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Integral representation of solutions of first-order linear partial differential equations, I

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Integral Representation of Solutions of First-Order Linear Partial Differential Equations, I (*).

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dedicated to Hans Lewy

Introduction.

Until now the proofs of the local solvability of the linear PDEs with simple real characteristics satisfying Condition (P) are strictly « existential » and based on a priori estimates (see [1], [5]). In the present paper we give an integral representation of solutions (in a small open set) of the equation

$$Lu = f, \tag{1}$$

in the (very) particular case where L is a first-order linear partial differential operator with analytic coefficients, nondegenerate, satisfying Condition (P) (which, in these circumstances, is equivalent to being locally solvable). The right-hand side f is assumed to be C^∞ and the found solution u is also C^∞ (all this is local, in a fixed open neighborhood of the central point). Thus we answer, in this particular case, the question of the solvability within C^∞-still open in the more general cases.

The techniques introduced in the present paper compel us to look at the local solvability of first-order linear PDEs from a new viewpoint, and to analyze in finer detail its geometric implications (see Ch. I). I believe they should lead to interesting results in more general set-ups. An outline of the main ideas of the paper can be found in [7].

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CHAPTER I

GENERAL CONSIDERATIONS

1. – A new formulation of the solvability condition (P) for first-order linear PDEs.

Let Ω be an open subset of $\mathbb{R}^N$ ($N \geqslant 2$); the coordinates in $\mathbb{R}^N$ are denoted (at least for the moment) by $y^1, \ldots, y^N$. We consider a first-order linear partial differential operator

$$L = \sum_{j=1}^{N} \gamma^j(y) \frac{\partial}{\partial y^j} + \gamma^0(y), \tag{1.1}$$

where the $\gamma^j (0 \leqslant j \leqslant N)$ are complex-valued C^∞ functions in Ω. We make throughout the hypothesis that L is nowhere degenerate, that is,

$$\forall y \in \Omega, \quad \sum_{j=1}^{N} |\gamma^j(y)| \neq 0. \tag{1.2}$$

We denote by L_0 the principal part of L, that is,

$$L_0 = \sum_{j=1}^{N} \gamma^j(y) \frac{\partial}{\partial y^j}. \tag{1.2'}$$

Let then y_0 be an arbitrary point of Ω. There is an open neighborhood $U \subset \Omega$ of y_0 and local coordinates in U, $x^1, \ldots, x^n, t$ (we set $n = N - 1$ and write $x = (x^1, \ldots, x^n)$) such that, in U,

$$L = \zeta(x, t)\left\{\frac{\partial}{\partial t} + i \sum_{j=1}^{n} b^j(x, t) \frac{\partial}{\partial x^j} + c(x, t)\right\}, \tag{1.3}$$

where ζ, b^j $(1 \leqslant j \leqslant n)$, c are C^∞ functions in U; furthermore, ζ does not vanish at any point of U and the b^j are real-valued. We may in fact suppose that U is a product-set

$$U = \{(x, t) \in \mathbb{R}^{n+1};\ x \in U_0,\ |t| < T\}, \tag{1.4}$$

where U_0 is an open neighborhood of the origin and T a number > 0 (we are assuming, for simplicity, that the coordinates $x^1, \ldots, x^n$, t all vanish at y_0). We set: $\boldsymbol{b}(x, t) = \big(b^1(x, t), \ldots, b^n(x, t)\big)$ $(\in \mathbb{R}^n)$.

DEFINITION 1.1. *We say that L satisfies Condition* (P) *at the point y_0 if there is an open neighborhood $V \subset U$ of y_0 such that the following is true:*

(1.5) *In V, the vector field $\boldsymbol{b}(x,t)/|\boldsymbol{b}(x,t)|$ is independent of t.*

Although we have formulated (P) at y_0 in terms of particular coordinates, it can be proved (see [4]) that it does not depend on our choice of the coordinates (provided that the expression of L be of the kind (1.3)). We say that *L satisfies Condition* (P) *in Ω* if it so does at every point of Ω.

Let us assume that (P) holds at y_0 and take the neighborhood U equal to V in Def. 1.1. Let us introduce the singular set

$$\mathcal{N}_0 = \{x \in U_0;\ \forall t,\ |t| < T,\ \boldsymbol{b}(x,t) = 0\}. \tag{1.6}$$

In $U_0 \setminus \mathcal{N}_0$, $\boldsymbol{b}/|\boldsymbol{b}|$ is a C^∞ vector field, nowhere zero; we shall regard it as a first-order linear partial differential operator, without zero-order term, which we denote by

$$X = \sum_{j=1}^{n} v^j(x) \frac{\partial}{\partial x^j}. \tag{1.7}$$

We extend $\boldsymbol{v} = (v^1, \dots, v^n)$ as an arbitrary unit-vector in $\mathcal{N}_0$ (in general, $\boldsymbol{v}$ will not be smooth throughout U_0). In U we may then write

$$L = \zeta(x,t)\left\{\frac{\partial}{\partial t} + i|\boldsymbol{b}(x,t)|X + c(x,t)\right\}. \tag{1.8}$$

We find ourselves in the following situation: U is the union of the « vertical » lines $x = x_0 \in \mathcal{N}_0$, along which $\zeta^{-1}L_0 = \partial/\partial t$, and of the cylinders $\Sigma = \Gamma \times]-T, T[$, where Γ is any integral curve of X in $U_0 \setminus \mathcal{N}_0$, cylinders on which $\zeta^{-1}L_0 = \partial/\partial t + i|\boldsymbol{b}(x,t)|X$.

Assuming now that (P) holds at every point of Ω, this leads to a *foliation* of Ω, which is best seen when the coefficients of L_0 are *analytic*. Let us write $L_0 = A + \sqrt{-1}B$, where A and B are real vector fields. Let us denote by $\mathfrak{g}(A, B)$ the real Lie algebra generated by A and B (for the commutation bracket $[A, B] = AB - BA$). When A and B are analytic vector fields, it is known (see [2]) that each point y_0 of Ω belongs to one, and only one, subset $\mathcal{M}$ of Ω having the following properties:

(1.9) *$\mathcal{M}$ is a connected analytic submanifold of Ω;*

(1.10) *the tangent space to $\mathcal{M}$ at anyone of its points, y, is exactly equal to the « freezing » of $\mathfrak{g}(A, B)$ at y;*

(1.11) *$\mathcal{M}$ is maximal for Properties* (1.9) *and* (1.10).

We shall refer to the submanifolds $\mathcal{M}$ as the *leaves defined by* L. When the coefficients of L_0 are analytic, Condition (P) is equivalent with the local solvability of L (here, at each point of Ω). It can be subdivided into two parts:

(P1) *the dimension of each leaf* $\mathcal{M}$, *defined by* L, *is either one or two.*

Observe that (P1) is always true when the dimension of the surrounding space, i.e., of Ω, is $\leqslant 2$—even when L is not locally solvable.

On an arbitrary two-dimensional leaf $\mathcal{M}$ the complex vector field L_0 defines a skew-symmetric real two-tensor, which it is natural to denote by

$$A \wedge B = -\frac{1}{2i} L_0 \wedge \bar{L}_0 \, . \tag{1.12}$$

(If we set $\alpha^j = \operatorname{Re} \gamma^j$, $\beta^j = \operatorname{Im} \gamma^j$, $j = 1, \ldots, N$, the coordinates of $A \wedge B$ on the canonical basis $\partial/\partial y^j \wedge \partial/\partial y^k$, $1 \leqslant j < k \leqslant N$, are $\alpha^j \beta^k - \alpha^k \beta^j$.) Locally on $\mathcal{M}$, we may always find a generator of the line bundle $\wedge^2 T\mathcal{M}$ (in the coordinates x, t used earlier we may take $\partial/\partial t \wedge X$). Let θ be such a generator, say over an open subset $\mathcal{O}$ of $\mathcal{M}$. Then $A \wedge B = \varrho\theta$, where ϱ is a real-valued (analytic) function in $\mathcal{O}$. Observe that the property that ϱ does not change sign in $\mathcal{O}$ is independent of our choice of θ; observe also that the zero-set of ϱ is a proper analytic subset of $\mathcal{O}$. This gives a meaning to the second « part » of Property (P):

(P2) *Restricted to any two-dimensional leaf* $\mathcal{M}$ (defined by L) *in* Ω, $-(1/2i) L_0 \wedge \bar{L}_0$ *does not change sign.*

Indeed, in the form (1.7), we have

$$-\frac{1}{2i} L_0 \wedge \bar{L}_0 = |\boldsymbol{b}(x, t)| \left(\frac{\partial}{\partial t} \wedge X \right) . \tag{1.13}$$

There is no need to point out that (P), stated as the conjunction of (P1) and (P2), is invariant under coordinates changes and also under multiplication of L by nowhere vanishing complex-valued (analytic) functions.

Along any one-dimensional leaf (parametrized by t) the principal part L_0 is essentially $\partial/\partial t$. We shall see, in the course of the proof of our main result, that, by virtue of (P2), on any two-dimensional leaf, L_0 can be transformed (up to a scalar factor) into the « Cauchy-Riemann operator » $\partial/\partial t + iX$, via a homeomorphism naturally associated with L_0.

In the case of C^∞ coefficients, the reformulation of Property (P) is more delicate. First of all it is not true, in general, that through each point y_0 of Ω passes a unique C^∞ submanifold $\mathcal{M}$ with property (1.10). There is another consequential difference: let us return to the local representation (1.8) and consider the restriction of L to a cylinder $\Sigma = \Gamma \times]-T, T[$, where Γ is an integral curve of X in $U_0 \setminus \mathcal{N}_0$. When the coefficients of L are analytic, $|\boldsymbol{b}(x, t)|$ is an analytic function on Σ; its zero-set intersects any vertical line $x = x_0 \in \Gamma$ only on a *locally finite* set, otherwise $|\boldsymbol{b}|$ would vanish identically on such a line and we should have $x_0 \in \mathcal{N}_0$, contrary to the definition of Γ. In particular the zero-set of $|\boldsymbol{b}|$ is a proper analytic subset of Σ. In the case of C^∞ coefficients the zero-set of $|\boldsymbol{b}|$ might have a nonempty interior. It is submitted to the sole condition of not containing any vertical segment $\{x_0\} \times]-T, T[$. Requirement which is of course dependent on the length T of the « time interval ».

Nevertheless we may formulate (P) as the conjunction of (P1) and (P2), provided we incorporate to (P1) a statement as to the existence of the foliation $\{\mathcal{M}\}$. Leaves cannot anymore be required to have property (1.10) (*example*: $L = \partial/\partial t + ib(x, t)(\partial/\partial x)$, where $b \in C^\infty(\mathbb{R}^2)$ is real-valued and does not vanish anywhere, except at the origin, where all derivatives of b vanish; there can be but one leaf, the whole plane, although $\mathfrak{g}(A, B)$ is one-dimensional at the origin).

DEFINITION 1.2. *Let k be any integer such that $0 \leqslant k \leqslant N$. A k-dimensional leaf, defined by L in Ω, is a subset $\mathcal{M}$ of Ω having the following properties*:

(1.14) *$\mathcal{M}$ is a connected C^∞ submanifold* (*) *of Ω, of dimension k;*

(1.15) *the tangent space to $\mathcal{M}$ at anyone of its points, y, contains the « freezing » of $\mathfrak{g}(A, B)$ at y;*

(1.16) *the boundary* (**) *of $\mathcal{M}$ is a union of leaves defined by L in Ω whose dimension is $< k$;*

(1.17) *$\mathcal{M}$ does not contain any leaf defined by L in Ω whose dimension is $< k$.*

With this definition we may now state:

(P1) *Ω is the union of leaves defined by L, whose dimension is either one or two.*

Observe that, because of Hypothesis (1.2), the dimension of $\mathfrak{g}(A, B)$ at every point is at least one; consequently there are no zero-dimensional leaves

(*) It is perhaps worth recalling what a C^∞ submanifold $\mathcal{M}$ of Ω is: every point of $\mathcal{M}$ (and not necessarily every point of Ω!) has a neighborhood in which $\mathcal{M}$ is defined by the vanishing of a number of smooth functions whose differentials are linearly independent.

(**) We use the word « boundary » as in « manifold with boundary », not with its point-set topology meaning.

and, in virtue of (1.16), the one-dimensional leaves defined by L must be without a boundary.

Let us prove that (P) is equivalent with the conjunction of (P1) and (P2). First, suppose (P) holds at every point of Ω. Let $\mathcal{M}$ denote a connected one-dimensional C^∞ submanifold of Ω, having property (1.15), and also the following one:

(1.18) *Let* $(U, x^1, \ldots, x^n, t)$ *be a local chart in* Ω, *yielding the representation* (1.8) *of* L. *Any segment* $\{x_0\} \times]-T, T[$, $x_0 \in U_0$, *which intersects* $\mathcal{M} \cap U$ *is contained in* $\mathcal{M}$.

The manifold $\mathcal{M}$ cannot have any boundary point y_0 in Ω, as one sees by taking a local chart $(U, x^1, \ldots, x^n, t)$ as in (1.18), containing y_0. Thus, by Def. 1.2, $\mathcal{M}$ is a leaf of L in Ω; and every one-dimensional leaf of L must have property (1.18). Let F denote the union of all the one-dimensional leaves of L in Ω. Since, in any local chart like the one in (1.18), such leaves are unions of vertical segments $\{x_0\} \times]-T, T[$ with $x_0 \in \mathcal{N}_0$, every point $y_0 \in U$ which belongs to the closure of F must lie on such a segment, from which it follows at once that F must be closed.

Let now $\mathcal{M}$ denote a connected two-dimensional C^∞ submanifold of $\Omega \setminus F$, satisfying (1.15) and (1.18), and maximal for these properties. Its boundary $\partial\mathcal{M}$ in Ω (see footnote (**) to p. 5) must also have property (1.18). By the maximality of $\mathcal{M}$, at no point of $\partial\mathcal{M}$ can the Lie algebra $\mathfrak{g}(A, B)$ be two-dimensional. Thus $\partial\mathcal{M}$ must be a union of one-dimensional leaves of L, in fact of at most two of them, since $\mathcal{M}$ is connected, and possibly of only one or of none. Thus $\mathcal{M}$ is a two-dimensional leaf of L. A moment of thought shows that, conversely, every two-dimensional leaf of L in Ω is such a submanifold of $\Omega \setminus F$. Note that every C^∞ submanifold of $\Omega \setminus F$, of dimension $\leqslant 2$, satisfying (1.15), must be entirely contained in some such leaf; and that any two such leaves cannot intersect without being identical. In particular, every point of $\Omega \setminus F$ is contained in one, and only one, two-dimensional leaf of L.

That (P2) is a consequence of (P) has already been explained.

Suppose now that (P1) holds. Let y_0 be an arbitrary point of Ω. By (1.2) we may assume that L has the representation (1.3) in a local chart $(U, x^1, \ldots, x^n, t)$ centered at y_0. This also implies that every leaf $\mathcal{M}$ defined by L in Ω (according to Def. 1.2), is equal, in U, to the union of vertical lines $x = x_0$: As a consequence, every two-dimensional leaf $\mathcal{M}$ intersects the piece of « hyperplane » $\{(x, t) \in U; t = 0\}$ along a smooth curve $\Gamma \subset U_0$; and $\mathcal{M}$ is the cylinder $\Sigma = \Gamma \times]-T, T[$. Let $\tilde{X}$ be a smooth vector field tangent to Γ and nowhere vanishing on Γ. From what we have just seen it follows that, on Σ, $\zeta^{-1}L_0 = \partial/\partial t + i\lambda(x,t)\tilde{X}$, and (P2) means exactly that λ does not change sign in Σ (if $\tilde{X}$ is a *unit* vector-field and if λ is nonnegative, in the notation of (1.8) we have $\lambda = |\boldsymbol{b}(x,t)|$ and $\tilde{X} = X$).

A final remark: in the C^∞ case, in contrast with what happens in the analytic case, the foliation defined by L in a proper open subset of Ω can be strictly finer than the one induced by the foliation of Ω.

2. – Reduction to flat right-hand sides.

We shall systematically use the following terminology:

DEFINITION 2.1. *Let M be a C^∞ manifold, ϱ a nonnegative continuous function in M. We say that a function $f \in C^\infty(M)$ is ϱ-flat in M if given any differential operator (with C^∞ coefficients) $\mathbb{P}$ in M and any integer $k \geq 0$, $\varrho^{-k}\mathbb{P}f$ is a continuous function in M, and we write then $f \underset{\varrho}{\sim} 0$.*

It is convenient to introduce the notation:

$$\mathcal{E}_{\varrho\text{-flat}}(M) = \{f \in C^\infty(M);\ f \text{ is } \varrho\text{-flat in } M\}\,, \tag{2.1}$$

$$\mathcal{D}_{\varrho\text{-flat}}(M) = \{f \in \mathcal{E}_{\varrho\text{-flat}}(M);\ \operatorname{supp} f \text{ is a compact subset of } M\}\,. \tag{2.2}$$

We shall comply throughout with the notation of Sect. 1. In particular, we reason locally, in the neighborhood U given by (1.4). We assume that the « central point » is the origin in $\mathbb{R}^{n+1}$ and that L is given by (1.3); we may even divide by ζ and take $L = L_0 + c(x,t)$, with

$$L_0 = \frac{\partial}{\partial t} + i\sum_{j=1}^{n} b^j(x,t)\frac{\partial}{\partial x^j}\,. \tag{2.3}$$

We suppose that $\boldsymbol{b} = (b^1, \dots, b^n)$ and c are C^∞ functions in an open neighborhood of $\overline{U}$ (which is compact), valued in $\mathbb{R}^n$ and $\mathbb{C}$ respectively.

The function ϱ we use (in a neighborhood of $\overline{U}$) is the following:

$$\varrho(x,t) = \left|\int_{-T}^{t} |\boldsymbol{b}(x,s)|\,ds\right|\,. \tag{2.4}$$

Following a suggestion of L. Nirenberg we begin by proving

THEOREM 2.1. *If U is small enough, there is a continuous linear operator $E\colon C_c^\infty(U) \to C^\infty(U)$ such that $R = LE - I$ maps $C_c^\infty(U)$ into $\mathcal{E}_{\varrho\text{-flat}}(U)$.*

REMARK 2.1. In Th. 2.1 it is *not* assumed that L is locally solvable.

REMARK 2.2. By Def. 2.1 it is clear that $\mathcal{E}_{\varrho\text{-flat}}(M)$ carries a natural Fréchet space topology: the coarsest one which renders all the mappings $f \mapsto \varrho^{-k}\mathbb{P}f$ continuous, from $\mathcal{E}_{\varrho\text{-flat}}(M)$ into $C^0(M)$, as k and $\mathbb{P}$ vary freely. The space $\mathcal{D}_{\varrho\text{-flat}}(M)$ carries the derived $\mathcal{LF}$-topology. It will be obvious that the operator R is continuous for these topologies.

PROOF. We apply Th. 1 of [6] (or else use almost-analytic extensions). If U is small enough, for each $j = 1, \ldots, n$, there is a complex-valued C^∞ function $z^j = z^j(x, t, t')$ such that

$$L_0 z^j \underset{\varrho}{\sim} 0 \quad \textit{in an open neighborhood of } \overline{U}\,, \tag{2.5}$$

$$z^j|_{t=t'} = x^j \tag{2.6}$$

$(t' \in [-T, T])$. If the coefficients of L_0 are analytic, the equivalence (2.5) can be replaced by an exact equality $L_0 z^j = 0$ (because of the Cauchy-Kovalevska theorem). Furthermore (according to Th. 1 of [6])

$$\left|z^j - x^j - i\int_{t'}^{t} b^j(x, s)\, ds\right| \leqslant C\varrho(x, t) \tag{2.7}$$

(where $C \to 0$ with T).

Application of Th. 1 of [6] yields (2.5) and (2.7) with $\varrho(x, t)$ replaced by $\left|\int_{t'}^{t} |\boldsymbol{b}(x, s)|\,ds\right|$; but if $-T \leqslant t' \leqslant t$, the latter quantity clearly does not exceed $\varrho(x, t)$. We write $z = (z^1, \ldots, z^n)$ and set, for arbitrary $f \in C_c^\infty(U)$,

$$E_0 f(x, t) = (2\pi)^{-n}\int_{-T}^{t}\int_{\xi\in\mathbf{R}_n} \exp[iz(x, t, t')\cdot\xi]\cdot g(\xi\cdot\operatorname{Im} z(x, t, t'))\hat{f}(\xi, t')\, dt'\, d\xi\,, \text{ (*)} \tag{2.8}$$

where $\hat{f}(\xi, t)$ is the Fourier transform of f with respect to the x-variables, and $g \in C^\infty(\mathbf{R}^1)$, $g(\tau) = 1$ for $\tau > -1$, $g(\tau) = 0$ for $\tau < -2$. It is checked at once that E_0 maps continuously $C_c^\infty(U)$ into $C^\infty(U)$. Furthermore, if $R_0 = L_0 E_0 - I$, we have

$$R_0 f(x, t) = (2\pi)^{-n}\int_{-T}^{t}\int_{\mathbf{R}_n} \exp[iz\cdot\xi] R_0(x, t, t', \xi)\hat{f}(\xi, t')\, d\xi\, dt'\,, \tag{2.9}$$

(*) This formula mimicks the construction of continuous almost-analytic extensions by Mather ([3]).

with

$$R_0(x,t,t',\xi) = \{ig(\xi\cdot\operatorname{Im} z)L_0 z + g'(\xi\cdot\operatorname{Im} z)L_0(\operatorname{Im} z)\}\cdot\xi \tag{2.10}$$

$\big(z = z(x,t,t')\big)$. Whatever $k\in\mathbb{Z}_+$, for a suitable $C_k>0$,

$$|g'(\xi\cdot\operatorname{Im} z)| \leqslant C_k|\xi\cdot\operatorname{Im} z|^k\,, \tag{2.11}$$

whence, combining (2.7) and (2.11),

$$|g'(\xi\cdot\operatorname{Im} z)L_0(\operatorname{Im} z)\cdot\xi| \leqslant C'_k|\xi|^{k+1}\varrho(x,t)^k\,. \tag{2.12}$$

On the other hand, by (2.5), we have

$$|L_0 z| \leqslant C''_k\varrho(x,t)^k\,. \tag{2.13}$$

Since on the support of $g(\xi\cdot\operatorname{Im} x)$, we have $|\exp[iz\cdot\xi]|\leqslant e^2$, we see that

$$|R_0 f(x,t)| \leqslant C'''_k\,\varrho(x,t)^k\int_{-T}^{t}\int(1+|\xi|)^{k+1}|\hat f(\xi,t')|\,d\xi\,dt'\,. \tag{2.14}$$

An analogous inequality can be obtained for every derivative of $R_0 f$ with respect to (x,t), which shows that $R_0 f$ is ϱ-flat.

Let now $c_1(x,t)\in C_c^\infty(U)$ equal $c(x,t)$ in a subneighborhood U_1 of the origin, and set

$$Ef = \exp[-E_0c_1]E_0\big(\exp[E_0c_1]f\big)\,,\qquad f\in C_c^\infty(U_1)\,. \tag{2.15}$$

We have, in U_1,

$$\begin{aligned} LEf &= (L_0+c_1)\{\exp[-E_0c_1]E_0\big(\exp[E_0c_1]f\big)\}\\ &= \exp[-E_0c_1](L_0+c_1-L_0E_0c_1)E_0\big(\exp[E_0c_1]f\big)\\ &= f-(R_0c_1)f+\exp[-E_0c_1]R_0\big(\exp[E_0c_1]f\big)\,, \end{aligned}$$

which shows that the statement of Th. 2.1 is verified, if U_1 is substituted for U and if we define R by

$$Rf = \exp[-E_0c_1]R_0\big(\exp[E_0c_1]f\big)-(R_0c_1)f\,. \qquad \text{Q.E.D.} \tag{2.16}$$

CHAPTER II

CONSTRUCTION OF FUNDAMENTAL SOLUTIONS WHEN THE COEFFICIENTS ARE ANALYTIC

1. – Statement of the theorem.

We use the notation of Ch. I. In particular, $L = L_0 + c(x, t)$ and L_0 has the form (2.3), Ch. I. The coefficients of L_0 are now analytic in an open neighborhood of the origin in $\mathbb{R}^{n+1}$, which we take to be Ω. In this chapter we prove:

THEOREM 1.1. *There is an open neighborhood $U \subset\subset \Omega$ of the origin in $\mathbb{R}^{n+1}$ and a continuous linear operator $K: C_c^\infty(U) \to C^\infty(U)$ such that $LK = I$, the identity of $C_c^\infty(U)$.*

Let us note right-away that it suffices to prove the result when $c(x, t) \equiv 0$. For if $L_0 K_0 = I$ and $h = K_0 c_1$, with $c_1 \in C_c^\infty(U)$, $c_1 = c$ in an open neighborhood $U' \subset U$ of 0, we may take $K = e^{-h} K_0 e^h$, which yields $LK = I$ in U'. From now on we assume that $L = L_0$, i.e.

$$L = \frac{\partial}{\partial t} + i \sum_{j=1}^{n} b^j(x, t) \frac{\partial}{\partial x^j} \tag{1.1}$$

in Ω.

If we take Th. 2.1, Ch. I, into account, we see that Th. 1.1 will follow from the result below. We introduce and shall use throughout the following function:

$$r(x) = \left\{ \int_{-T}^{T} |\boldsymbol{b}(x, s)|^2 \, ds \right\}^{\frac{1}{2}} . \tag{1.2}$$

If we compare with the function $\varrho(x, t)$ defined in (2.4), Ch. I, we see that

$$\varrho(x, t) \leqslant \sqrt{(T + t)}\, r(x) \leqslant \sqrt{2T}\, r(x) \tag{1.3}$$

for $(x, t) \in \Omega$, $|t| \leqslant T$. Thus if we regard r as a function in Ω, we see that any ϱ-flat (Def. 2.1, Ch. I) C^∞ function is also r-flat. In particular (see (2.1), Ch. I), $\mathcal{E}_{\varrho\text{-flat}}(U) \subset \mathcal{E}_{r\text{-flat}}(U)$. We shall prove the following result:

THEOREM 1.2. *If the open neighborhood $U \subset\subset \Omega$ of the origin is small enough, there is a linear operator $K_1: \mathcal{D}_{r\text{-flat}}(U) \to \mathcal{E}_{r\text{-flat}}(U)$* (see (2.1), (2.2), Ch. 1) *such that $LK_1 = I$, the identity of $\mathcal{D}_{r\text{-flat}}(U)$.*

Let then $\zeta(x, t) \in C_c^\infty(U)$, $\zeta(x, t) = 1$ in an open subneighborhood U' of 0. Let R be the operator in Th. 2.1, Ch. I, and set, for any $f \in C_1(U)$,

$$Kf = Ef - K_1[\zeta(x, t) Rf], \tag{1.4}$$

where E is the operator in Th. 2.1, Ch. I. Obviously K satisfies the requirements of Th. 1.1, if we substitute U' for U (inspection of the proofs shows easily that K_1, as well as R, are continuous when $\mathcal{D}_{r\text{-flat}}(U)$ and $\mathcal{E}_{r\text{-flat}}(U)$ carry their natural topologies, and this implies the continuity of K).

2. – Determination of an across-the-board phase function.

From now on we suppose that

$$U_0 = \{x \in \mathbb{R}^n;\ |x| < R\}, \qquad R > 0, \tag{2.1}$$

recalling that $U = U_0 \times]-T, T[$. We also recall that

$$\mathcal{N}_0 = \{x \in \Omega;\ \forall t,\ |t| < T,\ \boldsymbol{b}(x, t) = 0\}. \tag{2.2}$$

Let us set

$$\sigma(z, t) = \sum_{k=1}^{n} [b^k(z, t)]^2, \tag{2.3}$$

where $z = x + iy$ varies in an open neighborhood of $\overline{U} + i\{y \in \mathbb{R}^n;\ |y| \leqslant r_0\}$. Since $\sigma(x, t) \geqslant 0$ (for $x \in \mathbb{R}^n$ in a neighborhood of $\overline{U}$), there is a constant $M \geqslant 0$ such that

$$|\sigma_x(x, t)| \leqslant M\sqrt{\sigma(x, t)}, \qquad \forall x \in \overline{U},\ |t| < T. \tag{2.4}$$

Of course, $\sqrt{\sigma(x, t)} = |\boldsymbol{b}(x, t)|$. We may write

$$\sigma(x + iy, t) = \sigma(x, t) + iy \cdot \sigma_x(x, t) + \tau_2(x, y, t), \tag{2.5}$$

where $|\tau_2(x, y, t)| \leqslant M_1 |y|^2$. If $\varkappa > 0$ is small enough and if

$$|y| < \varkappa |\boldsymbol{b}(x, t)|, \tag{2.6}$$

we obtain:

$$|\sigma(x+iy,t)-\sigma(x,t)|<\tfrac{1}{2}\sigma(x,t)\,. \tag{2.7}$$

LEMMA 2.1. *If $\varkappa>0$ is small enough, $\beta(x,t)=|\boldsymbol{b}(x,t)|$ can be extended as a holomorphic function of $z=x+iy$ in the region:*

$$x\in U_0\setminus\mathcal{N}_0\,,\qquad |y|<\varkappa\sup_{|s|<T}|\boldsymbol{b}(x,s)|\,, \tag{2.8}$$

for all t, $|t|<T$ ($\varkappa$ is independent of t).

PROOF. According to what precedes the statement of Lemma 2.1, in the region

$$x\in U_0\setminus\mathcal{N}_0\,,\qquad |y|<\varkappa|\boldsymbol{b}(x,t)|\,, \tag{2.9}$$

we may (see (2.7)) set $\beta(z,t)=\sqrt{\sigma(z,t)}$ (the second inequality in (2.9) demands $|\boldsymbol{b}(x,t)|\neq 0$). If we set, in (2.9),

$$\boldsymbol{v}(z,t)=\beta(z,t)^{-1}\boldsymbol{b}(z,t)\,, \tag{2.10}$$

we see, by analytic continuation from $z=x$ real, that $\partial_t v(z,t)=0$, and we may write $\boldsymbol{v}(z)$ instead of $\boldsymbol{v}(z,t)$:

$$\beta(z,t)\boldsymbol{v}(z)=\boldsymbol{b}(z,t)\,. \tag{2.11}$$

Let $x_0\in U_0\setminus\mathcal{N}_0$ be arbitrary. Consider any number $\eta>0$ such that

$$\eta<\sup_{|t|<T}|\boldsymbol{b}(x,t)|\,. \tag{2.12}$$

There is t_0, $|t_0|<T$, such that $\beta(x_0,t_0)>\eta$. Actually, we may find $\eta_1>0$ such that

$$|x-x_0|<\eta_1\Rightarrow\beta(x,t_0)>\eta\,. \tag{2.13}$$

We apply (2.7) or, rather, its consequence, namely that

$$|y|<\varkappa\beta(x,t_0)\Rightarrow|\beta(x+iy,t_0)|>\frac{1}{\sqrt{2}}\beta(x,t_0)\,, \tag{2.14}$$

hence:

$$|x-x_0|<\eta_1\,,\qquad |y|<\varkappa\beta(x,t_0) \tag{2.15}$$

implies

$$\beta(x+iy, t_0) > \eta/\sqrt{2}\,. \tag{2.16}$$

But then, in the region (2.15),

$$\boldsymbol{v}(x+iy) = \boldsymbol{b}(x+iy, t_0)/\beta(x+iy, t_0) \tag{2.17}$$

is holomorphic and, since $|\boldsymbol{b}(z, t_0)| \geqslant |\beta(z, t_0)|$, nowhere vanishes. In particular, every point in the set (2.15) has an open neighborhood in which, for some $j = 1, \ldots, n$, $v^j(x+iy)$ is nowhere zero. Since

$$\beta(x+iy, t) = b^j(x+iy, t)/v^j(x+iy)\,,$$

we see that $\beta(z, t)$ can be extended as a holomorphic function to such a neighborhood. Letting

$$\eta \to \sup_{|s|<T} \beta(x_0, s)$$

proves the assertion in Lemma 2.1. Q.E.D.

REMARK 2.1. Actually $\beta(x+iy, t)$ is a C^∞ function of t, $|t| < T$, valued in the space of holomorphic functions of $x+iy$ in the set (2.8).

REMARK 2.2. We have also shown that $|\boldsymbol{v}(z+iy)| \geqslant 1$ in the set (2.8) provided that $\varkappa > 0$ is small enough.

Assume, as we may, that $T \leqslant \frac{1}{2}$. Then (cf. (1.2)):

$$r(x) = \left\{\int_{-T}^{T} |\boldsymbol{b}(x, s)|^2\, ds\right\}^{\frac{1}{2}} \leqslant \sup_{|s|\leqslant T} |\boldsymbol{b}(x, s)|\,. \tag{2.18}$$

LEMMA 2.2. *If $\varkappa > 0$ is small enough, the holomorphic function $\int_{-T}^{T} \beta(x+iy, s)^2\, ds$ does not vanish at any point $x+iy$ such that*

$$x \in U_0 \setminus \mathcal{N}_0\,, \qquad |y| < \varkappa r(x)\,. \tag{2.19}$$

PROOF. We have $\beta(z, s)^2 = \sigma(z, s)$ (see (2.3)). We have (cf. (2.4) and (2.5)):

$$\left|\int_{-T}^{T} \sigma(x+iy, s)\, ds - \int_{-T}^{T} \sigma(x, s)\, ds\right| \leqslant M\varkappa r(x) \int_{-T}^{T} \beta(x, s)\, ds + 2TM_1 \varkappa^2 r(x)^2\,. \tag{2.20}$$

On the other hand,

$$\int_{-T}^{T} \beta(x, s)\, ds \leqslant \sqrt{2T}\, r(x)\,,$$

whence, by (2.20),

$$\left| \int_{-T}^{T} \sigma(x + iy, s)\, ds - r(x)^2 \right| \leqslant M_2 \varkappa r(x)^2\,. \tag{2.21}$$

Lemma 2.2 follows from the fact that $r(x) > 0$ if $x \in U_0 \setminus \mathcal{N}_0$. Q.E.D.

REMARK 2.3. Actually, as we see in (2.21) where we take $M_2 \varkappa < \frac{1}{2}$, that is,

$$\left| \int_{-T}^{T} \sigma(x + iy, s)\, ds - r(x)^2 \right| \leqslant \tfrac{1}{2} r(x)^2\,, \tag{2.22}$$

we may define the square-root,

$$r(x + iy) = \left\{ \int_{-T}^{T} \beta(x + iy, s)^2\, ds \right\}^{\frac{1}{2}}, \tag{2.23}$$

which is holomorphic in the open set (2.19).

We are now going to study a Cauchy problem

$$\frac{\partial \varphi}{\partial t} + i \sum_{k=1}^{n} b^k(z, t) \frac{\partial \varphi}{\partial z^k} = g(z, t)\,, \tag{2.24}$$

in a region

$$x \in U_0 \setminus \mathcal{N}_0\,, \qquad |y| < \varkappa r(x)\,, \qquad |t| < T' \leqslant T\,, \tag{2.25}$$

with initial condition,

$$\varphi|_{t=0} = 0\,, \tag{2.26}$$

where g will be eventually chosen, but in any event, is a C^∞ function of (x, y, t) in the region

$$x \in U_0 \setminus \mathcal{N}_0\,, \qquad |y| < \varkappa r(x)\,, \qquad |t| < T\,, \tag{2.27}$$

holomorphic with respect to $x + iy$.

Let $x_0 \in U_0 \setminus \mathcal{N}_0$ be arbitrary. We denote by E_s $(0 < s < 1)$ the space of continuous functions in the ball $\{z \in \mathbf{C}^n;\ |z - x_0| \leqslant s\varkappa_0 r(x_0)\}$, holomorphic in the interior. Here $\varkappa_0$ is chosen so as to have, if $z = x + iy$,

$$|z - x_0| \leqslant \varkappa_0 r(x_0) \Rightarrow y < \varkappa r(x)\,, \tag{2.28}$$

where $\varkappa$ is the number in Lemma 2.2. Such a choice is indeed possible, for $|z - x_0| \leqslant \varkappa_0 r(x_0)$ implies $|x - x_0| \leqslant \varkappa_0 r(x_0)$ and $|y| \leqslant \varkappa_0 r(x_0)$ $(z = x + iy)$, therefore $|r(x) - r(x_0)| \leqslant \Big(\int\limits_{-T}^{T} |\boldsymbol{b}(x, s) - \boldsymbol{b}(x_0, s)|^2\, ds\Big)^{\frac{1}{2}} \leqslant M|x - x_0| \leqslant M\varkappa_0 r(x_0) \leqslant \frac{1}{2} r(x_0)$ if $M\varkappa_0 \leqslant \frac{1}{2}$, and thus it implies $r(x_0) \leqslant 2r(x)$, whence $|y| < \varkappa r(x)$ if $2\varkappa_0 < \varkappa$.

We shall denote by $h \mapsto \|h\|_s$ the (maximum) norm in E_s. We solve (2.24)-(2.26) by the standard iteration method: we set

$$\varphi(z, t) = \sum_{\nu=0}^{\infty} \varphi_\nu(z, t)\,, \tag{2.29}$$

with

$$\varphi_0(z, t) = \int\limits_0^t g(z, s)\, ds\,, \tag{2.30}$$

$$\varphi_\nu(z, t) = -\int\limits_0^t \sum_{k=1}^n b^k(z, s) \frac{\partial \varphi_{\nu-1}}{\partial z^k}(z, s)\, \mathrm{d}s\,, \qquad \textit{for } \nu > 0\,, \tag{2.31}$$

We shall prove that the series (2.29) converges in E_{s_0} provided that s_0 is small enough (and that the same is true of the t-derivatives of the series). We are going to use the notation

$$B(x, t) = \int\limits_0^t \{|\boldsymbol{b}(x, s)| + Mr(x)\}\, ds\,, \tag{2.32}$$

where $M > 0$ is chosen so as to have

$$|\boldsymbol{b}(z, t)| \leqslant \beta(x, t) + Mr(x) \tag{2.33}$$

if $|z - x| \leqslant \varkappa_0 r(x)$ (cf. (2.21)).

We shall prove by induction on $\nu = 0, 1, \ldots$, that for suitable constants C_0, $C > 0$, and all s, $0 < s < 1$,

$$\|\varphi_\nu(\cdot, t)\|_s \leqslant C_0 \left(\frac{CB(x_0, t)}{1 - s}\right)^\nu, \qquad |t| < T\,. \tag{2.34}$$

For $\nu = 0$, (2.34) holds trivially. By Cauchy's inequalities and (2.33), we have for $\nu > 0$,

$$\begin{aligned}(2.35)\qquad &\|\varphi_\nu(\cdot, t)\|_s \leqslant \\ &\leqslant \frac{1}{\varepsilon\varkappa_0 r(x_0)}\left|\int_0^t [\beta(x_0, t) + Mr(x_0)]\|\varphi_{\nu-1}(\cdot, t')\|_{s+\varepsilon}\,dt'\right| \leqslant \\ &\leqslant C_0 \frac{C^{\nu-1}}{\varkappa_0 r(x_0)} \frac{1}{\varepsilon} \frac{1}{(1-s-\varepsilon)^{\nu-1}}\left|\int_0^t B(x_0, t')^{\nu-1} \frac{\partial B}{\partial t}(x_0, t')\,dt'\right| \leqslant \\ &\leqslant C_0 \frac{C^{\nu-1}}{\varkappa_0 r(x_0)} \frac{1}{\nu\varepsilon(1-s-\varepsilon)^{\nu-1}} B(x_0, t)^\nu .\end{aligned}$$

We take $\varepsilon = (1-s)/(\nu+1)$. Then

$$(2.36)\qquad \|\varphi_\nu(\cdot, t)\|_s \leqslant C_0 \left(\frac{CB(x_0, t)}{1-s}\right)^\nu \frac{1}{C\varkappa_0 r(x_0)} \left(\frac{\nu+1}{\nu}\right)^\nu ,$$

and we take

$$(2.37)\qquad C \geqq \frac{e}{\varkappa_0 r(x_0)} .$$

This proves (2.34).

We note now that

$$B(x, t) \leqslant \sqrt{|t|}\Big(\int_0^t |b(x, s)|^2 ds\Big)^{\frac{1}{2}} + Mr(x)|t| \leqslant (1 + M\sqrt{T})\, r(x)\sqrt{|t|} ,$$

hence, by (2.34) and (2.37),

$$(2.38)\qquad \|\varphi_\nu(\cdot, t)\|_s \leqslant C_0 \left[\frac{e(1 + M\sqrt{T})}{(1-s)\varkappa_0}\right]^\nu |t|^{\nu/2}.$$

Finally we take $s \leqslant \frac{1}{2}$, and

$$(2.39)\qquad |t| \leqslant T' = \inf\left(T, \left[\frac{\varkappa_0}{2e(1+M\sqrt{T})}\right]^2\right),$$

and conclude that the series (2.29) converges in $C_0([-T', T']; E_s)$. From this and from the eq. (2.24) we derive easily that

$$\varphi(z, t) \in C^\infty([-T', T']; E_s) .$$

We apply what precedes to the case where $g(z, t) = \beta(z, t)/ir(z)$. By virtue of Lemma 2.1 (if $T \leqslant \frac{1}{2}$) we know that $\beta(z, t)$ is holomorphic in the set (2.19), and depends smoothly on (z, t) $\big(|t| \leqslant T\big)$. By Lemma 2.2 and Remark 2.3 we know that $r(z)$ is holomorphic and nowhere zero in (2.19). We may state:

LEMMA 2.3. *Suppose that $\varkappa > 0$ and $0 < T' \leqslant T$ are small enough. Then, in the region*

$$x \in U_0 \setminus \mathcal{N}_0\,, \qquad |y| < \varkappa r(x)\,, \qquad |t| < T'\,, \tag{2.40}$$

the Cauchy problem

$$\frac{\partial \varphi}{\partial t} + i \sum_{k=1}^{n} b^k(z, t) \frac{\partial \varphi}{\partial z^k} = \frac{1}{i} \frac{\beta(z, t)}{r(z)}\,, \tag{2.41}$$

$$\varphi|_{t=0} = 0\,, \tag{2.42}$$

has a unique solution which is C^∞ with respect to (x, y, t) and holomorphic with respect to $z = x + iy$. It satisfies in (2.40), *for $y = 0$,*

$$r(x) \left| \sum_{k=1}^{n} v^k(x) \frac{\partial \varphi}{\partial x^k}(x, t) \right| \leqslant C_1 |t|^{\frac{1}{2}}\,, \tag{2.43}$$

and if t' is any other point in the interval $]-T', T'[$,

$$\left| \varphi(x, t) - \varphi(x, t') - \frac{1}{ir(x)} \int_{t'}^{t} |\boldsymbol{b}(x, s)|\, ds \right| \leqslant C_1 \sup_{s \in [t, t']} \big(|s|^{\frac{1}{2}}\big) \left| \int_{t'}^{t} \frac{|\boldsymbol{b}(x, s)|}{r(x)}\, ds \right|. \tag{2.44}$$

We recall that $\beta(x, t) = |\boldsymbol{b}(x, t)|$ and $\boldsymbol{v}(x, t) = \boldsymbol{b}(x, t)/|\boldsymbol{b}(x, t)|$ $\big($see (2.10)$\big)$.

PROOF. The existence has already been proved; the uniqueness is standard. We shall prove (2.43) and (2.44).

If we apply (2.30) and the estimates (2.38) we see that

$$\left| \varphi(z, t) - \frac{1}{i} \int_0^t \frac{\beta(z, s)}{r(z)}\, ds \right| \leqslant C' |t|^{\frac{1}{2}}\,.$$

If $|z - x_0| < \varkappa_0 r(x_0)$ and if $\varkappa_0$ is small enough, we have

$$|\beta(z, s)| \leqslant \beta(x_0, s) \leqslant M^{\natural} r(x_0)\,,$$

$$|r(z)| \geqslant \tfrac{1}{2} r(x_0)\,,$$

whence:

$$|\varphi(z, t)| \leqslant 2\left|\int_0^t \frac{\beta(x_0, s)}{r(x_0)}\, ds\right| + 2M^{\natural}|t| + C'|t|^{\frac{1}{2}}.$$

But $\left|\int_0^t \beta(x_0, s)\, ds\right| \leqslant \sqrt{|t|}\, r(x_0)$, and therefore:

$$|\varphi(z, t)| \leqslant C''|t|^{\frac{1}{2}}. \tag{2.45}$$

By Cauchy's inequalities:

$$\left|\sum_{k=1}^{n} v^k(x) \frac{\partial\varphi}{\partial x^k}(x, t)\right| \leqslant \frac{1}{\varkappa_0 r(x)} \sup_{|z-x|\leqslant\varkappa_0 r(x)} |\varphi(z, t)|,$$

and if $\varkappa_0$ is small enough, we derive (2.43) from (2.45).

Let us denote by $w(x, t)$ the left-hand side in (2.43). By integration of (2.41) with respect to t, between t' and t, we derive:

$$\left|\varphi(x, t) - \varphi(x, t') + i\int_{t'}^{t} \frac{\beta(x, s)}{r(x)}\, ds\right| \leqslant \int_{t'}^{t} \frac{\beta(x, s)}{r(x)}\, w(x, s)\, ds.$$

Since, by (2.43), we have $w(x, s) \leqslant C_1|s|^{\frac{1}{2}}$, we get indeed (2.44). Q.E.D.

Appendix to Section 2

Estimates for later use.

Lemma 2.4. *To every $\alpha \in \mathbb{Z}_+^n$ there is a constant $C_\alpha > 0$ such that*

$$|\partial_x^\alpha \boldsymbol{v}(x)| \leqslant C_\alpha r(x)^{-|\alpha|}, \qquad \forall x \in U_0 \setminus \mathcal{N}_0. \tag{2.46}$$

Proof. Let $x_0 \in U_0 \setminus \mathcal{N}_0$ be arbitrary. By the mean value theorem there is t_0, $|t_0| < T$, such that $|\boldsymbol{b}(x_0, t_0)| = (1/\sqrt{2T})\, r(x_0)$ $(\neq 0)$.

For x sufficiently near x_0 we may write $\boldsymbol{v}(x) = \boldsymbol{b}(x, t_0)/|\boldsymbol{b}(x, t_0)|$. The numerator, and the square of the denominator are C^∞ functions in an open neighborhood of $\overline{U}_0$. Consequently, there is a constant $C'_\alpha > 0$, depending only on the derivatives of these functions (and not on x_0), such that, for x near x_0,

$$|\partial^\alpha \boldsymbol{v}(x)| \leqslant C'_\alpha |\boldsymbol{b}(x, t_0)|^{-|\alpha|}. \tag{2.47}$$

Making $x = x_0$ in (2.47) yields (2.46) with $C_\alpha = C'_\alpha (2T)^{|\alpha|/2}$. Q.E.D.

LEMMA 2.5. *To every* $\alpha \in \mathbb{Z}_+^n$ *there is a constant* $\tilde{C}_\alpha > 0$ *such that*

$$|\partial_x^\alpha r(x)| \leqslant \tilde{C}_\alpha r(x)^{1-|\alpha|}, \quad \forall x \in U_0 \setminus \mathcal{N}_0 . \tag{2.48}$$

In particular, r *is uniformly Lipschitz continuous in* U_0.

Indeed, $r^2 \in C^\infty(\overline{U}_0)$.

COROLLARY 2.1. *Given any* $\alpha \in \mathbb{Z}_+^n$ *there is a constant* $\tilde{\tilde{C}}_\alpha > 0$ *such that*

$$|\partial_x^\alpha[r(x)\boldsymbol{v}(x)]| \leqslant \tilde{\tilde{C}}_\alpha r(x)^{1-|\alpha|}, \quad \forall x \in U_0 \setminus \mathcal{N}_0 . \tag{2.49}$$

In particular, the function equal to $r(x)\boldsymbol{v}(x)$ *in* $U_0 \setminus \mathcal{N}_0$ *and to zero in* $\mathcal{N}_0$ *is uniformly Lipschitz continuous in* U_0.

In the statements below φ stands for the solution of (2.41)-(2.42).

LEMMA 2.6. *Given any* $\alpha \in \mathbb{Z}_+^n$, *there is a constant* $C'_\alpha > 0$ *such that*

$$|\partial_x^\alpha\{X\varphi(x, t)\}| \leqslant C'_\alpha |t|^{\frac{1}{2}} r(x)^{-\alpha-1}, \quad \forall x \in U_0 \setminus \mathcal{N}_0, \ |t| < T . \tag{2.50}$$

PROOF. By Cauchy's inequalities and (2.45) we have (provided that $\varkappa_0$ is small enough):

$$\frac{1}{\alpha!}|\partial_x^\alpha \varphi(x, t)| \leqslant [\varkappa_0 r(x)]^{-|\alpha|} \sup_{|z-x| \leqslant \varkappa_0 r(x)} |\varphi(z, t)| \leqslant C'' \varkappa_0^{-|\alpha|} |t|^{\frac{1}{2}} r(x)^{-|\alpha|} . \tag{2.51}$$

Combining (2.46) and (2.51) yields (2.50).

COROLLARY 2.2. — $\forall \alpha \in \mathbb{Z}_+^n$, *there is a constant* $\tilde{C}'_\alpha > 0$ *such that*

$$|\partial_x^\alpha\{r(x) X\varphi(x, t)\}| \leqslant C'_\alpha |t|^{\frac{1}{2}} \, r \, (x)^{-|\alpha|}, \quad \forall x \in U_0 \setminus \mathcal{N}_0, \ |t| < T . \tag{2.52}$$

Indeed, combine Lemmas 2.5 and 2.6.

COROLLARY 2.3. *Whatever* $f \in \mathfrak{D}_{r\text{-flat}}(U)$, *we also have* $(rX\varphi)f \in \mathfrak{D}_{r\text{-flat}}(U)$.

3. – Solution on the individual leaves.

In $U_0 \setminus \mathcal{N}_0$ the vector field X defined in (1.7), Ch. I,

$$X = \sum_{j=1}^n v^j(x) \frac{\partial}{\partial x^j} \tag{3.1}$$

is analytic and nowhere zero (see proof of Lemma 2.1). For any $x_0 \in U_0 \setminus \mathcal{N}_0$ we denote by Γ_{x_0} the *connected* integral curve of X through x_0, by Σ_{x_0} the two-dimensional « leaf » $\Gamma_{x_0}\times\,]-T, T[$. Writing $\boldsymbol{v}=(v^1, \ldots, v^n)$ as before, let us denote by $x(\chi, x_0)$ the solution of the problem

$$\frac{dx}{d\chi}=r(x)\boldsymbol{v}(x)\,, \qquad x|_{\chi=0}=x_0\,. \tag{3.2}$$

The function r is defined in (1.2). The mapping

$$\chi\mapsto x(\chi, x_0) \tag{3.3}$$

is a C^∞ mapping and, in fact, a local diffeomorphism of an open interval $J(x_0)$ of $\mathbb{R}^1$ onto Γ_{x_0}. Conversely we may regard χ as a smooth function in a sufficiently small open arc in Γ_{x_0}, centered at x_0, specifically the solution of

$$r(x)X\chi=1\,, \qquad \chi|_{x=x_0}=0\,. \tag{3.4}$$

Let us recall the standard relations, valid if $x, x_1\in\Gamma_{x_0}$:

$$\chi(x_0, x_1)=-\chi(x_1, x_0)\,, \tag{3.5}$$

$$\chi(x, x_1)=\chi(x, x_0)+\chi(x_0, x_1)\,. \tag{3.6}$$

It is perhaps worth distinguishing the various possibilities:

(I) Γ_{x_0} *is a compact subset of* $U_0\setminus\mathcal{N}_0$. This means that $x(\chi, x_0)$ is periodic with respect to χ; then $J(x_0)=\mathbb{R}^1$ and (3.3) is a covering map;

(II) Γ_{x_0} *is noncompact in* $U_0\setminus\mathcal{N}_0$; then (3.3) is a *global* diffeomorphism of $J(x_0)$ onto Γ_{x_0}.

In Case (II), $J(x_0)$ might have one finite boundary point in $\mathbb{R}^1$, two of them or none. But:

LEMMA 3.1. *If $J(x_0)$ has a finite boundary point χ_0, then as $\chi\to\chi_0$, $x(\chi, x_0)$ goes to the boundary of U_0* (i.e., exits from every compact subset of U_0).

PROOF. By Cor. 2.1 we know that $|r(x)\boldsymbol{v}(x)-r(x')\boldsymbol{v}(x')|\leqslant K|x-x'|$ for some $K<+\infty$ and all $x, x'\in U_0$. By the Picard iteration method applied to (3.2) we obtain

$$|x_1-x_0|\leqslant r(x_1)\left(\exp\left[K|\chi(x_0, x_1)|\right]-1\right) \tag{3.7}$$

(we have tacitly exploited (3.5)). Suppose then that $\chi(x_0, x_1) \to \chi_0 \in \partial J(x_0)$, $|\chi_0| < +\infty$; clearly we cannot have $r(x_1) \to 0$. On the other hand, since $r(x)X$ does not vanish in $U_0 \setminus \mathcal{N}_0$, x_1 must go to the boundary of $U_0 \setminus \mathcal{N}_0$, whence the conclusion.

An equivalent interpretation of Lemma 3.1 is that taking χ as the parameter on an integral curve Γ_{x_0} of X amounts to changing the metric along Γ_{x_0} in such a way that the distance (for the new metric) from x_0 to x tends to $+\infty$ whenever x tends to $\mathcal{N}_0$. Actually the reasons for using $r(x)$ are even subtler than this, for using $\int_{-T}^{T} |\boldsymbol{b}(x, s)|\, ds$ instead of $r(x) = \left(\int_{-T}^{T} |\boldsymbol{b}(x, s)|^2\, ds\right)^{\frac{1}{2}}$ would also have had the preceding implication, but would not have had the consequence, repeatedly needed (cf., e.g., (2.43) and its proof), that

$$\frac{1}{r(x)} \int_{-T}^{T} |\boldsymbol{b}(x, s)|\, ds \tag{3.8}$$

converges to zero with T.

Let then φ be the phase-function of Lemma 2.3. We set

$$z(x, t) = \chi(x, x_0) + \varphi(x, t), \qquad x \in \Gamma_{x_0}, \ |t| < T.$$

(We should rather write $z(x, x_0, t)$ for indeed z depends on x_0). Let us also write

$$\omega(x, t) = |\boldsymbol{b}(x, t)|/r(x),$$

whence

$$L = \frac{\partial}{\partial t} + i\omega(x, t) r(x) X. \tag{3.9}$$

If we take (2.41) and (3.4) into account we see that

$$Lz = 0, \tag{3.10}$$

and by (2.42), that

$$z|_{t=0} = \chi(x, x_0). \tag{3.11}$$

We consider then the mapping

$$(\chi, t) \mapsto z = \chi + \varphi\big(x(\chi, x_0), t\big). \tag{3.12}$$

It is a C^∞ mapping of the « slab »

$$S(x_0) = J(x_0) \times]-T, T[\tag{3.13}$$

into $\mathbb{R}^2$. The following result provides, in a sense, another interpretation of the local solvability condition (P):

LEMMA 3.2. *If $T > 0$ is small enough, the mapping* (3.12) *is a homeomorphism of $S(x_0)$ onto an open subset of $\mathbb{R}^2$. Its Jacobian determinant is given by*

$$\frac{D(\operatorname{Re} x, \operatorname{Im} z)}{D(\chi, t)} = -\omega |1 + rX\varphi|^2 . \tag{3.14}$$

PROOF. By (3.10) we have $z_t = -i|\boldsymbol{b}|Xz$, hence $\bar{z}_t = i|\boldsymbol{b}|X\bar{z}$, and therefore

$$(\operatorname{Re} z)_t = |\boldsymbol{b}|X(\operatorname{Im} z) , \qquad (\operatorname{Im} z)_t = - |\boldsymbol{b}|X(\operatorname{re} z) . \tag{3.15}$$

By (3.4) we have $r(x)X = \partial/\partial\chi$, hence

$$(\operatorname{Re} z)_t = \omega \frac{\partial}{\partial\chi}(\operatorname{Im} z) , \qquad (\operatorname{Im} z)_t = -\omega \frac{\partial}{\partial\chi}(\operatorname{Re} z) , \tag{3.16}$$

and therefore

$$\frac{D(\operatorname{Re} z, \operatorname{Im} z)}{D(\chi, t)} = -\omega \left| \frac{\partial z}{\partial\chi} \right|^2 . \tag{3.17}$$

By (3.8), $\partial z/\partial\chi = 1 + rX\varphi$, whence (3.14).

In order to prove that (3.12) is a homeomorphism, it suffices to prove that it is injective. Suppose that

$$\chi + \varphi(x(\chi, x_0), t) = \chi' + \varphi(x(\chi', x_0), t') , \tag{3.18}$$

that is:

$$\begin{aligned}(\chi - \chi') + [\varphi(x(\chi, x_0), t) - \varphi(x(\chi', x_0), t)] + \\ + [\varphi(x(\chi', x_0), t) - \varphi(x(\chi', x_0), t')] = 0 .\end{aligned} \tag{3.19}$$

By (3.4) and (2.43) we have:

$$|\varphi(x(\chi, x_0), t) - \varphi(x(\chi', x_0), t)| \leqslant \left| \int_{\chi'}^{\chi} (rX\varphi)(x(s, x_0), t)\, ds \right| \leqslant C_1 |t|^{\frac{1}{2}} |\chi - \chi'| . \tag{3.20}$$

Concerning the third term in the left-hand side of (3.19) we apply (2.44) with $x = x(\chi', x_0)$, obtaining

$$\left|\varphi(x, t) - \varphi(x, t') - \frac{1}{i}\int_{t'}^{t} \omega(x, s)\, ds\right| \leqslant C_1 \sqrt{T} \left|\int_{t'}^{t} \omega(x, s)\, ds\right|,$$

which, combined with (3.19) and (3.20), yields

$$\left|(\chi - \chi') + i\int_{t'}^{t} \omega(x, s)\, ds\right| \leqslant C_1 \sqrt{T}\left\{|\chi - \chi'| + \left|\int_{t'}^{t} \omega(x, s)\, ds\right|\right\}. \tag{3.21}$$

Therefore, if $C_1 \sqrt{T} < \frac{1}{2}$, we must have:

$$\chi = \chi', \quad \int_{t'}^{t} \omega(x, s)\, ds = 0. \tag{3.22}$$

But (by definition of $U_0 \setminus \mathcal{N}_0$ and by the analyticity of $\boldsymbol{b}(x, t)$ with respect to t) $\omega(x, s) > 0$ in any compact subinterval of $]-T, T[$, except possibly at a finite number of points. Therefore the second equation in (3.22) implies $t = t'$. Q.E.D.

We look now at the effect of the mapping (3.12) on the operator L. Since $L = (Lz)\partial_z + (L\bar{z})\partial_{\bar{z}}$, hence, by (3.10),

$$L = (L\bar{z})\partial_{\bar{z}}. \tag{3.23}$$

But (cf. (3.8)-(3.16)), $L\bar{z} = -2i\omega X\bar{z}$, hence

$$L = -2i\omega(1 + rX\bar{\varphi})\partial_{\bar{z}}. \tag{3.24}$$

Consider now an arbitrary function $f \in \mathfrak{D}_{r\text{-flat}}(U)$. If $x_0 \in U_0 \setminus \mathcal{N}_0$, by restriction f defines a smooth function $f^{\natural}(x, t; x_0)$ on the leaf Σ_x. Via the mapping (3.3), $\chi \mapsto x(\chi, x_0)$, $f^{\natural}(x, t; x_0)$ defines a function $f^{\natural\natural}(\chi, t; x_0)$ in the slab (3.13), $S(x_0) = J(x_0) \times]-T, T[$. We observe that $f^{\natural\natural} \in C^\infty(S(x_0))$ and we list a number of properties of $f^{\natural\natural}$:

(3.25) *the projection of the support of $f^{\natural\natural}$ on the t-axis is contained in a compact subset of* $]-T, T[$;

$$\sup_{(\chi, t) \in S(x_0)} |f^{\natural\natural}(\chi, t; x_0)| < +\infty; \tag{3.26}$$

(3.27) *if $J(x_0)$ has a finite boundary point χ_0, $f^{\natural\natural} \equiv 0$ in a full neighborhood of* $\{\chi_0\} \times]-T, T[$.

Suppose now that x_1 is another point on the orbit Γ_{x_0} of x_0: In view of (3.7) we have:

$$f^{\natural\natural}(\chi, t; x_1) = f^{\natural\natural}(\chi + \chi(x_0, x_1), t; x_0) . \tag{3.28}$$

Let now $O(x_0)$ be the open subset of $\mathbb{C}$ which is the image of the slab $S(x_0)$ $\big($in (3.13)$\big)$ under the mapping (3.12). By Lemma 3.1 we may define a *continuous bounded* function in $O(x_0)$,

$$\tilde{f}(z; x_0) = f^{\natural\natural}(\chi, t; x_0) \tag{3.29}$$

where $(\chi, t) \in S(x_0)$ corresponds to z by the inverse homeomorphism of (3.12).

We come now to the solution, in the leaf Σ_{x_0}, of the equation

$$Lu = f \in \mathfrak{D}_{r\text{-flat}}(U) . \tag{3.30}$$

Let us set $u(x, t) = v(x, t) + \int_{-T}^{t} f(x, s)\, ds$. The function v must be a solution of

$$Lv = -i \sum_{j=1}^{n} b^j(x, t) \int_{-T}^{t} \frac{\partial f}{\partial x^j}(x, s)\, ds . \tag{3.31}$$

We observe that the right-hand side in (3.31) belongs to $\mathcal{E}_{r\text{-flat}}(U)$ (it might fail to have a compact support with respect to t) but also that it vanishes wherever $|\boldsymbol{b}(x, t)|$ does. After multiplication by a cut-off function of t, we may assume that, in (3.31), the right-hand side is equal to

$$|\boldsymbol{b}| X g , \qquad g \in \mathfrak{D}_{r\text{-flat}}(U) . \tag{3.32}$$

We switch now to the coordinates $(\operatorname{Re} z, \operatorname{Im} z)$ on the leaf Σ_{x_0}. By virtue of (3.24) we must now solve

$$\partial_{\bar{z}} \tilde{v} = \frac{i}{2} [(1 + rX\bar{\varphi})^{-1}(rXg)]^{\sim} \qquad \textit{in } O(x_0) . \tag{3.33}$$

It follows at once from Cor. 2.3 that the right-hand side in (3.33) is of the form $\tilde{F}(z; x_0)$ for an obvious $F \in \mathfrak{D}_{r\text{-flat}}(U)$. We shall therefore write the integral expression of a solution $\tilde{v}$ of (3.33), that is, of

$$\partial_{\bar{z}} \tilde{v} = \tilde{F} \qquad \textit{in } O(x_0) . \tag{3.34}$$

In order to handle in a « uniform » manner all possible kinds of right-hand sides $\tilde{F}$, periodic, almost-periodic (at one end or at both), fastly decaying at infinity (also at one end or at both), etc., corresponding to the various kinds of orbits Γ_{x_0} (cf. Fig. 1), we shall use a special fundamental solution

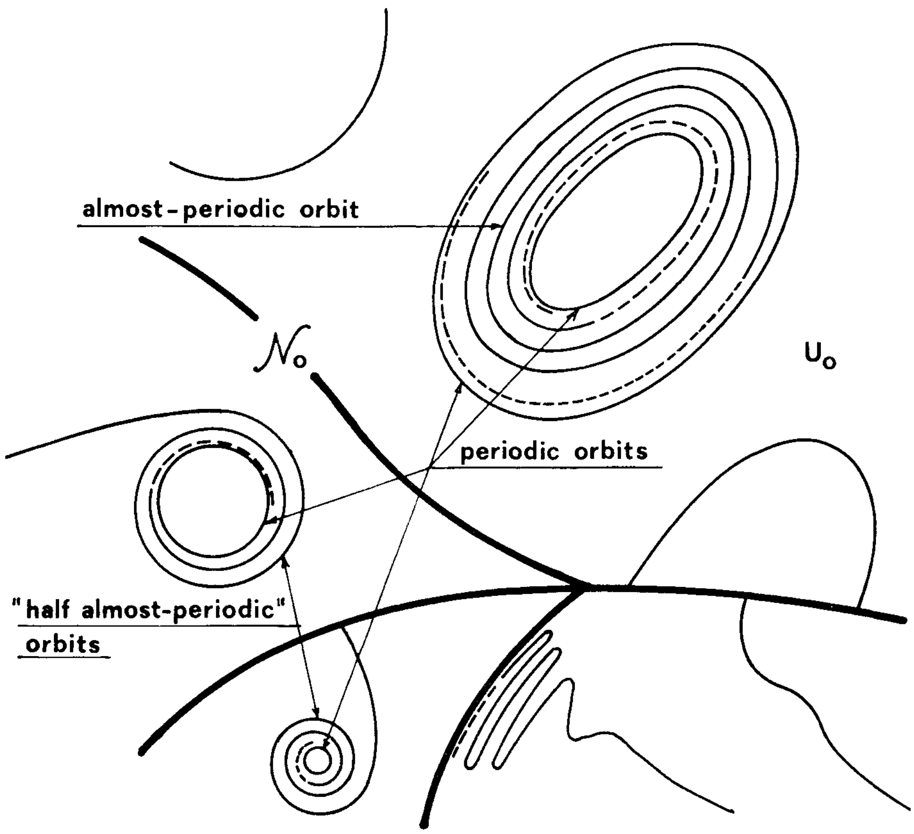

Fig. 1. – Examples of orbits of the vector field X in two space dimension.

of $\partial/\partial\bar{z}$. This will be possible thanks to Property (3.25) and to the fact that the open set $O(x_0)$ is contained in a slab $|\mathrm{Im}\, z| \leqslant \text{const.}\, T$. We observe that, if $h(z)$ is *any entire function such that* $h(0) = 1$, we have

$$\frac{\partial}{\partial\bar{z}}\left(\frac{h(z)}{\pi z}\right) = \delta \qquad \text{(the Dirac measure in } \boldsymbol{R}^2\text{)}, \tag{3.35}$$

and we choose

$$h(z) = \exp[-z^2],$$

solving then (3.34) by

$$\tilde{v}(z; x_0) = \frac{1}{\pi}\iint \frac{\exp[-(z-z')^2]}{z-z'}\tilde{F}(z'; x_0)\, d(\operatorname{Re} z')\, d(\operatorname{Im} z')\,. \tag{3.36}$$

1) *Convergence of the integral in* (3.36).

We take a closer look at the open set $O(x_0)$, image of $S(x_0) = J(x_0)\times\,]-T, T[$ under the map $(\chi, t) \mapsto z = \chi + \varphi(x(\chi, x_0), t)$. From (2.44) (where we take $t' = 0$) we derive

$$\left|\varphi(x, t) + i\int_0^t \omega(x, s)\, ds\right| \leqslant C_1\sqrt{|t|}\left|\int_0^t \omega(x, s)\, ds\right|. \tag{3.37}$$

Note also that, whatever $x \in U_0\setminus\mathcal{N}_0$ and t, $0 < |t| < T$,

$$0 < \left|\int_0^t \omega(x, s)\, ds\right| \leqslant \sqrt{|t|}\Big(\int_{-T}^{T}\omega(x, s)^2\, ds\Big)^{\frac{1}{2}} = \sqrt{|t|}\,. \tag{3.38}$$

We reach easily the following conclusions:

(3.39) *$O(x_0)$ is contained in the slab* $\{z;\ |\operatorname{Im} z| \leqslant \sqrt{T} + C_1 T\}$;

(3.40) *If* $J(x_0) \subset\,]\chi_0, +\infty[$, *$O(x_0)$ is contained in a half-space* $\operatorname{Re} z > \tilde{\chi}_0 > -\infty$; *if* $J(x_0) \subset\,]-\infty, \chi_0[$, $O(x_0) \subset \{z;\ \operatorname{Re} z < \tilde{\chi}_0 < +\infty\}$.

(3.41) *If* $J(x_0) = \mathbb{R}^1$, *$O(x_0)$ is an open neighborhood of the real axis in the z-plane.*

Suppose that z tends to the boundary of $O(x_0)$; this means (by Lemma 3.2) that (χ, t) tends to the boundary of $S(x_0)$. If then $F \in \mathfrak{D}_{r\text{-flat}}(U)$ and $|t| \to T$, $F(x(\chi, t), t)$ will eventually be zero. If $|t|$ remains $\leqslant T' < T$ then χ must tend either to $\pm\infty$ or to a finite boundary point χ_0 of $J(x_0)$. By (3.27) we see that, also in the latter case, $F(x(\chi, t), t)$ will eventually be zero. This is not necessarily so when $\chi_0 \to \pm\infty$ for then $x(\chi, x_0)$ might very well remain in a compact subset of $U_0\setminus\mathcal{N}_0$. But at any rate, if we denote by $\partial O(x_0)$ the boundary of $O(x_0)$ in the complex plane, that is, excluding the points at infinity, we see that

(3.42) *whatever* $F \in \mathfrak{D}_{r\text{-flat}}(U)$, $\tilde{F}(z, t)$ *vanishes in a neighborhood of* $\partial O(x_0)$.

We see that $\tilde{F}$ can then be extended as a C^∞ function in $\mathbb{R}^2$, which is bounded. Since $\exp[-z^2]/\pi z$ is integrable in every slab $|\operatorname{Im} z| < \text{const.}$ the convolution at the right in (3.36) defines a C^∞ function in $\mathbf{R}^2$, which is bounded in every horizontal slab.

2) *Formula* (3.36) *defines a function on the leaf* Σ_{x_0}.

In the double integral at the right, in (3.36), we switch from the coordinates $(\operatorname{Re} z, \operatorname{Im} z)$ to (χ, t). We set $\chi = \chi(x, x_0)$, $\chi' = \chi(x', x_0)$. The Jacobian determinant is given by (3.14) and we have $F = -(1/2i)\cdot \cdot(1 + rX\bar{\varphi})^{-1}(rXg)$; the integral under consideration can be rewritten as

$$(3.43)\qquad \frac{-1}{2\pi i}\iint \frac{\exp[-(z-z')^2]}{z-z'}(r\,X\,g)^{\natural\natural}(\chi', t'; x_0)\cdot \\ \cdot\big[1 + (r\,X\,g)\big(x(\chi', x_0), t'\big)\big]\,d\chi'\,dt',$$

where

$$(3.44)\qquad z = \chi + \varphi\big(x(\chi, x_0), t\big), \qquad z' = \chi' + \varphi\big(x(\chi', x_0), t'\big).$$

The integration in (3.43) is performed over $S(x_0)$ or, which amounts to the same, over $\mathbf{R}^2$ (since $(rXg)^{\natural\natural}$ vanishes identically in a neighborhood of $\partial S(x_0)$). Let us denote by $v^{\natural\natural}(\chi, t; x_0)$ the transfer of $\tilde{v}(z; x_0)$ under the homeomorphism (3.12); it is equal to (3.43). We must check that $v^{\natural\natural}(\chi, t; x_0)$ is indeed the image, under the map (3.3), of a function $v^{\natural}(x, t; x_0)$ on the leaf Σ_{x_0}.

For simplicity let us set

$$(3.45)\qquad E(z) = \frac{1}{2\pi i}\,\frac{\exp[-z^2]}{z}, \qquad x = x(\chi, x_0), \qquad x' = x(\chi', x_0).$$

We have:

$$(3.46)\qquad v^{\natural\natural}(\chi, t; x_0) = \\ -\iint E[\chi - \chi' + \varphi(x, t) - \varphi(x', t')](rXg)(x', t')[1 + r(x')X\varphi(x', t')]\,d\chi'\,dt'.$$

We apply (3.5)-(3.6):

$$(3.47)\qquad \chi - \chi' = \chi(x, x').$$

We may also write

$$(3.48)\qquad d\chi' = -\,d[\chi(x, x')],$$

regarding x as fixed (or as a parameter). Since $\Sigma_{x_0} = \Sigma_x$ in what precedes, we may define the following function of (x, t), $x \in U_0 \setminus \mathcal{N}_0$, $|t| < T$:

$$(3.49)\qquad v(x, t) = \int_{\Sigma_x}\!\!\int E[\chi(x, x') + \varphi(x, t) - \varphi(x', t')](rXg)(x', t')\cdot \\ \cdot[1 + r(x')X\varphi(x', t')]\,d\chi(x, x')\,dt'.$$

Then $v^{\natural\natural}(\chi, t; x_0)$ is the image under (3.3) of the restriction $v^{\natural}(x, t; x_0)$ of $v(x, t)$ to Σ_{x_0}.

In the next section we prove that, after setting $v(x, t) = 0$ when $x \in \mathcal{N}_0$, we have $v \in \mathcal{E}_{r\text{-flat}}(U)$. By taking then, in (3.49),

$$g(x, t) = -i\zeta(t)\int_{-T}^{t} f(x, s)\, ds$$

[$\zeta \in \mathrm{C}^\infty(\mathbb{R}^1)$, $\zeta(t) = 1$ if $t \leqq T' < T$, $\zeta(t) = 0$ if $t > \frac{1}{2}(T + T')$], we obtain the solution $u = v + g$ of (3.30) in $U' = U_0 \times]-T', T'[$, with the required properties ($f \mapsto u$ defines a linear operator $\mathcal{D}_{r\text{-flat}}(U') \to \mathcal{E}_{r\text{-flat}}(U')$, which can be given an explicit integral representation; the estimates in Sect. 4 easily imply the continuity of this operator).

4. – Smoothness and flatness of the solution.

We begin by simplifying the expression (3.49). We take $\chi = \chi(x, x')$ as one of the two integration variables at the right (the other remains t'), and regard x' as a function of (χ, x), defined as the solution of

$$\frac{dx'}{d\chi} = -r(x')\boldsymbol{v}(x')\,, \qquad x'|_{\chi=0} = x\,. \tag{4.1}$$

It is convenient to let (χ, t') vary in the whole plane, although x' is not defined for all χ but only for those belonging to $-J(x)$. But we have seen (Lemma 3.1) that if $J(x) \neq \mathbb{R}^1$ and if χ_0 is a boundary point of $J(x)$, then as $\chi \to -\chi_0$ (while x remains fixed), x' will exit from any compact subset of U_0, in particular from the support of $g(\cdot, t)$. And by the standard results on ODES, as long as x' remains away from the boundary of U_0, it is a C^∞ function of (χ, x) (x remains in $U_0 \setminus \mathcal{N}_0$). Let us set

$$\mathcal{W} = \mathcal{W}(x; \chi, t) = [1 + r(x')X\varphi(x', t)]r(x')Xg(x', t)\,; \tag{4.2}$$

also

$$Z = Z(x, t; \chi, t') = \chi + \varphi(x, t) - \varphi(x', t')\,. \tag{4.3}$$

With this notation (3.49) reads

$$v(x, t) = \iint E[Z(x, t; \chi, t')]\,\mathcal{W}(x; \chi, t')\, d\chi\, dt'\,. \tag{4.4}$$

The integration in (4.4) is performed over $\mathbb{R}^2$; (x, t) ranges over $(U_0\setminus\mathcal{N}_0)\times$ $\times]-T, T[$.

We avail ourselves of the following facts:

(4.5) $\mathcal{W}$ *is a* C^∞ *function of* (x, χ, t) *in* $(U_0\setminus\mathcal{N}_0)\times\mathbb{R}^1\times]-T, T[$;

(4.6) $\mathcal{W}$ *is bounded*;

(4.7) *the projection on the t-axis of* supp $\mathcal{W}$ *is contained in a compact subinterval of* $]-T, T[$.

On the other hand, we derive from (2.43)

$$|\varphi(x, t') - \varphi(x', t')| \leqslant C_0'|t|^{\frac{1}{2}}\chi(x, x'), \tag{4.8}$$

where $x\in U_0\setminus\mathcal{N}_0$, $x'\in\Gamma_x$, $|t'|<T$; we may rewrite (2.44) as

$$\left|\varphi(x, t) - \varphi(x, t') + i\int_{t'}^{t}\omega(x, s)\,ds\right| \leqslant C_1\sqrt{T}\left|\int_{t'}^{t}\omega(x, s)\,ds\right| \tag{4.9}$$

(here $|t|<T$), and by combining (4.8) and (4.9) we see that

$$\left|Z - \left\{\chi - i\int_{t'}^{t}\omega(x, s)\,ds\right\}\right| \leqslant C\sqrt{T}\left|\chi - i\int_{t'}^{t}\omega(x, s)\,ds\right|. \tag{4.10}$$

From all this it follows easily that v, given by (4.4), is a *continuous* function of (x, t) in $(U_0\setminus\mathcal{N}_0)\times]-T, T[$ (provided that $C\sqrt{T}$, in (4.10), be <1, which we may assume). We are now going to prove that it is a C^∞ function of (x, t) in that set and, at the end, that it is r-flat.

We shall need the following:

Lemma 4.1. *To every* $\alpha\in\mathbb{Z}_+^n$, $k\in\mathbb{Z}_+$, *there are constants* $C_{\alpha.k}$, $M_{\alpha.k}>0$ *and an integer* $d_{\alpha.k}\in\mathbb{Z}$ *such that*

$$|\partial_x^\alpha\partial_\chi^k x'| \leqslant C_{\alpha.k}\, r(x')^{-d_{\alpha.k}}\exp\left(M_{\alpha.k}|\chi|\right) \tag{4.11}$$

for all $x\in U_0\setminus\mathcal{N}_0$, $\chi\in -J(x)$.

Proof. The case $\alpha=0$, $k=0$ follows at once from (3.7), since

$$|x'-x| \leqslant r(x')\left(\exp[K|\chi|]-1\right). \tag{4.12}$$

Let us differentiate (4.1) with respect to χ and to x. If we set $\theta_{\alpha,k} = \partial_x^\alpha \partial_\chi^k x'$, we obtain

$$\partial_\chi \theta_{\alpha,k} = [\partial_x(r\boldsymbol{v})](x')\theta_{\alpha,k} + \sum_{|\beta|\leqslant|\alpha|+k} [\partial_x^\beta(r\boldsymbol{v})](x')\mathcal{T}_{\alpha,\beta,k}(\theta)\,, \tag{4.13}$$

where $\mathcal{T}_{\alpha,\beta,k}(\theta)$ denotes a *polynomial* with respect to the variables $\theta_{\gamma,l}$ with $\gamma_j \leqslant \alpha_j$ $(j = 1, \ldots, n)$, $l \leqslant k$, and $|\gamma| + l < |\alpha| + k$. We apply Cor. 2.1, obtaining

$$|\partial_\chi \theta_{\alpha,k}| \leqslant C|\theta_{\alpha,k}| + \sum_{|\beta|\leqslant|\alpha|+k} \tilde{\tilde{C}}\, r(x')^{1-|\beta|}|\mathcal{T}_{\alpha,\beta,k}(\theta)|\,. \tag{4.14}$$

We shall reason by induction on $|\alpha|$ and on k. First assume $k = 0$. Then to (4.14) we may adjoin the initial conditions

$$\theta_{\alpha,0}|_{\chi=0} = \partial_x^\alpha x\,. \tag{4.15}$$

Thus, if $|\alpha| = 1$ (in which case $\mathcal{T}_{\alpha,\beta,0}(\theta) \equiv 0$), we obtain at once

$$|\theta_{\alpha,0}| \leqslant \exp[C|\chi|]\,. \tag{4.16}$$

For $|\alpha| > 1$, the right-hand side in (4.15) is zero. By induction on $|\alpha|$ we derive from (4.14)

$$|\partial_\chi \theta_{\alpha,0}| \leqslant C|\theta_{\alpha,0}| + C'_\alpha r(x')^{-d_\alpha} \exp[M_\alpha|\chi|]\,. \tag{4.17}$$

We apply then Gronwall's inequality and obtain

$$|\theta_{\alpha,0}| \leqslant C_{\alpha,0} r(x')^{-d_{\alpha,0}} \exp[M_{\alpha,0}|\chi|]\,. \tag{4.18}$$

It suffices now to observe that the left-hand side in (4.14) can be rewritten as $|\theta_{\alpha,k+1}|$ and to reason by induction on k. Q.E.D.

We shall now differentiate v (given by (4.4)) with respect to (x, t). But since

$$\partial_t v - i|\boldsymbol{b}(x,t)|X v = |\boldsymbol{b}(x,t)|Xg \tag{4.19}$$

(see (3.31), (3.32)), it suffices to consider the derivatives of v with respect to x. We have

$$\partial_x^\alpha v = \sum_{\alpha'\leqslant\alpha} \binom{\alpha}{\alpha'} \iint \partial_{\alpha'}^x [E(Z)]\, \partial_x^{\alpha-\alpha'} \mathfrak{W}(x; Z, t')\, d\chi\, dt'\,, \tag{4.20}$$

where $\alpha' \leqslant \alpha$ means $\alpha_j' \leqslant \alpha_j$ for all $j = 1, \ldots, n$. First, if $|\alpha'| \geqslant 1$,

$$\partial_x^{\alpha'} E(Z) = \sum_{j=1}^{|\alpha'|} E^{(j)}(Z)\,\mathcal{T}_j(\partial_x^\beta Z)\,, \tag{4.21}$$

where $\mathcal{T}_j(\partial_x^\beta Z)$ is a polynomial with respect to the derivatives of Z with respect to x, of order β, $0 < |\beta| \leqslant |\alpha'|$. Since $Z = \chi + \varphi(x, t) - \varphi(x', t')$, we have

$$\partial_\chi = [1 - \partial_\chi \varphi(x', t')]\,\partial_Z\ , \tag{4.22}$$

and, by (4.1),

$$\partial_\chi \varphi(x', t') = \varphi_x(x', t')\,\frac{\partial x'}{\partial \chi} = -\,r(x') \sum v^j(x')\,\frac{\partial \varphi}{\partial x^j}(x', t')\,.$$

We apply once again (2.43); if $T > 0$ is small enough,

$$|\partial_\chi \varphi(x', t')| \leqslant C_1 \sqrt{T} < 1\,, \tag{4.23}$$

and $[1 - \partial_\chi \varphi(x', t')]$ is invertible. Thus, by iteration,

$$E^{(j)}(Z) = \left(\frac{1}{1 - \partial_\chi \varphi(x', t')}\,\frac{\partial}{\partial \chi}\right)^j E(Z)\,, \tag{4.24}$$

and thus, after integration by parts [which is legitimate: see remark following (4.33)], we see that $\partial_x^\alpha v$ is a linear combination of terms of the form

$$\iint E(Z)\left\{\frac{\partial}{\partial \chi}\left[\frac{-1}{1 - \partial_\chi \varphi(x', t')}\right]\right\}^j [\mathcal{T}_j(\partial_x^\beta Z)\,\partial_x^{\alpha-\alpha'}\mathcal{W}(x;\,\chi, t')]\,d\chi\,dt'\ . \tag{4.25}$$

We note that

$$\partial_x^\beta Z = \partial_x^\beta \varphi(x, t) - \sum_{\gamma \leqslant \beta} (\partial_x^\gamma \varphi)(x', t')\,\mathcal{Q}_\gamma(\partial_x^\lambda x')\,, \tag{4.26}$$

where $\mathcal{Q}_\gamma$ is a polynomial with respect to the derivatives $\partial_x^\lambda x'$ of x' with respect to x of appropriate orders λ. If $k > 0$,

$$\partial_x^\beta \partial_\chi^k Z = -\sum_{|\gamma| \leqslant |\beta| + k} (\partial_x^\gamma \varphi)(x', t')\,\mathcal{Q}_{\gamma, k}(\partial_x^\lambda \partial_\chi^l x')\,. \tag{4.27}$$

Applying Leibniz formula in (4.25) and taking (4.26)-(4.27) into account shows that (4.25), hence also $\partial_x^\alpha v$, is a linear combination of terms of the form

$$(4.28)\qquad \iint E(Z)\,\mathcal{R}_{\mu,j}[\partial_x^\beta\varphi(x,t),\,(\partial_x^\gamma\varphi)(x',t'),\,\partial_x^\lambda\partial_\chi^l x']\cdot \cdot\,\partial_x^\mu \mathfrak{W}(x;\,\chi,t')[1-\partial_\chi\varphi(x',t')]^{-2j}\,d\chi\,dt'\,;$$

in (4.28), $\mathcal{R}_{\mu,j}$ is a polynomial with respect to the indicated arguments. Let us rewrite (4.28) as

$$(4.29)\qquad v_{\mu,j}(x,t)=\iint E(Z)\,w_{\mu,j}(x,t;\,\chi,t')\,d\chi\,dt'\,.$$

Note that $w_{\mu,j}$ is a C^∞ function of (x,t,χ,t') in the region

$$(4.30)\qquad x\in U_0\diagdown\mathcal{N}_0\,,\qquad t,t'\in\,]-T,T[\,,\qquad \chi\in\mathbb{R}^1$$

(cf. the remarks following (4.1)). The t'-projection of $\operatorname{supp} w_{\mu,j}$ is contained in a compact subinterval of $]-T,T[$. Furthermore, as inspection of (4.28) shows, by virtue of (2.51) and (4.11), for suitable constants $C'_{\mu,j}$, $M'_{\mu,j}>0$ and an integer $d'_{\mu,j}\geqslant 0$,

$$(4.31)\qquad |w_{\mu,j}(x,t;\,\chi,t')|\leqslant \leqslant C'_{\mu,j}\left[1+\frac{1}{r(x)}+\frac{1}{r(x')}\right]^{d'_{\mu,j}}\exp\left[M'_{\mu,j}|\chi|\right]|\partial_x^\mu\mathfrak{W}(x;\,\chi,t')|\,.$$

By (4.2) and the results of the appendix of Sect. 2, we see immediately that W is $r(x')$-flat. If we combine this fact with Lemma 4.1 we deduce that, for a suitable constant $M_\mu>0$ (depending only on μ), for any $\nu\in\mathbb{Z}_+$ and a suitable constant $C''_{\mu,\nu}>0$,

$$(4.32)\qquad |\partial_x^\mu\mathfrak{W}(x;\,\chi,t')|\leqslant C''_{\mu,\nu}\,r(x')^\nu\exp\left[M_\mu|\chi|\right].$$

Combining this with (4.31) yields

$$(4.33)\qquad |w_{\mu,j}\,(x,t;\,\chi,t')|\leqslant C'''_{\mu,\nu,j}\,r(x')^\nu\left[1+\frac{1}{r(x)}\right]^{d'_{\mu,j}}\exp\left[M''_{\mu,j}|\chi|\right].$$

In (4.33) $\nu\in\mathbb{Z}_+$ is arbitrary.

The estimate (4.33) allows us to change slightly the integral representation (4.29) of $v_{\mu,j}$. Indeed, by (4.22),

$$\frac{1}{Z}=\left(\frac{1}{1-\partial_\chi\varphi(x',t')}\,\frac{\partial}{\partial\chi}\right)^2[Z(\log^\pm Z-1)]\,,$$

where $\log^{\pm}$ denotes suitable branches of the logarithmic function. Since $w_{\mu,j}$ grows at most exponentially with $|\chi|$, whereas $E(Z)$ behaves like $\exp[-\chi^2]$ at infinity, we may integrate by parts, obtaining

$$(4.34)\qquad v_{\mu,j}(x,t)=\iint \exp[-Z^2][Z(\log^{\pm} Z-1]\,w^{\natural}_{\mu,j}(x,t;\chi,t')\,d\chi\,dt',$$

where $w^{\natural}_{\mu,j}$ is a function analogous to $w_{\mu,j}$ (cf. (4.25) and subsequent reasoning), in particular satisfies an inequality similar to (4.33):

$$(4.35)\qquad |w^{\natural}_{\mu,j}(x,t;\chi,t')|\leqslant C^{\natural}_{\mu,\nu,j}\,r(x')^{\nu}\left[1+\frac{1}{r(x)}\right]^{a^{\natural}_{\mu,j}}\exp\,[M^{\natural}_{\mu,j}|\chi|]$$

($\nu\in\mathbb{Z}_+$ arbitrary).

For any $M>0$ we denote by $v^{M}_{\mu,j}(x,t)$ the value of the integral at the right in (4.34), when the integration with respect to χ is restricted to the region

$$(4.36)\qquad |\chi|>2\sqrt{M}(1+|\log r(x)|)^{\frac{1}{2}}\,.$$

If we choose M large enough, we see, by (4.10), that

$$(4.37)\qquad \operatorname{Re} Z^2\geqq \tfrac{3}{4}\chi^2-\text{const.}\geqslant \tfrac{1}{2}\chi^2+M|\log r(x)|$$

in the region (4.36), whatever $x\in U_0\setminus\mathcal{N}_0$, $|t|<T$. Thus, there,

$$(4.38)\qquad |\exp[-Z^2]|\leqslant r(x)^{\varepsilon M}\exp[-\chi^2/2]\,,$$

where $\varepsilon=+1$ if $r(x)\leqslant 1$, $\varepsilon=-1$ if $r(x)>1$.

Suppose first that x remains in a compact subset K of $U_0\setminus\mathcal{N}_0$. We have, by (4.34), (4.35), (4.38),

$$(4.39)\qquad |v^{M}_{\mu,j}(x,t)|\leqslant\text{const.}\int\limits_{|\chi|>2\sqrt{M}}\exp[-\tfrac{1}{2}\chi^2+M^{\natural}_{\mu,j}|\chi|]\,\chi^2\,d\chi\,.$$

Of course μ, j are fixed and so is $M^{\natural}_{\mu,j}$. We may choose M so large that the right-hand side in (4.39) will not exceed any number $\delta>0$ given in advance. Thus, in order to study the continuity of $v_{\mu,j}(x,t)$, $x\in K$, $|t|<T$, it suffices to study that of $(v_{\mu,j}-v^{M}_{\mu,j})(x,t)$. But the latter continuity is evident, as the integrand and the domain of integration, in the integral expression of $(v_{\mu,j}-v^{M}_{\mu,j})(x,t)$, depend continuously on (x,t), and the integrand is uniformly L^1 with respect to χ, t'.

This proves that v is a C^∞ function in $(U_0 \setminus \mathcal{N}_0) \times]-T, T[$. In order to complete the proof of Th. 1.2 we must prove that v is r-flat.

We suppose now that x is close to $\mathcal{N}_0$, in particular that $r(x) < 1$. Using once more (4.35) and (4.38) we get

$$|v^M_{\mu,j}(x,t)| \leq \text{const.}\, r(x)^{M-d^\natural_{\mu,j}} \int\limits_{|\chi|>2\sqrt{M}} \exp[-\tfrac{1}{2}\chi^2 + M^\natural_{\mu,j}]\chi^2\, d\chi, \tag{4.40}$$

thus:

$$|v^M_{\mu,j}(x,t)| \leq C(M,\mu,j)\, r(x)^{M'} \quad \text{if } M \geq M' + d^\natural_{\mu,j}. \tag{4.41}$$

We shall now estimate $v_{\mu,j} - v^M_{\mu,j}$. This difference is equal to the integral at the right in (4.34) but where the integration with respect to χ is restricted to the interval

$$|\chi| \leq 2\sqrt{M}(1 + |\log r(x)|)^{\frac{1}{2}}. \tag{4.42}$$

Lemma 4.2. *There is a constant $C > 0$ such that, for all $M > 0$, if* (4.42) *holds, then*

$$1 - \Theta \leq \left|\frac{\log r(x')}{\log r(x)}\right| \leq 1 + \Theta, \tag{4.43}$$

where $x' = x'(\chi, x)$ and $\Theta = 2\sqrt{M}(1 + |\log r(x)|)^{\frac{1}{2}}/|\log r(x)|$.

Proof. We have (Lemma 2.5)$|\partial_\chi r| = |rXr| \leq Cr$, hence

$$|\log r(x) - \log r(x')| = \left|\int_0^\chi \frac{1}{r}\frac{\partial r}{\partial \chi}\, d\chi\right| \leq C|\chi|.$$

Combining this with (4.42) yields (4.43). Q.E.D.

Keeping M fixed, we may find a neighborhood $\mathcal{V}_0$ of $\mathcal{N}_0$ such that if $x \in \mathcal{V}_0 \setminus \mathcal{N}_0$, $r(x)$ is so small that $\Theta < \frac{1}{2}$. But then, from the first inequality (4.43), we draw

$$r(x') \leq r(x)^{\frac{1}{2}} \quad \textit{if } x' = x'(\chi, x) \textit{ and } (4.42) \textit{ holds}. \tag{4.44}$$

By (4.35) we see that, in the region (4.42), we have

$$|w^\natural_{\mu,j}(x,t;\chi,t')| \leq C^{\natural\natural}_{\mu,\nu,j}\, r(x)^{\frac{1}{2}\nu - d^\natural_{\mu,j}} \exp[M^\natural_{\mu,j}|\chi|]. \tag{4.45}$$

Reporting this in the integral at the right of (4.34), where the integration in χ is restricted to (4.42), we obtain

$$
\begin{aligned}
(4.46) \qquad & |v_{\mu,j}(x,t) - v^{M}_{\mu,j}(x,t)| \leqslant \\
& \leqslant \text{const.}\, r(x)^{\frac{1}{2}\nu - d^{\natural}_{\mu,j}} \int_{-\infty}^{+\infty} \exp\left[-\chi^2 + M^{\natural}_{\mu,j}|\chi|\right] \chi^2 \, d\chi \\
& \leqslant C^{\natural}(\mu,j)\, r(x)^{M'} \quad \text{if } \nu \geqslant 2(M' + d^{\natural}_{\mu,j}) . \qquad \text{Q.E.D.}
\end{aligned}
$$

BIBLIOGRAPHICAL REFERENCES

[1] R. BEALS - CH. FEFFERMAN, *On local solvability of linear partial differential equations*, Ann. of Math., **97** (1973), 482-498.

[2] R. HERMANN, - *The Differential Geometry of Foliations*, II. Jr. of Math. and Mech., **2** (1972), 303-315.

[3] J. N. MATHER, *Proceedings of Liverpool Singularities*, Symposium 1, Lecture Notes in Mathematics 192, Springer-Verlag.

[4] L. NIRENBERG - F. TREVES, *Solvability of a linear first order partial differential equation*, Comm. Pure Appl. Math., **16** (1963), 331-351.

[5] L. NIRENBERG - F. TREVES, *On local solvability of linear partial differential equations*. Part II: *Sufficient conditions*, Comm. Pure Appl. Math., **924** (1971), 279-288.

[6] F. TREVES, *Solutions of Cauchy problems modulo flat functions*, Sémin. Equat. Dér. Part. Ecole Polytechnique Paris (1974-75).

[7] F. TREVES, *Solutions élémentaires d'équations linéaires aux dérivées partielles du premier ordre*, Sémin. Equat. Dér. Part. Ecole Polytechnique Paris (1974-75).