Integral Sets and Cayley Graphs of Finite Groups

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Abstract

Integral sets of finite groups are discussed and related to the integral Cayley graphs. The Boolean algebra of integral sets are determined for dihedral group and finite abelian groups. We characterize the finite abelian groups as those finite groups where the Boolean algebra generated by integral sets equals the Boolean algebra generated by its subgroups.

1 Introduction

Integral graphs were first introduced by Harary and Schwenk in 1974 [5]. These are simple graphs without loops with adjacency matrix having only integer eigenvalues. Since then the interest continues in finding integral graphs of certain types: trees, cubic graphs, Cayley graphs; see [1], [2] and their references. The integral Cayley graphs for a cyclic group were determined in [3]. This result was rediscovered [8]; recently, there is renewed interest in the case of abelian groups [6], [7].

We abstract some of the properties of integral Cayley graphs of abelian groups so that we can apply similar methods to any finite group. We introduce the notion of an integral set. The result then for the general finite *abelian* group is that the Boolean algebra generated by integral sets is the same as the Boolean algebras generated by subgroups (*cf.* Theorem 5.1). On the other hand for dihedral groups this algebra is the same as the power set of the group (*cf.* Theorem 6.1).

2 Boolean Algebra

Suppose X is a set and $\mathcal{F}$ is a family of subsets then the Boolean algebra $\mathbb{B}(\mathcal{F})$ generated by $\mathcal{F}$ is the lattice of subsets of X obtained by arbitrary finite intersections, unions, and complements of the sets in the family $\mathcal{F}$. The minimal non-empty subsets of this algebra are called the atoms. Each element of $\mathbb{B}(\mathcal{F})$ is expressible as a union of the atoms.

Consider the equivalence relation on X: $a \equiv b$ if for every $S \in \mathcal{F}$, then $a, b \in S$ or $a, b \notin S$.

Lemma 2.1 *The equivalence classes of this relation are the atoms of* $\mathbb{B}(\mathcal{F})$

Proof. If A is an atom then any set S either contains A or is disjoint from A. So elements of A are equivalent. Conversely if $a \in X$ and $\bar{a}$ is its equivalence class then whenever it meets the subset S it is contained in S; hence $\bar{a}$ must be a minimal set (an atom). $\square$

Theorem 2.2 *The atoms for the* $\mathbb{B}(\mathcal{F}_G)$ *where* $\mathcal{F}_G$ *is the family of subgroups of the finite group* G *are the sets* $[a] = \{b \mid \ <b> = <a>\}$, *the generators of the cyclic subgroup generated by* a, *for any* $a \in G$.

Proof. Let $a \in G$ and its equivalence class $\bar{a}$ from the relation afforded by $\mathcal{F}_G$. By Lemma 2.1 we see that $b \in \bar{a}$ iff every subgroup of G containing a also contains b so $<b> = <a>$. Hence $\bar{a} = [a]$. $\square$

3 Atomic Numbers

It is easy to see that the number of atoms of $\mathbb{B}(\mathcal{F}_G)$ (*atomic number*) is $\text{Atomic}(G) = \sum_{a \in G} \frac{1}{\phi(|a|)} = \sum_{d||G|} \frac{N_d}{\phi(d)}$, where N_d is the number of elements in G of order d and ϕ is Euler's totient function. For example for a cyclic group of order n, then there are $\phi(d)$ elements of order d, so $\text{Atomic}(\mathbb{Z}_n) = \sum_{d|n} \frac{\phi(d)}{\phi(d)} = \tau(n)$, the number of divisors of n.

Lemma 3.1 *If* $n_1 = |G_1|$ *and* $n_2 = |G_2|$ *are relatively prime, then*

$$\text{Atomic}(\text{G}_1 \times \text{G}_2) = \text{Atomic}(\text{G}_1)\text{Atomic}(\text{G}_2).$$

Proof. Every divisor $n = n_1 n_2$ is uniquely of the form $d_1 d_2$ where $d_i | n_i, i = 1, 2$; an element $(g_1, g_2) \in G_1 \times G_2$ has order $d = d_1 d_2$ iff $order(g_i) = d_i, i = 1, 2$. Thus $N_d(G_1 \times G_2) = N_{d_1}(G_1) N_{d_2}(G_2)$ and the result follows using the multiplicitivity of the ϕ function. $\square$

3.1 Symmetric Groups

Use the cycle decomposition of an element of S_n: an element decomposes into $e_i \geq 0$ cycles of length i; note that $\sum_{i=1}^{n} i e_i = n$. The order of the element is $g(e_1, \ldots, e_n) = lcm\{j : e_j > 0, 1 \leq j \leq n\}$; let $f(e_1, e_2, \ldots, e_n) = \phi(g(e_1, \ldots, e_n))$. Then

$$\text{Atomic}(S_n) = \sum_{(e_1, \ldots, e_n), e_i \geq 0, \sum_{i=1}^{n} i e_i = n} \frac{n!}{1^{e_1} e_1! 2^{e_2} e_2! \ldots n^{e_n} e_n! f(e_1, e_2, \ldots, e_n)}$$

For the alternating groups A_n we get the subsum:

$$\text{Atomic}(A_n) = \sum_{\sum_{i=1}^{\lfloor n/2 \rfloor} e_{2i} = 0 \bmod 2} \frac{n!}{1^{e_1} e_1! 2^{e_2} e_2! \dots n^{e_n} e_n! f(e_1, e_2, \dots, e_n)}$$

3.2 Dihedral Groups

The dihedral group D_n consists of the cyclic group of order n and n other elements of order 2. It follows that

$$\text{Atomic}(D_n) = \tau(n) + n.$$

3.3 Abelian Groups

By using Lemma 3.1 we may restrict to abelian p-groups, p a prime. Consider $G = \mathbb{Z}_{p^{e_1}} \oplus \mathbb{Z}_{p^{e_2}} \cdots \oplus \mathbb{Z}_{p^{e_r}}$, where $1 \leq e_1 \leq e_2 \cdots \leq e_r$. The orders of elements are p^f and $\phi(p^f) = p^f - p^{f-1}$ Therefore we have $\text{Atomic}(G) = 1 + \sum_{f=1}^{e_r} \frac{N_{p^f}(G)}{p^f - p^{f-1}}$. Suppose $e_1 \leq \cdots \leq e_s < f \leq e_{s+1} \cdots \leq e_r$. Let $s = s(f)$. Now an element $(g_1, g_2, \dots, g_r) \in G$ has order p^f iff among the components g_i all have order $\leq p^f$ and at least one has order equal to p^f. Thus the components in the first s slots are arbitrary, and the last $r - s$ slots have all element of order $\leq p^f$ and some element of order greater than p^{f-1}. Then $N_{p^f}(G) = (p^{e_1+e_2+\cdots+e_s})(p^{(r-s)f} - p^{(r-s)(f-1)})$. This gives the formula

$$\text{Atomic}(G) = 1 + \sum_{f=1}^{e_r} p^{e_1+e_2+\cdots+e_{s(f)}} \frac{p^{(r-s(f))f} - p^{(r-s(f))(f-1)}}{p^f - p^{f-1}}.$$

For example, $\text{Atomic}(\mathbb{Z}_2^r \oplus \mathbb{Z}_4^s \oplus \mathbb{Z}_3^t) = \text{Atomic}(\mathbb{Z}_3^t)\text{Atomic}(\mathbb{Z}_2^r \oplus \mathbb{Z}_4^s)$.

$$\text{Atomic}(\mathbb{Z}_3^t) = \frac{3^t + 1}{2}$$

$$\text{Atomic}(\mathbb{Z}_2^r \oplus \mathbb{Z}_4^s) = 2^{r+s-1}(1 + 2^s)$$

4 Boolean Algebra of Integral Sets

Suppose G is a finite group; let $\hat{G}$ be the set of characters of representations of G over the complex numbers.

For any subset $A \subseteq G$ and any $\chi \in \hat{G}$ let $\chi(A) = \sum_{a \in A} \chi(a)$. We call A *integral* if $\chi(A) \in \mathbb{Z}$ for every $\chi \in \hat{G}$. Since every character is an integer linear combination of irreducible characters, it suffices to check integrality of a set using only the irreducible characters.

Proposition 4.1 *Let G be a finite group. Then any atom of $\mathbb{B}(\mathcal{F}_G)$ is an integral subset of G.*

Proof. Diagonalize an element $a \in G$ using the representation ρ. The element a has order m so the eigenvalues are m-th roots of unity say ζ_j, $1 \leq j \leq n$, $n = \rho(1)$. The character on the atom $[a]$ has value $\sum_{r,(r,m)=1} \sum_{j=1}^{n} \zeta_j^r = \sum_j \sum_{r,(r,m)=1} \zeta_j^r$. Thus the spectrum of $[a]$ for this representation in the j-th component is $\sum_{r,(r,m)=1} \zeta_j^r$; let $\zeta_j = \zeta_m^s$ where ζ_m is a primitive m-th root of unity; so the eigenvalue at the j-th position is $z = \sum_{r,(r,m)=1} \zeta_m^{sr}$. Hence z is an algebraic integer. Moreover z is invariant under the action of the Galois group of $\mathbb{Q}(\zeta_m)$ over $\mathbb{Q}$ so z is rational. Thus z is an integer and hence any atom is integral. $\square$

Let $\mathbb{B}(\mathcal{I}_G)$ denote the Boolean algebra generated by the integral sets; let $\mathcal{P}(G)$ denote the power set of G.

Corollary 4.2 *For any finite group G, $\mathbb{B}(\mathcal{F}_G) \subseteq \mathbb{B}(\mathcal{I}_G) \subseteq \mathcal{P}(G)$*

Corollary 4.3 *Any subgroup of G is integral.*

Proof. Any subgroup H is a disjoint union of some atoms of $\mathbb{B}(\mathcal{F}_G)$. Each of these atoms is integral by Proposition 4.1. Hence H is integral since the union is disjoint. $\square$

Hence $\mathbb{B}(\mathcal{I}_G)$ has its atoms contained in the atoms of $\mathbb{B}(\mathcal{F}_G)$.

Theorem 4.4 *If H is a subgroup of the finite group G then $\mathbb{B}(\mathcal{I}_H) \subseteq \mathbb{B}(\mathcal{I}_G)$.*

Proof. If A is an integral set for H then for any $\chi \in \hat{G}$ its restriction to H is a character of H and so $\chi(A) \in \mathbb{Z}$. Hence any set in the Boolean algebra generated by integral subsets of H is a set in the Boolean algebra generated by integral subsets of G. $\square$

Theorem 4.5 *If all the irreducible representations of the finite group G are realized over $\mathbb{Q}$ then any non-empty subset of G is integral. Hence $\mathbb{B}(\mathcal{I}_G) = \mathcal{P}(G)$.*

Proof. Since every irreducible representation is rational all the characters are integer valued, so $\chi(a) \in \mathbb{Z}$ for all $\chi \in \hat{G}$ and $a \in G$. Thus every subset of G is integral. Every singleton set is an atom. $\square$

It is well-known that the irreducible representations of S_n are rational.

Corollary 4.6 *For all $n \geq 1$, every non-empty subset of S_n is integral, hence $\mathbb{B}(\mathcal{I}_{S_n}) = \mathcal{P}(S_n)$.*

4.1 Orthogonality of Characters in Finite Groups

As is well-known the irreducible characters form an orthonormal basis for the class functions on the group G, functions which are constant on conjugacy classes. The inner product between characters is given by

$$< \chi_1, \chi_2 >_G = \frac{1}{|G|} \sum_{g \in G} \chi_1(g) \bar{\chi}_2(g) = \frac{1}{|G|} \sum_{g \in G} \chi_1(g) \chi_2(g^{-1}).$$

For any subgroup $H \subseteq G$, the representations of G restrict to H to give representations of H and the representations of H induce to representations of G. A relation between these is captured by the Frobenius Reciprocity Law:

$$< \psi, Res(\chi) >_H = < Ind(\psi), \chi >_G .$$

Corollary 4.7 *Every non-empty subset of $\mathbb{Q}_8$, the quaternion group of order 8, is integral, hence $\mathbb{B}(\mathcal{I}_{\mathbb{Q}_8}) = \mathcal{P}(\mathbb{Q}_8)$.*

Proof. Here are the characters. One can determine this character table knowing the 1 dimensional representations; using orthogonality of the rows (weighted) we can determine the 2 dimensional character.

There is a central element and the other non-identity conjugacy classes of size 2. There are four 1 dimensional characters coming from the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ quotient of $\mathbb{Q}_8$ by its center. The first row represents the sizes of the conjugacy class.

	1	1	2	2	2
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

This group is not a rational group since its rational group algebra contains a copy of the quaternion skew-field. □

4.2 Dedekind Groups and Integrality

A group in which all subgroups are normal is called a Dedekind group. This set of groups includes the abelian groups. The non-abelian Dedekind groups are called Hamiltonian groups; the structure theorem for finite Hamiltonian groups (due to Dedekind) says that such a group is isomorphic to $\mathbb{Q}_8 \times D$ where D is an abelian group having no elements of order 4 [4].

We call G a $\mathcal{D}$-group when any non-normal cyclic subgroup of G is of order 2. Any product of dihedral group D_n, $n \geq 3$ is a $\mathcal{D}$-group.

A group is called *cyclotomic* if $\mathbb{B}(\mathcal{F}_G) = \mathbb{B}(\mathcal{I}_G)$.

Lemma 4.8 *A cyclotomic group is a $\mathcal{D}$-group.*

Proof. Suppose G has a non-normal cyclic subgroup H generated by an element a of order greater than 2. By Corollary 4.3, the subgroup H is integral. Without loss of generality we can assume $sas^{-1} \notin H$ for some $s \in G$. Let $K = H - \{a\} \cup \{sas^{-1}\}$. It is well-known that if χ is a character then $\chi(a) = \chi(sas^{-1})$. So K is integral. It is easy to see now that $H \cap K \cap [a]$ is also integral so $H \cap K \cap [a] \in \mathbb{B}(\mathcal{I}_G)$; thus, from the hypothesis,$H \cap K \cap [a] \in \mathbb{B}(\mathcal{F}_G)$. Since $H \cap K = H - \{a\}$ then $H \cap K \cap [a]$ is a proper subset of the atom $[a]$; this contradicts that $[a]$ is an atom. Thus G is a $\mathcal{D}$-group. $\square$

Theorem 4.9 *If a group G is cyclotomic then it is a $\mathcal{D}$-group which does not contain $\mathbb{Q}_8$, D_4 or D_3.*

Proof. By Lemma 4.8 G is a $\mathcal{D}$-group. If G contains $\mathbb{Q}_8$ then any character restricts to $\mathbb{Q}_8$ to give an integral character of $\mathbb{Q}_8$. By Corollary 4.7 for every $a \in \mathbb{Q}_8$, $\{a\}$ is integral. By Theorem 4.4 and $\mathbb{B}(\mathcal{I}_G) = \mathbb{B}(\mathcal{F}_G)$, every element of $\mathbb{Q}_8$ is an atom in $\mathbb{B}(\mathcal{F}_G)$. This is a contradiction since $\mathbb{Q}_8$ has elements of order 4.

It is easy to see that the character table of D_4 is the same as for $\mathbb{Q}_8$. Also the characters of D_3 are integral. Thus by a similar argument any group containing D_4 or D_3 can not be cyclotomic. $\square$

Corollary 4.10 *Any Dedekind group which is cyclotomic must be abelian.*

Proof. A non-abelian Dedekind group contains $\mathbb{Q}_8$. These can not be cyclotomic by Theorem 4.9.

5 Abelian Groups

Theorem 5.1 *For any finite abelian group D every atom of $\mathbb{B}(\mathcal{I}_D)$ is integral. Hence every subset of $\mathbb{B}(\mathcal{I}_D)$ is integral and D is cyclotomic.*

Proof. Consider the matrix $M = (m_{i,j}) = (\chi_i(a_j))$ where $a_j, j = 1, \ldots, n$ are the distinct elements of D and $\chi_i, i = 1, \ldots, n$, are the distinct characters, $n = |D|$. From the orthonormality of characters, the rows of M are independent. The entries of M belong to the field $\mathbb{Q}(\zeta_m)$, where ζ_m is a primitive m-th root of unity, where m is the exponent of D.

Consider the conjugate transpose $\bar{M}^t$; the orthogonality formulas for characters is the same as $M\bar{M}^t = |D|I_n$ or $M^{-1} = \frac{1}{n}\bar{M}^t$

For each atom $[a] \in \mathbb{B}(\mathcal{F}_D)$ let $v_{[a]}$ be the vector with a 1 in those locations which are in $[a]$, and 0s elsewhere. Thus $Mv_{[a]} \in \mathbb{Z}^n$ since atoms are integral. Let $V = \{v \in \mathbb{Q}^n \mid Mv \in \mathbb{Q}^n\}$. Hence $W = \mathrm{span}_{\mathbb{Q}}\{v_{[a]} \mid \text{all atoms } [a]\} \subseteq V$.

If $v \in V$ then $Mv = w \in \mathbb{Q}^n$; hence $v = \frac{1}{n}\bar{M}^t w$. Thus $v_i = \frac{1}{n}\sum_j \bar{\chi_j}(a_i)w_j \in \mathbb{Q}^n$.

Say $b = a^r$, where r is relatively prime to $|a|$. The Galois group of $\mathbb{Q}(\zeta_m)$ over $\mathbb{Q}$ acts transitively on roots of unity of the same order. Hence there is an automorphism σ taking $\chi_j(a)$ to $\chi_j(b)$ for every j.

We can index the components of v by the elements of D. Thus $v_a = \sigma(v_a) = \frac{1}{n}\sum_j \sigma(\overline{\chi_j}(a))w_j = \frac{1}{n}\sum_j \overline{\chi_j}(b)w_j$. Thus the b-component of v is the same as the a-component of v. Hence $v \in W$; so $V = W$.

It is also clear that the set of vectors $v_{[a]}$, $[a]$ an atom of $\mathbb{B}(\mathcal{F}_D)$, is a linearly independent set; hence it is also a basis for W. If A is integral then the vector v_A with a 1 in any component when $a \in A$, and 0s elsewhere satisfies $Mv_A \in \mathbb{Q}^n$ so v_A is a linear combination of the $v_{[a]}$. However it is clear that all coefficients in the linear combination are 0 or 1. It follows that A is a union of atoms for $\mathbb{B}(\mathcal{F}_D)$. Thus $\mathbb{B}(\mathcal{I}_D) = \mathbb{B}(\mathcal{F}_D)$. $\square$

6 Dihedral Groups

6.1 Representations of D_n

Let s be the generator the cyclic subgroup, $s^n = 1$, and t the basic reflection, $t^2 = (ts)^2 = 1$.

If n is odd then the abelianization is $\mathbb{Z}_2$ so there are two one dimensional representations and the characters take only integral values.

If n is even then the abelianization is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ so there are four one dimensional representations and the characters take only integral values.

The rest of the irreducible representations are 2-dimensional. Let ρ be the standard representation of D_n on the regular n-gon. The basic rotation for the generator of the cyclic subgroup is by $2\pi/n$. The other representations have basic rotations for the generator by $2k\pi/n$.

In case n is odd then the representations are different for $1 \leq k \leq \frac{n-1}{2}$. Thus there are a total of $\frac{n+3}{2}$ irreducible representations, which is the same as the number of conjugacy classes. There are $\frac{n-1}{2}$ conjugacy classes consisting of the pairs $\{s^i, s^{-i}\}$, $1 \leq i \leq \frac{n-1}{2}$. and the two singleton classes $\{1\}$ and $\{t\}$.

In case n is even then the representations are different and irreducible for $1 \leq k \leq \frac{n-2}{2}$. Thus there are a total of $\frac{n+6}{2}$ irreducible representations, which is the same as the number of conjugacy classes. There are $\frac{n-2}{2}$ conjugacy classes consist of the pairs $\{s^i, s^{-i}\}$, $1 \leq i \leq \frac{n-2}{2}$. and the four singleton classes $\{1\}$, $\{t\}$, $\{s^{\frac{n}{2}}\}$ and $\{ts^{\frac{n}{2}}\}$

It is easy to evaluate the characters χ_k for these two dimensional representations. The value of χ_k on any reflection is zero and the value on s^l is $2\cos(\frac{2kl\pi}{n})$. If n is even then the value of the character on the element of order 2 is $2\cos(k\pi) = \pm 2$.

6.2 Integral Sets

Theorem 6.1 *For the dihedral groups $D_n, n \geq 1$, then $\mathbb{B}(\mathcal{I}_{D_n}) = \mathcal{P}(D_n)$.*

Proof. Since the one dimensional representations are integral we can test integrality of a set using only the irreducible two dimensional characters. Any atom of $\mathbb{B}(\mathcal{I}_{D_n})$ is contained in an atoms $[a] \in \mathbb{B}(\mathcal{F}_{D_n})$. We can suppose that the size of $[a]$ is greater than 1.

The Galois group of the totally real subfield of $\mathbb{Q}(\zeta_m)$ over $\mathbb{Q}$, obtained as the invariants under complex conjugation, acts transitively on $\cos(\frac{2k\pi}{m})$ where $\gcd(k,m)=1$ and m is a divisor of n

Also any set $B \subseteq [s^d]$, $dm = n$, such that $B \cap B^{-1} = \phi$ and $B \cup B^{-1} = [s^d]$ is also integral since $2\cos(\frac{2k\pi}{m})$ is an algebraic integer of degree $\phi(m)/2$ for $\gcd(k,m)=1$. By taking intersections of sets of this type we can obtain any of the singleton sets, s^{dk}, $\gcd(k,m)=1$. Thus any set is in $\mathbb{B}(\mathcal{I}_{D_n})$. $\square$

Theorem 6.2 *If a group is cyclotomic then it is a $\mathcal{D}$-group which does not contain $\mathbb{Q}_8$ or any $D_n, n \geq 3$.*

Proof. This follows immediately from Theorem 6.1 and the methods of Theorem 4.9. $\square$

7 Spectra of Cayley Graphs

Suppose that A is a subset of the finite group G, which is closed under inversion and does not contain the identity. We use this as a set of generators for the Cayley graph $X = X(G,A)$. Let $\mathcal{A} = \sum_{a\in A} a$, an element of the group algebra $\mathbb{C}G$. It is easy to see that the adjacency matrix of X is obtained by calculating the matrix of the multiplication $\mathbb{C}G \to \mathbb{C}G$, $u \to \mathcal{A}u$. Let ρ denote the regular representation of G. The character of this represention is then χ_ρ and the trace of the adjacency matrix is $\chi_\rho(\mathcal{A}) = \sum_{a\in A}\chi_\rho(a)$.

The regular representation is the direct sum of $\chi_i(1) = n_i$ copies of the i-th irreducible representation, $1 \leq i \leq r$.

The *trace spectrum* of the Cayley graph $X(G,A)$ is the multi-set consisting of $\chi_1(A)$, $\chi_2(A), \ldots, \chi_r(A)$, each repeated according to its multiplicity $n_1, n_2, \ldots, n_r$ times, $[\chi_1(A)^{n_1}; \chi_2(A)^{n_2}; \ldots; \chi_r(A)^{n_r}]$.

For example, the group S_3 has irreducible characters are χ_1, χ_2, χ_3; these are the characters of the trivial representation, the sign representation and a two dimensional representation. The characters values on the non-trivial atoms $a_1 = (12)$, $a_2 = (13)$, $a_3 = (23)$ $a_4 = (123)$, $a_5 = (132)$, are

$$\chi_1(a_1) = 1, \chi_1(a_2) = 1, \chi_1(a_3) = 1, \chi_1(a_4) = 2, \chi_1(a_5) = 2$$

$$\chi_2(a_1) = -1, \chi_2(a_2) = -1, \chi_2(a_3) = -1, \chi_2(a_4) = 2, \chi_2(a_5) = 2$$

$$\chi_3(a_1) = 0, \chi_3(a_2) = 0, \chi_3(a_3) = 0, \chi_3(a_4) = 0, \chi_3(a_5) = 0$$

There are 16 possible choices for an integral set which does not include the identity and is closed under inversion. These are obtained by taking Boolean sums of the atomic trace spectra for the atoms: $a_1 : [1; -1; 0^2]$, $a_2 : [1; -1; 0^2]$, $a_3 : [1; -1; 0^2]$, $\{a_4, a_5\} : [4; 4; 0^2]$

Theorem 7.1 *If G is a finite group and the Cayley graph $X(G, A)$ has integer spectrum then A is an integral set in G.*

Proof. Every irreducible representation $\chi_i, 1 \leq i \leq r$, occurs as a sub-representation of the regular representation, and by hypothesis all the eigenvalues (of the regular representation) when summed over A are integers; hence also the irreducible sub-representation has integer eigenvalues, when summed over A; thus all $\chi_i(A), 1 \leq i \leq r$ are integers. Hence A is an integral set. $\square$

The following result confirms a conjecture of [6].

Corollary 7.2 *If G is a finite abelian group then the Cayley graph $X(G, A)$ has integral spectrum iff A is an integral set.*

Proof. All the characters of a finite abelian group are one dimensional. The result now follows from Theorem 5.1. $\square$

7.1 Cubic connected abelian Cayley graphs

Any set A with three non-identity elements will give a cubic Cayley graph; if the set is integral the possibilities for A (for a cyclotomic group) are:

1. three elements of order 2;
2. an element of order 2, an element of order 3 and its inverse;
3. an element of order 2, an element of order 4 and its inverse;
4. an element of order 2, an element of order 6 and its inverse.

If we also want the graph connected, then this set A will be a generating set for G. If G is an abelian group then it is one of the following:

1. $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$;
2. $\mathbb{Z}_6$;
3. $\mathbb{Z}_4$, $\mathbb{Z}_2 \oplus \mathbb{Z}_4$;
4. $\mathbb{Z}_6$, $\mathbb{Z}_2 \oplus \mathbb{Z}_6$.

The classification of non-abelian cubic connected Cayley graphs with integral spectra is given in [1] ; this can be somewhat streamlined using Theorem 7.1.

7.2 An integral dihedral graph

Let D_n be the dihedral group having generators a, b with $a^n = 1$, $b^2 = 1$, and $ba = a^{-1}b$. Consider the Cayley graph with the integral set $A = \{a, a^2, \ldots, a^{n-1}, b\}$. Use the representation of elements as a^i or $ba^i, i = 1, \ldots, n$

The adjacency matrix is given by the matrix $\begin{pmatrix} J-I & I \\ I & J-I \end{pmatrix}$ which is similar to $\begin{pmatrix} J & 0 \\ I & J-2I \end{pmatrix}$, where J is the matrix of all 1s. Since J is of rank 1, we can easily determine the eigenvalues: $(-2)^{n-1}, 0^{n-1}, n-2, n$

7.3 A non-integral dihedral graph

Let D_n be the dihedral group having generators a, b with $a^2 = 1$, $b^2 = 1$, and $(ab)^n = 1$. Consider the Cayley graph with the integral set $A = \{a, b\}$. Using the representation of elements as $1, a, ab, aba, \ldots, b, ba, bab, \ldots$ it is easy to calculate the adjacency matrix. The Cayley graph is a circuit of length $2n$; it is well-known that the eigenvalues are integral iff n=1, 2, 3, 4, 6 [1].

8 Cyclotomic Groups are Abelian

Dedekind classified finite groups where any proper subgroup is normal. A consequence of Dedekind's theorem is that either the group is abelian or it contains a $\mathbb{Q}_8$ subgroup. So to prove that a cyclotomic finite group is abelian it suffices to show that all proper subgroups are normal; then since there are no $\mathbb{Q}_8$ subgroups the group is abelian. To see that any subgroup is normal we notice that it suffices to show that cyclic subgroups are normal: for an element a in the subgroup H then $xax^{-1} \in < a > \subseteq H$ for any $x \in G$ so H is normal in G.

Lemma 8.1 *The group* $G_b = < x, y \mid x^2 = 1, y^{2^b} = 1, xyx = y^{1+2^{b-1}} >$ *is not cyclotomic for* $b \geq 3$*.*

Proof. The abelianization of G_b is generated by two independent elements X, Y with only the relations $2X = 0, 2^{b-1}Y = 0$; it follows that the abelianization of G_b is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_{2^{b-1}}$.

Consider the element $u = xy^{2^{b-2}}$, $u^2 = xy^{2^{b-2}}xy^{2^{b-2}} = y^{2^{b-1}}$ so u is an element of order 4 in G_b. Also the image of u in the abelianization is $X + 2^{b-2}Y$ and therefore every 1-dimensional character of G_b takes values ± 1 on u.

Using the character identity

$$\sum_r |\chi_r(g)|^2 = |C(g)|$$

with $g = u$ we get a contribution of 2^b on the left side from the 1-dimensional characters. The element u is not central so the righthand side is exactly 2^b and now again from the character identity above it follows that a higher dimensional character has value 0 on u. Thus u is an integral element of order 4 and G_b is not cyclotomic by Theorem 4.4. $\square$

Theorem 8.2 *Any cyclotomic 2-group is abelian.*

Proof. It suffices by Dedekind's Theorem to show that the subgroups of order 2 are normal. Say x is an element of order 2 and y is an element of order 2^b, $b \geq 1$. In case $b = 1$ the subgroup generated by x, y is dihedral so it must be D_2 and thus $yxy^{-1} = x$

Now suppose that $b \geq 2$. We do know that the subgroup generated by y is normal so $xyx = y^a$ and so also $y = x(xyx)x = xy^a x = (xyx)^a = y^{a^2}$; thus $a^2 = 1 \mod 2^b$. The case of $a = -1$ is impossible since then the subgroup generated by x, y is non-abelian dihedral. The case $a = 1$ shows that $< x >$ is normal.

If $b = 2$ then we also know from Theorem 4.9 that any group of order 8 that is cyclotomic must be abelian.

So now we consider the other two solution $a = \pm 1 + 2^{b-1}$ when $b \geq 3$ and show that these also are impossible for a cyclotomic group. Suppose first that $a = -1 + 2^{b-1}$, then $xy^2x = y^{2a} = y^{-2}$ since $2(1 + a) = 0 \mod 2^b$; this gives a non-abelian dihedral subgroup generated by x, y^2 of order 2^b if $b \geq 3$.

Thus we can assume that $a = 1 + 2^{b-1}, b \geq 3$. Consider the groups

$$G_b =< x, y \mid x^2 = 1, y^{2^b} = 1, xyx = y^{1+2^{b-1}} > .$$

This group is not cyclotomic by Lemma 8.1 and hence it can not be contained in G. Thus we have shown in all cases that G has all subgroups normal, and since it also does not contain $\mathbb{Q}_8$ by Theorem 4.9 it is abelian by Dedekind's Theorem. $\square$

Theorem 8.3 *Cyclotomic groups are abelian.*

Proof. For any Sylow p-subgroup S for p odd, its cyclic subgroups are normal by Lemma 4.8; then $x < a > x^{-1} =< a >$ for any $a \in S$, $x \in G$, so S is normal. Thus also S is abelian follows from Dedekind's Theorem since all subgroups are normal.

For $p = 2$ it follows from Theorem 8.2 the Sylow 2-subgroup must be abelian since subgroups of cyclotomic groups are cyclotomic. Let T be the subgroup generated by all elements of order 2 then any 2 of these generate a dihedral group so it is D_2 so the elements commute; thus T is elementary abelian and also normal ; say S is a Sylow 2 subgroup containing T then for any $x \in G$ if a is not of order 2, $x < a > x^{-1} \subseteq< a >\subseteq S$ and if a is of order 2 then $x < a > x^{-1} \subseteq T \subseteq S$ so S is normal.

Since all Sylow subgroups are normal and abelian, it now follows easily the group is the direct product of its Sylow subgroups and hence the group G is abelian. $\square$

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