = n$,

iii) for every $\alpha > 0$ C_α is non-singular and $\mathrm{Re}(C_\alpha^{-1}D_\alpha) > 0$.

In this report the decomposition of section 1.2 will be used frequently. Therefore we want to express the admissibility of a pair (C,D) of matrices in terms of this decomposition of C and D. We shall prove the following theorem.

1.3.7. Theorem. Let $C \in M_{n\times n}(\mathbb{C})$, $D \in M_{n\times n}(\mathbb{C})$ $(n \in \mathbb{N})$ and assume (C,D) is second matrix row of a symplectic matrix. Let furthermore $0 < r < n$, where r denotes the rank of C. Let us use the notation of section 1.2. Then $CT + D$ is non-singular for every $T \in H_n$ if and only if X is real and $\mathrm{Re}(C_{11}^{-1}D_{11}) \geq 0$.

Proof. Suppose X is real. Using the notation of section 1.2, we have

1.3.7.1. $$CT + D = U_1\left[\begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix} U_2^T T U_2 + \begin{pmatrix} D_{11} & 0 \\ 0 & I \end{pmatrix}\right] U_2^{-1} .$$

Since U_2 is real and non-singular, the mapping $\Psi_{S\in H_n}\ U_2^T S U_2$ maps H_n one-to-one onto H_n. Therefore CT+D is non-singular for every $T \in H_n$ if and only if $C_{11}T_1 + D_{11}$ is non-singular for every $T_1 \in H_r$ and this is true if and only if $\mathrm{Re}(C_{11}^{-1}D_{11}) \geq 0$ (see 1.3.2). (Note that $C_{11}^{-1}D_{11}$ is symmetric for (C_{11},D_{11}) is the second matrix row of a symplectic matrix of order r).

It only remains to prove now that $\mathrm{Im}(X) \neq 0$ implies the existence of a $T \in H_n$ such that $CT + D$ is singular. We first prove an auxiliary result.

1.3.7.2. Lemma. Let R be a symmetric matrix in $M_{n\times n}(\mathbb{R})$ $(n \in \mathbb{N})$ and let $S \in M_{k\times n}(\mathbb{R})$ $(k \in \mathbb{N})$ such that S is not the all zero matrix. Then there exists a $K \in M_{n\times k}(\mathbb{R})$ such that

$$R - KS - S^TK^T$$

is singular.

Proof. Let s denote the rank of S, then we have $s > 0$. Assume $s < \min(k,n)$. (In case $s = \min(k,n)$, we only need a slight change of the notation.) By elementary devisor theory (see [G], chapter VI), there are non-singular $P_1 \in M_{k\times k}(\mathbb{R})$ and $P_2 \in M_{n\times n}(\mathbb{R})$ such that

$$P_1SP_2 = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} .$$

It is easily seen that we only have to prove that there exists a $K' \in M_{n\times k}(\mathbb{R})$ such that

$$P_2^TRP_2 - K'\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}K'^T$$

is singular. Let us write

$$P_2^TRP_2 = \begin{pmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{pmatrix} ,$$

where $R_{11} \in M_{s\times s}(\mathbb{R})$, $R_{12} \in M_{s\times(n-s)}(\mathbb{R})$, $R_{22} \in M_{(n-s)\times(n-s)}(\mathbb{R})$, and

$$K' = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} ,$$

where $K_{11} \in M_{s\times s}(\mathbb{R})$, $K_{12} \in M_{s\times(k-s)}(\mathbb{R})$, $K_{21} \in M_{(n-s)\times s}(\mathbb{R})$ and $K_{22} \in M_{(n-s)\times(k-s)}(\mathbb{R})$. We now take $K_{11} := \frac{1}{2}R_{11}$, $K_{12} = R_{12}$, $K_{22} := 0$ and $K_{12} := 0$. □

We now are able to finish the proof of theorem 1.3.7. Suppose X is not real, then there exist matrices $V \in M_{(n-r)\times r}(\mathbb{R})$ and $W \in M_{(n-r)\times r}(\mathbb{R})$ such that

$$X^T = V + iW,\ W \neq 0 .$$

If we write

$$U_2 = Q(R,S)$$

(see section 1.2), where

$$R = \begin{pmatrix} I_r \\ X^T \end{pmatrix},\ S = \begin{pmatrix} 0 \\ I_{n-r} \end{pmatrix} ,$$

then, according to 1.3.7.1, we only have to prove the existence of a $T \in H_n$ such that

$$R^TTR + C_{11}^{-1}D_{11}$$

is singular.

Let

$$T = \begin{pmatrix} I_r + iY_{11} & iY_{12} \\ iY_{12}^T & I_{n-r} \end{pmatrix},$$

where $Y_{11} \in M_{r\times r}(\mathbb{R})$, $Y_{11} = Y_{11}^T$, $Y_{12} \in M_{r\times(n-r)}(\mathbb{R})$, then $T \in H_n$ and

$$R^T TR + C_{11}^{-1}D_{11} = \mathrm{Re}(C_{11}^{-1}D_{11}) + I_r + V^T V - W^T W - Y_{12}W - W^T Y_{12}^T$$

$$+ i(Y_{11} + \mathrm{Im}(C_{11}^{-1}D_{11}) + Y_{12}V + V^T Y_{12}^T + V^T W + W^T V) .$$

According to lemma 1.3.7.2, there exists a matrix $Y_{12} \in M_{r\times(n-r)}(\mathbb{R})$ such that the real part of this expression is singular since, by assumption, W is not the all zero matrix. In addition, it is easy to see that there exists a symmetric $Y_{11} \in M_{r\times r}(\mathbb{R})$ such that the imaginary part of this expression is the all zero matrix. We have therefore proved the existence of a $T \in H_n$ such that $CT + D$ is singular. □

1.4. Fractional linear transforms with range in H_n

In this section we are concerned with the question what conditions have to be imposed on the matrix $Z \in Sp(n)$ $(n \in \mathbb{N})$ in order to guarantee that $\mathcal{L}_Z$ is well defined (see 1.1.4) and maps H_n into H_n.

We restrict ourselves to matrices $Z \in Sp(n)$ $(n \in \mathbb{N})$ such that the second matrix row of Z is admissible. Then, of course, the mapping $\mathcal{L}_Z$ is well defined.

Let $Z \in Sp(n)$ $(n \in \mathbb{N})$ with block components $A,B,C,D \in M_{n\times n}(\mathbb{C})$, and assume that (C,D) is admissible and that C is non-singular. Using 1.1.4 and 1.1.2.1, we can write

1.4.1. $$\mathcal{L}_Z = \underset{S\in H_n}{\Psi} AC^{-1} - C^{-T}(S + C^{-1}D)^{-1}C^{-1} .$$

So, in this case, $\mathrm{Im}(C^{-1}D)$ and $\mathrm{Im}(AC^{-1})$ do not play any rôle in the question whether the image of H_n under $\mathcal{L}_Z$ is contained in H_n.

1.4.2. <u>Definition</u>. $G_0(n)$ $(n \in \mathbb{N})$ denotes the set of all $Z \in Sp(n)$ such that the second matrix row of Z is an admissible pair of elements of $M_{n\times n}(\mathbb{C})$ and such that the mapping $\mathcal{L}_Z$ maps H_n into H_n.

Obviously $G_0(n)$ is a semigroup under matrix multiplication (see 1.1.4.2). In particular, since $M_n(\alpha) \in G_0(n)$ for every $\alpha > 0$ (see 1.1.6), we see that $ZM_n(\alpha) \in G_0(n)$ $(\alpha > 0)$, where Z denotes an arbitrary element of $G_0(n)$. The converse of this statement is also true as we shall see in the following theorem.

1.4.3. <u>Theorem</u>. Let $Z \in Sp(n)$ $(n \in \mathbb{N})$ and assume that $ZM_n(\alpha) \in G_0(n)$ for every $\alpha > 0$. Then $Z \in G_0(n)$.

<u>Proof</u>. Let $A,B,C,D \in \mathcal{M}_{n\times n}(\mathbb{C})$ denote the block components of Z (according to 1.1.2). Let furthermore $T \in \mathcal{H}_n$. According to lemma 1.3.3, there exists an $\alpha > 0$ and a $T' \in \mathcal{H}_n$ such that

$$\mathcal{L}_{M_n(\alpha)}(T') = T .$$

Now if C_α and D_α (elements of $\mathcal{M}_{n\times n}(\mathbb{C})$) are defined by 1.3.4.1, then obviously (C_α, D_α) is the second matrix row of $ZM_n(\alpha)$ and $C_\alpha T' + D_\alpha$ is non-singular since (C_α, D_α) is admissible. Using 1.3.4.2, we infer that $CT + D$ is non-singular, and since T was an arbitrary element of $\mathcal{H}_n$, we see that (C,D) is admissible. Furthermore, from 1.1.4.2 we infer

$$\mathcal{L}_Z(T) = \mathcal{L}_Z(\mathcal{L}_{M_n(\alpha)}(T')) = \mathcal{L}_{ZM_n(\alpha)}(T') ,$$

so $Z \in G_0(n)$. □

1.4.4. Let $Z \in G_0(n)$. The reason why we consider the set $(ZM_n(\alpha))_{\alpha>0}$ lies in the fact that $ZM_n(\alpha)$ has some nice properties: If (C,D) denotes the second matrix row of Z, where $C \in \mathcal{M}_{n\times n}(\mathbb{C})$, $D \in \mathcal{M}_{n\times n}(\mathbb{C})$, and if C_α and D_α are defined by 1.3.4.1, then (C_α, D_α) is the second matrix row of $ZM_n(\alpha)$ and from theorem 1.3.6 we see that C_α is non-singular and $\mathrm{Re}(C_\alpha^{-1} D_\alpha) > 0$ $(\alpha > 0)$.

1.4.5. Let $Z \in Sp(n)$ $(n \in \mathbb{N})$ with block components $A,B,C,D \in \mathcal{M}_{n\times n}(\mathbb{C})$ (according to 1.1.2). Assume that C is non-singular and that $\mathrm{Re}(C^{-1}D) > 0$.
In this case a direct derivation of conditions for Z to be an element of $G_0(n)$ will be possible. Note that (C,D) is admissible since $\mathrm{Re}(C^{-1}D) > 0$ (see 1.3.2).

Let $U,V,L,M \in M_{n\times n}(\mathbb{R})$ be defined by

$$U := \mathrm{Re}(C^{-1}D),\ V := \mathrm{Re}(AC^{-1}),\ C^{-1} = L + iM .$$

Obviously $U > 0$. According to 1.4.1, we only have to deal with the mapping

1.4.5.1. $$\Psi_{S\in H_n}\ V - C^{-T}(S + U)^{-1}C^{-1} .$$

Now let $S \in H_n$ and let $X \in M_{n\times n}(\mathbb{R})$ and $Y \in M_{n\times n}(\mathbb{R})$ be defined by $S := X + iY$. Then, using 1.1.3, we see that image of S under the mapping given in 1.4.5.1 equals

1.4.5.2. $$V - (L^T + iM^T)(I_n - i(X+U)^{-1}Y)(X+U+Y(X+U)^{-1}Y)^{-1}(L+iM) .$$

If we abbreviate $P = (X+U)^{-1}$, $Q = ((X+U) + Y(X+U)^{-1}Y)^{-1}$, then we see that the real part of 1.4.5.2 equals

$$V - L^TPL + L^T(P-Q)L - L^TPYQM - M^TPYQL + M^TQM ,$$

and since

$$P - Q = P(Q^{-1} - P^{-1})Q = PYPYQ,\ PYQ = QYP$$

we see that the real part of 1.4.5.2 equals $E(X,Y)$, where $E(X,Y)$ is defined by

1.4.5.3. $$E(X,Y) := V - L^T(X+U)^{-1}L + (Y(X+U)^{-1}L - M)^T(X+U+Y(X+U)^{-1}Y)^{-1}(Y(X+U)^{-1}L - M) .$$

Obviously $Z \in G_0(n)$ if and only if

$$E(X,Y) > 0$$

for every positive definite $X \in M_{n\times n}(\mathbb{R})$ and every symmetric $Y \in M_{n\times n}(\mathbb{R})$.

It is clear from 1.4.5.3 and from the fact that

$$Y(X+U)^{-1}Y \le YU^{-1}Y ,$$

that $E(X,Y) \ge V - L^TU^{-1}L$ for every $X > 0$ and every symmetric Y.

Now suppose that there exist an $X > 0$, $X \in M_{n\times n}(\mathbb{R})$, and a symmetric $Y \in M_{n\times n}(\mathbb{C})$ and a vector $x \in M_{n\times 1}(\mathbb{R})$ such that

$$x^TE(X,Y)x = x^T(V - L^TU^{-1}L)x .$$

Then we have

$$(Lx)^T(U^{-1} - (X+U)^{-1})(Lx) +$$

$$+ (Y(X+U)^{-1}Lx - Mx)^T((X+U) + Y(X+U)^{-1}Y)^{-1}(Y(X+U)^{-1}Lx - Mx) = 0 ,$$

and since

$$U^{-1} > (X+U)^{-1}, \ (X+U+Y(X+U)^{-1}Y)^{-1} > 0$$

we obtain

$$Lx = 0, \ Mx = 0 ,$$

so $C^{-1}x = 0$. But C is non-singular and therefore $x = 0$. This implies that

$$E(X,Y) > V - L^TU^{-1}L$$

for every positive definite $X \in M_{n\times n}(\mathbb{R})$ and every symmetric $Y \in M_{n\times n}(\mathbb{R})$. So we have proved that $Z \in G_0(n)$ if $V - L^TU^{-1}L \geq 0$. We next show that the converse is also true.

Assume that $Z \in G_0(n)$ and that Z satisfies the conditions of 1.4.5. Let $x \in M_{n\times 1}(\mathbb{R})$.

i) If $Lx = 0$, then

$$x^TE(\alpha I_n, \alpha^{-1}I_n)x > 0 \quad (\alpha > 0) ,$$

so

$$0 \leq \lim_{\alpha \downarrow 0} x^TE(\alpha I_n, \alpha^{-1}I_n)x = x^T(V - L^TU^{-1}L)x .$$

ii) If $Lx \neq 0$, then there exists a symmetric $Y_\alpha \in M_{n\times n}(\mathbb{R})$ such that

$$Y_\alpha(\alpha I_n + U)^{-1}Lx = Mx \quad (\alpha > 0) .$$

Therefore

$$0 \leq \lim_{\alpha \downarrow 0} x^TE(\alpha I_n, Y_\alpha)x = x^T(V - L^TU^{-1}L)x .$$

From i) and ii) we see that

$$V - L^TU^{-1}L \geq 0 ,$$

and it is not hard to prove that this is equivalent with

$$\begin{pmatrix} V & L^T \\ L & U \end{pmatrix} \geq 0$$

(note that $U > 0$).

Summarizing, we have proved the following lemma.

1.4.6. Lemma. Let $Z \in Sp(n)$ $(n \in \mathbb{N})$ with block-components $A,B,C,D \in M_{n\times n}(\mathbb{C})$ (according to 1.1.2). Assume that C is non-singular and that $\mathrm{Re}(C^{-1}D) > 0$. Then $Z \in G_0(n)$ if and only if

$$\mathrm{Re}\begin{pmatrix} AC^{-1} & C^{-T} \\ C^{-1} & C^{-1}D \end{pmatrix} \geq 0 .$$

It may seem that this lemma deals with a rather special case. However, if we combine this lemma with theorem 1.3.6 and 1.4.3, we see that we can handle all cases, as expressed in the following theorem.

1.4.7. Theorem. Let $Z \in Sp(n)$ $(n \in \mathbb{N})$ and let $A_\alpha, B_\alpha, C_\alpha, D_\alpha \in M_{n\times n}(\mathbb{C})$ denote the block components of $ZM_n(\alpha)$ $(\alpha > 0)$, according to 1.1.2. Then $Z \in G_0(n)$ if and only if for every $\alpha > 0$ we have that C_α is non-singular and

$$\mathrm{Re}\begin{pmatrix} A_\alpha C_\alpha^{-1} & C_\alpha^{-T} \\ C_\alpha^{-1} & C_\alpha^{-1}D_\alpha \end{pmatrix} \geq 0 .$$

We shall see in section 1.5 that we can replace "$\geq$" by "$>$".

1.5. The case that C is non-singular

The condition of theorem 1.4.7 may seem very complicated: for every $\alpha > 0$ we have to check whether a given matrix is positive definite. The reader must be aware however of the fact that the elements of the matrices involved are rational functions of $\forall_{\alpha>0}\ \cosh\alpha$ and $\forall_{\alpha>0}\ \sinh\alpha$. In some cases the condition of theorem 1.4.7 can be reduced to an easier one, as we shall see in this section.

1.5.1. Lemma. Let $Z \in Sp(n)$ $(n \in \mathbb{N})$ with block components $A,B,C,D \in M_{n\times n}(\mathbb{C})$. Assume that C is non-singular and that

$$\mathrm{Re}\begin{pmatrix} AC^{-1} & C^{-T} \\ C^{-1} & C^{-1}D \end{pmatrix} \geq 0 .$$

Then $K(\alpha) > 0$ for every $\alpha > 0$, where $K(\alpha)$ is defined by

$$K(\alpha) \ := \mathrm{Re}\begin{pmatrix} A_\alpha C_\alpha^{-1} & C_\alpha^{-T} \\ C_\alpha^{-1} & C_\alpha^{-1} D_\alpha \end{pmatrix} \qquad (\alpha > 0) \ .$$

Here $A_\alpha, B_\alpha, C_\alpha$ and $D_\alpha \in M_{n\times n}(\mathbb{C})$ denote the block components of $ZM_n(\alpha)$.

Proof. First of all we remark that (C,D) is the second matrix row of a symplectic matrix and $\mathrm{Re}(C^{-1}D) \geq 0$. So (C,D) is admissible and this implies that C_α is non-singular (see 1.3.4.1). The proof of this lemma requires a tedious but straightforward matrix computation. We omit some of the details.

i) Assume $\alpha_0 > 0$ is such that $K(\alpha_0) \geq 0$. Let $x \in M_{2n\times 1}(\mathbb{R})$ be such that

$$x^T K(\alpha_0) x = 0, \ x \neq 0 \ .$$

If we decompose

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ x_1 \in M_{n\times 1}(\mathbb{R}), \ x_2 \in M_{n\times 1}(\mathbb{R}) \ ,$$

and if we define $L_\alpha, M_\alpha, U_\alpha, P_\alpha, V_\alpha \in M_{n\times n}(\mathbb{R})$ by

$$C_\alpha^{-1} D_\alpha = U_\alpha + iP_\alpha, \ C_\alpha^{-1} = L_\alpha + iM_\alpha, \ V_\alpha := \mathrm{Re}(A_\alpha C_\alpha^{-1}) \ (\alpha > 0) \ ,$$

then we have

1.5.1.1.
$$\begin{aligned} L_{\alpha_0} x_1 + U_{\alpha_0} x_2 &= 0 \ , \\ V_{\alpha_0} x_1 + L_{\alpha_0}^T x_2 &= 0 \ . \end{aligned}$$

Clearly $\forall_{\alpha>0}$ $K(\alpha)$ is differentiable, and with the standard rules of differentiation we infer by using 1.5.1.1

1.5.1.2.
$$[\frac{d}{d\alpha} x^T K(\alpha) x]_{\alpha=\alpha_0} = [M_{\alpha_0} x_1 + P_{\alpha_0} x_2]^T [M_{\alpha_0} x_1 + P_{\alpha_0} x_2] + x_2^T x_2 \geq 0 \ .$$

If $x \neq 0$, then we have

$$[\frac{d}{d\alpha} x^T K(\alpha) x]_{\alpha=\alpha_0} > 0 \ ,$$

for if 1.5.1.2 equals zero, then $x_2 = 0$ and $M_{\alpha_0} x_1 = 0$. Using 1.5.1.1 we see that $L_{\alpha_0} x_1 = 0$, hence

$$(L_{\alpha_0} + iM_{\alpha_0}) x_1 = 0 \ ,$$

and since $C_{\alpha_0}^{-1}$ is non-singular we have $x_1 = 0$. Contradiction.

ii) Let $U,V,L,M,P \in \mathcal{M}_{n\times n}(\mathbb{R})$ be defined by

$$U + iP = C^{-1}D,\ L + iM = C^{-1},\ V = \mathrm{Re}(AC^{-1}) \ .$$

It is not hard to expand U_α, V_α and L_α in power series:

1.5.1.3.
$$\begin{aligned} U_\alpha &= U + \tanh\alpha[I_n + P^2 - U^2] + O(\tanh^2\alpha) \quad (\alpha \downarrow 0) \ , \\ V_\alpha &= V - \tanh\alpha[L^TL - M^TM] + O(\tanh^2\alpha) \quad (\alpha \downarrow 0) \ , \\ L_\alpha &= L + \tanh\alpha[PM - UL] + O(\tanh^2\alpha) \quad (\alpha \downarrow 0) \ . \end{aligned}$$

If $x \in \mathcal{M}_{n\times 1}(\mathbb{R})$, $y \in \mathcal{M}_{n\times 1}(\mathbb{R})$, then we have

$$\begin{aligned} &x^TV_\alpha x + 2x^TL_\alpha^Ty + y^TU_\alpha y = x^TVx + 2x^TL^Tz_\alpha + z_\alpha^TUz_\alpha + \\ &\tanh\alpha[x^TM^TMx + 2x^TM^TPy + y^T(I_n + P^2 - U^2)y \\ &+ 2x^TL^TUy + x^TL^TLx + 2y^TU^2y] + O(\tanh^2\alpha) \quad (\alpha \downarrow 0) \ , \end{aligned}$$

where z_α is defined by

$$z_\alpha := y - Lx - \tanh\alpha\ Uy \qquad (\alpha > 0) \ .$$

Using the fact that

$$\begin{pmatrix} U & L^T \\ L & U \end{pmatrix} \geq 0 \ ,$$

we see that

$$\begin{pmatrix} V_\alpha & L_\alpha^T \\ L_\alpha & U_\alpha \end{pmatrix} \geq \tanh\alpha \begin{pmatrix} M^TM + L^TL & L^TU + M^TP \\ UL + PM & I_n + U^2 + P^2 \end{pmatrix} + O(\tanh^2\alpha) \quad (\alpha \downarrow 0) \ ,$$

and it is not hard to see that the matrix on the right-hand side is positive definite.

Combining i) and ii), we easily see that $K(\alpha) > 0$ for every $\alpha > 0$. □

This lemma enables us to prove the following theorem.

1.5.2. <u>Theorem</u>. Let $Z \in Sp(n)$, and assume that $A,B,C,D \in M_{n\times n}(\mathbb{C})$ are the block components of Z and that C is non-singular. Then $Z \in G_0(n)$ if and only if

$$\mathrm{Re}\begin{pmatrix} AC^{-1} & C^{-T} \\ C^{-1} & C^{-1}D \end{pmatrix} \geq 0 \ .$$

<u>Proof</u>. This is an easy consequence of 1.4.7 and 1.5.1. □

1.5.3. <u>Corollary</u>. Let $Z \in G_0(n)$ $(n \in \mathbb{N})$ and let $A_\alpha, B_\alpha, C_\alpha$ and $D_\alpha \in M_{n\times n}(\mathbb{C})$ denote the block components of $ZM_n(\alpha)$ $(\alpha > 0)$. Then

$$\mathrm{Re}\begin{pmatrix} A_\alpha C_\alpha^{-1} & C_\alpha^{-T} \\ C_\alpha^{-1} & C_\alpha^{-1}D_\alpha \end{pmatrix} > 0$$

for every $\alpha > 0$.

This is easily seen as follows:

From theorem 1.4.7 we have

$$\mathrm{Re}\begin{pmatrix} A_\alpha C_\alpha^{-1} & C_\alpha^{-T} \\ C_\alpha^{-1} & D_\alpha^{-1}D_\alpha \end{pmatrix} \geq 0 \ ,$$

and since

$$ZM_n(\alpha)M_n(\alpha) = ZM_n(2\alpha)$$

we have by lemma 1.5.1 that

$$\mathrm{Re}\begin{pmatrix} A_{2\alpha} C_{2\alpha}^{-1} & C_{2\alpha}^{-T} \\ C_{2\alpha}^{-1} & C_{2\alpha}^{-1}D_{2\alpha} \end{pmatrix} > 0$$

for every $\alpha > 0$.

1.6. <u>The group $G_1(n)$</u>

In this section we shall discuss an important subset of $G_0(n)$ $(n \in \mathbb{N})$.

1.6.1. <u>Definition</u>. $G_1(n)$ denotes the set of matrices $Z \in Sp(n)$ such that A and D are real, B and C are purely imaginary, where $A,B,C,D \in M_{n\times n}(\mathbb{C})$ denote the block components of Z (according to 1.1.2).

It is easily seen that $G_1(n)$ is a group (see 1.1.2.4).

In the following theorem we shall prove that $G_1(n)$ is a subset of $G_0(n)$.

1.6.2. <u>Theorem</u>. If $Z \in G_1(n)$ $(n \in \mathbb{N})$, then $\mathcal{L}_Z$ is well defined (see 1.1.4) and $\mathcal{L}_Z$ maps H_n one-to-one onto H_n. If we denote the inverse of $\mathcal{L}_Z$ by $\mathcal{L}_Z^{-1}$, then we have

$$\mathcal{L}_Z^{-1} = \mathcal{L}_{Z^{-1}} .$$

<u>Proof</u>. Let $A,B,C,D \in M_{n\times n}(\mathbb{C})$ denote the block components of Z (according to 1.1.2). Note that A and D are real, B and C are purely imaginary. We consider the three possible cases.

i) Assume that C is non-singular. Then

$$\mathrm{Re}\begin{pmatrix} AC^{-1} & C^{-T} \\ C^{-1} & C^{-1}D \end{pmatrix} = 0 ,$$

so, according to 1.5.2, we have that $Z \in G_0(n)$.

ii) Assume that C is the all zero matrix. Then $C_\alpha = \sinh\alpha\ D$ (see 1.3.4.1) and obviously this matrix is non-singular (see lemma 1.3.1). Furthermore, from 1.1.2.1 we have $A = D^{-T}$, so

$$\mathrm{Re}\begin{pmatrix} A_\alpha C_\alpha^{-1} & C_\alpha^{-T} \\ C_\alpha^{-1} & C_\alpha^{-1}D_\alpha \end{pmatrix} = \begin{pmatrix} \coth\alpha\ D^{-T}D^{-1} & \sinh^{-1}\alpha\ D^{-T} \\ \sinh^{-1}\alpha\ D^{-1} & \coth\alpha\ I_n \end{pmatrix} ,$$

and it is not hard to check that this matrix is positive definite. So, according to theorem 1.4.7, $Z \in G_0(n)$.

iii) Now assume $0 < r < n$, where r denotes the rank of C. Let us use the decomposition and notations of section 1.2. From the fact that C is purely imaginary we see that C_{11} is purely imaginary and X is real. If C_α is defined by 1.3.4.1 $(\alpha > 0)$, then it is not hard to see that

$$C_\alpha = U_1 \begin{pmatrix} C_{11}(I + XX^T)\cosh\alpha + D_{11}\sinh\alpha & C_{11}X\cosh\alpha \\ 0 & \sinh\alpha\ I_{n-r} \end{pmatrix} U_2 ,$$

so C_α is non-singular if and only if

$$I + XX^T + \tanh\alpha\ C_{11}^{-1}D_{11}$$

is non-singular. Since X is real and $\mathrm{Re}(C_{11}^{-1}D_{11}) = 0$ this matrix is an element of H_r, hence it is non-singular (see 1.1.3). Since U_1 and U_2

are real and non-singular we see that $\mathcal{L}_Z$ is well defined and maps H_n into H_n if and only if $\mathcal{L}_{Z_1}$ has this property (see section 1.2, in particular 1.2.6). Now let $A_{11\alpha}, B_{11\alpha}, C_{11\alpha}$ and $D_{11\alpha} \in M_{r\times r}(\mathbb{C})$ denote the block components of $Z_{11}M_r(\alpha)$. Then $C_{11\alpha}$ is non-singular and we have

1.6.2.1. $$\mathrm{Re}\begin{pmatrix} A_{11\alpha}C_{11\alpha}^{-1} & C_{11\alpha}^{-T} \\ C_{11\alpha}^{-1} & C_{11\alpha}^{-1}D_{11\alpha} \end{pmatrix} > 0 \qquad (\alpha > 0) .$$

This can easily be seen by combining the results of i), 1.4.7 and 1.5.3. If $A_{1\alpha}, B_{1\alpha}, C_{1\alpha}$ and $D_{1\alpha} \in M_{n\times n}(\mathbb{C})$ denote the block components of $Z_1M_n(\alpha)$ $(\alpha > 0)$, then

$$\mathrm{Re}\begin{pmatrix} A_{1\alpha}C_{1\alpha}^{-1} & C_{1\alpha}^{-T} \\ C_{1\alpha}^{-1} & C_{1\alpha}^{-1}D_{1\alpha} \end{pmatrix} =$$

$$\mathrm{Re}\begin{pmatrix} \mathrm{diag}(A_{11\alpha}C_{11\alpha}^{-1}, \coth\alpha\ I_{n-r}) & \mathrm{diag}(C_{11\alpha}^{-T}, \sinh^{-1}\alpha\ I_{n-r}) \\ \mathrm{diag}(C_{11}^{-1}\alpha, \sinh^{-1}\alpha\ I_{n-r}) & \mathrm{diag}(C_{11\alpha}^{-1}D_{11\alpha}, \coth\alpha\ I_{n-r}) \end{pmatrix},$$

and using 1.6.2.1 we see that this matrix is positive definite. So, according to theorem 1.4.7, $Z_1 \in G_0(n)$, hence $Z \in G_0(n)$.

Since $G_1(n)$ is a group, $\mathcal{L}_{Z^{-1}}$ is well defined and maps H_n into H_n. From 1.1.4.3 we see that

$$\mathcal{L}_Z(\mathcal{L}_{Z^{-1}}(T)) = T \qquad (T \in H_n) ,$$

so $\mathcal{L}_Z$ maps H_n one-to-one onto H_n with inverse $\mathcal{L}_{Z^{-1}}$. □

In 1.6.1 we have described the set $G_1(n)$ $(n \in \mathbb{N})$ algebraicly. We can also describe the set from a geometric point of view, as we shall show in the next theorem.

1.6.3. Theorem. Let $Z \in Sp(n)$ $(n \in \mathbb{N})$. If $\mathcal{L}_Z$ is well defined (see 1.1.4) and if $\mathcal{L}_Z$ maps H_n onto H_n, then $Z \in G_1(n)$.

Proof. Let A, B, C and $D \in M_{n\times n}(\mathbb{C})$ denote the block components of Z (according to 1.1.2). We consider the three possible cases.

i) Assume that C is non-singular. Since $Z \in G_0(n)$ we have

1.6.3.1. $$\mathrm{Re}\begin{pmatrix} AC^{-1} & C^{-T} \\ C^{-1} & C^{-1}D \end{pmatrix} \geq 0$$

(see 1.5.2). Using 1.1.4.1 and 1.1.4.2, we see that $\mathcal{L}_Z{-1}$ is well defined and maps $\mathcal{H}_n$ onto $\mathcal{H}_n$, so

$$\mathrm{Re}\begin{pmatrix} -D^TC^{-T} & -C^{-1} \\ -C^{-T} & -C^{-T}A^T \end{pmatrix} \geq 0$$

(see 1.1.2.4 and 1.5.2). Combining this with 1.6.3.1 and using the symmetry of AC^{-1} and $C^{-1}D$, we easily infer that A and D are real and that C is purely imaginary. Since

$$B = C^{-T}A^TD - C^{-T}$$

(see 1.1.2.3) the matrix B clearly is purely imaginary.

ii) Assume that $C = 0$. Then A is non-singular and $D = A^{-T}$ (see 1.1.2). From theorem 1.4.7 we obtain

$$\mathrm{Re}\begin{pmatrix} AA^T\coth\alpha + BA^T & \sinh^{-1}\alpha\ A \\ \sinh^{-1}\alpha\ A^T & \coth\alpha\ I_n \end{pmatrix} \geq 0$$

for every $\alpha > 0$, so, by multiplying with $\sinh\alpha$ and taking the limit for α tends to zero, we obtain

$$\begin{pmatrix} XX^T - YY^T & X \\ X^T & I_n \end{pmatrix} \geq 0 \ ,$$

where $X \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $Y \in \mathcal{M}_{n\times n}(\mathbb{R})$ are defined by $A = X + iY$. It is easily seen now that $Y = 0$, hence A and D are real. The mapping $\mathcal{L}_Z$ equals $\forall_{S\in\mathcal{H}_n}\ ASA^T + BA^T$ and since $\forall_{S\in\mathcal{H}_n}\ ATA^T$ maps $\mathcal{H}_n$ one-to-one onto $\mathcal{H}_n$ we have $\mathrm{Re}(BA^T) \geq 0$ (see lemma 1.3.2 and note that BA^T is symmetric (see 1.1.2)). Using the same argument for Z^{-1} we obtain $\mathrm{Re}(-B^TA^{-T}) \geq 0$, so AB^T is purely imaginary, hence B is purely imaginary.

iii) Assume that $0 < r < n$, where r denotes the rank of C. Let us use the decomposition and the notation of section 1.2. If we define for every $\alpha > 0$ the matrix $W(\alpha)$ by

$$W(\alpha) := \begin{pmatrix} I_n & 0 \\ U_2\begin{pmatrix} 0 & 0 \\ 0 & i\alpha I_{n-r}\end{pmatrix}U_2^T & I_n \end{pmatrix},$$

then, using the fact that U_2 is real (see lemma 1.3.7), we see that $W(\alpha) \in G_1(n)$, hence $\mathcal{L}_{ZW(\alpha)}$ is well defined and maps $\mathcal{H}_n$ onto $\mathcal{H}_n$ (see 1.1.4.3 and theorem 1.6.2) for every $\alpha > 0$. If $A_\alpha, B_\alpha, C_\alpha$ and D_α are the block components of $ZW(\alpha)$, then we have

$$B_\alpha = B,\ D_\alpha = D,\ A_\alpha = U_1^{-T}\begin{pmatrix} A_{11} & i\alpha B_{12} \\ A_{21} & I_{n-r} + i\alpha B_{22}\end{pmatrix}U_2^T,$$

$$C_\alpha = U_1\begin{pmatrix} C_{11} & 0 \\ 0 & i\alpha I_{n-r}\end{pmatrix}U_2^T,$$

so C_α is non-singular. According to i) A_α and D_α are real and C_α and B_α are purely imaginary. Since the entries of $ZW(\alpha)$ are continuous functions of α and $ZW(\alpha)$ tends to Z if α tends to zero we infer that A and D are real and that C and B are purely imaginary. □

1.7. The semigroup $G_2(n)$

From the correspondence between $\mathcal{C}_n$ and $\mathbb{C}^{\frac{1}{2}n(n+1)}$ $(n \in \mathbb{N})$ we already know the notion of bounded set in $\mathcal{C}_n$. The following lemma will be useful.

1.7.1. Lemma. A subset F of $\mathcal{C}_n$ $(n \in \mathbb{N})$ is bounded if there exists a $\beta > 0$ such that

1.7.1.1. $$\beta^2 I_n - \bar{W}W > 0$$

for every W in F.

The lemma is obvious: 1.7.1.1 means that the inner product norm of W is less than β.

Note that $\overline{W}W$ is hermitian if $W \in \mathcal{C}_n$ and that 1.7.1.1 can be considered as a generalization of the circle with centre 0 and radius β. Obviously if F is a bounded subset of $\mathcal{C}_n$, so are the following sets:

$$\{Q^T W Q \mid W \in F\} \qquad (Q \in \mathcal{M}_{n\times n}(\mathbb{C}))$$

and

$$\{Q + W \mid W \in F\} \qquad (Q \in \mathcal{C}_n) \ .$$

We now consider a special subset of $G_0(n)$ $(n \in \mathbb{N})$ denoted by $G_2(n)$.

1.7.2. Definition. $G_2(n)$ denotes the set of all $Z \in G_0(n)$ $(n \in \mathbb{N})$ such that the image of $\mathcal{H}_n$ under $\mathcal{L}_Z$ is a bounded subset of $\mathcal{H}_n$ which is entirely contained in $\{T \mid T \in \mathcal{H}_n, \mathrm{Re}(T) \geq \alpha I_n\}$ for some positive number α.

Obviously $G_2(n)$ is a semigroup under matrix multiplication.

For $n = 1$, $G_2(n)$ consists of those elements of $SL(2)$ (see 1.1.2.5) such that the corresponding fractional linear transform maps the right half-plane into an open circular disc such that the closure of this disc is entirely contained in the right half-plane.

We want to derive conditions for $Z \in G_0(n)$ to be an element of $G_2(n)$. We first proof a lemma.

1.7.3. Lemma. Let $Z \in G_2(n)$ $(n \in \mathbb{N})$ with block components $A,B,C,D \in \mathcal{M}_{n\times n}(\mathbb{C})$ (according to 1.1.2). Then C is non-singular and $\mathrm{Re}(C^{-1}D) > 0$.

Proof. Suppose C is the all zero matrix. Then, from 1.1.1, we easily infer that

$$\mathcal{L}_Z = \Psi_{S\in\mathcal{H}_n}\ ASA^T + BA^T \ ,$$

so the image of $\mathcal{H}_n$ under $\mathcal{L}_Z$ is not bounded. Therefore C is not the all zero matrix.

Suppose $0 < r < n$, where r denotes the rank of C. Using the notation of section 1.2, in particular 1.2.6, we infer

$$\mathcal{L}_Z(U_2^{-T}\mathrm{diag}(T_1,T_2)U_2^{-1}) = U_1^{-T}\begin{pmatrix}(A_{11}T_1+B_{11})(C_{11}T_1+D_{11})^{-1} & B_{12} \\ B_{12}^T & B_{22}+T_2\end{pmatrix}U_1^{-1} ,$$

where $T_1 \in H_r$ and $T_2 \in H_{n-r}$, and this implies that the image of H_n under $\mathcal{L}_Z$ is not bounded.

Summarizing we see that C is non-singular.

From 1.4.1 we obtain

$$(S + C^{-1}D)^{-1} = C^T(AC^{-1} - \mathcal{L}_Z(S))C \quad (S \in H_n) ,$$

so the image of H_n under the map

$$\forall_{S\in H_n} (S + C^{-1}D)^{-1}$$

is bounded. It is not hard to see that this implies

$$\mathrm{Re}(C^{-1}D) > 0 .$$

□

The result of this lemma enables us to prove the following theorem.

1.7.5. Theorem. Let $Z \in Sp(n)$ $(n \in \mathbb{N})$ with block components $A,B,C,D \in M_{n\times n}(\mathbb{C})$. Then $Z \in G_2(n)$ if and only if C is non-singular and

$$\mathrm{Re}\begin{pmatrix}AC^{-1} & C^{-T} \\ C^{-1} & C^{-1}D\end{pmatrix} > 0 .$$

Proof. The proof of this theorem is similar to that of 1.4.7. Instead of $E(X,Y) > 0$ we have to consider now $E(X,Y) > \alpha I_n$ for some $\alpha > 0$ and for every positive definite $X \in M_{n\times n}(\mathbb{R})$ and every symmetric $Y \in M_{n\times n}(\mathbb{R})$ (see the proof of theorem 1.4.7). □

1.8. Some additional properties of $G_2(n)$

In this section we shall discuss some additional properties of $G_2(n)$ elements.

First of all we shall show that if $Z \in G_2(n)$ $(n \in \mathbb{N})$, the image of H_n under $\mathcal{L}_Z$ can be denoted in a way that generalizes the nice circular disc we have in the case $n = 1$.

1.8.1. Let $Z \in G_2(n)$ $(n \in \mathbb{N})$ and let $A,B,C,D \in M_{n\times n}(\mathbb{C})$ denote the block components of Z (according to 1.1.2). From 1.7.3 we see that C is non-singular and that $U > 0$, where $U := \mathrm{Re}(C^{-1}D)$. According to 1.4.1, we only have to deal with the mapping

$$\forall_{S\in H_n}\ AC^{-1} - C^{-T}(S + U)^{-1}C^{-1} .$$

i) We consider the mapping

1.8.1.1. $$\forall_{S\in H_n}\ (S + U)^{-1} .$$

This mapping maps H_n one-to-one into the set K_1 defined by

$$K_1 := \{W \in C_n \mid U^{-1} - \overline{(2W - U^{-1})}U(2W - U^{-1}) > 0\} ,$$

for

$$U^{-1} - \overline{(2(S + U)^{-1} - U^{-1})}U(2(S + U)^{-1} - U^{-1}) =$$

$$(\overline{S+U})^{-1}[(\overline{S+U})U^{-1}(S+U) - (2I_n - (\overline{S+U})U^{-1})U(2I_n - U^{-1}(S+U))](S+U)^{-1} =$$

$$(\overline{S+U})^{-1}(S+\bar{S})(S+U)^{-1} > 0 \quad (S \in H_n) .$$

Furthermore, if $W \in K_1$ and if $x \in M_{n\times 1}(\mathbb{C})$ such that $Wx = 0$, then

$$x^H[U^{-1} - (2\bar{W} - U^{-1})U(2W - U^{-1})]x = x^H U^{-1}x - x^H U^{-1}x = 0 ,$$

so $x = 0$. Therefore W is non-singular.

Now

$$U^{-1} - (2\bar{W} - U^{-1})U(2W - U^{-1}) = 2\bar{W}(\mathrm{Re}(W^{-1} - U))W > 0 ,$$

so if we define S by

$$S := W^{-1} - U ,$$

then $S \in H_n$ and

$$(S + U)^{-1} = W .$$

This implies that the mapping 1.8.1.1 maps H_n onto K_1.

ii) The mapping

$$\forall_{W\in K_1}\ C^{-T}WC^{-1}$$

maps K_1 one-to-one onto the set K_2 defined by

$$K_2 := \{W \in \mathcal{C}_n \mid C^{-H}UC^{-1} - \overline{(2W - C^{-T}U^{-1}C^{-1})}(C^{-T}\bar{U}\bar{C}^{-1})^{-1}(2W - C^{-T}U^{-1}C^{-1}) > 0\}.$$

and finally

iii) the mapping

1.8.1.3. $$\Psi_{W \in K_2} \; AC^{-1} - W$$

maps K_2 one-to-one onto the set

1.8.1.4. $$\{W \in \mathcal{C}_n \mid R - (\overline{W - M})\bar{R}^{-1}(W - M) > 0\} ,$$

where

$$M := AC^{-1} - \tfrac{1}{2}C^{-T}U^{-1}C^{-1}$$

and

$$R := \tfrac{1}{2}C^{-H}U^{-1}C^{-1} .$$

Note that R is hermitian and positive definite and that $M \in \mathcal{C}_n$. The set 1.8.1.4 is the image of $\mathcal{H}_n$ under the mapping $\mathcal{L}_Z$ and this set can be considered as the generalization of the open circular disc with centre M and radius R.

1.8.2. Note that if $Z \in Sp(n)$ $(n \in \mathbb{N})$ with block components A,B,C and $D \in \mathcal{M}_{n\times n}(\mathbb{C})$ such that C is non-singular and $\mathrm{Re}(C^{-1}D) > 0$, then $\mathcal{L}_Z$ is well defined (see lemma 1.3.2.1) and the image of $\mathcal{H}_n$ under $\mathcal{L}_Z$ is bounded and can be represented in the same way as was done for $Z \in G_2(n)$ in 1.8.1.
Assume that $R \in \mathcal{M}_{n\times n}(\mathbb{C})$ is positive definite and that $M \in \mathcal{C}_n$. Then there exists a non-singular $P \in \mathcal{M}_{n\times n}(\mathbb{C})$ such that $P^HP = R$. Now the fractional linear transform related to the symplectic matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} P^T + MP^{-1} & MP^{-1} - P^T \\ P^{-1} & P^{-1} \end{pmatrix}$$

clearly is well defined (see lemma 1.3.2.1) and maps $\mathcal{H}_n$ one-to-one onto the set

$$\{W \in \mathcal{C}_n \mid R - (\overline{W - M})\bar{R}^{-1}(W - M) > 0\}$$

(see 1.8.1). So every generalized open circular disc (see 1.8.1.4) is the image of $\mathcal{H}_n$ under a fractional linear transform.

1.8.3. From 1.1.6 we obtain that $M_n(\alpha) \in G_2(n)$ $(\alpha > 0,\ n \in \mathbb{N})$. Application of 1.8.1.4 with $Z = M_n(\alpha)$ gives

1.8.3.1. $$\mathcal{L}_{M_n(\alpha)}(H_n) = \{W \in C_n \mid R^2(\alpha) I_n - \overline{(W - N(\alpha) I_n)}\,(W - N(\alpha) I_n) > 0\} \ ,$$

where

$$R(\alpha) \ := \ (2 \sinh\alpha \cosh\alpha)^{-1}$$

and

$$N(\alpha) \ := \ (\cosh^2\alpha + \sinh^2\alpha)(2 \sinh\alpha \cosh\alpha)^{-1} \ .$$

Expression 1.8.3.1 enables us to prove the following theorem.

1.8.4. <u>Theorem</u>. If $Z \in G_2(n)$ $(n \in \mathbb{N})$, then there exists a $\beta > 0$ and a $Z' \in G_2(n)$ such that $Z = M_n(\beta) Z'$.

<u>Proof</u>. The mapping $\mathcal{L}_{M_n(\beta)}$ $(\beta > 0)$ maps H_n one-to-one onto $\mathcal{L}_{M_n(\beta)}(H_n)$ and its inverse is given by

$$\forall_{S \in \mathcal{L}_{M_n(\beta)}(H_n)} \ (\cosh\beta\ S - \sinh\beta\ I_n)(-\sinh\beta\ S + \cosh\beta\ I_n)^{-1}$$

(see 1.1.4.2). From 1.7.1 and 1.7.2 we infer that there exist positive numbers α and γ such that

$$W + \bar{W} \geq \gamma I_n$$

and such that

$$\alpha^2 I_n - \bar{W}W > 0$$

for every $W \in \mathcal{L}_Z(H_n)$. So if $W \in \mathcal{L}_Z(H_n)$, we have

$$R^2(\beta) I_n - \overline{(W - N(\beta) I_n)}\,(W - N(\beta) I_n) =$$

$$\alpha^2 I_n - \bar{W}W + N(\beta)(W + \bar{W}) + (R^2(\beta) - \alpha^2 - N^2(\beta)) I_n > 0$$

for β sufficiently small $(0 < \beta < \delta$ say) since in that case

$$\gamma N(\beta) + R^2(\beta) - N^2(\beta) > \alpha^2 \ .$$

So $\mathcal{L}_Z(H_n)$ is entirely contained in $\mathcal{L}_{M_n(\beta)}$ $(0 < \beta < \delta)$ and therefore the mapping

$$\forall_{S\in H_n} \quad (\cosh\beta\ \mathcal{L}_Z(S) - \sinh\beta\ I_n)(-\sinh\beta\ \mathcal{L}_Z(S) + \cosh\beta\ I_n)^{-1}$$

is well defined and maps H_n into H_n. Since this mapping corresponds with the matrix $M_n^{-1}(\beta)Z$ we have

$$M_n^{-1}(\beta)Z \in G_0(n) \qquad (0 < \beta < \delta) .$$

If we take $\beta := \frac{1}{2}\delta$ and $Z' := M_n^{-1}(\frac{\delta}{2})Z$, then $Z' = M_n(\frac{\delta}{4})[M_n^{-1}(\frac{3\delta}{4})Z]$, so $Z' \in G_2(n)$ and

$$Z = M_n(\frac{\delta}{2})Z' \in G_2(n) . \qquad \square$$

If $Z \in G_2(n)$ $(n \in \mathbb{N})$ with block components $A,B,C,D \in M_{n\times n}(\mathbb{C})$, then B and C play a symmetric rôle.

1.8.5. <u>Theorem</u>. If $Z \in G_2(n)$ $(n \in \mathbb{N})$, then $Z^T \in G_2(n)$.

<u>Proof</u>. Let $A,B,C,D \in M_{n\times n}(\mathbb{C})$ denote the block components of Z (according to 1.1.2). From 1.7.5 we infer that C is non-singular and that $\mathrm{Re}(C^{-1}D) > 0$, $\mathrm{Re}(AC^{-1}) > 0$. Since $C^{-1}D$ and AC^{-1} are symmetric (see 1.1.2.5) we obtain

$$AC^{-1} \in H_n,\ C^{-1}D \in H_n ,$$

hence A and D are non-singular (see 1.1.3).
Suppose now there is an $x \in M_{n\times 1}(\mathbb{C})$ such that $Bx = 0$. If we define $y := -Dx$, then we have

$$A^Ty + x = 0$$
$$y + Dx = 0$$

(see 1.1.2.1), so

$$\begin{pmatrix} C^{-T}A^T & C^{-T} \\ C^{-1} & C^{-1}D \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

Using the symmetry of AC^{-1}, 1.7.5 and 1.1.3, we see that $x = 0$, hence B is non-singular.

Furthermore, from 1.1.2.1 we have

$$\begin{pmatrix} AC^{-1} & C^{-T} \\ C^{-1} & C^{-1}D \end{pmatrix} \begin{pmatrix} DB^{-1} & -B^{-T} \\ -B^{-1} & B^{-1}A \end{pmatrix} = I_{2n}$$

and since the first matrix is an element of H_{2n} (see 1.7.5) the second matrix also is an element of H_{2n} and therefore

$$\mathrm{Re} \begin{pmatrix} DB^{-1} & B^{-T} \\ B^{-1} & B^{-1}A \end{pmatrix} > 0 \ .$$

From the symmetry of DB^{-1} and $B^{-1}A$ and from 1.7.5 we easily see now that $Z^T \in G_2(n)$. □

1.8.6. Corollary 1. If $Z \in G_2(n)$ ($n \in \mathbb{N}$) with block components $A,B,C,D \in M_{n\times n}(\mathbb{C})$, then A,B,C and D are non-singular.

1.8.7. Corollary 2. If $Z \in G_2(n)$ ($n \in \mathbb{N}$), then there exists an $\alpha > 0$ and a $Z' \in G_2(n)$ such that

$$Z = Z'M_n(\alpha) \ .$$

This is an easy consequence of theorem 1.8.4, theorem 1.8.5 and the symmetry of $M_n(\alpha)$ ($\alpha > 0$).

1.8.8. Corollary 3. If $Z \in G_2(n)$, $V \in G_0(n)$, $W \in G_0(n)$ ($n \in \mathbb{N}$), then the matrix Z_1 defined by $Z_1 := VZW$ belongs to $G_2(n)$.

This can be seen as follows.
According to theorem 1.8.4 there exists an $\alpha > 0$ and a $Z' \in G_2(n)$ such that $Z = M_n(\alpha)Z'$. According to corollary 1.5.3 and theorem 1.7.5, $VM_n(\alpha) \in G_2(n)$, so by using 1.8.4 we infer the existence of a $\beta > 0$ and a $V' \in G_2(n)$ such that

$$Z_1 = M_n(\beta)[V'Z'W] \ ,$$

and since $V'Z'W \in G_0(n)$ we obviously have that $Z_1 \in G_2(n)$.

1.9. Fixed point of the fractional linear transform

1.9.1. In this section we shall prove that the fractional linear transform related to a matrix $Z \in G_2(n)$ $(n \in \mathbb{N})$ has a unique fixed point, i.e. there exists exactly one $T \in \mathcal{H}_n$ such that

1.9.1.1. $$\mathcal{L}_Z(T) = T .$$

We first prove a lemma.

1.9.2. Lemma. Let $Z \in G_2(n)$ $(n \in \mathbb{N})$. Then the closure of $\mathcal{L}_Z(\mathcal{H}_n)$ is homeomorphic to the set E_n, where E_n $(n \in \mathbb{N})$ is defined by

$$E_n := \{W \in \mathcal{C}_n \mid I_n - \bar{W}W \geq 0\} .$$

Proof. From 1.8.1 we obtain the existence of a non-singular $P \in \mathcal{M}_{n\times n}(\mathbb{C})$ and an $M \in \mathcal{C}_n$ such that

$$\overline{\mathcal{L}_Z(\mathcal{H}_n)} = \{W \in \mathcal{C}_n \mid P^H P - (\overline{W-M})\bar{P}^{-1}P^{-T}(W-M) \geq 0\} .$$

Let $W \in \overline{\mathcal{L}_Z(\mathcal{H}_n)}$. Then

$$P^H(I_n - (\overline{P^{-T}WP^{-1} - P^{-T}MP^{-1}})(P^{-T}WP^{-1} - P^{-T}MP^{-1}))P \geq 0 ,$$

so

$$\bar{P}^{-T}WP^{-1} - P^{-T}MP^{-1} \in E_n .$$

Conversely, if $V \in E_n$ and $W := P^T VP + M$, then $W \in \overline{\mathcal{L}_Z(\mathcal{H}_n)}$. Therefore we see that the mapping

$$\Psi_{V \in E_n} \; P^T VP + M$$

maps E_n one-to-one onto $\overline{\mathcal{L}_Z(\mathcal{H}_n)}$ and it is easily seen that this mapping and its inverse are continuous, thus we have established a homeomorphism. □

1.9.3. The set E_n of lemma 1.9.2 can be considered as a compact subset of $\mathbb{C}^{\frac{1}{2}n(n+1)}$. If furthermore $x \in \mathcal{M}_{n\times 1}(\mathbb{C})$ and if $W \in E_n$, then, using the symmetry of W, we have

$$x^H x - (Wx)^H(Wx) \geq 0 ,$$

so

1.9.3.1. $$\|Wx\| \leq \|x\| \ ,$$

where $\|.\|$ denotes the usual norm in $\mathbb{C}^n$:

$$\|x\| := (x^H x)^{\frac{1}{2}} \qquad (x \in M_{n\times 1}(\mathbb{C})) \ .$$

Now assume that W_1 and W_2 are elements of E_n and that $0 \leq \lambda \leq 1$. Then

$$I_n - \overline{(\lambda W_1 + (1-\lambda)W_2)}(\lambda W_1 + (1-\lambda)W_2) =$$

$$\lambda^2(I_n - \bar{W}_1 W_1) + (1-\lambda)^2(I_n - \bar{W}_2 W_2) + \lambda(1-\lambda)(2I_n - \bar{W}_1 W_2 - \bar{W}_2 W_1) \geq 0$$

which can easily be seen by using 1.9.3.1. So E_n is convex.

Summarizing we see that E_n is a convex compact subset of $\mathbb{C}^{\frac{1}{2}n(n+1)}$, and since the mapping $\mathcal{L}_Z$ maps $\overline{\mathcal{L}_Z(H_n)}$ continuously into itself and $\overline{\mathcal{L}_Z(H_n)}$ is homeomorphic to E_n we infer from a theorem of Schauder (see [DS], page 456) that $\mathcal{L}_Z$ has a fixed point. We have thus proved the following lemma.

1.9.4. Lemma. If $Z \in G_2(n)$ $(n \in \mathbb{N})$, then there exists a $T \in H_n$ such that

$$\mathcal{L}_Z(T) = T \ .$$

Next we shall show the uniqueness of the fixed point of lemma 1.9.4.

We first prove three lemmas.

1.9.5. Lemma. If $U \in M_{n\times n}(\mathbb{C})$ $(n \in \mathbb{N})$ is a unitary matrix (i.e. $U^H U = I_n$), then

$$\begin{pmatrix} \mathrm{Re}(U) & i\ \mathrm{Im}(U) \\ i\ \mathrm{Im}(U) & \mathrm{Re}(U) \end{pmatrix} \in G_1(n) \ .$$

Proof. If we abbreviate $L = \mathrm{Re}(U)$, $M = \mathrm{Im}(U)$, we easily obtain from $U^H U = I_n$ that

$$L^T L + M^T M = I_n, \ L^T M = M^T L \ ,$$

so the matrix

$$\begin{pmatrix} L & iM \\ iM & L \end{pmatrix}$$

is symplectic. Furthermore L and M both are real. □

1.9.6. Lemma. If $T \in H_n$ ($n \in \mathbb{N}$), then the matrix $M(T)$ defined by

$$M(T) := \begin{pmatrix} \mathrm{Im}(T)P^{-1} & -iP^T \\ -iP^{-1} & 0 \end{pmatrix},$$

where $P \in M_{n\times n}(\mathbb{R})$ is such that $\mathrm{Re}(T) = P^TP$, belongs to $G_1(n)$ and $\mathcal{L}_{M(T)}(I_n) = T$.

The proof of this lemma is obvious.

1.9.7. Lemma. Let $Z \in G_2(n)$ ($n \in \mathbb{N}$) with block components A,B,C and $D \in M_{n\times n}(\mathbb{C})$. Assume that

$$A + B = C + D = \Lambda ,$$

where $\Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ with positive real numbers $\lambda_1,\ldots,\lambda_n$. Then $\Lambda > I_n$.

Proof. From $A + B = C + D = \Lambda$ and the relations of 1.1.2 we infer

$$I_n = (A + B)^TD - B^TD - C^TB = \Lambda(D - B) ,$$

so

$$D - B = \Lambda^{-1} .$$

Therefore

$$Z = \begin{pmatrix} A & \Lambda - A \\ A - \Lambda^{-1} & \Lambda + \Lambda^{-1} - A \end{pmatrix} .$$

Since $Z \in G_2(n)$ we infer that $A - \Lambda^{-1}$ is non-singular (see lemma 1.7.3) and that the matrix K defined by

$$K := \begin{pmatrix} I_n + \Lambda^{-1}U & U^T \\ U & U\Lambda - I_n \end{pmatrix},$$

where U denotes $\mathrm{Re}((A - \Lambda^{-1})^{-1})$, is positive definite (see theorem 1.7.5). Hence the quadratic form corresponding to K is positive definite: for every nonzero $x \in M_{n\times 1}(\mathbb{R})$ we have

$$(x^T, -x^T\Lambda^{-1})K\begin{pmatrix} x \\ \\ -\Lambda^{-1}x \end{pmatrix} > 0 ,$$

and since $U\Lambda$ is symmetric (note that K is symmetric) we obtain

$$x^T(I_n - \Lambda^{-2})x > 0$$

whenever $x \in M_{n\times 1}(\mathbb{R})$, $x \neq 0$. It follows that $\Lambda > I_n$. □

1.9.8. Let $Z \in G_2(n)$ ($n \in \mathbb{N}$) with block components A,B,C and $D \in M_{n\times n}(\mathbb{C})$ and let $T \in H_n$ denote a fixed point of $\mathcal{L}_Z$ (see theorem 1.9.4). Then there exists a non-singular $P \in M_{n\times n}(\mathbb{R})$ such that

$$T = P^TP + i\ \mathrm{Im}(T) \ .$$

If we define $Z_1 := M(T)ZM^{-1}(T)$, then $Z_1 \in G_1(n)$ (see lemma 1.9.6 and corollary 1.8.8) and we have

$$\mathcal{L}_{Z_1}(I_n) = \mathcal{L}_{M(T)}(\mathcal{L}_Z(\mathcal{L}_{M^{-1}(T)}(I_n))) = I_n$$

(see 1.4.3, lemma 1.9.6 and theorem 1.6.2). So if A_1,B_1,C_1 and $D_1 \in M_{n\times n}(\mathbb{C})$ denote the block components of Z_1, we have

$$A_1 + B_1 = C_1 + D_1 = P(CT + D)P^{-1} \ .$$

Since $C_1 + D_1$ is non-singular there exist matrices L_1,L_2,M_1 and $M_2 \in M_{n\times n}(\mathbb{R})$ such that $L_1 + iM_1$ and $L_2 + iM_2$ are unitary and such that

$$(L_1 + iM_1)(C_1 + D_1)(L_2 + iM_2) = \Lambda ,$$

where $\Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ with positive numbers $\lambda_1,\ldots,\lambda_n$. If we define

$$S_j := \begin{pmatrix} L_j & iM_j \\ iM_j & L_j \end{pmatrix} \quad (j = 1,2) ,$$

then, according to lemma 1.9.5, $S_j \in G_1(n)$ ($j = 1,2$) and Z_2 defined by $Z_2 := S_1Z_1S_2$ belongs to $G_2(n)$ (see 1.8.8). We easily see that I_n is a fixed point of $\mathcal{L}_{Z_2}$. An easy computation furthermore leads to $A_2 + B_2 = C_2 + D_2 = \Lambda$, where A_2,B_2,C_2 and D_2 denote the block components of Z_2. Application of lemma 1.9.7 gives the following lemma.

1.9.9. Lemma. If the conditions of 1.9.8 are satisfied, then

$$P[CT + D]P^{-1}$$

is equivalent to a real diagonal matrix Λ under the group of unitary matrices, and we have $\Lambda > I_n$.

We shall use this lemma to prove the uniqueness of the fixed point of theorem 1.9.4.

1.9.10. Theorem. If $Z \in G_2(n)$ $(n \in \mathbb{N})$, then the fractional linear transform $\mathcal{L}_Z$ has exactly one fixed point in H_n.

Proof. Lemma 1.9.4 establishes the existence of atleast one fixed point. Suppose there are two fixed points T_1 and T_2 in H_n, $T_1 \neq T_2$. There exist non-singular matrices $P_1 \in M_{n\times n}(\mathbb{R})$ and $P_2 \in M_{n\times n}(\mathbb{R})$ such that $\mathrm{Re}(T_j) = P_j^T P_j$ $(j = 1,2)$. Let (C,D) denote the second matrix row of Z $(C \in M_{n\times n}(\mathbb{C})$, $D \in M_{n\times n}(\mathbb{C}))$. According to lemma 1.9.9 there exist unitary matrices $U_1, V_1, U_2, V_2 \in M_{n\times n}(\mathbb{C})$ and real diagonal matrices $\Lambda_1, \Lambda_2 \in M_{n\times n}(\mathbb{R})$ such that

$$\Lambda_j > I_n,\ CT_j + D = P_j^{-1} U_j \Lambda_j V_j P_j \qquad (j = 1,2)\ .$$

We now have

$$T_1 - T_2 = \mathcal{L}_Z(T_1) - \mathcal{L}_Z(T_2) = (\mathcal{L}_Z(T_1))^T - \mathcal{L}_Z(T_2) =$$

$$= (CT_1 + D)^{-T}(T_1 - T_2)(CT_2 + D)^{-1}$$

(see 1.1.2.1 and 1.1.4.3), so if we define $Q \in M_{n\times n}(\mathbb{C})$ by

$$Q := U_1^T P_1^{-T}(T_1 - T_2)P_2^{-1}U_2\ ,$$

then we have

$$Q = \Lambda_1^{-1} W_1 Q W_2 \Lambda_2^{-1}\ ,$$

where W_1 and W_2 are unitary matrices. Using the submultiplicativity of the matrixnorm

$$\|\ \|_2 := \forall_{A \in M_{n\times n}(\mathbb{C})}\ [\max \sigma(A^H A)]^{\frac{1}{2}}\ ,$$

where $\sigma(B)$ denotes the set of eigenvalues of B $(B \in M_{n\times n}(\mathbb{C}))$, we obtain

$$\|Q\|_2 \leq \|\Lambda_1^{-1}\|_2 \|W_1\|_2 \|Q\|_2 \|W_2\|_2 \|\Lambda_2^{-1}\| = \rho \|Q\|_2 ,$$

where $0 < \rho < 1$, hence $Q = 0$ and therefore $T_1 = T_2$. □

2. Function space transforms related to symplectic matrices

2.0. Introduction

2.0.1. In this chapter we intend to apply the theory about fractional linear transforms developped in chapter one in order to study certain linear transforms of both the space of smooth functions and the space of generalized functions of n variables. The class of operators that we consider consists of integral operators related to symplectic matrices and a vector in the phase space: if $n \in \mathbb{N}$ and if Z is a symplectic matrix of order n with block components A,B,C,D (with non-singular C) and if $\binom{e}{f} \in M_{(n+n)\times 1}(\mathbb{C})$ is a vector in the 2n-dimensional phase space, then we consider the integral operator

2.0.1.1. $$\Psi_{g\in S^n} \Psi_{z\in\mathbb{C}^n} \int_{\mathbb{R}^n} K(z,t)g(t)dt \, ,$$

where

$$K(z,t) := \det(C)^{-\frac{1}{2}}\exp(-\pi[(AC^{-1}z,z) - 2(C^{-1}z,t)$$

$$+ (C^{-1}Dt,t) + 2(C^{-1}f,t) + 2(e - AC^{-1}f,z)$$

$$+ (AC^{-1}f - e,f)]) \quad (z \in \mathbb{C}^n,\ t \in \mathbb{C}^n) \, ,$$

and $(\det(C))^{-\frac{1}{2}}$ is a complex number which square equals $(\det(C))^{-1}$ and which sign is explained in 2.1. Compare [B], 27.3.9.

In the general case, where C might be singular, we are still able to study operators of this type, but then we consider

2.0.1.2. $$\Psi_{g\in S^n} \Psi_{z\in\mathbb{C}^n} [F_z,\bar{g}]_n \, ,$$

where F_z is a generalized function of n variables for every $z \in \mathbb{C}^n$ which is (apart from its sign) uniquely determined by Z and $\binom{e}{f}$. This F_z equals $\Psi_{t\in\mathbb{C}^n} K(z,t)$ for every $z \in \mathbb{C}^n$ if C is non-singular.

2.0.2. We mention one result in particular: if $Z \in G_1(n)$ $(n \in \mathbb{N})$ and if r denotes the rank of C, then the operator of type 2.0.1.2 can (after suitable transformations) be reduced to an integral operator of the type 2.0.1.1, in which only r integrations occur.

2.0.3. We shall show that it is possible to extend the operators of 2.0.1.2 to linear operators of the class of generalized functions of n variables such that the extended operator and the original one coincide on S^n.

2.0.4. Furthermore we shall devote some attention to applications of our theory in quantum mechanics. In particular we study the classical limit in quantum mechanics in relation with the Wigner distribution (this was already discussed in [B], 27.26.3, but here we consider the case that more particles are involved).

2.1. Analytic square-root of determinant functions

In this section the notion of analytic functions of several variables will be used. For the definition of analytic functions of several variables we refer to [D], chapter IX. Furthermore we use a theorem of Hartogs ([BT], III, satz 15) which states that a function of several variables is analytic in all variables if and only if it is analytic in every variable separately.

2.1.1. Let χ be a mapping on $\mathcal{H}_n$ ($n \in \mathbb{N}$) that maps $\mathcal{H}_n$ into $\mathbb{C}$ such that χ is an analytic function of $\frac{1}{2}n(n+1)$ complex variables (see section 1.1), and assume that χ is nonzero on $\mathcal{H}_n$.

2.1.1.1. An example of such a function is

$$\Upsilon_{T\in\mathcal{H}_n} \det(CT + D) \ ,$$

where (C,D) is an admissible pair of matrices of $\mathcal{M}_{n\times n}(\mathbb{C})$.

One of the aims of this section is to define an analytic function φ on $\mathcal{H}_n$ such that

$$\varphi^2 = \chi \ .$$

2.1.2. Let χ be as in 2.1.1 and let $T_0 \in \mathcal{H}_n$. It is easily seen that $\mathcal{H}_n$ may be considered as an open subset of $\mathbb{C}^{\frac{1}{2}n(n+1)}$ and combining this with the continuity of the function χ, we see that there exists a polydisc P with center T_0 (see [D], IX, section 1) such that P is entirely contained in $\mathcal{H}_n$ and such that

$$|\chi(T) - \chi(T_0)| < \tfrac{1}{2}|\chi(T_0)|$$

for every $T \in P$.

Now we easily infer the existence of exactly two analytic function ψ_1 and ψ_2 on P satisfying

2.1.2.1. $$\psi_1 = -\psi_2,\ \psi_1^2 = \chi \ .$$

If $T \in H_n$, we define $\psi_1(T)$ as follows: $\psi_1(T)$ is the analytic continuation of ψ_1 along the curve

2.1.2.2. $$\Psi_{t\in[0,1]}\ \chi((1-t)T_0 + tT) \ .$$

Note that $(1-t)T_0 + tT \in H_n$ for every $t \in [0,1]$. We shall define $\psi_2(T)$ similarly.

It is not hard to see that ψ_1 and ψ_2 are analytic functions on H_n and that

2.1.2.3. $$\psi_1 = -\psi_2,\ \psi_1^2 = \chi$$

on H_n.

Next suppose φ is an analytic function on H_n such that $\varphi^2 = \chi$. Then φ is analytic on the polydisc P, hence φ equals ψ_1 or ψ_2 on P and thus φ equals ψ_1 or ψ_2 on H_n.
So we have proved the following theorem.

2.1.3. Theorem. If χ is a mapping on H_n satisfying the conditions of 2.1.1, then there exist exactly two analytic functions ψ_1 and ψ_2 defined on H_n such that 2.1.2.3 holds.

2.1.4. Let $C \in M_{n\times n}(\mathbb{C})$ and $D \in M_{n\times n}(\mathbb{C})$ $(n \in \mathbb{N})$ be such that (C,D) is an admissible pair. Combining 2.1.1.1 and theorem 2.1.3, we infer the existence of exactly two analytic functions ψ_1 and ψ_2 defined on H_n such that

$$\psi_1 = -\psi_2,\ \psi_1^2 = \Psi_{T\in H_n}\ \det(CT + D) \ .$$

2.1.5. Applying theorem 2.1.3 to

$$\chi = \Psi_{T\in H_n}\ \det(T) \ ,$$

we see that there exists exactly one analytic function φ_n such that

2.1.5.1. $$\varphi_n^2 = \Psi_{T\in H_n}\ \det(T),\ \varphi_n(I_n) = 1 \ .$$

2.1.6. We shall use this function φ_n for evaluating the integral

2.1.6.1. $$F(T) := \int_{\mathbb{R}^n} \exp(-\pi(Tx,x))\,dx \quad (T \in H_n) .$$

Let $T \in H_n$. Since $\mathrm{Re}(T) > 0$ there exists a non-singular $P \in M_{n\times n}(\mathbb{R})$ such that $\mathrm{Re}(T) = P^T P$. From the symmetry of $P^{-T}\mathrm{Im}(T)P^{-1}$ we infer the existence of a matrix $S \in M_{n\times n}(\mathbb{R})$ and a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ ($\lambda_i \in \mathbb{R}$, $1 \le i \le n$) such that $S^T S = I_n$ and $P^{-T}\mathrm{Im}(T)P^{-1} = S^T\Lambda S$. This implies that

2.1.6.2. $$T = (SP)^T(I_n + i\Lambda)(SP) .$$

If we transform variables according to $y = SPx$, 2.1.6.1 becomes

$$F(T) = |\det(SP)|^{-1} \prod_{j=1}^{n} (1 + i\lambda_j)^{-\frac{1}{2}} ,$$

where $z^{\frac{1}{2}}$ ($z \in \mathbb{C}$) denotes the principal value of the square-root. Therefore we have

2.1.6.3. $$F^2(T) = \det(T)^{-1} \quad (T \in H_n) .$$

Since F is obviously a continuous function on H_n and $F^2 = \varphi_n^2$ (see 2.1.5.1) F is an analytic function on H_n. Using 2.1.5 and 2.1.6.3 we infer from $F(I_n) > 0$ that

2.1.6.4. $$F = \varphi_n^{-1} .$$

With every $Z \in G_0(n)$ ($n \in \mathbb{N}$) we can associate <u>two</u> analytic functions ψ_1 and ψ_2 defined on H_n (according to 2.1.4) satisfying $\psi_1 = -\psi_2$, $\psi_1^2 = \det(CT + D)$, where (C,D) denotes the second matrix row of Z ($C \in M_{n\times n}(\mathbb{C})$, $D \in M_{n\times n}(\mathbb{C})$). These functions will be used frequently in this report.

2.1.7. <u>Definition</u>. $X_0(n)$ ($n \in \mathbb{N}$) denotes the set of all pairs $(Z(e,f),\psi)$ such that

i) $$Z(e,f) := \begin{pmatrix} Z & \binom{e}{f} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & B & e \\ C & D & f \\ 0 & 0 & 1 \end{pmatrix} ,$$

where $Z \in G_0(n)$ with block components A,B,C and $D \in M_{n\times n}(\mathbb{C})$ and $e,f \in M_{n\times 1}(\mathbb{C})$,

ii) ψ is an analytic function defined on H_n such that

$$\psi^2 = \Psi_{T \in H_n} \det(CT + D) \ .$$

The vectors $e \in M_{n\times 1}(\mathbb{C})$ and $f \in M_{n\times 1}(\mathbb{C})$ will often be called the vector components of $Z(e,f)$. If $e = f = 0$, then the matrix $Z(0,0)$ will often be denoted by Z, and $(Z(e,f), \psi)$ by (Z, ψ).

2.1.8. Definition 2.1.7 is based on the set $G_0(n)$. We have a similar definition of $X_2(n)$ (based on the set $G_2(n)$). The set $X_1(n)$ is defined similarly (based on the set $G_1(n)$), but here we have one extra condition: if $e \in M_{n\times 1}(\mathbb{C})$ and $f \in M_{n\times 1}(\mathbb{C})$ denote the vector components, we require that e is purely imaginary and that f is real.

2.1.9. It is possible to define a product in $X_0(n)$ $(n \in \mathbb{N})$ such that $X_0(n)$ becomes a semigroup. We shall denote this product by $\otimes$.
Let $(Z_1(e_1,f_1), \psi_1)$ and $(Z_2(e_2,f_2), \psi_2)$ be elements of $X_0(n)$. Then

$$(Z_1(e_1,f_1), \psi_1) \otimes (Z_2(e_2,f_2), \psi_2) := (Z_3(e_3,f_3), \psi_3) \ ,$$

where

$$Z_3 := Z_1 Z_2, \quad \begin{pmatrix} e_3 \\ f_3 \end{pmatrix} := \begin{pmatrix} e_1 \\ f_1 \end{pmatrix} + Z_1 \begin{pmatrix} e_2 \\ f_2 \end{pmatrix}$$

and

$$\psi_3 := \Psi_{T \in H_n} \psi_1(\mathcal{L}_{Z_2}(T))\psi_2(T) \ .$$

Note that

$$Z_1(e_1,f_1)Z_2(e_2,f_2) = Z_3(e_3,f_3) \ .$$

Obviously $Z_3 \in G_0(n)$ since $G_0(n)$ is a semigroup (see 1.4.2). It is easily seen that ψ_3 is well defined and that ψ_3 is an analytic function defined on H_n satisfying

$$\psi_3^2 = \Psi_{T \in H_n} \det(C_3 T + D_3) \ ,$$

where (C_3,D_3) denotes the second matrix row of Z_3. Furthermore, using 1.1.4.3, we easily infer that $\otimes$ is associative.

2.1.10. With every $(Z(e,f),\psi) \in X_0(n)$ $(n \in \mathbb{N})$ we can associate the number

$$\rho := \lim_{t\to\infty} \frac{\psi(tI_n)}{|\psi(tI_n)|} .$$

The existence of this limit is easily proved in case the C component of Z (see 2.1.7) is non-singular or the all zero matrix. In all other cases we can use 1.3.7.1.

This ρ satisfies $|\rho| = 1$. Now to every $(Z(e,f),\psi) \in X_0(n)$ we make correspond the $4n^2+2n+1$-dimensional complex vector consisting of all non-trivial entries of $Z(e,f)$ and the number ρ, and this gives an injection of $X_0(n)$ into the $4n^2+2n+1$-dimensional complex space. The ordinary topology of this space induces a topology on $X_0(n)$. With this topology $X_0(n)$ is a topological semigroup (in the sense that the product $\otimes$ is a continuous function of the factors). It is not hard to see that $X_2(n)$ is also a topological semigroup and that $X_1(n)$ is a topological group (the inverse of an element is a continuous function of that element): If $(Z(e,f),\psi) \in X_1(n)$, then the inverse is given by:

$$(Z(e,f),\psi)^{-1} = (Z(e,f)^{-1}, \Psi_{T\in H_n} [\psi(\mathcal{L}_{Z^{-1}}(T))]^{-1})$$

(see section 1.6), where $Z(e,f)^{-1}$ denotes the inverse of the matrix $Z(e,f)$ (which is an element of $M_{(2n+1)\times(2n+1)}(\mathbb{C})$). Note that

$$Z(e,f)^{-1} = Z^{-1}(e',f') ,$$

where $e' \in M_{n\times 1}(\mathbb{C})$ and $f' \in M_{n\times 1}(\mathbb{C})$ are defined by

$$\begin{pmatrix} e' \\ f' \end{pmatrix} = -Z^{-1}\begin{pmatrix} e \\ f \end{pmatrix} .$$

2.1.11. Let $(Z(e,f),\psi) \in X_2(n)$ and let (C,D) denote the second matrix row of Z ($C \in M_{n\times n}(\mathbb{C})$, $D \in M_{n\times n}(\mathbb{C})$). According to lemma 1.7.3, C is non-singular. It is possible to define the square-root of $\det(C)$ in a natural way, based on the function ψ:

2.1.11.1. $$\det(C)^{\frac{1}{2}} := \lim_{t\to\infty} \psi(tI_n)t^{-\frac{n}{2}} .$$

We denote $(\det(C)^{\frac{1}{2}})^{-1}$ as usually by $\det(C)^{-\frac{1}{2}}$.

2.1.12. The set $X_0(n)$ describes a double covering of the set of all $Z(e,f)$ $(Z \in G_0(n),\ e \in M_{n\times 1}(\mathbb{C}),\ f \in M_{n\times 1}(\mathbb{C}))$. We have to answer the question whether this double covering is really necessary. It is indeed, since the set $X_0(n)$ is a connected topological space. This depends essentially on the fact that $G_0(n)$ is connected (in the sense of the topology induced by the ordinary topology of C^{4n^2} (see 2.1.10); we omit the proof) and that $G_0(n)$ contains closed curves such that the C component is non-singular for all points of the curve and such that $\det(C)^{\frac{1}{2}}$ changes sign after a full revolution: If we define

$$Z_c := \begin{pmatrix} 2cI_n & (2c - c^{-1})I_n \\ cI_n & cI_n \end{pmatrix},\ \psi_c := \Psi_{T\in H_n}\ c^{\frac{1}{2}}\varphi_n(I_n + T)$$

(see 2.1.5), where c denotes an arbitrary element of the unit circle in C, we infer from

$$\mathrm{Re}\begin{pmatrix} 2I_n & c^{-1}I_n \\ c^{-1}I_n & I_n \end{pmatrix} > 0,\ \psi_c^2 = \Psi_{T\in H_n}\ \det(cT + cI_n)$$

that $(Z_c(0,0),\psi_c) \in X_0(n)$ (see theorem 1.7.5).
Now if c runs through the unit circle, then after a full revolution we have the same Z_c, but ψ_c has changed sign.

2.1.13. We conclude this section with a property that will be used in the next section.

Let $(Z_1(e_1,f_1),\psi_1)$, $(Z_2(e_2,f_2),\psi_2)$ and $(Z_3(e_3,f_3),\psi_3)$ be elements of $X_2(n)$. Let (C_1,D_1), (C_2,D_2) and (C_3,D_3) denote the second matrix row of Z_1, Z_2 and Z_3, and assume that

$$(Z_1(e_1,f_1),\psi_1) \otimes (Z_2(e_2,f_2),\psi_2) = (Z_3(e_3,f_3),\psi_3)\ .$$

We shall prove that

2.1.13.1. $$\det(C_3)^{\frac{1}{2}}\det(C_1)^{-\frac{1}{2}}\det(C_2)^{-\frac{1}{2}}$$

(see 2.1.11) does not depend on the choice of ψ_1 and ψ_2 (see 2.1.7) and equals

2.1.13.2. $$\varphi_n(C_1^{-1}C_3C_2^{-1})$$

(see 2.1.5).

It is not hard to check that

$$C_1^{-1}C_3C_2^{-1} \in H_n ,$$

so 2.1.13.2 is well defined. Furthermore, using 2.1.11.1 and 2.1.9, we see that 2.1.13.1 equals

2.1.13.3. $$\lim_{t\to\infty} \psi_1(\mathcal{L}_{Z_2}(tI_n))\psi_1(tI_n)^{-1}t^{n/2} ,$$

and indeed this does not depend on the choice of ψ_1 and ψ_2. If we define

2.1.13.4. $$\theta := \forall_{T\in H_n}\ \psi_1(\mathcal{L}_{Z_2}(T))\psi_1(T)^{-1} ,$$

then obviously θ is an analytic function defined on H_n and

2.1.13.5. $$\theta^2 = \forall_{T\in H_n}\ \det((C_1T + C_1)^{-1}(C_3T + D_3)(C_2T + D_2)^{-1}) .$$

As a result of a matrix computation we have

2.1.13.6. $$(C_1T + C_1)^{-1}(C_3T + D_3)(C_2T + D_2)^{-1} = (T + C_1^{-1}D_1)^{-1}(\mathcal{L}_{Z_2}(T) + C_1^{-1}D_1)$$

$(T \in H_n)$.

From theorem 1.9.10 we infer the existence of a $T_0 \in H_n$ such that $\mathcal{L}_{Z_2}(T_0) = T_0$, so, according to 2.1.13.4, we have $\theta(T_0) = 1$. Since

2.1.13.7. $$\forall_{T\in H_n}\ \varphi_n(\mathcal{L}_{Z_2}(T) + C_1^{-1}D_1)\varphi_n(T + C_1^{-1}D_1)^{-1}$$

is analytic on H_n we infer from 2.1.3, using 2.1.13.5, 2.1.13.6 and $\theta(T_0) = 1$, that θ equals 2.1.13.7 and therefore

$$\lim_{t\to\infty} \theta(tI_n)t^{n/2} = \varphi_n(\lim_{t\to\infty} \mathcal{L}_{Z_2}(tI_n) + C_1^{-1}D_1) .$$

It is not hard to check that this equals 2.1.13.2.

2.2. Operators related to elements of $X_0(n)$

In this section we shall relate to every element of $X_0(n)$ $(n \in \mathbb{N})$ an operator of S^n. Before dealing with $X_0(n)$ however, we restrict ourselves to $X_2(n)$.

2.2.1. <u>Definition</u>. Let $(Z(e,f),\psi) \in X_2(n)$ and let A,B,C and $D \in M_{n\times n}(\mathbb{C})$ denote the block components of Z (according to 1.1.2). Then we define

$$\begin{aligned} V(z,t;(Z(e,f),\psi)) := {}& \det(C)^{-\frac{1}{2}}\exp(-\pi[(AC^{-1}z,z) \\ & - 2(C^{-1}z,t) + (C^{-1}Dt,t) + 2(C^{-1}f,t) \\ & + 2(e - AC^{-1}f,z) + (AC^{-1}f - e,f)]) \quad (z \in \mathbb{C}^n,\ t \in \mathbb{C}^n). \end{aligned}$$

Note that C is non-singular (see 1.7.3) and that $\det(C)^{-\frac{1}{2}}$ is defined in 2.1.11. Obviously $(Z(e,f),-\psi) \in X_2(n)$, and we have

$$V(z,t;(Z(e,f),-\psi)) = -V(z,t;(Z(e,f),\psi)) \qquad (z \in \mathbb{C}^n,\ t \in \mathbb{C}^n) .$$

2.2.2. Let $(Z(e,f),\psi) \in X_2(n)$ and assume Z has block components A,B,C and $D \in M_{n\times n}(\mathbb{C})$ (according to 1.1.2). From lemma 1.7.3 we infer that C is non-singular and that $\mathrm{Re}(C^{-1}D) > 0$. Therefore

$$\Psi_{t\in\mathbb{C}^n} V(z,t;(Z(e,f),\psi))$$

is a smooth function of n variables for every fixed $z \in \mathbb{C}^n$. Furthermore, for every $g \in S^n$

$$\Psi_{z\in\mathbb{C}^n} \int_{\mathbb{R}^n} V(z,t;(Z(e,f),\psi))g(t)dt$$

is also an element of S^n: the analiticity can be shown by routine methods and the inequality of 0.2 by using theorem 1.7.5.

2.2.3. An important property of the integral transformation with kernel

$$\Psi_{(z,t)\in\mathbb{C}^n\times\mathbb{C}^n} V(z,t;(Z(e,f),\psi)) ,$$

where $(Z(e,f),\psi) \in X_2(n)$, is <u>multiplicativity</u>: If $(Z_1(e_1,f_1),\psi_1)$, $(Z_2(e_2,f_2),\psi_2)$ and $(Z_3(e_3,f_3),\psi_3)$ are elements of $X_2(n)$ such that

$$(Z_1(e_1,f_1),\psi_1) \otimes (Z_2(e_2,f_2),\psi_2) = (Z_3(e_3,f_3),\psi_3) ,$$

then

$$\int_{\mathbb{R}^n} V(z,\eta;(Z_1(e_1,f_1),\psi_1))V(\eta,t;(Z_2(e_2,f_2),\psi_2))d\eta =$$

$$\exp([(e_3,f_1) - (e_1,f_3)])V(z,t;(Z_3(e_3,f_3),\psi_3))$$

for every $z \in \mathbb{C}^n$ and every $t \in \mathbb{C}^n$.

The proof of this property is tedious but straightforward: it follows by evaluating the integral and by using 1.1.2 and 2.1.13.

2.2.4. As an example of 2.2.1 we consider ($M_n(\alpha)(0,0), \varphi_\alpha$) ($\alpha > 0$, $n \in \mathbb{N}$), where $M_n(\alpha)$ is defined in 1.1.6 and φ_α is the (unique) analytic function defined on H_n such that

$$\varphi^2 = \Psi_{T \in H_n} \det(\sinh\alpha\ T + \cosh\alpha\ I_n),\ \varphi_\alpha(I_n) > 0$$

(see 2.1.3 and 2.1.4).

From 1.1.6 and 2.1.7 we infer that ($M_n(\alpha), \varphi_\alpha$) is an element of $X_2(n)$, and using 2.1.9 and the fact that $\mathcal{L}_{M_n(\alpha)}(I_n) = I_n$ for every $\alpha > 0$ leads to

$$(M_n(\alpha), \varphi_\alpha) \otimes (M_n(\beta), \varphi_\beta) = (M_n(\alpha + \beta), \varphi_{\alpha+\beta}) \qquad (\alpha > 0,\ \beta > 0)\ .$$

Furthermore, as a consequence of 2.2.1 and 0.4, we see that

$$\Psi_{(z,t) \in \mathbb{C}^n \times \mathbb{C}^n} V(z,t;(M_n(\alpha), \varphi_\alpha))$$

is the kernel of the integral operator $N_{\alpha,n}$ ($\alpha > 0$). Combining this with the multiplicativity property (see 2.2.3) and the fact that

$$V(z,t;(M_n(\alpha), \varphi_\alpha)) = V(t,z;(M_n(\alpha), \varphi_\alpha)) \qquad (z \in \mathbb{C}^n,\ t \in \mathbb{C}^n)$$

we see that for every fixed $z \in \mathbb{C}^n$ and for every ($Z(e,f), \psi$) $\in X_2(n)$

$$N_{\alpha,n}(\Psi_{t \in \mathbb{C}^n} V(z,t;(Z(e,f), \psi))) =$$

$$\Psi_{t \in \mathbb{C}^n} V(z,t;(Z(e,f), \psi) \otimes (M_n(\alpha), \varphi_\alpha))\ .$$

If ($Z(e,f), \psi$) $\in X_0(n)$, then it is easily seen from 1.4.7 and 1.5.1 that ($Z(e,f), \psi$) $\otimes$ ($M_n(\alpha), \varphi_\alpha$) $\in X_2(n)$ for every $\alpha > 0$. Summarizing we see that for every ($Z(e,f), \psi$) $\in X_0(n)$

2.2.4.1. $$\Psi_{\alpha > 0} \Psi_{t \in \mathbb{C}^n} V(z,t;(Z(e,f), \psi) \otimes (M_n(\alpha), \varphi_\alpha))$$

is a generalized function of n variables ($z \in \mathbb{C}^n$).

Using the generalized function of 2.2.4.1, we are able to generalize the operators described in [B], 27.3.8 and 27.3.9 to the higher dimensional case.

2.2.5. <u>Definition</u>. If $(\ Z(e,f),\psi\ ) \in X_0(n)$ $(n \in \mathbb{N})$, then we define the operator $\Gamma_{(\ Z(e,f),\psi\ )}$ of S^n by

$$\Gamma_{(\ Z(e,f),\psi\ )} :=$$

$$\Psi_{g\in S^n}\ \Psi_{z\in\mathbb{C}^n}\ [\Psi_{\alpha>0}\ \Psi_{t\in\mathbb{C}^n}\ V(z,t;(\ Z(e,f),\psi\ ) \otimes (\ M_n(\alpha),\varphi_\alpha\ )),\bar{g}]_n\ .$$

For the definition of the inner product of an element of S^n and a generalized function of n variables we refer to 0.7. If $g \in S^n$, then there exists an $\alpha > 0$ and an $h \in S^n$ such that $N_{\alpha,n}h = g$, so if $(\ Z(e,f),\psi\ ) \in X_0(n)$, we have

$$\Gamma_{(\ Z(e,f),\psi\ )}g = \Psi_{z\in\mathbb{C}^n} \int_{\mathbb{R}^n} V(z,t;(\ Z(e,f),\psi\ ) \otimes (\ M_n(\alpha),\varphi_\alpha\ ))h(t)dt,$$

and from 2.2.2 we know that this is an element of S^n. So $\Gamma_{(\ Z(e,f),\psi\ )}$ maps S^n (linearly) into S^n.

2.2.6. <u>Theorem</u>. Let $(\ Z_1(e_1,f_1),\psi_1\ )$, $(\ Z_2(e_2,f_2),\psi_2\ )$ and $(\ Z_3(e_3,f_3),\psi_3\ )$ be elements of $X_0(n)$ $(n \in \mathbb{N})$. If

$$(\ Z_1(e_1,f_1),\psi_1\ ) \otimes (\ Z_2(e_2,f_2),\psi_2\ ) = (\ Z_3(e_3,f_3),\psi_3\ )\ ,$$

then

$$\Gamma_{(\ Z_1(e_1,f_1),\psi_1\ )}\ \Gamma_{(\ Z_2(e_2,f_2),\psi_2\ )} = \exp(\pi[(e_3,f_1)-(f_3,e_1)]) \times$$
$$\times\ \Gamma_{(\ Z_3(e_3,f_3),\psi_3\ )}\ .$$

This will be called the multiplicativity property.

<u>Proof</u>. Let $g \in S^n$. If $\alpha > 0$ and $k \in S^n$ are such that $g = N_{\alpha,n}k$ (see 0.5 iii)), then we have

$$h = \Psi_{z\in\mathbb{C}^n} \int_{\mathbb{R}^n} V(z,t;(\ Z_2(e_2,f_2),\psi_2\ ) \otimes (\ M_n(\alpha),\varphi_\alpha\ ))k(t)dt\ ,$$

where

$$h := \Gamma_{(\ Z_2(e_2,f_2),\psi_2\ )}g\ .$$

Using theorem 1.8.4 we easily see that there exists a $\beta > 0$ and a $Z' \in G_2(n)$ such that $M_n(\beta)Z' = Z_2M_n(\alpha)$, so

$$(\ M_n(\beta),\varphi_\beta\ ) \otimes (\ Z'(e',f'),\psi'\ ) = (\ Z_2(e_2,f_2),\psi_2\ ) \otimes (\ M_n(\alpha),\varphi_\alpha\ )\ ,$$

where

$$\begin{pmatrix} e' \\ f' \end{pmatrix} = M_n^{-1}(\beta)\begin{pmatrix} e_2 \\ f_2 \end{pmatrix},\ \psi' = \Upsilon_{T\in H_n}\varphi_\alpha(T)\psi_2(\mathcal{L}_{M_n(\alpha)}(T))\varphi_\beta(\mathcal{L}_{Z'}(T))^{-1}\ .$$

(It is not hard to check that $(Z'(e',f'),\psi') \in X_2(n))$.
Application of 2.2.3 gives

$$h = \Upsilon_{z\in C^n} \int_{\mathbb{R}^n} [\int_{\mathbb{R}^n} V(z,\eta;(M_n(\beta),\varphi_\beta))V(\eta,t;(Z'(e',f'),\psi'))d\eta]k(t)dt,$$

and changing the order of integration in an absolutely convergent repeated integral we get

$$h = N_{\beta,n}(\Upsilon_{z\in C^n} \int_{\mathbb{R}^n} V(z,t;(Z'(e',f'),\psi'))k(t)dt)\ ,$$

so

$$\Gamma_{(Z_1(e_1,f_1),\psi_1)}h = \Upsilon_{z\in\mathbb{C}^n} \int_{\mathbb{R}^n} V(z,\eta;(Z_1(e_1,f_1),\psi_1)\otimes(M_n(\beta),\varphi_\beta))$$

$$[\int_{\mathbb{R}^n} V(\eta,t;(Z'(e',f'),\psi'))k(t)dt]d\eta\ ,$$

and again changing the order of integration and applying 2.2.3 and the fact that $\otimes$ is associative (see 2.1.9) we obtain

$$\Gamma_{(Z_1(e_1,f_1),\psi_1)}\Gamma_{(Z_2(e_2,f_2),\psi_2)}g =$$

$$\exp(\pi[(e_3,f_1)-(f_3,e_1)])\Gamma_{(Z_3(e_3,f_3),\psi_3)}g\ . \qquad \square$$

For the sake of completeness we mention another property of the operators defined in 2.2.5. We shall not prove it however, for it does not play a rôle in this report.

2.2.7. <u>Theorem</u>. Let $(Z(e,f),\psi) \in X_0(n)$ $(n \in \mathbb{N})$ and let $(Z_m(e_m,f_m),\psi_m)_{m\in\mathbb{N}}$ be a sequence in $X_0(n)$ such that

$$(Z_m(e_m,f_m),\psi_m) \to (Z(e,f),\psi) \qquad (m \to \infty)$$

in the sense of the topology of $X_0(n)$. Let furthermore $g \in S^n$ and let $(g_m)_{m\,\mathbb{N}}$ be a sequence in S^n such that

$$g_m \xrightarrow{S^n} \breve{g} \qquad (m \to \infty)$$

(see 0.9). Then

$$\Gamma_{(Z_m(e_m,f_m),\psi_m)}\, g_m \xrightarrow{S^n} \Gamma_{(Z(e,f),\psi)}\, g \qquad (m \to \infty) .$$

2.2.10. If $(Z(e,f),\psi) \in X_2(n)$, then the operator $\Gamma_{(Z(e,f),\psi)}$ may be considered as a smoothing operator (compare 1.8.4 and [B], theorem 3.1). The smoothing effect depends on the fact that in this case the real part of

$$(AC^{-1}z,z) - 2(C^{-1}z,t) + (C^{-1}Dt,t) \qquad (z \in \mathbb{R}^n,\ t \in \mathbb{R}^n)$$

is a positive definite quadratic form (see 1.7.5), where A,B,C and $D \in M_{n\times n}(\mathbb{C})$ denote the block components of Z (according to 1.1.2).

2.2.11. We conclude this section with two examples.

i) Let for every $T \in H_n$, h^T be defined by

$$h^T := \Psi_{z\in\mathbb{C}^n} \exp(-\pi(Tz,z))$$

and let $(Z(0,0),\psi) \in X_0(n)$. Since $\operatorname{Re}(T) > 0$ we obviously have that $h^T \in S^n$. If $T \in H_n$, then there exists an $\alpha > 0$ and a $T' \in H_n$ such that $\mathcal{L}_{M_n(\alpha)}(T') = T$ (see lemma 1.3.3). From an easy computation we now infer that

$$h^T = N_{\alpha,n}(\varphi_n(\sinh\alpha\ T' + \cosh\alpha\ I_n)h^{T'})$$

(see 2.1.6 and 1.1.6), hence (see 2.2.5)

$$\Gamma_{(Z,\psi)}h^T = \Psi_{z\in\mathbb{C}^n} \int_{\mathbb{R}^n} V(z,t;(Z,\psi)\otimes(M_n(\alpha),\varphi_\alpha))h^{T'}(t)\,dt\ \times$$

$$\times\, \varphi_n(\sinh\alpha\ T' + \cosh\alpha\ I_n) .$$

Evaluating this integral and using 2.1.6.1 and 2.1.6.4 we find

$$\Gamma_{(Z,\psi)}h^T = \frac{\varphi_n(\sinh\alpha\ T' + \cosh\alpha\ I_n)\sinh\alpha^{-n/2}}{\varphi_n(T' + C_\alpha^{-1}D_\alpha)\psi(\coth\alpha\ I_n)}\, h^{\mathcal{L}_Z(T)} ,$$

where (C_α,D_α) denotes the second matrix row of $ZM_n(\alpha)$ (see 1.7.3) $(\alpha > 0)$. It is not hard to show now that

$$\Gamma_{(Z,\psi)} h^T = \psi(T)^{-1} h^{\mathcal{L}_Z(T)} .$$

(Compare [B], 4.1 and 4.2.)

ii) The operator $N_{\alpha,n}$ has been defined for $\alpha \gg 0$ only. We shall now define $N_{\xi,n}$ for every $\xi \in \mathbb{C}$, $\mathrm{Re}(\xi) \geq 0$.
Let $\xi \in \mathbb{C}$, $\mathrm{Re}(\xi) \geq 0$, be fixed and define

$$M_n(\xi) := \begin{pmatrix} \cosh\xi\ I_n & \sinh\xi\ I_n \\ \sinh\xi\ I_n & \cosh\xi\ I_n \end{pmatrix} .$$

Obviously $M_n(\xi) \in G_0(n)$ since

$$\mathrm{Re} \begin{pmatrix} \coth(\alpha + \xi) I_n & \sinh^{-1}(\alpha + \xi) I_n \\ \sinh^{-1}(\alpha + \xi) I_n & \coth(\alpha + \xi) I_n \end{pmatrix} > 0$$

for every $\alpha > 0$ (see 1.4.7). Evidently $M_n(\xi)$ is an element of $G_2(n)$ if $\mathrm{Re}(\xi) > 0$ and it is an element of $G_1(n)$ if $\mathrm{Re}(\xi) = 0$.
Now let φ_ξ (see 2.1.4 and 2.2.4) be the analytic function defined on H_n satisfying

$$\varphi_\xi^2 = \Psi_{T \in H_n} \det(\sinh\xi\ T + \cosh\xi\ I_n) , \quad \varphi_\xi(I_n) = e^{\frac{n}{2}\xi} .$$

Then clearly $(M_n(\xi), \varphi_\xi) \in X_0(n)$ and this definition coincides with the one given in 2.2.4 in case ξ is real and positive. Furthermore we have for $\xi_1 \in \mathbb{C}$, $\xi_2 \in \mathbb{C}$, $\mathrm{Re}(\xi_1) \geq 0$, $\mathrm{Re}(\xi_2) \geq 0$

$$(M_n(\xi_1), \varphi_{\xi_1}) \otimes (M_n(\xi_2), \varphi_{\xi_2}) = (M_n(\xi_1 + \xi_2), \varphi_{\xi_1+\xi_2}) ,$$

hence the set of operators $(N_{\xi,n})_{\mathrm{Re}(\xi)\geq 0}$ defined by

$$N_{\xi,n} := \Gamma_{(M_n(\xi), \varphi_\xi)} \qquad (\xi \in \mathbb{C}, \mathrm{Re}(\xi) \geq 0)$$

is a semigroup of operators of S^n (see 2.2.6).

2.3. Some special cases

Definition 2.2.5 is not always easy to handle. If the generalized function of 2.2.4.1 is the embedding of a function of the class S^{n+} ($n \in \mathbb{N}$; the definition of S^{n+} is similar to the one of S^{+} given in [B], section 20), then there is an nicer way to represent the operator of 2.2.5, We have the following theorem.

2.3.1. Theorem. Let $(Z(e,f),\psi) \in X_0(n)$ $(n \in \mathbb{N})$ and assume that the C component of Z (see 1.1.2) is non-singular. Then

$$\Upsilon_z \Upsilon_t V(z,t;(Z(e,f),\psi))$$

is well defined by 2.2.1 and

$$\Gamma_{(Z(e,f),\psi)} = \Upsilon_{g\in S^n} \Upsilon_{z\in\mathbb{C}^n} \int_{\mathbb{R}^n} V(z,t;(Z(e,f),\psi))g(t)dt \ .$$

Proof. From 1.5.2 we easily infer that for every fixed $z \in \mathbb{C}^n$

$$\Upsilon_{t\in\mathbb{C}^n} V(z,t;(Z(e,f),\psi)) \in S^{n+} ,$$

and from 1.3.6 and an easy extension of the multiplicativity property (see 2.2.3) we see that for every fixed $z \in \mathbb{C}^n$

$$\Upsilon_{\alpha>0} \Upsilon_{t\in\mathbb{C}^n} V(z,t;(Z(e,f),\psi) \otimes (M_n(\alpha),\varphi_\alpha)) =$$

$$= \mathrm{emb}(\Upsilon_{t\in\mathbb{C}^n} V(z,t;(Z(e,f),\psi))) \ .$$

The proof now follows from [B], 20.3. (The property mentioned there is also valid for functions of several variables.) □

We now give a survey of some other special cases. The results are established by application of definition 2.2.5 and except for the first case we omit proofs.

2.3.2. Theorem. Let $(Z(e,f),\psi) \in X_0(n)$ $(n \in \mathbb{N})$ and let A,B,C and $D \in M_{n\times n}(\mathbb{C})$ denote the block components of Z. If C is the all zero matrix, then

$$\Gamma_{(Z(e,f),\psi)} = \Upsilon_{g\in S^n} \Upsilon_{z\in\mathbb{C}^n} \psi(I_n)^{-1}g(D^{-1}(z-f)) \ \times$$

$$\times \exp(-\pi[(BD^{-1}z,z) + 2(e-BD^{-1}f,z) + (BD^{-1}f-e,f)]) \ .$$

Proof. Since $\psi^2 = \Psi_{T\in H_n} \det(D)$ we infer from the analyticity of ψ that ψ is constant (note that D is non-singular (see theorem 1.3.5 ii))). Let $g \in S^n$. Then there exists a $h \in S^n$ and an $\alpha > 0$ such that $g = N_{\alpha,n}h$ (see 0.5 iii)). If furthermore χ is defined by

$$\chi := \Psi_{T\in H_n} \psi(\mathcal{L}_{M_n(\alpha)}(T))\varphi_\alpha(T) ,$$

then

$$\lim_{t\to\infty} \chi(tI_n)t^{-n/2} = \psi(I_n)\sinh\alpha^{n/2} .$$

From definition 2.2.5 and some computational work (see 2.2.1) we infer

$$\Gamma_{(Z(e,f),\psi)} = \Psi_{g\in S^n} \Psi_{z\in\mathbb{C}^n} \psi(I_n)^{-1}\exp(-\pi[(BD^{-1}z,z) + 2(e - BD^{-1}f,z)$$

$$+ (BD^{-1}f - e,f)]) \int_{\mathbb{R}^n} V(D^{-1}(z - f),t;(M_n(\alpha),\varphi_\alpha))h(t)dt .$$

Our assertion now follows from 2.2.4. $\square$

2.3.3. Theorem. Let $(Z(e,f),\psi) \in X_0(n)$ $(n \in \mathbb{N})$ and assume that $Z(e,f)$ is of the form

$$Z(e,f) = \begin{pmatrix} \operatorname{diag}(A_1,A_2) & \operatorname{diag}(B_1,B_2) & \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \\ \operatorname{diag}(C_1,C_2) & \operatorname{diag}(D_1,D_2) & \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\ 0 & 0 & 1 \end{pmatrix},$$

where A_1, B_1, C_1 and D_1 are elements of $M_{n_1\times n_1}(\mathbb{C})$, A_2, B_2, C_2 and D_2 are elements of $M_{n_2\times n_2}(\mathbb{C})$ $(n_1 \in \mathbb{N},\ n_2 \in \mathbb{N},\ n_1 + n_2 = n)$ and $e_1,f_1 \in M_{n_1\times 1}(\mathbb{C})$, e_2, $f_2 \in M_{n_2\times 1}(\mathbb{C})$. If we define

$$Z_i := \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \quad (i = 1,2) ,$$

then there exist analytic functions ψ_1 defined on H_{n_1} and ψ_2 defined on H_{n_2} such that

$$(Z_1(e_1,f_1),\psi_1) \in X_0(n_1),(Z_2(e_2,f_2),\psi_2) \in X_0(n_2) ,$$

$$\psi(\mathrm{diag}(T_1,T_2)) = \psi_1(T_1)\psi_2(T_2) \qquad (T_1 \in H_{n_1},\ T_2 \in H_{n_2}) .$$

Furthermore we have

$$\Gamma_{(\ Z(e,f),\psi\ )} = \Gamma^{(n_1)}_{(\ Z_1(e_1,f_1),\psi_1\ )}\ \Gamma^{(n_2)}_{(\ Z_2(e_2,f_2),\psi_2\ )} =$$

$$= \Gamma^{(n_2)}_{(\ Z_2(e_2,f_2),\psi_2\ )}\ \Gamma^{(n_1)}_{(\ Z_1(e_1,f_1),\psi_1\ )} .$$

(If T is a linear operator of S^n, then $T^{(n_1)} := \Psi_{f\in S^n}\ \Psi_{z\in\mathbb{C}^n}\ T(\Psi_{(z_1,\ldots,z_{n_1})}f)$ and $T^{(n_2)}$ is similarly defined).

This theorem can be generalized to the case of more than two blocks.

2.3.4. Theorem. Let $(\ Z(e,f),\psi\ ) \in X_0(n)$ $(n \in \mathbb{N})$ and assume that $Z(e,f)$ is of the form

$$Z(e,f) = \begin{pmatrix} \begin{pmatrix} A_{11} & 0 \\ A_{21} & I_{n-r} \end{pmatrix} & \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} & \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \\ \begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} D_{11} & 0 \\ 0 & I_{n-r} \end{pmatrix} & \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\ 0 & 0 & 1 \end{pmatrix},$$

where $0 < r < n$, A_{11}, B_{11}, C_{11} and $D_{11} \in M_{r\times r}(\mathbb{C})$ such that C_{11} is non-singular, A_{21} and $B_{21} \in M_{(n-r)\times r}(\mathbb{C})$, $B_{12} \in M_{r\times(n-r)}(\mathbb{C})$ and $B_{22} \in M_{(n-r)\times(n-r)}(\mathbb{C})$ (see section 1.2, in particular 1.2.2 and 1.2.4), $e_1, f_1 \in M_{r\times 1}(\mathbb{C})$ and $e_2, f_2 \in M_{(n-r)\times 1}(\mathbb{C})$. Let ψ_1 and ψ_2 be defined by

$$\psi_1 := \Psi_{T\in H_r}\ \psi(\mathrm{diag}(T,I_{n-r})),\quad \psi_2 := \Psi_{T\in H_{n-r}}\ 1 .$$

Then $(\ Z_1(e_1,f_1),\psi_1\ ) \in X_0(r)$ and $(\ Z_2(e_2,f_2),\psi_2\ ) \in X_0(n-r)$, where

$$Z_1 := \begin{pmatrix} A_{11} & B_{11} \\ C_{11} & D_{11} \end{pmatrix},\quad Z_2 := \begin{pmatrix} I_{n-r} & B_{12} \\ 0 & I_{n-r} \end{pmatrix},$$

and

$$\Gamma_{(\ Z(e,f),\psi\ )} = \Psi_{g\in S^n} \Psi_{z\in\mathbb{C}^n} \exp(-2\pi(B_{12}(z_2 - f_2),(z_1 - f_1)) \times$$

$$\times\ \Gamma^{(r)}_{(\ Z_1(e_1,f_1),\psi_1\ )}\ \Gamma^{(n-r)}_{(\ Z_2(e_2,f_2),\psi_2\ )}{}^{g}\ .$$

(Here $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ $(z_1 \in \mathbb{C}^r,\ z_2 \in \mathbb{C}^{n-r})$).
Note that theorem 2.3.1 can be applied to $\Gamma_{(\ Z_1(e_1,f_1),\psi_1\ )}$ and that theorem 2.3.2 can be used for $\Gamma_{(\ Z_2(e_2,f_2),\psi_2\ )}$. Therefore $\Gamma_{(\ Z(e,f),\psi\ )}$ can be reduced to an integral operator in which only r integrations occur.

2.3.5. The latter theorem seems to deal with a rather special case, but comparing section 1.2, in particular 1.2.3, and using the multiplicativity property (see theorem 2.2.6) we see it does not.

If $(\ Z(e,f),\psi\ ) \in X_1(n)$ such that the C component of Z has rank r ($0 < r < n$; see 1.1.2), then both U_1 and U_2 are real and non-singular (notation of section 1.2), so there exist analytic functions ν_1,ν_2 (both constant) and ψ_1 defined on H_n such that

$$(\ \mathrm{diag}(U_1^{-T},U_1)(e,f),\nu_1\ ) \in X_0(n),\ (\ \mathrm{diag}(U_2^{T},U_2^{-1}),\nu_2\ ) \in X_0(n)\ ,$$

$$(\ Z_1,\psi_1\ ) \in X_0(n)$$

(the matrix Z_1 is defined in section 1.2), and

2.3.5.1. $$(\ \mathrm{diag}(U_1^{-T},U_1)(e,f),\nu_1\ ) \otimes (\ Z_1,\psi_1\ ) \otimes (\ \mathrm{diag}(U_2^{T},U_2^{-1}),\nu_2\ ) = (\ Z(e,f),\psi\ )$$

(see 1.2.3). It is easily seen (see theorem 2.3.2) that

2.3.5.2. $$\Gamma_{(\ \mathrm{diag}(U_2^{T},U_2^{-1}),\nu_2\ )} = \Psi_{g\in S^n} \Psi_{z\in\mathbb{C}^n} \nu_2(I_n)^{-1} g(U_2 z)\ ,$$

2.3.5.3. $$\Gamma_{(\ \mathrm{diag}(U_1^{-T},U_1)(e,f),\nu_1\ )} = \Psi_{g\in S^n} \Psi_{z\in\mathbb{C}^n} \nu_1(I_n)^{-1} g(U_1^{-1} z) \times$$

$$\times \exp(-\pi[2(e,z) - (e,f)])\ .$$

Using the multiplicativity property (theorem 2.2.6) and theorem 2.3.4 we see that (after suitable transformations of coordinates according to 2.3.5.2 and 2.3.5.3) $\Gamma_{(\ Z(e,f),\psi\ )}$ can be reduced to an integral operator in which only r integrations occur:

$$\Gamma_{(\ Z(e,f),\psi\ )} = \underset{g\in S^n}{\Psi} \ \underset{z\in\mathbb{C}^n}{\Psi} \ k(z) \int_{\mathbb{R}^r} V(z,t;(\ Z_{11},\psi_{11}\ ))g(t,z_2-f_2)dt\ ,$$

where Z_{11} is defined in section 1.2, ψ_{11} is an analytic function defined on H_r such that $(Z_{11},\psi_{11}) \in X_0(r)$ and k is an analytic function satisfying $|k(z)| = 1$ for $z \in \mathbb{R}^n$. Furthermore $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ $(z_1 \in \mathbb{C}^r,\ z_2 \in \mathbb{C}^{n-r})$ and $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ $(f_1 \in M_{r\times 1}(\mathbb{C}),\ f_2 \in M_{(n-r)\times 1}(\mathbb{C}))$.

In case $(\ Z(e,f),\psi\ ) \in X_0(n)$, the matrix U_1 (see section 1.2, in particular 1.2.3) in general is not real, so we can not repeat the arguments used above. It is possible however to prove that in this case the operator $\Gamma_{(\ Z(e,f),\psi\ )}$ can also be reduced to an integral operator in which only r integrations occur (where r denotes the rank of the C component of Z (see 1.1.2)). We omit the proof.

2.4. Operators related to elements of the group $X_1(n)$

In this section we shall discuss operators of type 2.2.5 related to elements of $X_1(n)$ $(n \in \mathbb{N})$. We already mentioned that $X_1(n)$ is a group (see 2.1.10). The fact that the fractional linear transforms related to elements of $G_1(n)$ is a group of transformations of H_n has an important consequence, as we shall see in the following theorem.

2.4.1. Theorem. If $(\ Z(e,f),\psi\ ) \in X_1(n)$, then the operator $\Gamma_{(\ Z(e,f),\psi\ )}$ of S^n has an inverse $\Gamma^{-1}_{(\ Z(e,f),\psi\ )}$, and

$$\Gamma^{-1}_{(\ Z(e,f),\psi\ )} = \Gamma_{(\ Z(e,f),\psi\ )^{-1}}\ .$$

Furthermore

$$[\Gamma_{(\ Z(e,f),\psi\ )}g,h]_n = [g,\Gamma_{(\ Z(e,f),\psi\ )^{-1}}h]_n$$

for every $g \in S^n$ and every $h \in S^n$.

Before we prove this theorem we need a lemma.

2.4.1.1. <u>Lemma</u>. Let $(\ Z(e,f),\psi\ ) \in X_2(n)$ and assume that $A,B,C,D \in \mathcal{M}_{n\times n}(\mathbb{C})$ denote the block components of Z (according to 1.1.2). If we define

$$W := \begin{pmatrix} D^H & B^H \\ C^H & A^H \end{pmatrix} ,$$

then $W \in G_2(n)$ and if φ is the (unique) analytic function defined on $\mathcal{H}_n$ satisfying

$$\varphi^2 = \psi_{T\in \mathcal{H}_n} \det(C^H T + A^H) \ ,$$

$$\lim_{t\to\infty} \varphi(tI_n)t^{-n/2} = \lim_{t\to\infty} \overline{\psi(tI_n)}\,t^{-n/2} \ ,$$

then $(\ W(e',f'),\varphi\ ) \in X_2(n)$ and

$$[\Gamma_{(\ Z(e,f),\psi\ )}g,h]_n = [g,\Gamma_{(\ W(e',f'),\varphi\ )}h]_n \ ,$$

where

$$\begin{pmatrix} e' \\ f' \end{pmatrix} := W \begin{pmatrix} \bar{e} \\ -\bar{f} \end{pmatrix} .$$

<u>Proof</u>. Since C is non-singular (see lemma 1.7.3) and

$$\mathrm{Re}\begin{pmatrix} D^H C^{-H} & \bar{C}^{-1} \\ \bar{C}^{-T} & C^{-H}A^H \end{pmatrix} = \mathrm{Re}\begin{pmatrix} C^{-1}D & C^{-1} \\ C^{-T} & AC^{-1} \end{pmatrix} > 0$$

(see 1.1.2.5 and 1.7.5), we infer by using 1.7.5 that $W \in G_2(n)$, hence $(\ W(e',f'),\varphi\ ) \in X_2(n)$. It is easily seen that for every $z \in \mathbb{R}^n$ and every $t \in \mathbb{R}^n$

$$V(z,t;(\ Z(e,f),\psi\ )) = V(t,z;(\ W(e',f'),\varphi\ )) \ .$$

The remainder of the proof is just a matter of changing the order of integration in an absolutely convergent repeated integral. $\square$

<u>Proof of theorem 2.4.1</u>. It is a result of the multiplicativity property (2.2.6) that

$$\Gamma_{(\ Z(e,f),\psi\ )}\,\Gamma_{(\ Z(e,f),\psi\ )^{-1}}g = g \quad (g \in S^n) \ .$$

(The factor $\exp(\pi[(e_3,f_1) - (f_3,e_1)])$ of 2.2.6 drops out since $e_3 = f_3 = 0$). Let $g \in S^n$, $h \in S^n$. There exists an $\alpha > 0$ and $k \in S^n$, $\ell \in S^n$ such that $N_{\alpha,n}k = g$ and $N_{\alpha,n}\ell = h$ (see 0.5 iii)). From the symmetry of $N_{\alpha,n}$ (see 0.5 ii)) and from 2.2.3 and 2.2.4 we obtain

$$[\Gamma_{(\ Z(e,f),\psi\ )}f,g]_n =$$

$$= \int_{\mathbb{R}^n} [\int_{\mathbb{R}^n} V(z,t;(\ M_n(\alpha),\varphi_\alpha\ ) \otimes (\ Z(e,f),\psi\ ) \otimes (\ M_n(\alpha),\varphi_\alpha\ ))k(t)dt]\ell(z)dz.$$

Applying lemma 2.4.1.1 to

$$(\ M_n(\alpha),\varphi_\alpha\ ) \otimes (\ Z(e,f),\psi\ ) \otimes (\ M_n(\alpha),\varphi_\alpha\ )$$

and using again the symmetry of $N_{\alpha,n}$, 2.2.3 and 2.2.4 we finally obtain

$$[\Gamma_{(\ Z(e,f),\psi\ )}g,h]_n = [g,\Gamma_{(\ Z(e,f),\psi\ )^{-1}}h]_n \ . \qquad \square$$

2.4.2. Corollary. Operators of type 2.2.5 related to elements of $X_1(n)$ are unitary: if $(\ Z(e,f),\psi\ ) \in X_1(n)$ and $g,h \in S^n$, we have

$$[\Gamma_{(\ Z(e,f),\psi\ )}g,\ \Gamma_{(\ Z(e,f),\psi\ )}h]_n = [g,h]_n \ .$$

2.5. Extension to S^{n*}

2.5.1. Let $(\ Z(e,f),\psi\ ) \in X_0(n)$ $(n \in \mathbb{N})$. In this section we shall show that it is possible to extend $\Gamma_{(\ Z(e,f),\psi\ )}$ to a linear operator of S^{n*} (which is again denoted by $\Gamma_{(\ Z(e,f),\psi\ )}$) such that

2.5.1.1. $$\Gamma_{(\ Z(e,f),\psi\ )}(\mathrm{emb}(g)) = \mathrm{emb}(\Gamma_{(\ Z(e,f),\psi\ )}g) \quad (g \in S^n) \ .$$

(See 0.7).

2.5.2. Let $\alpha > 0$. Using theorem 1.8.4 and corollary 1.8.7 we easily see that for every sufficiently small $\beta > 0$ there exists a $Z_{\alpha,\beta} \in G_0(n)$ and an analytic function $\psi_{\alpha,\beta}$ defined on $\mathcal{H}_n$ such that $(\ Z_{\alpha,\beta}(e_\alpha,f_\alpha),\psi_{\alpha,\beta}\ ) \in X_0(n)$ and

2.5.2.1. $$(\ M_n(\alpha),\varphi_\alpha\ ) \otimes (\ Z(e,f),\psi\ ) = (\ Z_{\alpha,\beta}(e_\alpha,f_\alpha),\psi_{\alpha,\beta}\ ) \otimes (\ M_n(\beta),\varphi_\beta\ ) \ ,$$

where

$$\begin{pmatrix} e_\alpha \\ f_\alpha \end{pmatrix} := M_n(\alpha)\begin{pmatrix} e \\ f \end{pmatrix} .$$

Let $g \in N_{\alpha,n}(S^{n*})$. If $\beta > 0$ is sufficiently small (we can restrict β to the set $0 < \beta \leq \alpha$), we define

$$Y_\alpha g := \Gamma_{(\ Z_{\alpha,\beta}(e_\alpha,f_\alpha),\psi_{\alpha,\beta}\ )}h\ ,$$

where $h \in S^{n*}$ is such that $g = N_{\alpha-\beta,n}h$. Note that since $g \in N_{\alpha,n}(S^{n*})$ and

$$N_{\alpha,n}(S^{n*}) = \bigcap_{0<\gamma<\alpha} N_{\gamma,n}(S^n)$$

(see [B], theorem 19.1) such a function $h \in S^n$ always exists if we restrict β to the set $0 < \beta \leq \alpha$. It is now a matter of routine to show that the definition of Y_α ($\alpha > 0$) does not depend on the particular choice of β and that the following identities are satisfied.

$$Y_{\alpha+\beta}N_{\beta,n}g = N_{\beta,n}Y_\alpha g \qquad (g \in N_{\alpha,n}(S^{n*}),\ \alpha > 0,\ \beta > 0)\ ,$$

$$N_{\alpha,n}\Gamma_{(\ Z(e,f),\psi\ )}g = Y_\alpha N_\alpha g \qquad (g \in S^n,\ \alpha > 0)\ .$$

So, according to 0.8, $\Gamma_{(\ Z(e,f),\psi\ )}$ is extendable by means of $(Y_\alpha)_{\alpha>0}$ and the extension is given by

2.5.2.2. $$\Gamma_{(\ Z(e,f),\psi\ )}F := \Psi_{\alpha>0}\ Y_\alpha(F(\alpha)) \qquad (F \in S^{n*})\ .$$

2.5.3. We conclude this section with two examples.

i) Consider the shift operators T_q and R_p ($q \in \mathcal{M}_{n\times 1}(\mathbb{C})$, $p \in \mathcal{M}_{n\times 1}(\mathbb{C})$) (see 0.6 iii)). Using theorem 2.3.2 it is easily seen that

$$T_q = \Gamma_{(\ I_{2n}(0,-q),\Psi_{T\in\mathcal{H}_n} 1\ )}\ ,$$

$$R_p = \Gamma_{(\ I_{2n}(ip,0),\Psi_{T\in\mathcal{H}_n} 1\ )}\ .$$

Note that both $(\ I_{2n}(0,-q),\Psi_{T\in\mathcal{H}_n} 1\ )$ and $(\ I_{2n}(ip,0),\Psi_{T\in\mathcal{H}_n} 1\ )$ are elements of $X_1(n)$ if $q \in \mathcal{M}_{n\times 1}(\mathbb{R})$ and $p \in \mathcal{M}_{n\times 1}(\mathbb{R})$. Equation 2.5.2.1 (with $(\ I_{2n}(0,-q),\Psi_{T\in\mathcal{H}_n} 1\ )$ substituted for $(\ Z(e,f),\psi\ )$) has a solution in $X_0(n)$ for $\beta = \alpha$ ($\alpha > 0$):

$$(\ Z_{\alpha,\beta}(e_\alpha,f_\alpha),\psi_{\alpha,\beta}\ ) = (\ I_{2n}(-\sinh\alpha\ q,-\cosh\alpha\ q),\Psi_{T\in\mathcal{H}_n} 1\ )\ .$$

So from 2.5.2.2 and the multiplicativity property we obtain

$$T_q F = \Upsilon_{\alpha>0}\, R_{i\,\sinh\alpha\, q^T\cosh\alpha\, q} F(\alpha) \times \exp(\pi \sinh\alpha \cosh\alpha (q,q)) \quad (F \in S^n).$$

This result coincides with [B], section 19, example (v) in case n = 1.

ii) Consider the Fourier transform F_n of S^n (see 0.6 iii)). Using theorem 2.3.1 we easily obtain

$$F_n = e^{\frac{\pi i n}{4}} \Gamma_{(Z,\psi)} ,$$

where

$$Z := \begin{pmatrix} 0 & iI_n \\ iI_n & 0 \end{pmatrix}, \quad \psi := \Upsilon_{T\in H_n}\, e^{\frac{\pi i n}{4}} \varphi_n(T)$$

(see 2.1.5). In a similar way as in example i) we find

$$F_n F = \Upsilon_{\alpha>0}\, F_n F(\alpha) \qquad (F \in S^{n*}) .$$

2.6. Application in quantum mechanics

In this section we shall discuss applications of our theorems in quantum mechanics. We start with a short survey of some notions in classical and quantum mechanics. For more details we refer to [B] and [P].

2.6.1. In classical mechanics a system of n moving points (degree n) can be described by the Hamilton equations

2.6.1.1. $$\dot{q}_j = \frac{\partial H}{\partial p_j} , \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} \qquad (j = 1,\dots,n) ,$$

where q_j and p_j denote the components of the position vector $q \in M_{n\times 1}(\mathbb{R})$ and the momentum vector $p \in M_{n\times 1}(\mathbb{R})$ (the dot over q_j and p_j denotes differentiation with respect to the time) and H is a function of q and p, also called the Hamiltonian. At any time t the system is totally described by q and p. The q-p-space $(\mathbb{R}^n \times \mathbb{R}^n)$ is usually called the phase space.

A transformation $u = \varphi(p,q)$, $v = \psi(p,q)$ of coordinates in phase space is a canonical transformation if the equations of 2.6.1.1 expressed in the new coordinates have the form

$$\dot{u}_j = \frac{\partial H^*}{\partial v_j} , \quad \dot{v}_j = -\frac{\partial H^*}{\partial u_j} \qquad (j = 1,\dots,n) ,$$

where H^* is the Hamiltonian of 2.6.1.1 expressed in u and v (see [TH], p.98).

Let us consider a linear transformation of coordinates in phase space.

2.6.1.2. $$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix},$$

where $A,B,C,D \in M_{n\times n}(\mathbb{R})$.
The equations 2.6.1.1 can be written as

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix},$$

where $\frac{\partial H}{\partial q}$ and $\frac{\partial H}{\partial p}$ denote the vectors

$$\begin{pmatrix} \frac{\partial H}{\partial q_1} \\ \vdots \\ \frac{\partial H}{\partial q_n} \end{pmatrix} \text{ and } \begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \vdots \\ \frac{\partial H}{\partial p_n} \end{pmatrix}$$

respectively. So

2.6.1.3. $$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = ZJ_n \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} = ZJ_n Z^T \begin{pmatrix} \frac{\partial H}{\partial u} \\ \frac{\partial H}{\partial v} \end{pmatrix},$$

where

$$Z := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and J_n is defined in 1.1.1.1. Therefore 2.6.1.2 is a canonical transformation if and only if Z is symplectic (see 1.1.2.2).

2.6.2. Before we discuss the quantum mechanical approach we need some definitions.

If $f \in S^n$, $g \in S^n$, we define

2.6.2.1. $$W(q,p;f,g) := \int_{\mathbb{R}^n} \exp(-2\pi i(p,t))f(q+\tfrac{1}{2}t)\bar{g}(q-\tfrac{1}{2}t)dt \quad (q \in \mathbb{C}^n,\ p \in \mathbb{C}^n)$$

(see [B], 12.1.)

Considered as a function of the variables $q \in \mathbb{C}^n$ and $p \in \mathbb{C}^n$ it is called the <u>mixed Wigner distribution</u> of f and g; if f = g, it is called the <u>Wigner distribution</u> of f. From an easy generalization of [B], 13.1 we have

$$\Upsilon_{q\in\mathbb{C}^n} \Upsilon_{p\in\mathbb{C}^n} W(q,p;f,g) \in S^{2n} .$$

This function will often be denoted by $W(f,g)$.

Generalization of [B], theorem 26.1 gives the following theorem.

2.6.2.2. <u>Theorem</u>. If H is a generalized function of 2n variables (i.e. $H \in S^{2n*}$), then there exists a unique linear mapping ω_H of S^n into S^{n*} such that

$$[f,\omega_H g]_n = [W(f,g),H]_{2n}$$

for every $f \in S^n$ and every $g \in S^n$ (see 0.7). The correspondence between H and ω_H is called <u>Weyl's correspondence</u>.

2.6.3. In quantum mechanics a system of n moving particles with Hamiltonian H is described by a wave function

$$\Upsilon_{q\in\mathbb{C}^n} \Psi_t(q) \qquad (t > 0)$$

that satisfies the Schrödinger equation

$$\omega_H \Psi_t = \frac{i}{2\pi} \frac{\partial}{\partial t} \Psi_t ,$$

where ω_H is the operator related to H by Weyl's correspondence. (Scalings are assumed to have been made such that Planck's constant disappears from the formulas).

We now discuss a theorem on Wigner distributions.

2.6.4. <u>Theorem</u>. Let $A,B,C,D \in M_{n\times n}(\mathbb{R})$ be such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n) \qquad (n \in \mathbb{N}) .$$

Let $e,f \in M_{n\times n}(\mathbb{R})$ and consider the transformation

2.6.4.1. $$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} \qquad (q \in M_{n\times 1}(\mathbb{R}),\ p \in M_{n\times 1}(\mathbb{R}))\ .$$

Now $Z \in G_1(n)$, where $Z := \begin{pmatrix} D & -iC \\ iB & A \end{pmatrix}$, and

$$W(q',p';\Gamma_{(Z(-if,e),\psi)}g,\ \Gamma_{(Z(-if,e),\psi)}h) =$$

$$W(q,p;g,h) \qquad (q \in \mathbb{C}^n,\ p \in \mathbb{C}^n)$$

for every $g \in S^n$ and every $h \in S^n$. (Here ψ denotes one of the square-roots corresponding with Z (see 2.1.4)).

Proof. From 1.1.2 and 1.6.1 we infer that $Z \in G_1(n)$. Let $g \in S^n$ and $h \in S^n$. According to a generalization of [B], 12.3, the Wigner distribution can be written as an inner product:

2.6.4.2. $$W(a,b;g,h) = 2^n[T_aR_bg, F_n^2T_aR_bh]_n \qquad (a \in \mathbb{C}^n,\ b \in \mathbb{C}^n)\ .$$

Note that $F_n^2 = i^n\Gamma_{(-I_{2n}, \Upsilon_{T\in H_n} i^n)}$ (see 2.5.3 ii)), so $F_n^2 g = \Upsilon_{z\in\mathbb{C}^n}\, g(-z)$ $(q \in S^n)$ (see theorem 2.3.2).

Using 2.5.3 i), 2.6.4.2 and the multiplicativity property (see 2.2.6), we have for every $q \in \mathbb{R}^n$ and $p \in \mathbb{R}^n$ (q' and p' defined in 2.6.4.1)

2.6.4.3. $$W(q',p';\Gamma_{(Z(-if,e),\psi)}g,\ \Gamma_{(Z(-if,e),\psi)}h) =$$

$$= 2^n[T_{q'}R_{p'}\Gamma_{(Z(-if,e),\psi)}g,\ F_n^2T_{q'}R_{p'}\Gamma_{(Z(-if,e),\psi)}h]_n$$

$$= 2^n[\Gamma_{(V(k,l),\chi)}g,\ F_n^2\Gamma_{(V(k,l),\chi)}h]_n\ ,$$

where

$$(V(k,l),\chi) := (I_{2n}(0,-q'), \Upsilon_{T\in H_n} 1) \otimes (I_{2n}(ip',0), \Upsilon_{T\in H_n} 1)$$

$$\otimes\ (Z(-if,e),\psi)\ .$$

(Note that the factor $\exp(\pi[(e_3,f_1) - (f_3,e_1)])$ of 2.2.6 drops out in this case since it is an element of the unit circle and it occurs on both sides of the inner product). Using theorem 2.4.1 we infer that 2.6.4.3 equals

$$2^n[g, i^n\Gamma_{(V(k,l),\chi)^{-1}}\Gamma_{(-I_{2n}, \Upsilon_{T\in H_n} i^n)}\Gamma_{(V(k,l),\chi)}h]_n\ .$$

It is just a matter of a simple computation to see that $V = Z$ and that $k = iCq + iDp$, $l = -Aq - Bp$. Since $Z^{-1}\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} ip \\ -q \end{pmatrix}$ we easily obtain from 2.2.6 that 2.6.4.3 equals

2.6.4.5. $$2^n[g, R_{-p}T_{-q}F_n^2T_qR_ph]_n = W(q,p;g,h) \qquad (q \in \mathbb{R}^n,\ p \in \mathbb{R}^n) ,$$

and since both function are analytic (see 2.6.2) the equality also holds for every $q \in \mathbb{C}^n$ and $p \in \mathbb{C}^n$. □

2.6.5. Assume now $H \in S^{2n}$ and consider the transformation

2.6.5.1. $$K := \Psi_{(q,p)\in\mathbb{C}^n\times\mathbb{C}^n} H(Aq + Bp + e,\ Cq + Dp + f) ,$$

where $A,B,C,D \in M_{n\times n}(\mathbb{R})$ such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n)$$

and where $e \in M_{n\times 1}(\mathbb{R})$, $f \in M_{n\times 1}(\mathbb{R})$. We want to express ω_K in terms of ω_H.

It is easily seen that for every $g \in S^n$ and $h \in S^n$

$$[W(g,h),K]_{2n} = [\Psi_{(q,p)\in\mathbb{C}^n\times\mathbb{C}^n} W(D^T(q-e) - B^T(p-f),$$

$$-C^T(q-e) + A^T(p-f);g,h),H]_{2n} ,$$

and using theorem 2.6.4 gives

$$[W(g,h),K]_{2n} = [\Psi_{(q,p)\in\mathbb{C}^n\times\mathbb{C}^n} W(q,p;\Gamma_{(Z(-if,e),\psi)}g,$$

$$\Gamma_{(Z(-if,e),\psi)}h),H]_{2n} ,$$

where

$$Z := \begin{pmatrix} D & -iC \\ iB & A \end{pmatrix}$$

and ψ is one of the square-roots related to Z by 2.1.4. So we obtain (see 2.6.2.2 and 2.4.1)

$$[g,\omega_K h]_n = [\Gamma_{(Z(-if,e),\psi)}g,\ \omega_H\Gamma_{(Z(-if,e),\psi)}h]_n =$$

$$[g,\Gamma_{(Z(-if,e),\psi)^{-1}}\omega_H\Gamma_{(Z(-if,e),\psi)}h]_n ,$$

hence

$$\omega_K = \Gamma_{(\ Z(-if,e),\psi\ )}^{-1}\omega_H\Gamma_{(\ Z(-if,e),\psi\ )}\ .$$

We have restricted ourselves to the case $H \in S^{2n}$. We have the same result however in case $H \in S^{2n*}$, but in that case we have to replace the transformation 2.6.5.1 by the extension of the operator

$$\Upsilon_{(q,p)\in\mathbb{C}^n\times\mathbb{C}^n}\ \Upsilon_{H\in S^{2n}}\ H(Aq + Bp + e,\ Cq + Dp + f)$$

of S^{2n} to S^{2n*}. (Using [J], appendix 1, theorem 3.2, we easily see that this operator indeed is extendable to S^{2n*}).

2.6.6. We shall say a few things about the rôle of the Wigner distribution in quantum mechanics. As said before scalings are assumed to have been made such that **Planck's** constant disappears from the formulas.

The general idea (see [B], 27.26.3) is that for any fixed t we consider the Wigner distribution $W(\Psi_t,\Psi_t)$, where $(\Psi_t)_{t>0}$ denotes the solution of the Schrödinger equation (see 2.6.3). This distribution can be seen as a kind of cloud that moves if t runs though the real numbers. Seen from afar, the cloud may often be idealized as a point in the phase space, and then it can be said that this point moves according to the Hamilton equations.

We shall give an example.

Let $a \in M_{n\times 1}(\mathbb{R})$, $b \in M_{n\times 1}(\mathbb{R})$ and $c \in \mathbb{R}$ be fixed. If $H = \Upsilon_{(q,p)}\ c + (a,q) + $ $+ (b,p)$ (this of course will be the case if we linearize), then

$$\omega_H = cI + \sum_{i=1}^{n} q_iQ_i + b_iP_i$$

(see 0.6 iv)), where a_i and b_i are the components of a and b $(1 \le i \le n)$ and I denotes the identity (see [B], 27.26.3). Schrödinger's equation (see 2.6.3) is now linear, and can be solved explicitly:

$$\Psi_t = \Upsilon_{q\in\mathbb{C}^n}\ \exp(-2\pi ict)R_{at}T_{-bt}\Psi_0(q) \qquad (t > 0)\ ,$$

where Ψ_0 is an arbitrary element of S^n, say (Ψ_0 describes the initial position of the system). (See [B], 27.26.3). By means of theorem 2.6.4 it is easily shown that

$$W(q,p;\Psi_t,\Psi_t) = W(q-bt,\ q+at;\Psi_0,\Psi_0) \qquad (q \in \mathbb{C}^n,\ p \in \mathbb{C}^n) \ .$$

This means that the entire cloud is just shifted if time proceeds. If the cloud can be considered as a point, that point moves according to the Hamilton equations (see 2.6.1.1). By this procedure the situation of classical physics appears as a limit of the quantum mechanical description. Thus we get a nice insight in what is usually called the classical limit.

As a further example we take the two-dimensional harmonic oscillator. As in the case of the linear function H we shall find a moving cloud (in this case it rotates) that does not change its shape.

We take $H = \Upsilon_{(q,p)}\ \frac{1}{2}(q_1^2 + q_2^2 + p_1^2 + p_2^2 + c)$, where c is a constant. Now (see [B], 27.27.4) $\omega_H = \frac{1}{2}(Q_1^2 + Q_2^2 + P_1^2 + P_2^2 + cI)$ (see 0.6 iv)), where I again denotes the identity. The Schrödinger equation can be solved in terms of Hermite expansions (similar to the case n = 1 (see [B], 27.26.4); we omit details):

$$\Psi_t = e^{-\pi i c t} N_{it,2} \Psi_0 \qquad (t > 0) \ ,$$

where Ψ_0 is an arbitrary element of S^2, say (Ψ_0 describes the initial position of the system). The operator $N_{it,2}$ was discussed in 2.2.11 ii). Since $N_{it,2} = \Gamma_{(\ M_2(it),\varphi_{it}\ )}$ (see 2.2.11 ii)) we infer from theorem 2.6.4 that

$$W(q,p;\Psi_t,\Psi_t) = W(q(t),p(t);\Psi_0,\Psi_0) \ ,$$

where

$$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \cos t\ I_2 & -\sin t\ I_2 \\ \sin t\ I_2 & \cos t\ I_2 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \qquad (q \in M_{2\times 1}(\mathbb{R}),\ p \in M_{2\times 1}(\mathbb{R}),\ t > 0) \ .$$

This shows that the cloud moves according to the Hamilton equations.

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