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INTEGRAL REPRESENTATIONS WITH TRIVIAL FIRST COHOMOLOGY GROUPS

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Let Π be a finite group and denote by $\boldsymbol{M}_\Pi$ the class of finitely generated $\boldsymbol{Z}$-free $\boldsymbol{Z}\Pi$-modules. In [2] we defined a certain equivalence relation on $\boldsymbol{M}_\Pi$ and constructed the abelian semigroup $T(\Pi)$, which was studied in [3] (see [1] and [5], too). In this paper we will define a certain subsemigroup $\tilde{T}(\Pi)$ of $T(\Pi)$ and using this will give a complete answer to a problem raised by H. W. Lenstra, Jr. (for the precise statement see Theorem 2.1 in Section 2).

The contents of this paper were obtained in 1974 and briefly anounced in [4].

§1.

Let Π be a finite group and $\boldsymbol{M}_\Pi$ the class of all $\boldsymbol{Z}\Pi$-lattices, namely, finitely generated $\boldsymbol{Z}$-free $\boldsymbol{Z}\Pi$-modules. We further define:

$$\boldsymbol{H}_\Pi^i = \{M \in \boldsymbol{M}_\Pi \mid H^i(\Pi', M) = 0 \text{ for every subgroup } \Pi' \text{ of } \Pi\}$$
$$\tilde{\boldsymbol{H}}_\Pi = \boldsymbol{H}_\Pi^1 \cap \boldsymbol{H}_\Pi^{-1}$$
$$\boldsymbol{S}_\Pi = \{\text{permutation } \boldsymbol{Z}\Pi\text{-modules}\}$$
$$\boldsymbol{D}_\Pi = \{\text{direct summands of permutation } \boldsymbol{Z}\Pi\text{-modules}\}.$$

Define $M^* = \mathrm{Hom}_{\boldsymbol{Z}}(M, \boldsymbol{Z})$. If $M \in \boldsymbol{H}_\Pi^1$, then $M^* \in \boldsymbol{H}_\Pi^{-1}$.

LEMMA 1.1. *For every $M \in \boldsymbol{M}_\Pi$ there exist two exact sequences*

(1) $0 \longrightarrow N \longrightarrow S \longrightarrow M \longrightarrow 0, \quad N \in \boldsymbol{H}_\Pi^1,\ S \in \boldsymbol{S}_\Pi$

(2) $0 \longrightarrow M \longrightarrow L \longrightarrow T \longrightarrow 0, \quad L \in \boldsymbol{H}_\Pi^1,\ T \in \boldsymbol{S}_\Pi.$

Proof. (1) is Lemma 1.1 in [3].

(2) M can be imbedded in a suitable free $\boldsymbol{Z}\Pi$-module F such that F/M is $\boldsymbol{Z}$-free. By (1) there is an exact sequence $0 \to N' \to T \to F/M \to 0$ with

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$N' \in \boldsymbol{H}_\Pi^1$ and $T \in \boldsymbol{S}_\Pi$. Taking the pullback of $\begin{array}{c} F \to F/M \\ \uparrow \\ T \end{array}$ we have an exact sequence $0 \to M \to F \oplus N' \to T \to 0$. This completes the proof.

LEMMA 1.2. *Let Π_0 be a subgroup of Π. Consider an exact sequence*

$$0 \longrightarrow M \longrightarrow N \xrightarrow{\phi} Z\Pi/\Pi_0 \oplus L \longrightarrow 0, \qquad M,\ N,\ L \in \boldsymbol{M}_\Pi .$$

If $H^1(\Pi_0, M) = 0$, then N has a direct sum decomposition $N = N_1 \oplus N_2$ such that the restriction ϕ to N_1 is an isomorphism onto $Z\Pi/\Pi_0$.

Especially $\mathrm{Ext}^1_{Z\Pi}(L, M) = 0$ *for $L \in \boldsymbol{D}_\Pi$ and $M \in \boldsymbol{H}_\Pi^1$.*

Proof. Let n be a fixed element of $\phi^{-1}(\Pi_0/\Pi_0)$. Then $\sigma n - n \in M$ for every $\sigma \in \Pi_0$, i.e., $\sigma n - n$, $\sigma \in \Pi_0$ is a cocycle of Π_0 with values in M. By the assumption there is an $m \in M$ such that $\sigma n - n = \sigma m - m$ for all $\sigma \in \Pi_0$. Set $N_1 = Z\Pi \cdot (n - m)$ and $N_2 = \phi^{-1}(L)$. Clearly $N = N_1 \oplus N_2$ and ϕ: $N_1 \to Z\Pi/\Pi_0$ is an isomorphism.

Remark 1.3. Let p be a prime. Z_p denotes the completion of Z at p. For a p-group Π, a more precise statement than Lemma 1.2 holds. Namely, consider an exact sequence: $0 \to M \to S \xrightarrow{\phi} Z_p\Pi/\Pi_0 \oplus T \to 0$ with M, a $Z_p\Pi$-lattice and S, T, permutation $Z_p\Pi$-modules. If $H^1(\Pi_0, M) = 0$, then S has a direct sum decomposition $S = S_1 \oplus S_2$ such that S_1 and S_2 are permutation $Z_p\Pi$-modules and $\phi: S_1 \to Z_p\Pi/\Pi_0$ is an isomorphism.

The proof follows from the similar argument as above, Krull-Schmidt's theorem and the fact that $Z_p\Pi/\Pi'$ is an indecomposable $Z_p\Pi$-module for an arbitrary subgroup Π' of Π.

For M, $N \in \boldsymbol{M}_\Pi$ we define $M \equiv N$ by the existence of two exact sequences:

$$\begin{array}{c} 0 \longrightarrow M \longrightarrow X \longrightarrow S \longrightarrow 0 \\ 0 \longrightarrow N \longrightarrow X \longrightarrow T \longrightarrow 0 \end{array}$$

with $X \in \boldsymbol{M}_\Pi$ and S, $T \in \boldsymbol{S}_\Pi$. Lemmas 1.1 and 1.2 show that "$\equiv$" is an equivalence relation on $\boldsymbol{M}_\Pi$. Now we define

$$T(\Pi) = \boldsymbol{M}_\Pi/(\equiv)$$

where the addition is introduced to $T(\Pi)$ by the direct sum. It is easy to check that this semigroup coincides with our old $T(\Pi)$ defined in [2]. By Lemma 1.1 $T(\Pi)$ is generated by $\boldsymbol{H}_\Pi^1$. $\tilde{T}(\Pi)$ is defined to be the subsemigroup of $T(\Pi)$ which is generated by $\boldsymbol{H}_\Pi^{-1}$ (note that this is already generated by $\tilde{\boldsymbol{H}}_\Pi$). We denote by $T^g(\Pi)$ the set of invertible elements in $T(\Pi)$. This is clearly an abelian group and is known to be generated by $\boldsymbol{D}_\Pi$. Hence $T^g(\Pi)$ is finitely generated by [3], (1.5).

§ 2.

The aim of this paper is to prove the following

THEOREM 2.1. *Let Π be a finite group and let Π^p be a Sylow p-subgroup of Π for each prime p. Then the following statements are equivalent*:

(1) $\tilde{\boldsymbol{H}}_\Pi = \boldsymbol{D}_\Pi$, *i.e.*, $\tilde{T}(\Pi)$ *is a group,*

(2) Π^p *is cyclic for each odd prime p, and* Π^2 *is cyclic or dihedral* (*including Klein's four group*).

This gives a complete answer to a problem which was raised by H. W. Lenstra, Jr.

In this section we will give an outline of the proof of the theorem and postpone proofs of technical lemmas to later sections.

LEMMA 2.2. (1) *If* $\tilde{\boldsymbol{H}}_\Pi = \boldsymbol{D}_\Pi$, *then* $\tilde{\boldsymbol{H}}_{\Pi'} = \boldsymbol{D}_{\Pi'}$ *for any subgroup* Π' *of* Π.

(2) $\tilde{\boldsymbol{H}}_\Pi = \boldsymbol{D}_\Pi$ *if and only if* $\tilde{\boldsymbol{H}}_{\Pi^p} = \boldsymbol{D}_{\Pi^p}$ *for every prime p.*

Proof. (1) If $M \in \tilde{\boldsymbol{H}}_{\Pi'}$, then $\boldsymbol{Z}\Pi \otimes_{\boldsymbol{Z}\Pi'} M \in \tilde{\boldsymbol{H}}_\Pi = \boldsymbol{D}_\Pi$. Since M is a direct summand of $\boldsymbol{Z}\Pi \otimes_{\boldsymbol{Z}\Pi'} M$ as a $\boldsymbol{Z}\Pi'$-module, we see that $M \in \boldsymbol{D}_{\Pi'}$.

(2) If $M \in \tilde{\boldsymbol{H}}_\Pi$, then for any prime p, $M \in \tilde{\boldsymbol{H}}_{\Pi^p}$. If $\tilde{\boldsymbol{H}}_{\Pi^p} = \boldsymbol{D}_{\Pi^p}$ for every prime p, then clearly $M \in \boldsymbol{D}_\Pi$ by [3], (1.4). The converse follows from (1).

By this lemma it suffices to prove the theorem for a p-group Π.

Let Π be a finite group and let I_Π be the augmentation ideal of $\boldsymbol{Z}\Pi$, i.e., $I_\Pi = \operatorname{Ker} \varepsilon_\Pi$, where $\varepsilon_\Pi \colon \boldsymbol{Z}\Pi \to \boldsymbol{Z}$ denotes the augmentation map. We have an exact sequence

$$0 \longrightarrow I_\Pi \otimes_{\boldsymbol{Z}} I_\Pi \longrightarrow \boldsymbol{Z}\Pi^{(n-1)} \longrightarrow I_\Pi \longrightarrow 0, \qquad n = |\Pi| .$$

We define $L_\Pi = (I_\Pi \otimes_{\boldsymbol{Z}} I_\Pi)^* = \operatorname{Hom}_{\boldsymbol{Z}}(I_\Pi \otimes_{\boldsymbol{Z}} I_\Pi, \boldsymbol{Z})$, then $[L_\Pi] \in \tilde{T}(\Pi)$. It is routine to show

LEMMA 2.3. $L_\Pi \in \boldsymbol{H}_\Pi^{-1} \cap \boldsymbol{H}_\Pi^{-3}$, $H^{-2}(\Pi, L_\Pi) \cong \boldsymbol{Z}/|\Pi|\,\boldsymbol{Z}$, $\hat{H}^0(\Pi, L_\Pi) \cong \Pi/[\Pi, \Pi]$ ($[\Pi, \Pi]$ *denotes the commutator subgroup of* Π) *and* $H^1(\Pi, L_\Pi) \cong H^2(\Pi, \boldsymbol{Q}/\boldsymbol{Z})$ (*the Schur multiplier of* Π).

This lemma will be used in Section 3.

LEMMA 2.4. *Let Π be one of the following groups*:

(1) *$Z/2Z \times Z/2Z \times Z/2Z$,*
(2) *the quarternion group H_2 of order 8,*
(3) *$Z/pZ \times Z/pZ$, where p is an odd prime,*
(4) *$Z/4Z \times Z/2Z$.*

Then $[L_\Pi] \notin T^g(\Pi)$. Especially, $\tilde{T}(\Pi)$ is not a group.

This lemma will be proved in Section 3. It is easy to show the following

LEMMA 2.5. *Let Π be a finite 2-group of order $\geqq 8$. Assume that there exists no subgroup of Π isomorphic to one of the following:*
(1) *$Z/2Z \times Z/2Z \times Z/2Z$,*
(2) *the quarternion group H_2 of order 8,*
(3) *$Z/4Z \times Z/2Z$,*

then Π is cyclic or dihedral.

For an odd prime p, if a p-group Π has no subgroup isomorphic to $Z/pZ \times Z/pZ$, then Π is cyclic. Hence the implication (1) $\Rightarrow$ (2) in Theorem 2.1 follows from Lemmas 2.2, 2.4 and 2.5.

We denote by D_{2^ℓ}, $\ell = 1$, the dihedral group of order $2^{\ell+1}$, i.e.,

$$D_{2^\ell} = \langle \sigma, \tau \mid \sigma^{2^\ell} = \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma^{-1} \rangle$$

D_2 is Klein's four group $Z/2Z \times Z/2Z$. From now on we assume $\ell \geqq 2$. The key point in proving the implication (2) $\Rightarrow$ (1) in Theorem 2.1 is that a ZD_{2^ℓ}-lattice with no non-zero element fixed by $\langle \sigma^{2^{\ell-1}} \rangle$, the center of D_{2^ℓ}, can be completely classified locally. To state the classification explicitly we will prepare a few more notations.

Denote by ζ a primitive 2^ℓ-th root of unity and put $R_\ell = Z[\zeta]$ and $\mathscr{P}_\ell = (\zeta - 1)R_\ell$. Define the action of $\langle \tau \rangle$ on R_ℓ by $\tau(\zeta) = \zeta^{-1}$, i.e., identify τ with the complex conjugation. Then $\mathscr{P}_\ell$ is the unique ambiguous prime ideal of R_ℓ ramified over Z. We define $\Lambda_\ell = ZD_{2^\ell}/(\sigma^{2^{\ell-1}} + 1)$. Then Λ_ℓ is isomorphic to the trivial crossed product of R_ℓ and $\langle \tau \rangle$. R_ℓ and $\mathscr{P}_\ell$ clearly can be regarded as Λ_ℓ-modules naturally. Λ_ℓ, R_ℓ and $\mathscr{P}_\ell$ are quasipermutation modules ($M \in M_\Pi$ is a quasi-permutation module if there exists permutation $Z\Pi$-modules S_1, S_2 such that $0 \to M \to S_1 \to S_2 \to 0$ is exact).

LEMMA 2.6. *Let M be a finitely generated Z-free Λ_ℓ-module. Then M has the same genus as $\Lambda_\ell^{(r)} \oplus R_\ell^{(s)} \oplus \mathscr{P}_\ell^{(t)}$ for some r, s, $t \geqq 0$, i.e., these two modules are locally isomorphic and hence $[M] \in T^g(D_{2^\ell})$.*

Using the lemma we can prove

LEMMA 2.7. *$\tilde{T}(D_{2^\ell})$ is a group.*

In [3] we proved that let Π be a p-group with p odd prime, then $T(\Pi)$ is a group if and only if Π is cyclic. Lemma 2.4 shows that the following statements are equivalent for a p-group Π with p odd prime:

(i) Π is cyclic,

(ii) $T(\Pi)$ is a group,

(iii) $\tilde{T}(\Pi)$ is a group.

The implication (2) $\Rightarrow$ (1) in Theorem 2.1 follows from this observation and Lemmas 2.2 and 2.7.

Remark 2.8. In [3] we proved that $T(\Pi)$ is a group if and only if $[I_\Pi^*] \in T^g(\Pi)$. An analogous statement for $\tilde{T}(\Pi)$ can also be proved, i.e., $\tilde{T}(\Pi)$ is a group if and only if $[L_\Pi] = [(I_\Pi \otimes_Z I_\Pi)^*] \in T^g(\Pi)$.

§ 3.

Proof of Lemma 2.4. Throughout this section we assume that Π is a p-group. It suffices to show that if $[L_\Pi] \in T^g(\Pi)$, then Π is not isomorphic to any group listed in Lemma 2.4. Assume that $[L_\Pi] \in T^g(\Pi)$. Since M belongs to $\boldsymbol{D}_\Pi$ if and only if $\boldsymbol{Z}_p \otimes_{\boldsymbol{Z}} M$ is a permutation $\boldsymbol{Z}_p$-module Lemma 1.1 shows that there is an exact sequence:

$$(3.1) \qquad 0 \longrightarrow \boldsymbol{Z}_p \otimes_{\boldsymbol{Z}} L_\Pi \longrightarrow S_1 \longrightarrow S_2 \longrightarrow 0$$

where S_1 and S_2 are permutation $\boldsymbol{Z}_p\Pi$-modules. Remark 1.3 allows us to assume that S_2 has no direct summand isomorphic to $\boldsymbol{Z}_p\Pi/\Pi_0$ for any cyclic subgroup Π_0 of Π (note that $L_\Pi \in \boldsymbol{H}_\Pi^{-1}$ and hence $H^1(\Pi_0, L_\Pi) = 0$ by periodicity). Taking a long cohomology sequence and noting Lemma 2.3, we obtain an exact sequence:

$$(3.2) \qquad 0 \longrightarrow H^{-3}(\Pi, S_1) \longrightarrow H^{-3}(\Pi, S_2) \longrightarrow \boldsymbol{Z}/|\Pi|\boldsymbol{Z} \\ \longrightarrow H^{-2}(\Pi, S_1) \longrightarrow H^{-2}(\Pi, S_2) \longrightarrow 0 .$$

(1) Let Π be $\boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z}$. In this case all cohomology groups appearing in (3.2) are annihilated by 2. This is a contradiction.

(2) Let Π be the quaternion group of order 8. Then $H^{-2}(\Pi, S)$ is annihilated by 4 and $H^{-3}(\Pi, S) \cong H^{-1}(\Pi, S) = 0$ for all permutation $\boldsymbol{Z}_2\Pi$-modules S. This contradicts (3.2).

(3) Let Π be $\boldsymbol{Z}/p\boldsymbol{Z} \times \boldsymbol{Z}/p\boldsymbol{Z}$, where p is an odd prime. In this case we

can put $S_2 = Z_p^{(n)}$, $S_1 = Z_p^{(m)} \oplus T$, where T is a permutation $Z_p\Pi$-module having no direct summand isomorphic to Z_p since a proper subgroup of Π is cyclic. We derive another cohomology sequence from (3.1):

$$0 \longrightarrow \hat{H}^0(\Pi, Z_p \otimes_Z L_\Pi) \longrightarrow \hat{H}^0(\Pi, Z_p^{(m)} \oplus T) \longrightarrow \hat{H}^0(\Pi, Z_p^{(n)}) \longrightarrow H^1(\Pi, Z_p \otimes_Z L_\Pi) \longrightarrow 0 .$$

From this we obtain an exact sequence:

$$0 \longrightarrow (Z/pZ)^{(2)} \longrightarrow (Z/p^2Z)^{(m)} \oplus \hat{H}^0(\Pi, T) \longrightarrow (Z/p^2Z)^{(n)} \longrightarrow Z/pZ \longrightarrow 0 .$$

Hence $n-1=m$ or $m=n$ because $\hat{H}^0(\Pi, T)$ is annihilated by p. Let $\Pi' \neq 1$ be a cyclic subgroup of Π. From (3.1) we get an exact sequence:

$$0 \longrightarrow Z/pZ \longrightarrow (Z/pZ)^{(m)} \oplus \hat{H}^0(\Pi', T) \longrightarrow (Z/pZ)^{(n)} \longrightarrow 0 .$$

Hence $\hat{H}^0(\Pi', T) \cong (Z/pZ)^{(2)}$ if $n-1=m$ or $\cong Z/pZ$ if $n=m$. On the other hand since T has no direct summand isomorphic to Z_p, we see that $\mathrm{rank}_{Z_p} \hat{H}^0(\Pi', T)$ is divisible by p. This is a contradiction.

(4) Let Π be $Z/4Z \times Z/2Z = \langle\sigma\rangle \times \langle\tau\rangle$. In this case (3.1) looks like

$$(3.3) \qquad 0 \longrightarrow Z_p \otimes_Z L_\Pi \longrightarrow Z_p^{(\ell)} \oplus (Z_p\Pi/\langle\sigma^2, \tau\rangle)^{(m)} \oplus T \longrightarrow Z_p^{(\ell')} \oplus (Z_p\Pi/\langle\sigma^2, \tau\rangle)^{(m')} \longrightarrow 0$$

where $T = \oplus_{\Pi' \leqq \Pi, \Pi' : \text{cyclic}} (Z_p\Pi/\Pi')^{(n_{\Pi'})}$.

Set $\Pi_0 = \langle\sigma^2, \tau\rangle$. Then it is easy to check that

$$(3.4) \quad 0 \longrightarrow Z_p \otimes_Z L_{\Pi_0} \longrightarrow Z_p\Pi_0/\langle\sigma^2\rangle \oplus Z_p\Pi_0/\langle\tau\rangle \oplus Z_p\Pi_0/\langle\sigma^2\tau\rangle \xrightarrow{\varepsilon} Z_p \longrightarrow 0$$

is exact, where ε is the sum of augmentation maps. It is also easy to see that

$$(Z_p \otimes_Z L_\Pi) \oplus F_1 \cong (Z_p \otimes_Z L_{\Pi_0}) \oplus F_2$$

as $Z_p\Pi_0$-modules with suitable free $Z_p\Pi_0$-modules F_1, F_2. Taking the push-out of (3.3) and (3.4), we have

$$(3.5) \qquad \begin{aligned} &Z_p^{(\ell)} \oplus (Z_p\Pi/\langle\sigma^2, \tau\rangle)^{(m)} \oplus T \oplus F_1 \oplus Z_p \\ &\cong Z_p^{(\ell')} \oplus (Z_p\Pi/\langle\sigma^2, \tau\rangle)^{(m')} \oplus F_2 \oplus Z_p\Pi_0/\langle\sigma^2\rangle \\ &\qquad \oplus Z_p\Pi_0/\langle\tau\rangle \oplus Z_p\Pi_0/\langle\sigma^2\tau\rangle \end{aligned}$$

as $Z_p\Pi_0$-modules. Simple computations show that as a $Z_p\Pi_0$-module T has even number of direct summands isomorphic to $Z_p\Pi_0/\langle\tau\rangle$. This contradicts (3.5).

§ 4.

Proofs of Lemmas 2.6 *and* 2.7. First we consider proof of Lemma 2.6.

LEMMA 4.1. *Let R be a ring and let A_1, A_2 and X be R-modules with the following properties*:

(1) $0 \to A_i \to X \to A_i \to 0$ $(i = 1, 2)$ *are non-split exact sequences.*

(2) $\mathrm{Ext}^1_R(A_i, A_i) \cong Z/2Z$, $\mathrm{Ext}^1_R(A_i, X) \cong \mathrm{Ext}^1_R(X, A_i) = 0$ $i = 1, 2$ *and* $\mathrm{Ext}^1_R(A_1, A_2) \cong \mathrm{Ext}^1_R(A_2, A_1) \cong \mathrm{Ext}^1_R(X, X) = 0$.

Consider an extension

$$0 \longrightarrow A_1^{(s_1)} \oplus A_2^{(s_2)} \oplus X^{(t)} \longrightarrow Y \longrightarrow A_1^{(s_1')} \oplus A_2^{(s_2')} \oplus X^{(t')} \longrightarrow 0\,,$$

then Y is of the same type, i.e., $Y = A_1^{(s_1'')} \oplus A_2^{(s_2'')} \oplus X^{(t'')}$.

Proof is easy.

We have the exact sequences:

$$0 \longrightarrow \Lambda_\ell(\tau - 1) \longrightarrow \Lambda_\ell \longrightarrow \Lambda_\ell(\tau + 1) \longrightarrow 0$$
$$0 \longrightarrow \Lambda_\ell(\tau - \zeta^{-1}) \longrightarrow \Lambda_\ell \longrightarrow \Lambda_\ell(\tau + \zeta) \longrightarrow 0\,.$$

Define the homomorphism $\phi\colon R_\ell \to \Lambda_\ell(\tau + 1)$ (resp. $\phi'\colon \mathscr{P}_\ell \to \Lambda_\ell(\tau + \zeta)$) by $\phi(x) = x(\tau + 1)$ (resp. $\phi'(x(\zeta + 1)) = x(\tau + \zeta)$) for any $x \in R_\ell$. Then ϕ and ϕ' are Λ_ℓ-homomorphisms. Put $\omega = (1 + \zeta)(1 - \zeta)^{-1}$ and $\omega' = (1 + \zeta)(1 - \zeta^{-1})^{-1} \in U(R_\ell)$ and define the homomorphism $\psi\colon R_\ell \to \Lambda_\ell(\tau - 1)$ (resp. $\psi'\colon \mathscr{P}_\ell \to \Lambda_\ell(\tau - \zeta^{-1})$) by $\psi(x) = x\omega(\tau - 1)$ (resp. $\psi'(x(\zeta + 1)) = x\omega'(\tau - \zeta^{-1})$) for any $x \in R_\ell$. We easily show that both ψ and ψ' are Λ_ℓ-isomorphisms. Therefore, we have

$$\Lambda_\ell(\tau + 1) \cong \Lambda_\ell(\tau - 1) \cong R_\ell \quad \text{and} \quad \Lambda_\ell(\tau + \zeta) \cong \Lambda_\ell(\tau - \zeta^{-1}) \cong \mathscr{P}_\ell$$

as Λ_ℓ-modules. Thus we get the non-split exact sequences:

$$\begin{aligned} &0 \longrightarrow R_\ell \longrightarrow \Lambda_\ell \xrightarrow{\;f\;} R_\ell \longrightarrow 0 \\ &0 \longrightarrow \mathscr{P}_\ell \longrightarrow \Lambda_\ell \xrightarrow{\;f'\;} \mathscr{P}_\ell \longrightarrow 0 \end{aligned} \tag{4.1}$$

where f (resp. f') is defined by $f(x + y\tau) = x + y$ (resp. $f'(x + y\tau) = (x + y\zeta^{-1})(\zeta + 1)$). From (4.1) we get the exact sequence:

$$\mathrm{Hom}_{\Lambda_\ell}(R_\ell, \Lambda_\ell) \xrightarrow{\;\tilde f\;} \mathrm{Hom}_{\Lambda_\ell}(R_\ell, R_\ell) \longrightarrow \mathrm{Ext}^1_{\Lambda_\ell}(R_\ell, R_\ell) \longrightarrow 0\,.$$

Every $g \in \mathrm{Hom}_{\Lambda_\ell}(R_\ell, R_\ell)$ can be identified with $g(1) \in R_\ell$. Then we have

$$\mathrm{Hom}_{\Lambda_\ell}(R_\ell, R_\ell) = \{x \in R_\ell \mid \bar x = x\} \quad \text{and} \quad \mathrm{Im}\,\tilde f = \{x + \bar x \mid x \in R_\ell\}$$

where $\bar{x}$ is the complex conjugation of $x \in R_\ell$. By a direct computation it is seen that

$$\mathrm{Ext}^1_{\Lambda_\ell}(R_\ell, R_\ell) \cong \mathrm{Hom}_{\Lambda_\ell}(R_\ell, R_\ell)/\mathrm{Im}\,\tilde{f} \cong Z/2Z .$$

Similarly we can show that

$$\mathrm{Ext}^1_{\Lambda_\ell}(\mathscr{P}_\ell, \mathscr{P}_\ell) \cong Z/2Z \text{ and } \mathrm{Ext}^1_{\Lambda_\ell}(R_\ell, \mathscr{P}_\ell) \cong \mathrm{Ext}^1_{\Lambda_\ell}(\mathscr{P}_\ell, R_\ell) = 0 .$$

This shows that R_ℓ, $\mathscr{P}_\ell$ and Λ_ℓ satisfy the conditions in Lemma 4.1.

We localize everything at 2 and denote them by the same notations. Let Ω_ℓ be a maximal order in $Q_2\Lambda_\ell$ containing Λ_ℓ and let M be a Λ_ℓ-lattice. Then we can write uniquely $\Omega_\ell M \cong \Omega_\ell R_\ell^{(n)}$, $n \geqq 0$. We call n the rank of M. We will prove the assertion by induction on n. It is noted that any ambiguous ideal of R_ℓ is isomorphic to R_ℓ or $\mathscr{P}_\ell$. If $n = 1$, M is isomorphic to an ambiguous ideal of R_ℓ and so $M \cong R_\ell$ or $\mathscr{P}_\ell$. Now we assume that $n \geqq 2$ and the assertion is true for N with rank $N \leqq n-1$. We can write $\Omega_\ell M = L_1 \oplus L_2$, where $L_1 \cong \Omega_\ell R_\ell^{(n-1)}$ and $L_2 \cong \Omega_\ell R_\ell$. Put $N = M \cap L_1$ and $M' = M/N$. Then by the induction hypothesis we have $N \cong \Lambda_\ell^{(r)} \oplus R_\ell^{(s)} \oplus \mathscr{P}_\ell^{(t)}$ for some r, s, and t. Since $M' \cong R_\ell$ or $\mathscr{P}_\ell$, we have $M \cong \Lambda_\ell^{(r')} \oplus R_\ell^{(s')} \oplus \mathscr{P}_\ell^{(t')}$ by Lemma 4.1. This completes the proof.

Finally we shall prove Lemma 2.7. If Π is cyclic this assertion was proved in [3]. Therefore we only need to consider the case where Π is dihedral, i.e.,

$$\Pi = D_{2^\ell} = \langle \sigma, \tau \mid \sigma^{2^\ell} = \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma^{-1} \rangle, \qquad \ell \geqq 1 .$$

We will prove the assertion by induction on ℓ.

We first assume that $\ell = 1$. In this case D_2 is Klein's four group. Let $L \in \tilde{H}_\Pi$. Define the homomorphism $N_\sigma \colon L \to L$ by $N_\sigma(u) = (1+\sigma)u$ for $u \in L$. Then $H^{-1}(\langle\sigma\rangle, L) = \mathrm{Ker}\, N_\sigma/(\sigma-1)L = 0$ and hence we have the exact sequence

$$0 \longrightarrow (\sigma-1)L \longrightarrow L \longrightarrow \mathrm{Im}\, N_\sigma \longrightarrow 0 .$$

Since $\hat{H}^0(\Pi, (\sigma-1)L) = 0$, we have $H^{-1}(\Pi, \mathrm{Im}\, N_\sigma) = 0$. Since $\mathrm{Im}\, N_\sigma$ can be regarded as a $Z\Pi/\langle\sigma\rangle$-module and Z-free, we easily see that $H^{-1}(\Pi/\langle\sigma\rangle, \mathrm{Im}\, N_\sigma) = 0$. This shows that $\mathrm{Im}\, N_\sigma$ is a permutation $Z\Pi/\langle\sigma\rangle$-module. We obtain that

$$(\sigma-1)L \cong (Z\Pi/(\sigma+1, \tau-1))^{(r)} \oplus (Z\Pi/(\sigma+1, \sigma\tau-1))^{(s)} \oplus (Z\Pi/(\sigma+1))^{(t)}$$

for some $r, s, t \geqq 0$ and therefore $[(\sigma-1)L]$ is zero in $T(\Pi)$. Hence $[L] =$

$[(\sigma - 1)L] = 0$. This shows $\tilde{T}(\Pi) = 0$, or $\tilde{\boldsymbol{H}}_\Pi = \boldsymbol{D}_\Pi$. Next assume that $\ell \geqq 2$ and the assertion is true for D_{2^j}, $j \leqq \ell - 1$. Let $L \in \tilde{\boldsymbol{H}}_{D_{2^\ell}}$. Then we have the exact sequence

$$0 \longrightarrow (\sigma^{2^{\ell-1}} - 1)L \longrightarrow L \longrightarrow (\sigma^{2^{\ell-1}} + 1)L \longrightarrow 0 .$$

Put $L' = (\sigma^{2^{\ell-1}} - 1)L$ and $L'' = (\sigma^{2^{\ell-1}} + 1)L$. If Π' is a subgroup of $\Pi = D_{2^\ell}$ containing $\langle \sigma^{2^{\ell-1}} \rangle$, then $\hat{H}^0(\Pi', L') = 0$ hence $H^{-1}(\Pi', L'') = 0$. Since L'' can be regarded as $\boldsymbol{Z}$-free $\boldsymbol{Z}\Pi/\langle \sigma^{2^{\ell-1}} \rangle$-module, we have $H^{-1}(\Pi'/\langle \sigma^{2^{\ell-1}} \rangle, L'') = 0$. Hence $L'' \in \boldsymbol{H}_\Pi^{-1}/\langle \sigma^{2^{\ell-1}} \rangle$. By Lemma 2.6 there are a permutation $\boldsymbol{Z}\Pi$-module S and a $\boldsymbol{Z}\Pi$-lattice T locally isomorphic to a permutation module such that

$$0 \longrightarrow L' \longrightarrow S \longrightarrow T \longrightarrow 0$$

is exact. Taking the pushout of $L' \to L$ we get the following commutative
$\searrow s$

diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L' & \longrightarrow & L & \longrightarrow & L'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & s & \longrightarrow & X & \longrightarrow & L'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & T & = & T & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $L'' \in \boldsymbol{H}^{-1}_{\Pi/\langle \sigma^{2^{\ell-1}} \rangle} \subseteqq \boldsymbol{H}_\Pi^{-1}$ and $L \in \boldsymbol{H}_\Pi^1$, we have $X \cong L \oplus T \cong S \oplus L''$. This shows that $L'' \in \boldsymbol{H}_\Pi^1$ and hence $L'' \in \tilde{\boldsymbol{H}}_{\Pi/\langle \sigma^{2^{\ell-1}} \rangle}$. By the induction hypothesis $L'' \in \boldsymbol{D}_{\Pi/\langle \sigma^{2^{\ell-1}} \rangle} \subseteqq \boldsymbol{D}_\Pi$, thus $L \in \boldsymbol{D}_\Pi$. This completes the proof.

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