Integral Relations for Bessel Functions and Analytical Solutions for Fourier Transform in Elliptic Coordinates

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Abstract: - The work continues the development of Fourier transform in elliptic coordinates. Several analytical results for elementary and special functions have been obtained. In fact these are the new relations for the integral representation for the multiplication of two Bessel functions of zero order. We analyze the natural restrictions of application of these formulae. Also we provide the recommendations based on the numerical analysis for the using of obtained results.

Key-Words: - Integral transform, Integral representation, Axial symmetry, Numerical integration, Numerical analysis, Rapidly oscillating functions

1 Introduction

The scope of Fourier transform studies is generally devoted to rectangular and polar coordinates [1]-[5]. The range of applications is wide, for example [6]-[8]. The methods based on the Fourier transform have a good advantage in comparison with classical finite-difference schemes. This is a low calculation time saving the high accuracy [2], [5].

Previously we have introduced the Fourier transform in elliptical coordinates [9]. The resulted formulae of direct and inverse transforms include the product of two Bessel functions of zero order. However, using of these results is complicated enough due to the highly oscillating integrand of the proposed formulae.

The purpose of this work is obtaining the analytical examples of Fourier transform in elliptic coordinates. Such examples are very useful for testing of calculating algorithms. Moreover, new integral relationships for the Bessel functions can be obtained.

One important remark is that the formulae whose will be presented relate to the case of axial symmetry so the integrand function depends on the quasi radial elliptic coordinate.

At first we briefly reproduce the main steps of Fourier transform obtaining in elliptic coordinates. After this we consider the integration method. We make change of variables to apply the well-known relationships collected in literature. Combination of different formulae allows cutting out the initial "time" variable leaving just "frequency" dependence for Fourier image.

Finally, we include the numerical analysis of application of each formula and conclude the considered study.

2 Fourier Transform in Elliptic Coordinates

We begin with definition of coordinates. The relation of rectangular coordinates (x,y) and elliptic coordinates (μ,ν) is:

$$\begin{aligned} x &= a\cosh\mu\cos\nu; \\ y &= a\sinh\mu\sin\nu. \end{aligned} \quad (1)$$

The coordinates (μ,ν) vary within the following ranges:

$$\mu \geq 0; \quad 0 \leq \nu < 2\pi. \quad (2)$$

In general form the Fourier transform in two dimensions is:

$$F(\omega) = \int_D f(\mathbf{r}) e^{-i\mathbf{r}\boldsymbol{\omega}} d\mathbf{r}, \quad (3)$$

where $\mathbf{r} = \overrightarrow{\{x,y\}}$ and $\boldsymbol{\omega} = \overrightarrow{\{\omega_x,\omega_y\}}$. Here the integral is taken through the whole area D, which is for rectangular coordinates

$$D = \{(x,y) : x \in (-\infty,+\infty), y \in (-\infty,+\infty)\}$$

and

$$D = \{(\mu,\nu) : \mu \in [0,+\infty), \nu \in [0,2\pi)\}$$

for elliptic coordinates. And the $\mathbf{r\omega}$ is the scalar product of vectors $\mathbf{r}$ and $\boldsymbol{\omega}$ [8]. Expanding this scalar product in the exponent, we receive the formula in elliptic coordinates. Let us to do this in details. The scalar product in rectangular coordinates is:

$$\mathbf{r\omega} = x\omega_x + y\omega_y. \qquad (4)$$

Then we change coordinates taking into account (1). Continuing with vector $\boldsymbol{\omega}$ we have:

$$\omega_x = b\cosh\omega_\mu\cos\omega_\nu; \quad \omega_y = b\sinh\omega_\mu\sin\omega_\nu. \qquad (5)$$

After small manipulations with hyperbolic functions we receive instead of (4):

$$x\omega_x + y\omega_y = 0.5\cosh(\mu+\omega_\mu)\cos(\nu-\omega_\nu) + 0.5\cosh(\mu-\omega_\mu)\cos(\nu+\omega_\nu). \qquad (6)$$

Thus, we have the main expression for Fourier transform in elliptic coordinates using the (3):

$$F(\omega_\mu,\omega_\nu) = a^2\int_0^\infty\int_0^{2\pi} f(\mu,\nu)(\sinh^2\mu+\sin^2\nu)\times \exp[i0.5ab(\cosh(\mu+\omega_\mu)\cos(\nu-\omega_\nu))]\times \exp[i0.5ab(\cosh(\mu-\omega_\mu)\cos(\nu+\omega_\nu))]d\mu d\nu, \qquad (7)$$

where $f(\mu,\nu)$ - function of the space $L^2(D)$.

Inverse transform has a similar form:

$$f(\mu,\nu) = b^2\int_0^\infty\int_0^{2\pi} F(\omega_\mu,\omega_\nu)(\sinh^2\omega_\mu+\sin^2\omega_\nu)\times \exp[-i0.5ab(\cosh(\mu+\omega_\mu)\cos(\nu-\omega_\nu))]\times \exp[-i0.5ab(\cosh(\mu-\omega_\mu)\cos(\nu+\omega_\nu))]d\omega_\mu d\omega_\nu. \qquad (8)$$

The systems that possess axial symmetry are often the most interesting, especially in optics. If $f(\mu,\nu) = f(\mu)$ then it is possible to obtain more applicable formulae instead of (7)-(8).

The final result is:

$$F(\omega_\mu) = a^2\pi\int_0^\infty f(\mu)J_0(c\cosh(\mu+\omega_\mu))\times J_0(c\cosh(\mu-\omega_\mu))\cosh(2\mu)d\mu, \qquad (9)$$

where $c = 0.5ab$. We also used the identity $2\sinh^2\mu + 1 = \cosh(2\mu)$.

The formula (9) represents Fourier transform in elliptic coordinates for the case of axial symmetry.

The inverse formula is:

$$f(\mu) = b^2\pi\int_0^\infty F(\omega_\mu)J_0(c\cosh(\mu+\omega_\mu))\times J_0(c\cosh(\mu-\omega_\mu))\cosh(2\omega_\mu)d\omega_\mu, \qquad (10)$$

which is the inverse Fourier transform in elliptic coordinates for the case of axial symmetry.

For more details see [9].

The integral transform (9)-(10) generalizes the Fourier-Bessel transform. Major axis A and minor axis B of ellipse is equal correspondingly to $A = a\cosh\mu$ and $B = a\sinh\mu$, where a is focus distance. Eccentricity is $\varepsilon = 1/\cosh\mu$. If the major axis A is constant and $\varepsilon \to 0$ then $\mu \to \infty$ and $a \to 0$. So $a\cosh\mu = a\sinh\mu \to r$. As a result we have transformation from elliptic coordinate μ to polar coordinate r. In $\boldsymbol{\omega}$ - space we have the similar transformations. Therefore, ω_μ becomes ω_r.

Thus, we have well-known Fourier-Bessel transform instead (9)-(10):

$$F(\omega_r) = 2\pi\int_0^\infty f(r)J_0(r\omega_r)rdr,$$

$$f(r) = 2\pi\int_0^\infty F(\omega_r)J_0(r\omega_r)\omega_r d\omega_r.$$

Parseval's formula [7] also is valid:

$$a^2\int_0^\infty f(\mu)^2\cosh(2\mu)d\mu = b^2\int_0^\infty F(\omega_\mu)^2\cosh(2\omega_\mu)d\omega_\mu. \qquad (11)$$

3 Analytical Formulae

We can present now the integrals which could be calculated analytically and consequently analyzed in terms of the elliptic Fourier transform. These formulae extremely important for the verification process of numerical integration.

For the starting point we make a change of variables as:

$$\alpha = \sinh\mu; \quad \beta = \sinh\omega_\mu. \qquad (12)$$

Correspondingly, we have $d\mu = d\alpha/\sqrt{1+\alpha^2}$, $d\omega_\mu = d\beta/\sqrt{1+\beta^2}$ and instead of (9):

$$F(\beta) = a^2\pi\int_0^\infty f(\alpha)J_0(c\sqrt{(\alpha^2+1)(\beta^2+1)}+c\alpha\beta) J_0(c\sqrt{(\alpha^2+1)(\beta^2+1)}-c\alpha\beta))\frac{(1+2\alpha^2)}{\sqrt{1+\alpha^2}}d\alpha, \qquad (13)$$

Expression (13) allows us to provide some analytical possibilities.

For the convenience we insert the exact formulae taken from the literature source and provide the needed parameters.

3.1 Combination of 2.12.27 from [10] and 6.788 from [11]

The initial integral is 2.12.27 from [10]:

$$\int_0^\pi \cos(2mz) J_0\left(\sqrt{p^2 - q^2 \sin z^2}\right) dz = (-1)^m \pi J_m\left(\frac{p+\sqrt{p^2-q^2}}{2}\right) J_m\left(\frac{p-\sqrt{p^2-q^2}}{2}\right), \quad (14)$$

where $0 \le q \le p$. Taking $m = 0$, $p^2 = 4c^2(\alpha^2+1)(\beta^2+1)$ and $q^2 = 4c^2(\alpha^2+\beta^2+1)$ we have:

$$J_0\left(c\sqrt{(\alpha^2+1)(\beta^2+1)} + c\alpha\beta\right) \times J_0\left(c\sqrt{(\alpha^2+1)(\beta^2+1)} - c\alpha\beta\right) = \frac{1}{\pi}\int_0^\pi J_0\left(2c\sqrt{\alpha^2(\beta^2+\cos^2 z) + (\beta^2+1)\cos^2 z}\right) dz. \quad (15)$$

Substitution of (15) into (13) results in:

$$F(\beta) = a^2 \int_0^\infty f(\alpha) \frac{(1+2\alpha^2)}{\sqrt{1+\alpha^2}} \times \int_0^\pi J_0\left(2c\sqrt{\alpha^2(\beta^2+\cos^2 z) + (\beta^2+1)\cos^2 z}\right) dz d\alpha, \quad (16)$$

Special case of 6.596.1 [11]:

$$\int_0^\infty J_m\left(p\sqrt{z^2+q^2}\right) \frac{z^{2k+1}}{(z^2+q^2)^{m/2}} dz = \frac{2^k \Gamma(k+1)}{p^{k+1} x^{m-k-1}} J_{m-k-1}(pq) \quad (17)$$

with $m = 0$, $k = -1/2$, $p = 2c\sqrt{\beta^2+\cos^2 z}$ and after changing of the order of integration allows to receive:

$$F(\beta) = \frac{a^2}{2c}\int_0^\pi \frac{\cos\left(2c\sqrt{\beta^2+1}\cos z\right) dz}{\sqrt{\beta^2+\cos^2 z}}. \quad (18)$$

Formula (18) is a Fourier image for the function $f(\alpha)$:

$$f(\alpha) = \frac{\sqrt{1+\alpha^2}}{1+2\alpha^2}. \quad (19)$$

3.2 Combination of 2.12.27 from [10] and 6.726.2 from [11]

As a start we use the previous result (16). Applying the 6.726.2 from [11]:

$$\int_0^\infty (y^2+q^2)^{-m/2} J_m\left(p\sqrt{y^2+q^2}\right) \cos(sy) dy = \sqrt{\frac{\pi}{2}} p^{-m} q^{-m+1/2} (p^2-s^2)^{m/2-1/4} J_{m-1/2}\left(q\sqrt{p^2-s^2}\right) \quad (20)$$

with $m = 0, y = \alpha$, $q^2 = (\beta^2+1)\cos^2 z / (\beta^2+\cos^2 z)$, $p^2 = 4c^2(\beta^2+\cos^2 z)$ we have:

$$F(\beta) = \frac{a^2}{\pi} \times \int_0^\pi \frac{\cos\left(\frac{\sqrt{(\beta^2+1)\cos^2 z}}{\sqrt{w(z)^2}} \sqrt{4c^2 w(z)^2 - s^2}\right) dz}{(4c^2 w(z)^2 - s^2)^{-1/2}} \quad (21)$$

with $w(z)^2 = \beta^2 + \cos^2 z$ and $0 < s < p$.

Formula (21) is a Fourier image for the function $f(\alpha)$:

$$f(\alpha) = \frac{\sqrt{1+\alpha^2}}{1+2\alpha^2} \cos s\alpha. \quad (22)$$

3.3 Combination of 2.12.27 from [10] and 6.596.5 from [11]

Once again we use the result of (16). The formula 6.596.5 from [11] is:

$$\int_0^\infty J_n(sz) \frac{J_m(p\sqrt{y^2+q^2})}{\sqrt{(y^2+q^2)^m}} y^{n-1} dz = \frac{2^{n-1}\Gamma(n)}{s^n} \frac{J_m(pq)}{q^m}, \quad (23)$$

where $\mathrm{Re}(m+2) \ge \mathrm{Re} n \ge 0$ and $p > s > 0$.

Applying this with $m = 0$, $p = 2c\sqrt{\beta^2+\cos^2 z}$ and $q = \sqrt{(\beta^2+1)\cos^2 z} / \sqrt{\beta^2+\cos^2 z}$ after changing of the order of integration we receive:

$$F(\beta) = \frac{a^2 2^{n-1}}{s^n} \int_0^\pi J_0\left(2c\sqrt{\beta^2+1}\cos z\right) dz \quad (24)$$

with $0 < p < s$.

Formula (24) is the Fourier image of the function $f(\alpha)$:

$$f(\alpha) = \frac{\sqrt{1+\alpha^2}}{1+2\alpha^2} J_n(s\alpha)\alpha^{n-1}. \quad (25)$$

2.4 Combination of 2.12.27 from [10] and 6.726.2 from [11]

Finally, we introduce the fully analytical formula with completed integration. The formula 6.597 [11] is:

$$\int_0^\infty y^{n+1} J_k\left(h\sqrt{y^2+v^2}\right)\frac{J_n(sy)}{\sqrt{y^2+v^2}\,(y^2+\sigma^2)}dy = \sigma^n J_k\left(h\sqrt{v^2-\sigma^2}\right)\frac{K_n(s\sigma)}{\sqrt{(v^2-\sigma^2)^k}}, \quad (26)$$

where $-1<\mathrm{Re}(n)<2+\mathrm{Re}(k)$, $s\ge h$, $\mathrm{Re}(\sigma)>0$. K_n is Bessel function of third type of order *n*.

Using the (16) and applying (26) with $y=\alpha$, $k=0$, $v=\sqrt{(\beta^2+1)\cos^2 z}\,/\sqrt{\beta^2+\cos^2 z}$ and $h=2c\sqrt{\beta^2+\cos^2 z}$ we receive the result after changing of the order of integration:

$$F(\beta)=a^2\sigma^n K_n(s\sigma)\times \int_0^\pi J_0\left(2cv(z)^2\sqrt{\frac{(\beta^2+1)\cos^2 z}{v(z)^2}-\sigma^2}\right)dz \quad (27)$$

with $v(z)^2=\beta^2+\cos^2 z$.
Here we take $f(\alpha)$ as:

$$f(\alpha)=\frac{\sqrt{1+\alpha^2}}{1+2\alpha^2}\frac{J_n(s\alpha)\alpha^{n+1}}{\alpha^2+\sigma^2}. \quad (28)$$

The integral in (27) is evaluated in terms of expression (14) in reverse manner. If we take $p=2c\sqrt{(\beta^2+1)(1-\sigma^2)}$, $q=2c\sqrt{\beta^2+1-\sigma^2}$ and $m=0$ we obtain the final result:

$$F(\beta)=a^2\sigma^n K_n(s\sigma)\pi\times J_0\left(c\sqrt{(\beta^2+1)(1-\sigma^2)}-c\sqrt{-\beta^2\sigma^2}\right)\times J_0\left(c\sqrt{(\beta^2+1)(1-\sigma^2)}+c\sqrt{-\beta^2\sigma^2}\right). \quad (29)$$

This is a Fourier image for the function (28).

3.5 Results with elliptic variables

Turning back to the μ and ω_μ variables using (12) we obtain the following pairs.

$$f(\mu)=\frac{\cosh\mu}{\cosh 2\mu}\rightarrow F(\omega_\mu)=\frac{a^2}{2c}\int_0^\pi\frac{\cos(2c\cosh(\omega_\mu)\cos z)}{\sqrt{\sinh^2\omega_\mu+\cos^2 z}}dz \quad (30)$$

with $\omega_\mu>0$.

$$f(\mu)=\frac{\cosh\mu}{\cosh 2\mu}\cos(s\sinh\mu)\rightarrow F(\omega_\mu)=a^2\times \int_0^\pi\frac{\cos\left(2cw(z)^{-1}\sqrt{w(z)^2-s^2}\cosh\omega_\mu\cos z\right)dz}{\sqrt{4c^2(w(z)^2-s^2)}} \quad (31)$$

where $w(z)^2=\sinh^2\omega_\mu+\cos^2 z$, $0<s<c\cosh\omega_\mu$.

$$f(\mu)=\frac{\cosh\mu}{\cosh 2\mu}J_n(s\sinh\mu)(\sinh\mu)^{n-1}\rightarrow F(\omega_\mu)=a^2 2^{n-1}s^{-n}\int_0^\pi J_0\left(2c\cosh\omega_\mu\cos z\right)dz, \quad (32)$$

where $0<n<2$ and $s>2c\cosh\omega_\mu$.

$$f(\mu)=\frac{\cosh\mu}{\cosh 2\mu}\frac{J_n(s\sinh\mu)(\sinh\mu)^{n+1}}{\sinh^2\mu+\sigma^2}\rightarrow F(\omega_\mu)=a^2\sigma^n K_n(s\sigma)\pi\times J_0\left(c\cosh\omega_\mu\sqrt{1-\sigma^2}-ic\sinh\omega_\mu\sigma\right)\times J_0\left(c\cosh\omega_\mu\sqrt{1-\sigma^2}+ic\sinh\omega_\mu\sigma\right), \quad (33)$$

where $\sigma>0, -1<n<2$ and $s\ge 2c\cosh\omega_\mu$.

The results in this form are preferable since the numerical integration is more stable: the integrand function behaves less oscillatory. See the next section for details.

Unfortunately, no one of these formulae are not fully applicable due to limitations risen in derivation. So we cannot define these pairs as Fourier transformation pairs because the reverse transform is impossible to define.

4 Numerical Analysis

Now we analyze the previously obtained results in terms of numerical calculation. Due to the restrictions of application every formula must be tested separately.

For the tests we use a laptop PC Asus N56V with Processor Intel Core i7-3610QM CPU 2.3 GHz and 8GB of RAM. The calculation has been performed by Wolfram Mathematica.

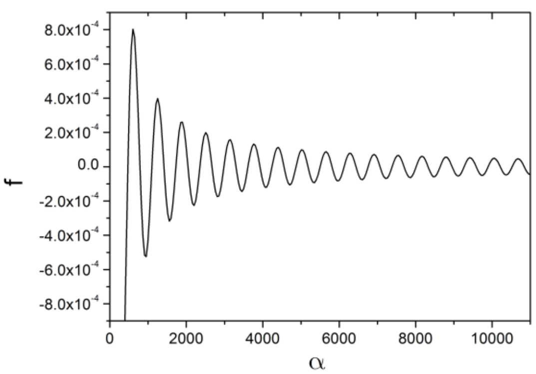
We compare the results of numerical integration using the built-in capabilities of Mathematica package for the direct approximation of (9) with the one for the direct approximation of results in section 3.5, excepting the last formula which is fully analytical. The built-in integration methods include Levin-type collocation [12], Clenshaw-Curtis [13] rule and many others. About the work of built-in

"Numerical integration" function you can see the Manual.

The purpose of this comparison is to show how the incoming parameters influence to the accuracy and the calculation time for the approximation analog of (9). The results from section 3.5 are used as a sort of quasi-analytical or benchmark result.

We also notice that the numerical analysis of the (13) could be useful. But on practice this form is not applicable since it has low accuracy and bigger calculation time in comparison with (9). This fact can be easily explained using the Fig.1. We compare function $f(\alpha)$ from (22) with function $f(\mu)$ from (31). The (31) has almost non-oscillatory behavior. In contrast the (22) oscillates rapidly even for s=0.01. These oscillations cost an expensive calculation "price".

a)



b)

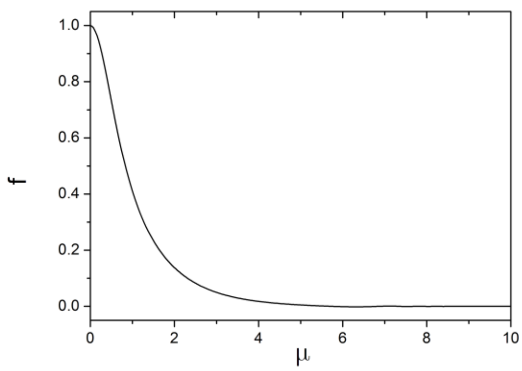


Figure 1: Comparison of a) $f(\alpha)$ (22) with b) $f(\alpha)$ (22). The parameter s=0.01. The range of μ corresponds the one of $\alpha = \sinh\mu$

We use the following parameters for $F(\omega_\mu)$ calculation. $\omega_\mu = 0.01 + (i-1)\omega_{max}/(N-1)$ where $i = 1,2,..N$, N=128 and ω_{max} is taken individually for each formula. The parameters of Mathematica *NIntegration function* are {Accuracy goal→8, Method → "Symbolic Preprocessing", "Oscillatory Selection" → True} for all formulae.

We estimate how fast the $F_1(\omega_\mu)$ calculated by (9) numerically $F_2(\omega_\mu)$ coincides with results from section 3.5 in respect to incoming parameters. It is possible to call this as "accuracy" with some reservations. The estimated parameter is:

$$A = \left| \frac{F_1(\omega_\mu) - F_2(\omega_\mu)}{NF_1(\omega_\mu)} \right| 100\%. \quad (34)$$

The calculation time using quasianalytical results $F_1(\omega_\mu)$ spends 1.5-3.0 seconds of CPU time in average and never exceeds 6 seconds. The time calculation for $F_2(\omega_\mu)$ will be presented for each result independently.

4.1 Result 1 (30)

We take $a = 0.5;1;5$, $b = 0.4;2;4$ and $c = 0.1;1;10$ correspondingly.

Since the upper limit of integral (9) is infinity we need to break off somewhere the numerical integration. And for the upper limit we take value of $\mu_{\max} = 10;20;30$. See the results for CPU time and accuracy A in Table 1.

Table 1. The influence of $\mu_{\max}$ and c to the CPU time t and accuracy A for the numerical integration of (9) in relation to (30)

$\mu_{\max}$	c	t, s	A, %
10	0.1	16.6	0.24
	1.0	19.6	9.67E-2
	10.0	26.5	1.35E-2
20	0.1	22.0	8.17E-6
	1.0	23.8	2.89E-6
	10.0	32.1	6.83E-7
30	0.1	26.6	1.81E-9
	1.0	28.6	5.26E-10
	10.0	29.8	8.95E-10

According to the Table 1 the $\mu_{\max} = 20$ allows to have optimal combination of accuracy and time. And smaller c results in smaller CPU time, but the accuracy becomes worse simultaneously.

4.2 Result 2 (31)

Since the limiting condition is $0 < s < c\cosh\omega_\mu$, the using of (35) is defined mainly by the relation between s and c parameters.

We try to define the lowest boundary ω_0 where the results of numerical integration of (9) are closed to that of (31). The parameters A < 10E-3 and $\mu_{max} = 30$. The CPU time varies in 43-53 seconds for numerical integration of (9) and 2-6 seconds for the (35).

The results for μ_0 are presented in Table 2. For example, if c=2 and s=1, then we can argue that function from (35) differs from that of (9) no more than 0.1% in the interval [2.6, ∞).

Table 2. The lowest boundary ω_0 for the different parameters c and s

$c = 0.5ab$	s	ω_0
0.2	0.01	1.0
a = 0.5	0.1	2.2
b = 0.8	0.2	2.6
2	0.1	0.5
a = 2	1	2.6
b = 2	2	3.2
20	1	1.3
a = 5	10	3.2
b = 8	20	3.7

The analysis of data from the Table 2 permits to conclude that the significant increasing of area of application for the formula (31) is possible when the ratio c/s=20 at least.

Also we need to notice that in resulted function for Fourier image $F(\omega_\mu)$ there are oscillations if we take c > 1. For the c=20 $F(\omega_\mu)$ is a highly oscillating function. So the number of points N of discretization of ω_μ influences to the accuracy parameter A (34). However, N does not change the general trend of that significantly smaller s relative to c is needed for better results of ω_0.

4.3 Result 3 (32)

Since the limitation $s > 2c\cosh\omega_\mu$ the possible values of ω_μ cannot exceed the initially defined value ω_{max}. We set up this value equal to four.

The main feature of (32) is that the Fourier image depends on parameter n as a factor before the integral. The integral itself is the same for all possible values of n and s.

We analyze different $\Delta s = s - 2c\cosh\omega_{max}$ values and μ_{max} values. We try to understand what parameters should be taken to provide the best matching of the results of the numerical integration of (9) and that of (32).

We take a=0.5, b=0.4, so c=0.1. It is needed to have relatively small oscillating function $f(\mu)$.

The time of calculation for the analytical formula (32) is approximately 3 seconds.

The scope of calculation results is in Table 3.

Table 3. The influence of μ_{max} and Δs to the CPU time t and accuracy A for the numerical integration of (9) in relation to (32)

n =1/2	μ_{max} = 10		μ_{max} = 20	
Δs	t, s	A, %	t, s	A, %
0.1	19.6	1.8E-4	22.9	4.6E-11
1	21.1	1.4E-4	25.4	1.3E-10
10	25.0	1.9E-4	25.3	2.7E-10
n =1	μ_{max} = 10		μ_{max} = 20	
Δs	t, s	A, %	t, s	A, %
0.1	35.1	0.2	41.9	3.9E-8
1	36.6	0.2	40.8	5.5E-8
10	37.1	0.9E-1	43.3	8.2E-9
n =3/2	μ_{max} = 10		μ_{max} = 20	
Δs	t, s	A, %	Δs	t, s
0.1	93.7	0.1	110.6	7.5E-4
1	92.9	0.2	109.9	1.8E-3
10	92.3	1.9E-2	104.8	3.9E-3

We recommend to take μ_{max} = 20 at least.

We notice that the higher value of parameter Δs allows to have the better accuracy on the one hand but in other hand it increases the number of oscillations in given function $f(\mu)$ (see (32)) that rises the problems in numerical calculation and consequently reduces the accuracy.

4.4 Result 4 (33)

Finally we consider the comparison of the numerical integration (9) with completely analytical formula (33)

Once again we analyze CPU time t and accuracy A (see (34)) for different values of μ_{max}, n and σ.

We take $c = 0.5ab = 0.5 \cdot 0.5 \cdot 0.4 = 0.1$, $\omega_{max} = 6$ and Δs = 1. The μ_{max} is not exceeds the 10 since the function $f(\mu)$ from (33) is damping fast.

The possibilities of Mathematica permit to analyze relatively small values of σ. In some cases the calculation is not finished and failed. See the Figure 2.

The results for numerical integration using (9) are in Table 4.

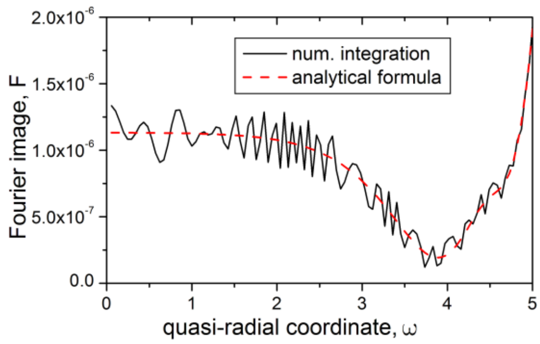


Figure 2. The numerical integration result $F(\omega_\mu)$ compared with analytical one (33). The parameters taken are n=0, σ=0.3, μ_{max}=5, $\omega_{max} = 6$, Δs=1.

Table 4. The influence of μ_{max} and σ to the CPU time t and accuracy A for the numerical integration of (9) in relation to (33)

n =-1/2	μ_{max} = 2.5		μ_{max} = 5	
Δs	t, s	A, %	t, s	A, %
0.001	21.6	4.0E-4	33.2	6.3E-5
0.01	21.3	3.2E-3	25.4	6.1E-6
0.1	23.6	0.2	33.2	2.8E-4
n =0	μ_{max} = 5		μ_{max} = 10	
Δs	t, s	A, %	t, s	A, %
0.001	61.9	2.8E-3	75.8	1.2E-9
0.01	63.6	4.7E-4	78.3	2.8E-9
0.1	84.9	5.6E-3	105.4	2.0E-8
n =1/2	μ_{max} = 5		μ_{max} = 10	
Δs	t, s	A, %	Δs	t, s
0.001	79.8	1.6E-2	112.6	4.2E-6
0.01	79.3	1.7E-2	130.8	3.4E-6
0.1	94.5	4.7E-2	135.0	9.2E-6
n=1	μ_{max} = 5		μ_{max} = 10	
Δs	t, s	A, %	Δs	t, s
0.001	207.4	2.6	304.4	5.6E-3
0.01	202.0	1.8	302.6	3.4E-3
0.1	-	-	340.9	4.5E-3

For Δs =0.1, μ_{max} = 5 and n=1 the calculation result provided by Mathematica has a very low accuracy (A > 50%), so we do not include it into the table. For n=3/2 the calculation provided by Mathematica failed for all test parameters tried.

Thus, the importance of analytical result (33) increases.

5 Conclusion

For previously received Fourier transformation formula (9) in elliptical coordinates we introduce four analytical examples. These results can be considered as inverse Fourier transformation examples changing a to b, μ to ω_μ and f(μ) to $F(\omega_\mu)$.

Three of these examples are also the integrals of more simple type. And one more formula is fully analytical. There are also natural restrictions for the parameters used. So it is needed to take them into account. Due to these limitations we cannot construct even one complete Fourier transformation pair.

However, the obtained formulae are the new relationships for the products of two Bessel functions of zero order with arguments of hyperbolic functions.

The analytical formulae like (30-33) are extremely important for the development of calculating algorithms. We provide a detailed comparison for the different parameters used in every case. Also we formulate the recommendations for the better choice of these parameters. Thus, we prepare a base for further development of calculating algorithms.

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