Integral Sliding-Based Robust Control

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1. Introduction

In this chapter we will study the robust performance control based-on integral sliding-mode for system with nonlinearities and perturbations that consist of external disturbances and model uncertainties of great possibility time-varying manner. Sliding-mode control is one of robust control methodologies that deal with both linear and nonlinear systems, known for over four decades (El-Ghezawi et al., 1983; Utkin & Shi, 1996) and being used extensively from switching power electronics (Tan et al., 2005) to automobile industry (Hebden et al., 2003), even satellite control (Goeree & Fasse, 2000; Liu et al., 2005). The basic idea of sliding-mode control is to drive the sliding surface s from $s \neq 0$ to $s = 0$ and stay there for all future time, if proper sliding-mode control is established. Depending on the design of sliding surface, however, $s = 0$ does not necessarily guarantee system state being the problem of control to equilibrium. For example, sliding-mode control drives a sliding surface, where $s = Mx - Mx_0$, to $s = 0$. This then implies that the system state reaches the initial state, that is, $x = x_0$ for some constant matrix M and initial state, which is not equal to zero. Considering linear sliding surface $s = Mx$, one of the superior advantages that sliding-mode has is that $s = 0$ implies the equilibrium of system state, i.e., $x = 0$. Another sliding surface design, the integral sliding surface, in particular, for this chapter, has one important advantage that is the improvement of the problem of reaching phase, which is the initial period of time that the system has not yet reached the sliding surface and thus is sensitive to any uncertainties or disturbances that jeopardize the system. Integral sliding surface design solves the problem in that the system trajectories start in the sliding surface from the first time instant (Fridman et al., 2005; Poznyak et al., 2004). The function of integral sliding-mode control is now to maintain the system's motion on the integral sliding surface despite model uncertainties and external disturbances, although the system state equilibrium has not yet been reached.

In general, an inherent and invariant property, more importantly an advantage, that all sliding-mode control has is the ability to completely nullify the so-called matched-type uncertainties and nonlinearities, defined in the range space of input matrix (El-Ghezawi et al., 1983). But, in the presence of unmatched-type nonlinearities and uncertainties, the conventional sliding-mode control (Utkin et al., 1999) can not be formulated and thus is unable to control the system. Therefore, the existence of unmatched-type uncertainties has the great possibility to endanger the *sliding dynamics*, which identify the system motion on the sliding surface after matched-type uncertainties are nullified. Hence, another control action simultaneously stabilizes the sliding dynamics must be developed.

Next, a new issue concerning the performance of integral sliding-mode control is addressed, that is, we develop a performance measure in terms of $\mathcal{L}_2$-gain of *zero dynamics*. The concept of zero dynamics introduced by (Lu & Spurgeon, 1997) treats the sliding surface s as the controlled output of the system. The role of integral sliding-mode control is to reach and maintain $s = 0$ while keeping the performance measure within bound. In short, the implementation of integral sliding-mode control solves the influence of matched-type nonlinearities and uncertainties while, in the meantime, maintaining the system on the integral sliding surface and bounding a performance measure without reaching phase. Simultaneously, not subsequently, another control action, i.e. robust linear control, must be taken to compensate the unmatched-type nonlinearities, model uncertainties, and external disturbances and drive the system state to equilibrium.

Robust linear control (Zhou et al., 1995) applied to the system with uncertainties has been extensively studied for over three decades (Boyd et al., 1994) and reference therein. Since part of the uncertainties have now been eliminated by the sliding-mode control, the rest unmatched-type uncertainties and external disturbances will be best suitable for the framework of robust linear control, in which the stability and performance are the issues to be pursued. In this chapter the control in terms of $\mathcal{L}_2$-gain (van der Schaft, 1992) and $\mathcal{H}_2$ (Paganini, 1999) are the performance measure been discussed. It should be noted that the integral sliding-mode control signal and robust linear control signal are combined to form a composite control signal that maintain the system on the sliding surface while simultaneously driving the system to its final equilibrium, i.e. the system state being zero.

This chapter is organized as follows: in section 2, a system with nonlinearities, model uncertainties, and external disturbances represented by state-space is proposed. The assumptions in terms of norm-bound and control problem of stability and performance issues are introduced. In section 3, we construct the integral sliding-mode control such that the stability of zero dynamics is reached while with the same sliding-mode control signal the performance measure is confined within a bound. After a without reaching phase integral sliding-mode control has been designed, in the section 4, we derive robust control scheme of $\mathcal{L}_2$-gain and $\mathcal{H}_2$ measure. Therefore, a composite control that is comprised of integral sliding-mode control and robust linear control to drive the system to its final equilibrium is now completed. Next, the effectiveness of the whole design can now be verified by numerical examples in the section 5. Lastly, the chapter will be concluded in the section 6.

2. Problem formulation

In this section the uncertain systems with nonlinearities, model uncertainties, and disturbances and control problem to be solved are introduced.

2.1 Controlled system
Consider continuous-time uncertain systems of the form

$$\dot{x}(t) = A(t)x(t) + B(t)(u(x,t) + h(x)) + \sum_{i=1}^{N} g_i(x,t) + B_d w(t) \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(x,t) \in \mathbb{R}^m$ is the control action, and for some prescribed compact set $\mathcal{S} \in \mathbb{R}^p$, $w(t) \in \mathcal{S}$ is the vector of (time-varying) variables that represent exogenous inputs which includes disturbances (to be rejected) and possible references (to be tracked). $A(t) \in \mathbb{R}^{n \times n}$ and $B(t) \in \mathbb{R}^{n \times m}$ are time-varying uncertain matrices. $B_d \in \mathbb{R}^{n \times p}$

is a constant matrix that shows how $w(t)$ influences the system in a particular direction. The matched-type nonlinearities $h(x) \in \mathbb{R}^m$ is continuous in x. $g_i(x,t) \in \mathbb{R}^n$, an unmatched-type nonlinearity, possibly time-varying, is piecewise continuous in t and continuous in x. We assume the following:

1. $A(t) = A + \Delta A(t) = A + E_0 F_0(t) H_0$, where A is a constant matrix and $\Delta A(t) = E_0 F_0(t) H_0$ is the unmatched uncertainty in state matrix satisfying

$$\|F_0(t)\| \leq 1, \tag{2}$$

where $F_0(t)$ is an unknown but bounded matrix function. E_0 and H_0 are known constant real matrices.

2. $B(t) = B(I + \Delta B(t))$ and $\Delta B(t) = F_1(t) H_1$. $\Delta B(t)$ represents the input matrix uncertainty. $F_1(t)$ is an unknown but bounded matrix function with

$$\|F_1(t)\| \leq 1, \tag{3}$$

H_1 is a known constant real matrix, where

$$\|H_1\| = \beta_1 < 1, \tag{4}$$

and the constant matrix $B \in \mathbb{R}^{n \times m}$ is of full column rank, i.e.

$$\text{rank}(B) = m. \tag{5}$$

3. The exogenous signals, $w(t)$, are bounded by an upper bound $\bar{w}$,

$$\|w(t)\| \leq \bar{w}. \tag{6}$$

4. The $g_i(x,t)$ representing the unmatched nonlinearity satisfies the condition,

$$\|g_i(x,t)\| \leq \theta_i \|x\|, \ \ \forall \, t \geq 0, \ \ i = 1, \cdots, N, \tag{7}$$

where $\theta_i > 0$.

5. The matched nonlinearity $h(x)$ satisfies the inequality

$$\|h(x)\| \leq \eta(x), \tag{8}$$

where $\eta(x)$ is a non-negative known vector-valued function.

Remark 1. *For the simplicity of computation in the sequel a projection matrix M is such that $MB = I$ for $\text{rank}(B) = m$ by the singular value decomposition:*

$$B = (U_1 \ \ U_2) \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} V,$$

where $(U_1 \ \ U_2)$ and V are unitary matrices. $\Sigma = diag(\sigma_1, \cdots, \sigma_m)$. Let

$$M = V^T \left(\Sigma^{-1} \ 0 \right) \begin{pmatrix} U_1^T \\ U_2^T \end{pmatrix}. \tag{9}$$

It is seen easily that

$$MB = I. \tag{10}$$

2.2 Control problem

The control action to (1) is to provide a feedback controller which processes the full information received from the plant in order to generate a composite control signal

$$u(x,t) = u_s(t) + u_r(x,t), \qquad (11)$$

where $u_s(t)$ stands for the sliding-mode control and $u_r(x,t)$ is the linear control that robustly stabilize the system with performance measure for all admissible nonlinearities, model uncertainties, and external disturbances. Taking the structure of sliding-mode control that completely nullifies matched-type nonlinearities is one of the reasons for choosing the control as part of the composite control (11). For any control problem to have satisfactory action, two objectives must achieve: *stability* and *performance*. In this chapter sliding-mode controller, $u_s(t)$, is designed so as to have asymptotic stability in the Lyapunov sense and the performance measure in $\mathcal{L}_2$ sense satisfying

$$\int_0^T \|s\|^2 dt \le \rho^2 \int_0^T \|w\|^2 dt, \qquad (12)$$

where the variable s defines the sliding surface. The mission of $u_s(t)$ drives the system to reach $s = 0$ and maintain there for all future time, subject to zero initial condition for some prescribed $\rho > 0$. It is noted that the asymptotic stability in the Lyapunov sense is saying that, by defining the sliding surface s, sliding-mode control is to keep the sliding surface at the condition, where $s = 0$. When the system leaves the sliding surface due to external disturbance reasons so that $s \neq 0$, the sliding-mode control will drive the system back to the surface again in an asymptotic manner. In particular, our design of integral sliding-mode control will let the system on the sliding surface without reaching phase. It should be noted that although the system been driven to the sliding surface, the unmatched-type nonlinearities and uncertainties are still affecting the behavior of the system. During this stage another part of control, the robust linear controller, $u_r(x,t)$, is applied to compensate the unmatched-type nonlinearities and uncertainties that robust stability and performance measure in $\mathcal{L}_2$-gain sense satisfying

$$\int_0^T \|z\|^2 dt \le \gamma^2 \int_0^T \|w\|^2 dt, \qquad (13)$$

where the controlled variable, z, is defined to be the linear combination of the system state, x, and the control signal, u_r, such that the state of sliding dynamics will be driven to the equilibrium state, that is, $x = 0$, subject to zero initial condition for some $\gamma > 0$. In addition to the performance defined in (13), the $\mathcal{H}_2$ performance measure can also be applied to the sliding dynamics such that the performance criterion is finite when evaluated the energy response to an impulse input of random direction at w. The $\mathcal{H}_2$ performance measure is defined to be

$$J(x_0) = \sup_{x(0)=x_0} \|z\|_2^2. \qquad (14)$$

In this chapter we will study both performance of controlled variable, z. For the composite control defined in (11), one must aware that the working purposes of the control signals of $u_s(t)$ and $u_r(x,t)$ are different. When applying the composite control simultaneously, it should be aware that the control signal not only maintain the sliding surface but drive the system toward its equilibrium. These are accomplished by having the asymptotic stability in the sense of Lyapunov.

3. Sliding-mode control design

The integral sliding-mode control completely eliminating the matched-type nonlinearities and uncertainties of (1) while keeping $s = 0$ and satisfying $\mathcal{L}_2$-gain bound is designed in the following manner.

3.1 Integral sliding-mode control

Let the switching control law be

$$u_s(t) = -\alpha(t) \frac{s(x,t)}{\|s(x,t)\|}. \tag{15}$$

The integral sliding surface inspired by (Cao & Xu, 2004) is defined to be

$$s(x,t) = Mx(t) + s_0(x,t), \tag{16}$$

where $s_0(x,t)$ is defined to be

$$s_0(x,t) = -M \left(x_0 + \int_0^t (Ax(\tau) + Bu_r(\tau)d\tau \right); \quad x_0 = x(0). \tag{17}$$

The switching control gain $\alpha(t)$ being a positive scalar satisfies

$$\alpha(t) \geq \frac{1}{1 - \beta_1} \left(\lambda + \beta_0 + (1 + \beta_1)\eta(x) + \beta_1 \|u_r\| \right) \tag{18}$$

where

$$\beta_0 = \kappa\|ME_0\|\|H_0\| + \kappa\|M\| \sum_{i=1}^{N} \theta_i + \|MB_d\|\bar{w}. \tag{19}$$

λ is chosen to be some positive constant satisfying performance measure. It is not difficult to see from (16) and (17) that

$$s(x_0, 0) = 0, \tag{20}$$

which, in other words, from the very beginning of system operation, the controlled system is on the sliding surface. Without reaching phase is then achieved. Next to ensure the sliding motion on the sliding surface, a Lyapunov candidate for the system is chosen to be

$$V_s = \frac{1}{2} s^T s. \tag{21}$$

It is noted that in the sequel if the arguments of a function is intuitively understandable we will omit them. To guarantee the sliding motion of the sliding surface, the following differentiation of time must hold, i.e.

$$\dot{V}_s = s^T \dot{s} \leq 0. \tag{22}$$

It follows from (16) and (17) that

$$\dot{s} = M\dot{x} + M(Ax + Bu_r) \tag{23}$$

Substituting (1) into (23) and in view of (10), we have

$$\dot{s} = M\Delta A(t)x + (I + \Delta B(t))(u + h(x)) + M \sum_{i=1}^{N} g_i(x,t) + MB_d w - u_r. \tag{24}$$

Thus the following inequality holds,

$$\dot{V}_s = s^T \left(M\Delta A(t)x + (I + \Delta B(t))(u + h(x)) + M\sum_{i=1}^{N} g_i(x,t) + MB_d w - u_r \right)$$
$$\leq \|s\|(\beta_0 + (1 + \beta_1)\eta(x) + \beta_1\|u_r\| + (\beta_1 - 1)\alpha(t)). \tag{25}$$

By selecting $\alpha(t)$ as (18), we obtain

$$\dot{V}_s \leq -\|s\|\lambda \leq 0, \tag{26}$$

which not only guarantees the sliding motion of (1) on the sliding surface, i.e. maintaining $s = 0$, but also drives the system back to sliding surface if deviation caused by disturbances happens. To illustrate the inequality of (25), the following norm-bounded conditions must be quantified,

$$s^T(M\Delta A(t)x) \leq \|s\|\|M\Delta A(t)x\| = \|s\|\|ME_0 F_0(t)H_0 x\|$$
$$\leq \|s\|\|ME_0 F_0(t)H_0\|\|x\| \leq \|s\|\|ME_0\|\|H_0\|\kappa, \tag{27}$$

by the assumption (2) and by asymptotic stability in the sense of Lyapunov such that there exists a ball, $\mathcal{B}$, where $\mathcal{B} \triangleq \{x(t) : \max_{t \geq 0} \|x(t)\| \leq \kappa, \text{ for } \|x_0\| < \delta\}$. In view of (3), (4), (68), and the second term of parenthesis of (25), the following inequality holds,

$$s^T(I + \Delta B(t))h(x) \leq \|s\|\|(I + \Delta B)h\| = \|s\|\|(I + F_1(t)H_1)h\|$$
$$\leq \|s\|(1 + \|H_1\|)\eta(x) = \|s\|(1 + \beta_1)\eta(x). \tag{28}$$

By the similar manner, we obtain

$$s^T\Delta B(t)u \leq \|s\|\|\Delta Bu\| = \|s\|\|F_1(t)H_1(u_s + u_r)\|$$
$$\leq \|s\|\|H_1\|(\|u_s\| + \|u_r\|) = \|s\|\beta_1(\alpha(t) + \|u_r\|), \tag{29}$$

where $\|u_s\| = \| - \alpha(t)\frac{s}{\|s\|}\| = \alpha(t)$. As for the disturbance w, we have

$$s^T MB_d w \leq \|s\|\|MB_d w\| \leq \|s\|\|MB_d\|\bar{w}, \tag{30}$$

by using the assumption of (6). Lastly,

$$s^T M\sum_{i}^{N} g_i(x,t) \leq \|s\|\|M\|\|\sum_{i=1}^{N} g_i(x,t)\| \leq \|s\|\|M\|\sum_{i=1}^{N}\|g_i(x,t)\|$$
$$\leq \|s\|\|M\|\left(\sum_{i=1}^{N}\theta_i\|x\|\right) \leq \|s\|\|M\|\left(\kappa\sum_{i=1}^{N}\theta_i\right), \tag{31}$$

for the unmatched nonlinearity $g_i(x,t)$ satisfies (7). Applying (27)-(31) to (22), we obtain the inequality (25). To guarantee the sliding motion on the sliding surface right from the very beginning of the system operation, i.e. $t = 0$, and to maintain $s = 0$ for $t \geq 0$, are proved by having the inequality (26)

$$\dot{V}_s = \frac{dV_s}{dt} \leq -\lambda\|s\| = -\lambda\sqrt{V_s} \leq 0.$$

This implies that

$$\frac{dV_s}{\sqrt{V_s}} \leq -\int_0^t \lambda dt$$

Integrating both sides of the inequality, we have

$$\int_{V_s(0)}^{V_s(t)} \frac{dV_s}{\sqrt{V_s}} = 2\sqrt{V_s(t)} - 2\sqrt{V_s(0)} \leq -\lambda t.$$

Knowing that (20) and thus $V_s(0) = 0$, this implies

$$0 \leq 2\sqrt{V_s(t)} = 2\sqrt{s^T(x,t)s(x,t)} \leq 0. \tag{32}$$

This identifies that $s = 0$, which implies that $\dot{s} = 0$ for $t \geq 0$, from which and (24), we find

$$u = -(I + \Delta B(t))^{-1} \left(M\Delta A(t)x + (I + \Delta B(t))h(x) + \sum_{i=1}^N g_i(x,t) + MB_dw - u_r \right), \tag{33}$$

where (4) guarantees the invertibility of (33) to exist. Substituting (33) into (1) and in view of (6), we obtain the sliding dynamics

$$\dot{x} = Ax + G\left(\Delta A(t)x + \sum_{i=1}^N g_i(x,t) \right) + GB_dw + Bu_r, \tag{34}$$

where $G = I - BM$. It is seen that the matched uncertainties, $\Delta B(t)u$ and $(I + \Delta B(t))h(x)$ are completely removed.

3.2 Performance measure of sliding-mode control

The concept of *zero dynamics* introduced by (Lu & Spurgeon, 1997) in sliding-mode control treats the sliding surface s as the controlled output in the presence of disturbances, nonlinearities and uncertainties. With regard to (1) the performance measure similar to (van der Schaft, 1992) is formally defined:

Let $\rho \geq 0$. The system (1) and zero dynamics defined in (16) is said to have $\mathcal{L}_2$-gain less than or equal to ρ if

$$\int_0^T \|s\|^2 dt \leq \rho^2 \int_0^T \|w\|^2 dt, \tag{35}$$

for all $T \geq 0$ and all $w \in \mathcal{L}_2(0, T)$. The inequality of (35) can be accomplished by appropriately choosing the sliding variable λ that satisfies

$$\lambda \geq 2\zeta + 2\rho\bar{w}, \tag{36}$$

where the parameter ζ is defined in (40). To prove this the following inequality holds,

$$-(\rho w - s)^T(\rho w - s) \leq 0. \tag{37}$$

With the inequality (37) we obtain

$$\|s\|^2 - \rho^2\|w\|^2 \leq 2\|s\|^2 - 2\rho s^T w. \tag{38}$$

It is noted that

$$
\begin{aligned}
\int_0^T (\|s\|^2 - \rho^2\|w\|^2)dt &\le \int_0^T 2(\|s\|^2 - \rho s^T w)dt \\
&\le \int_0^T \left(2(\|s\|^2 - \rho s^T w) + \dot{V} \right) dt - (V(T) - V(0)) \\
&\le \int_0^T \left(2(\|s\|^2 - \rho s^T w) - \lambda\|s\| \right) dt \\
&\le \int_0^T \|s\|(2\|s\| + 2\rho\bar{w} - \lambda)dt
\end{aligned}
\tag{39}
$$

The above inequalities use the fact (20), (26), and (32). Thus to guarantee the inequality we require that the λ be chosen as (36). In what follows, we need to quantify $\|s\|$ such that finite λ is obtained. To show this, it is not difficult to see, in the next section, that $u_r = Kx$ is so as to $A + BK$ Hurwitz, i.e. all eigenvalues of $A + BK$ are in the left half-plane. Therefore, for $x(0) = x_0$

$$
\begin{aligned}
\|s\| &= \left\| Mx - M\left(x_0 + \int_0^\infty (Ax + Bu_r)d\tau \right) \right\| \\
&\le \|M\|\|x - x_0\| + \|M\| \left\| \int_0^\infty (A + BK)x d\tau \right\| \\
&\le \|M\|(\|x\| + \|x_0\|) + \|M\|\|A + BK\| \left\| \int_0^\infty x d\tau \right\| \\
&\le \|M\|(\kappa + \|x_0\|) + \|M\|\|A + BK\| \left\| \int_0^T x d\tau + \int_T^\infty x d\tau \right\| \\
&\le \|M\|(\kappa + \|x_0\|) + \|M\|\|A + BK\| \left\| \int_0^T x d\tau \right\| \\
&\le \|M\|(\kappa + \|x_0\|) + \|A + BK\|\kappa T \triangleq \zeta,
\end{aligned}
\tag{40}
$$

where the elimination of $\int_T^\infty x d\tau$ is due to the reason of asymptotic stability in the sense of Lyapunov, that is, when $t \ge T$ the state reaches the equilibrium, i.e. $x(t) \to 0$.

4. Robust linear control design

The foregoing section illustrates the sliding-mode control that assures asymptotic stability of sliding surface, where $s = 0$ is guaranteed at the beginning of system operation. In this section we will reformulate the sliding dynamics (34) by using linear fractional representation such that the nonlinearities and perturbations are lumped together and are treated as uncertainties from linear control perspective.

4.1 Linear Fractional Representation (LFR)
Applying LFR technique to the sliding dynamics (34), we have LFR representation of the following form

$$
\begin{aligned}
\dot{x} &= Ax + Bu_r + B_p p + B_w w \\
z &= C_z x + D_z u_r
\end{aligned}
\tag{41}
$$

and

$$
p_i = g_i(x, t), \quad i = 0, 1, \cdots, N
$$

where $z \in \mathcal{R}^{n_z}$ is an additional artificial controlled variable to satisfy robust performance measure with respect to disturbance signal, w. In order to merge the uncertainty $\Delta A(t)x$ with nonlinearities $\sum_{i=1}^{N} g_i(x,t)$, the variable p_0 is defined to be

$$p_0 = g_0(x,t) = F_0(t)H_0 x = F(t)q_0,$$

where $q_0 = H_0 x$. Thus, by considering (2), p_0 has a norm-bounded constraint

$$\|p_0\| = \|F_0(t)q_0\| \leq \theta_0 \|q_0\|, \tag{42}$$

where $\theta_0 = 1$. Let $p_i = g_i(x,t)$, $i = 1, \cdots, N$ and $q_i = x$, then in view of (7)

$$\|p_i\| = \|g_i(x,t)\| \leq \theta_i \|x\| = \theta_i \|q_i\|, \quad \forall \ i = 1, \cdots, N. \tag{43}$$

Let the vector $p \in \mathcal{R}^{(N+1)n}$ and $q \in \mathcal{R}^{(N+1)n}$ lumping all p_is be defined to be

$$p^T = \begin{pmatrix} p_0^T & p_1^T & \cdots & p_N^T \end{pmatrix}, \quad q^T = \begin{pmatrix} q_0^T & q_1^T & \cdots & q_N^T \end{pmatrix},$$

through which all the uncertainties and the unmatched nonlinearities are fed into the sliding dynamics. The matrices, B_p, B_w, and C_q are constant matrices as follows,

$$B_p = G \underbrace{\begin{pmatrix} E_0 & I & \cdots & I \end{pmatrix}}_{(N+1) \text{ matrix}}, \quad B_w = GB_d \quad \text{and} \quad C_q = \begin{pmatrix} H_0 \\ I \\ \vdots \\ I \end{pmatrix}.$$

Since full-state feedback is applied, thus

$$u_r = Kx. \tag{44}$$

The overall closed-loop system is as follows,

$$\begin{aligned} \dot{x} &= \mathcal{A}x + B_p p + B_w w \\ q &= C_q x \\ z &= \mathcal{C}x \\ \text{and} \\ p_i &= g_i(q_i, t), \quad i = 0, 1, \cdots, N, \end{aligned} \tag{45}$$

where $\mathcal{A} = A + BK$ and $\mathcal{C} = C_z + D_z K$. This completes LFR process of the sliding dynamics. In what follows the robust linear control with performance measure that asymptotically drive the overall system to the equilibrium point is illustrated.

4.2 Robust performance measure
4.2.1 Robust $\mathcal{L}_2$-gain measure
In this section the performance measure in $\mathcal{L}_2$-gain sense is suggested for the robust control design of sliding dynamics where the system state will be driven to the equilibrium. We will be concerned with the stability and performance notion for the system (45) as follows:

Let the constant $\gamma > 0$ be given. The closed-loop system (45) is said to have a *robust $\mathcal{L}_2$-gain measure* γ if for any admissible norm-bounded uncertainties the following conditions hold.
(1) The closed-loop system is uniformly asymptotically stable.
(2) Subject to the assumption of zero initial condition, the controlled output z satisfies

$$\int_0^\infty \|z\|^2 dt \leq \gamma^2 \int_0^\infty \|w\|^2 dt. \tag{46}$$

Here, we use the notion of quadratic Lyapunov function with an $\mathcal{L}_2$-gain measure introduced by (Boyd et al., 1994) and (van der Schaft, 1992) for robust linear control and nonlinear control, respectively. With this aim, the characterizations of robust performance based on quadratic stability will be given in terms of matrix inequalities, where if LMIs can be found then the computations by finite dimensional convex programming are efficient. Now let quadratic Lyapunov function be

$$V = x^T X x^T, \tag{47}$$

with $X \succ 0$. To prove (46), we have the following process

$$\int_0^\infty \|z\|^2 dt \leq \gamma^2 \int_0^\infty \|w\|^2 dt$$
$$\Leftrightarrow \int_0^\infty \left(z^T z - \gamma^2 w^T w \right) dt \leq 0 \tag{48}$$
$$\Leftrightarrow \int_0^\infty \left(z^T z - \gamma^2 w^T w + \frac{d}{dt} V \right) dt - V(x(\infty)) \leq 0.$$

Thus, to ensure (48), $z^T z - \gamma^2 w^T w + \dot{V} \leq 0$ must hold. Therefore, we need first to secure

$$\frac{d}{dt} V(x) + z^T z - \gamma^2 w^T w \leq 0, \tag{49}$$

subject to the condition

$$\|p_i\| \leq \theta_i \|q_i\|, \quad i = 0, 1, \cdots, N, \tag{50}$$

for all vector variables satisfying (45). It suffices to secure (49) and (50) by *S*-procedure (Boyd et al., 1994), where the quadratic constraints are incorporated into the cost function via Lagrange multipliers σ_i, i.e. if there exists $\sigma_i > 0$, $i = 0, 1, \cdots, N$ such that

$$z^T z - \gamma^2 w^T w + \dot{V} - \sum_{i=0}^N \sigma_i (\|p_i\|^2 - \theta_i^2 \|q_i\|^2) \leq 0. \tag{51}$$

To show that the closed-loop system (45) has a robust $\mathcal{L}_2$-gain measure γ, we integrate (51) from 0 to ∞, with the initial condition $x(0) = 0$, and get

$$\int_0^\infty \left(z^T z - \gamma^2 w^T w + \dot{V} + \sum_{i=0}^N \sigma_i \left(\theta_i^2 \|q_i\|^2 - \|p_i\|^2 \right) \right) dt - V(x(\infty)) \leq 0. \tag{52}$$

If (51) hold, this implies (49) and (46). Therefore, we have robust $\mathcal{L}_2$-gain measure γ for the system (45). Now to secure (51), we define

$$\Theta = \begin{pmatrix} \theta_0 I & 0 & \cdots & 0 \\ 0 & \theta_1 I & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \theta_N I \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_0 I & 0 & \cdots & 0 \\ 0 & \sigma_1 I & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sigma_N I \end{pmatrix}, \tag{53}$$

where the identity matrix $I \in \mathbb{R}^{n_{q_i} \times n_{q_i}}$. It is noted that we require that $\theta_i > 0$ and $\sigma_i > 0$ for all i. Hence the inequality (51) can be translated to the following matrix inequalities

$$\Pi(X, \Sigma, \gamma) \prec 0, \tag{54}$$

where

$$\Pi(X, \Sigma, \gamma) = \begin{pmatrix} \Xi & XB_p & XB_w \\ \star & -\Sigma & 0 \\ \star & 0 & -\gamma^2 I \end{pmatrix}, \tag{55}$$

with $\Xi = \mathcal{A}^T X + X \mathcal{A} + \mathcal{C}^T \mathcal{C} + C_q^T \Theta^T \Sigma \Theta C_q$. Then the closed-loop system is said to have robust $\mathcal{L}_2$-gain measure γ from input w to output z if there exists $X > 0$ and $\Sigma > 0$ such that (54) is satisfied. Without loss of generality, we will adopt only strict inequality. To prove uniformly asymptotic stability of (45), we expand the inequality (54) by Schur complement,

$$\mathcal{A}^T X + X \mathcal{A} + \mathcal{C}^T \mathcal{C} + C_q^T \Theta^T \Sigma \Theta C_q + X(B_p \Sigma^{-1} B_p^T + \gamma^{-2} B_w B_w^T)X \prec 0. \tag{56}$$

Define the matrix variables

$$\mathcal{H} = \begin{pmatrix} \mathcal{C} \\ \Sigma^{1/2} \Theta C_q \end{pmatrix}, \quad \mathcal{G} = \begin{pmatrix} B_p \Sigma^{-1/2} & \gamma B_w \end{pmatrix}. \tag{57}$$

Thus, the inequality (56) can be rewritten as

$$\mathcal{A}^T X + X \mathcal{A} + \mathcal{H}^T \mathcal{H} + X \mathcal{G} \mathcal{G}^T X \prec 0. \tag{58}$$

Manipulating (58) by adding and subtracting $j\omega X$ to obtain

$$-(-j\omega I - \mathcal{A}^T)X - X(j\omega I - \mathcal{A}) + \mathcal{H}^T \mathcal{H} + X \mathcal{G} \mathcal{G}^T X \prec 0. \tag{59}$$

Pre-multiplying $\mathcal{G}^T(-j\omega I - \mathcal{A}^T)^{-1}$ and post-multiplying $(j\omega I - \mathcal{A})^{-1}\mathcal{G}$ to inequality (59), we have

$$\begin{aligned} -\mathcal{G}^T X(j\omega I - \mathcal{A})^{-1}\mathcal{G} - \mathcal{G}^T(-j\omega I - \mathcal{A}^T)^{-1}X\mathcal{G} \\ + \mathcal{G}^T(-j\omega I - \mathcal{A}^T)^{-1}X\mathcal{G}\mathcal{G}^T X(j\omega I - \mathcal{A})^{-1}\mathcal{G} \\ + \mathcal{G}^T(-j\omega I - \mathcal{A}^T)^{-1}\mathcal{H}^T \mathcal{H}(j\omega I - \mathcal{A})^{-1}\mathcal{G} \prec 0. \end{aligned} \tag{60}$$

Defining a system

$$\begin{aligned} \dot{x} &= \mathcal{A}x + \mathcal{G}w \\ z &= \mathcal{H}x \end{aligned} \tag{61}$$

with transfer function $T(s) = \mathcal{H}(sI - \mathcal{A})^{-1}\mathcal{G}$ and thus $T(j\omega) = \mathcal{H}(j\omega I - \mathcal{A})^{-1}\mathcal{G}$ and a matrix variable $\bar{M}(j\omega) = \mathcal{G}^T X(j\omega I - \mathcal{A})^{-1}\mathcal{G}$. The matrix inequality (60) can be rewritten as

$$T^*(j\omega)T(j\omega) - \bar{M}(j\omega) - \bar{M}^*(j\omega) + \bar{M}^*(j\omega)\bar{M}(j\omega) \prec 0,$$

or

$$\begin{aligned} T^*(j\omega)T(j\omega) &\prec \bar{M}(j\omega) + \bar{M}^*(j\omega) - \bar{M}^*(j\omega)\bar{M}(j\omega) \\ &= -(I - \bar{M}^*(j\omega))(I - \bar{M}(j\omega)) + I \\ &\preceq I, \quad \forall \ \omega \in \mathbb{R}. \end{aligned} \tag{62}$$

Hence, the maximum singular value of (62)

$$\sigma_{\max}(T(j\omega)) < 1, \quad \forall \ \omega \in \mathbb{R}.$$

By small gain theorem, we prove that the matrix $\mathcal{A}$ is Hurwitz, or equivalently, the eigenvalues of $\mathcal{A}$ are all in the left-half plane, and therefore the closed-loop system (45) is uniformly asymptotically stable.

Next to the end of the robust $\mathcal{L}_2$-gain measure γ is to synthesize the control law, K. Since (54) and (56) are equivalent, we multiply both sides of inequality of (56) by $Y = X^{-1}$. We have

$$Y\mathcal{A}^T + \mathcal{A}Y + Y\mathcal{C}^T\mathcal{C}Y + YC_q^T\Theta^T\Sigma\Theta C_q Y + B_p\Sigma^{-1}B_p^T + \gamma^{-2}B_w B_w^T \prec 0.$$

Rearranging the inequality with Schur complement and defining a matrix variable $W = KY$, we have

$$\begin{pmatrix} \Omega_L & YC_z^T + W^T D_z^T & YC_q^T\Theta^T & B_w \\ \star & -I & 0 & 0 \\ \star & 0 & -V & 0 \\ \star & 0 & 0 & -\gamma^2 I \end{pmatrix} < 0, \tag{63}$$

where $\Omega_L = YA^T + AY + W^T B^T + BW + B_p V B_p^T$ and $V = \Sigma^{-1}$. The matrix inequality is linear in matrix variables Y, W, V and a scalar γ, which can be solved efficiently.

Remark 2. *The matrix inequalities (63) are linear and can be transformed to optimization problem, for instance, if $\mathcal{L}_2$-gain measure γ is to be minimized:*

$$\begin{aligned} &\text{minimize } \gamma^2 \\ &\text{subject to (63), } Y \succ 0, \ V \succ 0 \ \text{and} \ W. \end{aligned} \tag{64}$$

Remark 3. *Once from (64) we obtain the matrices W and Y, the control law $K = WY^{-1}$ can be calculated easily.*

Remark 4. *It is seen from (61) that with Riccati inequality (56) a linear time-invariant system is obtained to fulfill $\|T\|_\infty < 1$, where $\mathcal{A}$ is Hurwitz.*

Remark 5. *In this remark, we will synthesize the overall control law consisting of $u_s(t)$ and $u_r(t)$ that perform control tasks. The overall control law as shown in (22) and in view of (15) and (44),*

$$u(t) = u_s(t) + u_r(x,t) = -\alpha(t)\frac{s(x,t)}{\|s(x,t)\|} + Kx(t) \tag{65}$$

where $\alpha(t) > 0$ satisfies (18), integral sliding surface, $s(x,t)$, is defined in (16) and gain K is found using optimization technique shown in (64).

4.2.2 Robust $\mathcal{H}_2$ measure

In this section we will study the $\mathcal{H}_2$ measure for the system performance of (45). The robust stability of which in the presence of norm-bounded uncertainty has been extensively studied Boyd et al. (1994) and reference therein. For self-contained purpose, we will demonstrate robust stability by using quadratic Lyapunov function (47) subject to (45) with the norm-bounded constraints satisfying (7) and (42). To guarantee the asymptotic stability with respect to (47) (or called *storage function* from dissipation perspective), we consider the a quadratic supply function

$$\int_0^\infty (w^T w - z^T z)dt, \tag{66}$$

and incorporate the quadratic norm-bounded constraints via Lagrange multipliers σ_i through S-procedure, it is then said that the system is dissipative if, and only if

$$\dot{V} + \sum_{i=0}^{N} \sigma_i(\theta_i^2\|q_i\|^2 - \|p_i\|^2) \leq w^T w - z^T z. \tag{67}$$

It is worth noting that the use of dissipation theory for (47), (69), and (67) is for the quantification of $\mathcal{H}_2$ performance measure in the sequel. It is also shown easily by plugging (45) into (67) that if there exist $X \succ 0$, $\Sigma \succ 0$, then (67) implies

$$\begin{pmatrix} \Omega_H & XB_p & XB_w \\ (XB_p)^T & -\Sigma & 0 \\ (XB_w)^T & 0 & -I \end{pmatrix} \prec 0, \tag{68}$$

where $\Omega_H = \mathcal{A}^T X + X\mathcal{A} + \mathcal{C}^T\mathcal{C} + C_q^T\Theta^T\Sigma\Theta C_q$ and Θ and Σ are defined exactly the same as (53). Then the system is robustly asymptotically stabilized with the norm-bounded uncertainty if (68) is satisfied. This is shown by the fact, Schur complement, that (68) is equivalent to

$$\Omega_H \prec 0 \tag{69}$$

$$\Omega_H + \begin{pmatrix} XB_p & XB_w \end{pmatrix}\begin{pmatrix} \Sigma & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} B_p^T X \\ B_w^T X \end{pmatrix} \prec 0 \tag{70}$$

If (69) and (70) are both true, then $\mathcal{A}^T X + X\mathcal{A} \prec 0$. This implies that $\mathcal{A}$ is Hurwitz. In addition to robust stability, the robust performance of the closed-loop uncertain system (45) on the sliding surface that fulfils the $\mathcal{H}_2$ performance requirement is suggested for the overall robust design in this section. We will show that the $\mathcal{H}_2$ performance measure will also guarantee using the inequality (68).

Given that the $\mathcal{A}$ is stable, the closed-loop map $T_{zw}(g_i(q_i,t))$ from w to z is bounded for all nonlinearities and uncertainties $g_i(q_i,t)$; we wish to impose an $\mathcal{H}_2$ performance specification on this map. Consider first the nominal map $T_{zw0} = T_{zw}(0)$, this norm is given by

$$\|T_{zw0}\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}(T_{zw0}(j\omega)^* T_{zw0}(j\omega))d\omega \tag{71}$$

This criterion is classically interpreted as a measure of transient response to an impulse applied to $w(t)$ and it gives the bound of output energy of z. The approach of $\mathcal{H}_2$ performance criterion as the evaluation of the energy response to an impulse input of random direction at $w(t)$ is

$$\|T_{zw}(\Delta)\|_{2,imp}^2 \triangleq \mathbb{E}_{w_0}(\|z\|_2^2), \tag{72}$$

where $z(t) = T_{zw}(g_i(q_i,t))w_0\delta(t)$, and w_0 satisfies random vector of covariance $\mathbb{E}(w_0 w_0') = I$. The above definition of $\mathcal{H}_2$ performance can also be equivalently interpreted by letting the initial condition $x(0) = B_w w_0$ and $w(t) = 0$ in the system, which subsequently responds autonomously. Although this definition is applied to the case where $g_i(x,t)$ is LTI and standard notion of (71), we can also apply it to a more general perturbation structure, nonlinear or time-varying uncertainties. Now to evaluate the energy bound of (72), consider first the index $J(x_0)$ defined to be

$$J(x_0) = \sup_{x(0)=x_0} \|z\|^2 \tag{73}$$

The next step is to bound $J(x_0)$ by an application of so-called S-procedure where quadratic constraints are incorporated into the cost function (73) via Lagrange Multipliers σ_i. This leads to

$$J(x_0) \leq \inf_{\sigma_i > 0} \sup_{x_0} \left(\|z\|^2 + \sum_{i=0}^{i=1} \sigma_i(\theta_i^2 \|q_i\|^2 - \|p_i\|^2) \right) \tag{74}$$

To compute the right hand side of (74), we find that for fixed σ_i we have an optimization problem,

$$\sup_{x(0)=x_0,(45)} \int_0^\infty \left(z^T z + q^T \Theta^T \Sigma \Theta q - p^T \Sigma p \right) dt. \tag{75}$$

To compute the optimal bound of (75) for some $\Sigma \succ 0$ satisfying (68), the problem (75) can be rewritten as

$$J(x_0) \leq \int_0^\infty \left(z^T z + q^T \Theta^T \Sigma \Theta q - p^T \Sigma p + \frac{d}{dt} V(x) \right) dt + V(x_0), \tag{76}$$

for $x(\infty) = 0$. When (68) is satisfied, then it is equivalent to

$$\begin{pmatrix} x^T & p^T & w^T \end{pmatrix} \begin{pmatrix} \Omega & XB_p & XB_w \\ (XB_p)^T & -\Sigma & 0 \\ (XB_w)^T & 0 & -I \end{pmatrix} \begin{pmatrix} x \\ p \\ w \end{pmatrix} < 0, \tag{77}$$

or,

$$\begin{pmatrix} x^T & p^T & w^T \end{pmatrix} \begin{pmatrix} \Omega & XB_p & XB_w \\ (XB_p)^T & -\Sigma & 0 \\ (XB_w)^T & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ p \\ w \end{pmatrix} < w^T w. \tag{78}$$

With (78), we find that the problem of performance $J(x_0)$ of (76) is

$$J(x_0) \leq \int_0^\infty w^T w \, dt + V(x_0). \tag{79}$$

It is noted that the matrix inequality (68) is jointly affine in Σ and X. Thus, we have the index

$$J(x_0) \leq \inf_{X \succ 0, \Sigma \succ 0, (77)} x_0^T X x_0, \tag{80}$$

for the alternative definition of robust $\mathcal{H}_2$ performance measure of (71), where $w(t) = 0$ and $x_0 = B_w w_0$. Now the final step to evaluate the infimum of (80) is to average over each impulsive direction, we have

$$\sup_{g_i(q_i,t)} \mathbb{E}_{w_0} \|z\|_2^2 \leq \mathbb{E}_{w_0} J(x_0) \leq \inf_X \mathbb{E}_{w_0} (x_0^T X x_0) = \inf_X \text{Tr}(B_w^T X B_w).$$

Thus the robust performance design specification is that

$$\text{Tr}(B_w^T X B_w) \leq \vartheta^2 \tag{81}$$

for some $\vartheta > 0$ subject to (77). In summary, the overall robust $\mathcal{H}_2$ performance control problem is the following convex optimization problem:

$$\begin{aligned} &\text{minimize } \vartheta^2 \\ &\text{subject to (81), (68), } X \succ 0, \ \Sigma \succ 0. \end{aligned} \tag{82}$$

Next to the end of the robust $\mathcal{H}_2$ measure is to synthesize the control law, K. Since (68) and (70) are equivalent, we multiply both sides of inequality of (70) by $Y = X^{-1}$. We have

$$Y\mathcal{A}^T + \mathcal{A}Y + Y\mathcal{C}^T\mathcal{C}Y + YC_q^T\Theta^T\Sigma\Theta C_q Y + B_p\Sigma^{-1}B_p^T + B_w B_w^T \prec 0.$$

Rearranging the inequality with Schur complement and defining a matrix variable $W = KY$, we have

$$\begin{pmatrix} \Omega & YC_z^T + W^T D_z^T & YC_q^T\Theta^T & B_w \\ \star & -I & 0 & 0 \\ \star & 0 & -V & 0 \\ \star & 0 & 0 & -I \end{pmatrix} < 0, \tag{83}$$

where $\Omega = YA^T + AY + W^T B^T + BW + B_p V B_p^T$ and $V = \Sigma^{-1}$. The matrix inequality is linear in matrix variables Y, W, and V, which can be solved efficiently.

Remark 6. *The trace of (81) is to put in a convenient form by introducing the auxiliary matrix U as*

$$U \succ B_w^T X B_w$$

or, equivalently,

$$\begin{pmatrix} U & B_w^T \\ B_w & X^{-1} \end{pmatrix} = \begin{pmatrix} U & B_w^T \\ B_w & Y \end{pmatrix} \succ 0. \tag{84}$$

Remark 7. *The matrix inequalities (83) are linear and can be transformed to optimization problem, for instance, if robust $\mathcal{H}_2$ measure is to be minimized:*

$$\begin{aligned} &\text{minimize } \; \vartheta^2 \\ &\text{subject to } \; (83), (84), \; \mathbf{Tr}(U) \le \vartheta^2, \; Y \succ 0, \; V \succ 0 \; and \; W. \end{aligned} \tag{85}$$

Remark 8. *Once from (85) we obtain the matrices W and Y, the control law $K = WY^{-1}$ can be calculated easily.*

Remark 9. *To perform the robust $\mathcal{H}_2$ measure control, the overall composite control of form (65) should be established, where the continuous control gain K is found by using optimization technique shown in (85).*

5. Numerical example

A numerical example to verify the integral sliding-mode-based control with $\mathcal{L}_2$-gain measure and $\mathcal{H}_2$ performance establishes the solid effectiveness of the whole chapter. Consider the system of states, x_1 and x_2, with nonlinear functions and matrices:

$$A(t) = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} 1.4 \\ -2.3 \end{pmatrix} 0.8\sin(\omega_0 t)\,(-0.1\;0.3)\,, \quad B(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}(1 + 0.7\sin(\omega_1 t)) \tag{86}$$

$$B_d = \begin{pmatrix} 0.04 \\ 0.5 \end{pmatrix}, \; g_1(x,t) = x_1, \; g_2(x,t) = x_2, \text{ and } g_1(x,t) + g_2(x,t) \le 1.01(\|x_1\| + \|x_2\|) \tag{87}$$

$$h(x) = 2.1(x_1^2 + x_2^2) \le \eta(x) = 2.11(x_1^2 + x_2^2), \text{ and } w(t) = \varepsilon(t-1) + \varepsilon(t-3), \tag{88}$$

where the necessary parameter matrices and functions can be easily obtained by comparison (86), (87), and (88) with assumption 1 through 5, thus we have

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \; E_0 = \begin{pmatrix} 1.4 \\ -2.3 \end{pmatrix}, \; H_0 = (-0.1 \; 0.03), \; B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, H_1 = 0.7, \; \theta_1 = \theta_2 = 1.01.$$

It should be noted that $\varepsilon(t - t_1)$ denotes the pulse centered at time t_1 with pulse width 1 sec and strength 1. So, it is easy to conclude that $\bar{w} = 1$. We now develop the integral sliding-mode such that the system will be driven to the designated sliding surface $s(x, t)$ shown in (16). Consider the initial states, $x_1(0) = -0.3$ and $x_2(0) = 1.21$, thus, the ball, $\mathcal{B}$, is confined within $\kappa = 1.2466$. The matrix M such that $MB = I$ is $M = (0 \; 1)$, hence, $\|M\| = 1$, $\|ME_0\| = 2.3$, and $\|MB_d\| = 0.5$. To compute switching control gain $\alpha(t)$ of sliding-mode control in (18), we need (19), which $\beta_0 = 5.8853$. We then have

$$\alpha(t) = \frac{1}{0.3}(5.8853 + \lambda + 3.587(x_1^2 + x_2^2) + 0.7\|u_r\|), \tag{89}$$

where λ is chosen to be any positive number and $u_r = Kx$ is the linear control law to achieve performance measure. It is noted that in (89) the factor $\frac{1}{0.3}$ will now be replace by a control factor, α_1, which the approaching speed of sliding surface can be adjusted. Therefore, the (89) is now

$$\alpha(t) = \alpha_1(5.8853 + \lambda + 3.587(x_1^2 + x_2^2) + 0.7\|u_r\|). \tag{90}$$

It is seen later that the values of α_1 is related to how fast the system approaches the sliding surface, $s = 0$ for a fixed number of $\lambda = 0$.

To find the linear control gain, K, for performance $\mathcal{L}_2$-gain measure, we follow the computation algorithm outlined in (64) and the parametric matrices of (41) are as follows,

$$G = I - BM = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; B_w = GB_d = \begin{pmatrix} 0.04 \\ 0 \end{pmatrix}, \; B_p = G(E_0 \; I \; I) = \begin{pmatrix} 1.4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C_q = \begin{pmatrix} -0.1 & 0.03 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \; C_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; D_z = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The simulated results of closed-loop system for integral sliding-mode with $\mathcal{L}_2$-gain measure are shown in Fig.1, Fig.2, and Fig.3 under the adjust factor $\alpha_1 = 0.022$ in (90). The linear control gain $K = [-18.1714 \quad -10.7033]$, which makes the eigenvalues of $(A + BK)$ being $-4.3517 \pm 0.4841j$. It is seen in Fig.1(b) that the sliding surface starting from $s = 0$ at $t = 0$, which matches the sliding surface design. Once the system started, the values of s deviate rapidly from the sliding surface due to the integral part within it. Nevertheless, the feedback control signals soon drive the trajectories of s approaching $s = 0$ and at time about $t = 2.63$ the values of s hit the sliding surface, $s = 0$. After that, to maintain the sliding surface the sliding control u_s starts chattering in view of Fig.2(b). When looking at the Fig.2(a) and (b), we see that the sliding-mode control, u_s, dominates the feedback control action that the system is pulling to the sliding surface. We also note that although the system is pulling to the sliding surface, the states x_2 has not yet reached its equilibrium, which can be seen from Fig.1(a). Not until the sliding surface reaches, do the states asymptotically drive to their equilibrium. Fig.3 is the phase plot of states of x_1 and x_2 and depicts the same phenomenon. To show different
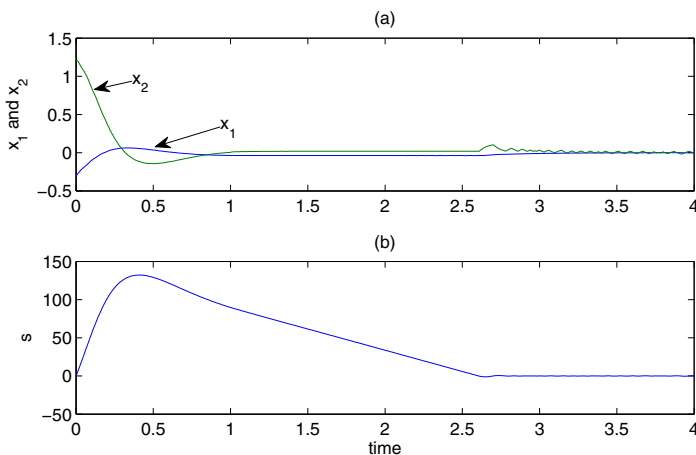
Fig. 1. Integral sliding-mode-based robust control with $\mathcal{L}_2$-gain measure (a) the closed-loop states - x_1 and x_2, (b) the chattering phenomenon of sliding surface $s(x, t)$. $\alpha_1 = 0.022$.
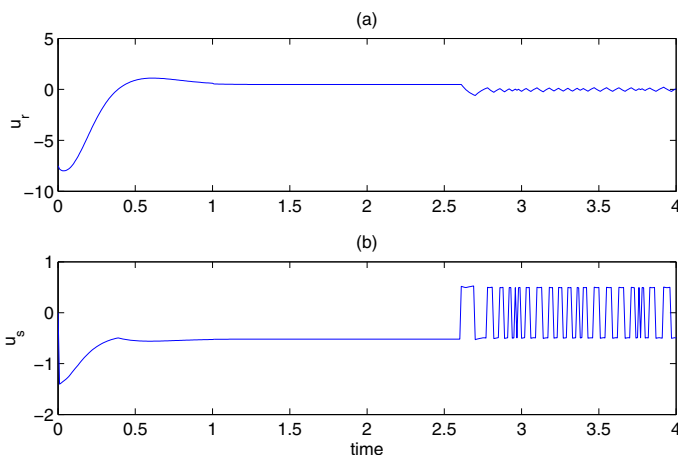

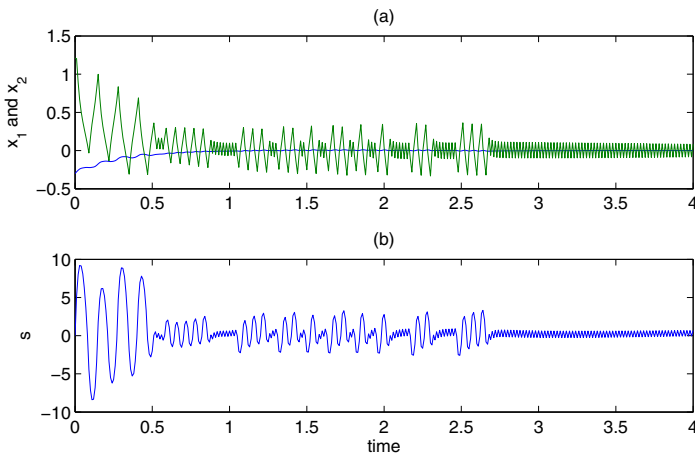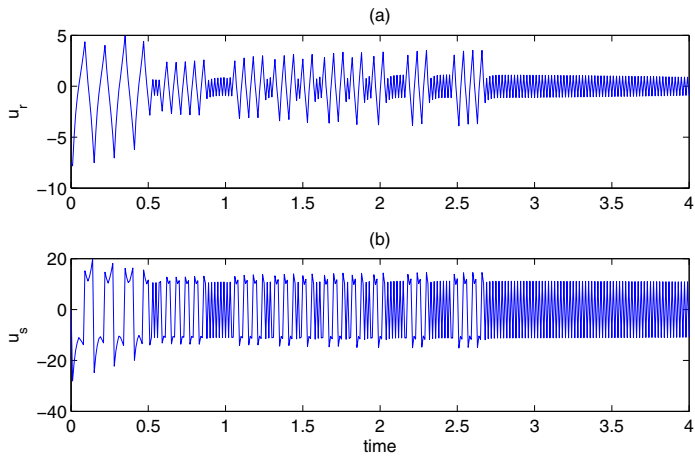Fig. 2. The control signals of (a) linear robust control, u_r, (b) integral sliding-mode control, u_s of integral sliding-mode-based robust control with $\mathcal{L}_2$-gain measure. $\alpha_1 = 0.022$.

approaching speed due to control factor $\alpha_1 = 0.5$, we see chattering phenomenon in the Fig.4, Fig.5, and Fig.6. This is because of inherent property of sliding-mode control. We will draw the same conclusions as for the case $\alpha_1 = 0.022$ with one extra comment that is we see the
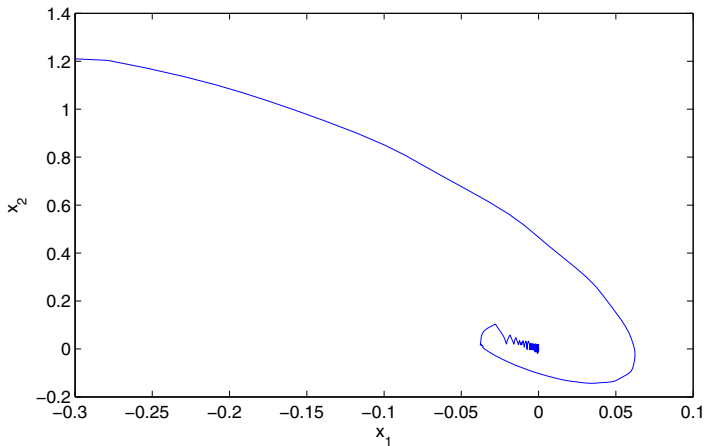
Fig. 3. The phase plot of state x_1 and x_2 of integral sliding-mode-based robust control with $\mathcal{L}_2$-gain measure. $\alpha_1 = 0.022$.

trajectory of state x_1 is always smoother that of x_2. The reason for this is because the state x_1 is the integration of the state x_2, which makes the smoother trajectory possible.

Next, we will show the integral sliding-mode-based control with $\mathcal{H}_2$ performance. The integral sliding-mode control, u_s is exactly the same as previous paragraph. The linear control part satisfying (85) will now be used to find the linear control gain K. The gain K computed is $K = [-4.4586 - 5.7791]$, which makes eigenvalues of $(A + BK)$ being $-1.8895 \pm 1.3741j$. From Fig.7, Fig.8, and Fig.9, we may draw the same conclusions as Fig.1 to Fig.6 do. We should be aware that the $\mathcal{H}_2$ provides closed-loop poles closer to the imaginary axis than $\mathcal{L}_2$-gain case, which slower the overall motion to the states equilibrium.

Fig. 4. Integral sliding-mode-based robust control with $\mathcal{L}_2$-gain measure (a) the closed-loop states - x_1 and x_2, (b) the chattering phenomenon of sliding surface $s(x, t)$. $\alpha_1 = 0.5$.

Fig. 5. The control signals of (a) linear robust control, u_r, (b) integral sliding-mode control, u_s of integral sliding-mode-based robust control with $\mathcal{L}_2$-gain measure. $\alpha_1 = 0.5$.
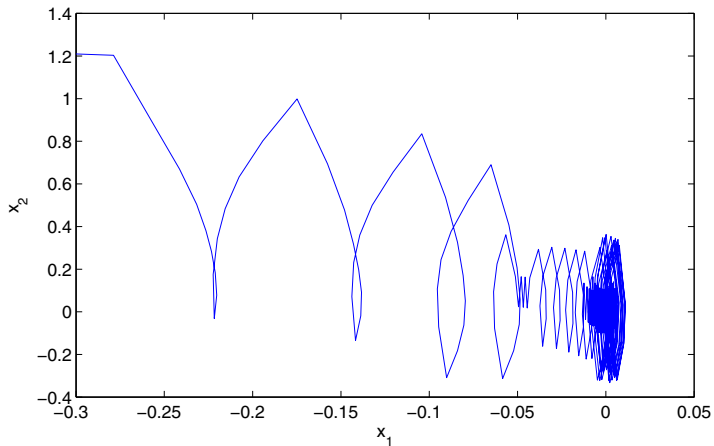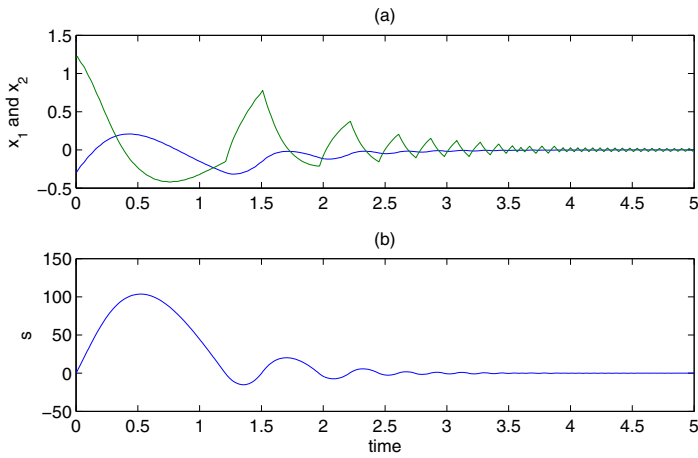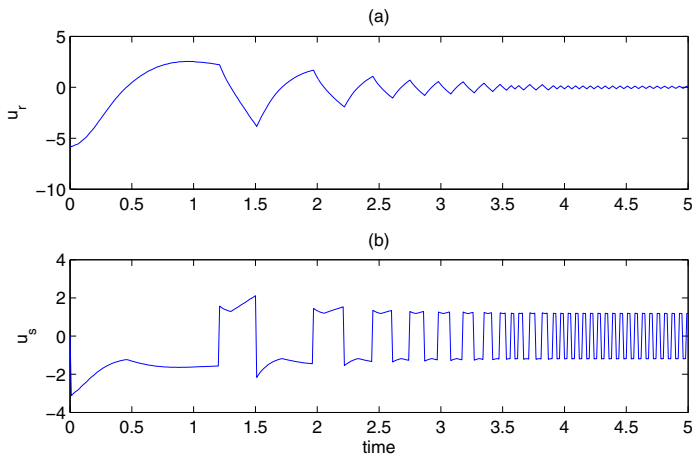

Fig. 6. The phase plot of state x_1 and x_2 of integral sliding-mode-based robust control with $\mathcal{L}_2$-gain measure. $\alpha_1 = 0.5$.

Fig. 7. Integral sliding-mode-based robust control with $\mathcal{H}_2$ performance (a) the closed-loop states - x_1 and x_2, (b) the chattering phenomenon of sliding surface $s(x, t)$. $\alpha_1 = 0.06$.

Fig. 8. The control signals of (a) linear robust control, u_r, (b) integral sliding-mode control, u_s of integral sliding-mode-based robust control with $\mathcal{H}_2$ performance. $\alpha_1 = 0.06$.
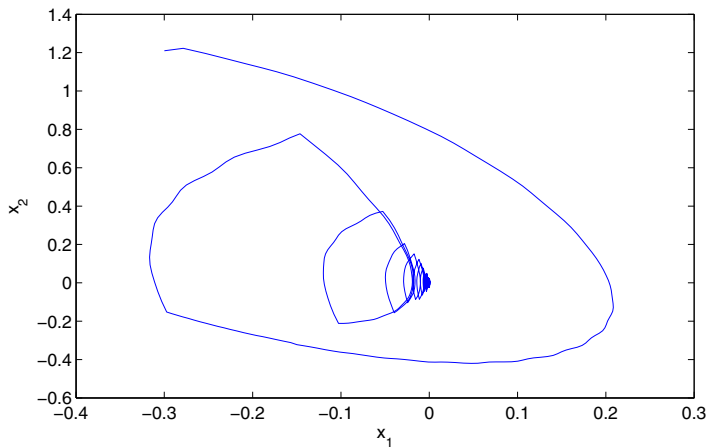
Fig. 9. The phase plot of state x_1 and x_2 of integral sliding-mode-based robust control with $\mathcal{H}_2$ performance. $\alpha_1 = 0.06$.

6. Conclusion

In this chapter we have successfully developed the robust control for a class of uncertain systems based-on integral sliding-mode control in the presence of nonlinearities, external disturbances, and model uncertainties. Based-on the integral sliding-mode control where reaching phase of conventional sliding-mode control is eliminated, the matched-type nonlinearities and uncertainties have been nullified and the system is driven to the sliding surface where sliding dynamics with unmatched-type nonlinearities and uncertainties will further be compensated for resulting equilibrium. Integral sliding-mode control drives the system maintaining the sliding surface with $\mathcal{L}_2$-gain bound while treating the sliding surface as zero dynamics. Once reaching the sliding surface where $s = 0$, the robust performance control for controlled variable z in terms of $\mathcal{L}_2$-gain and $\mathcal{H}_2$ measure with respect to disturbance, w, acts to further compensate the system and leads the system to equilibrium. The overall design effectiveness is implemented on a second-order system which proves the successful design of the methods. Of course, there are issues which can still be pursued such as we are aware that the control algorithms, say integral sliding-mode and $\mathcal{L}_2$-gain measure, apply separate stability criterion that is integral sliding-mode has its own stability perspective from Lyapunov function of integral sliding-surface while $\mathcal{L}_2$-gain measure also has its own too, the question is: is it possible produce two different control vectors that jeopardize the overall stability? This is the next issue to be developed.

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Edited by Dr. Andreas Mueller

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