# Integral transforms characterized by convolution 

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#### Abstract

Inspired by Jaming's characterization of the Fourier transform on specific groups via the convolution property, we provide a novel approach which characterizes the Fourier transform on any locally compact abelian group. In particular, our characterization encompasses Jaming's results. Furthermore, we demonstrate that the cosine transform as well as the Laplace transform can also be characterized via a suitable convolution property.


Keywords : Fourier transform, cosine transform, Laplace transform, convolution property
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## 1 Introduction

It is well-known that the Fourier transform satisfies the convolution property. More precisely, let $G$ be a locally compact abelian group with a Haar measure $\mu$ and a dual group $\widehat{G}$. The Fourier transform is a $\operatorname{map} \mathcal{F}: L^{1}(G) \longrightarrow C_{0}(\widehat{G})$ given by the formula

$$
\begin{equation*}
\forall_{f \in L^{1}(G)}^{\chi \in \bar{G}} \mathfrak{\mathcal { F }} \mathcal{F}(f)(\chi):=\int_{G} f(x) \overline{\chi(x)} d \mu(x), \tag{1}
\end{equation*}
$$

whereas the Fourier convolution $\star_{\mathcal{F}}: L^{1}(G) \times L^{1}(G) \longrightarrow L^{1}(G)$ is given by

$$
\begin{equation*}
\left.\forall_{f, g \in L^{1}(G)}^{x \in G}\right\} \tag{2}
\end{equation*}
$$

For the sake of convenience, from this point onwards we will write $d x$ and $d u$ instead of " $d \mu(x)$ " and " $d \mu(u)$ ", respectively. The following Fourier convolution property (see Lemma 1.7.2 in [4, p. 30) holds:

$$
\forall_{f, g \in L^{1}(G)} \mathcal{F}\left(f \star_{\mathcal{F}} g\right)=\mathcal{F}(f) \mathcal{F}(g) .
$$

Jaming proved that the convolution property characterizes (to a certain degree) the Fourier transform if $G=\mathbb{R}, S^{1}, \mathbb{Z}$ or $\mathbb{Z}_{n}$ (see [6]). His paper inspired Lavanya and Thangavelu to show that any continuous *homomorphism of $L^{1}\left(\mathbb{C}^{d}\right)$ (with twisted convolution as multiplication) into $B\left(L^{2}\left(\mathbb{R}^{d}\right)\right)$ is essentially a Weyl transform and deduce a similar characterization for the group Fourier transform on the Heisenberg group (see 9 and [10). Furthermore, Kumar and Sivananthan went on to demonstrate that the convolution property characterizes the Fourier transform on compact groups (see [7), while Alesker, Artstein-Avidan, Faifman and Milman studied the Fourier transform in terms of product preserving maps (see [1).

Studying the topic we realized two things that prompted us to write this paper. Firstly, Jaming's characterization of the Fourier transform need not be restricted to particular cases $G=\mathbb{R}, S^{1}, \mathbb{Z}$ or $\mathbb{Z}_{n}$. There is a unified approach for all locally compact abelian groups, which we demonstrate in the first part of Section 2

Secondly, we discovered that the Fourier transform is not the only one that can be characterized via the convolution property. In the second part of Section 2 and in Section 3 we explain that cosine and Laplace convolutions characterize cosine and Laplace transforms, respectively.

## 2 Fourier and cosine transform

As we have agreed in the Introduction, let $G$ stand for a locally compact abelian group with Haar measure $\mu$ and dual group $\widehat{G}$. The formula for the Fourier transform and Fourier convolution are given by (11) and 2 respectively.

Lemma 1. For every function $g \in L^{1}(G)$ and $x, y \in G$ the following equality holds

$$
L_{x} g \star_{\mathcal{F}} L_{y} g=g \star_{\mathcal{F}} L_{x+y} g,
$$

where for every $z \in G$ the operator $L_{z}: L^{1}(G) \longrightarrow L^{1}(G)$ is given by

$$
\forall_{u \in G} L_{z} f(u):=f(u-z) .
$$

Proof. For every $u \in G$ we have

$$
\begin{aligned}
L_{x} g \star_{\mathcal{F}} L_{y} g(u) & =\int_{G} L_{x} g(v) L_{y} g(u-v) d v=\int_{G} g(v-x) g(u-v-y) d v \\
& \stackrel{v \mapsto v+x}{=} \int_{G} g(v) g(u-v-(x+y)) d v=\int_{G} g(v) L_{x+y} g(u-v) d v=g \star_{\mathcal{F}} L_{x+y} g(u),
\end{aligned}
$$

which completes the proof.
The following theorem generalizes Theorems 2.1 and 3.1 in Jaming's paper (see [6]).
Theorem 2. Let $T: L^{1}(G) \longrightarrow L^{\infty}(\widehat{G})$ be a linear and bounded operator. If it satisfies the Fourier convolution property

$$
\forall_{f, g \in L^{1}(G)} T\left(f \star_{\mathcal{F}} g\right)=T(f) T(g)
$$

then there exists a function $\theta_{\mathcal{F}}: \widehat{G} \longrightarrow \widehat{G} \cup\{0\}$ such that

$$
\begin{equation*}
\forall_{f \in L^{1}(G)} T(f)=\mathcal{F}(f) \circ \theta_{\mathcal{F}} \tag{3}
\end{equation*}
$$

where $\theta_{\mathcal{F}}(\phi)=0$ if and only if $T(f)(\phi)=0$ for every $f \in L^{1}(G)$.

Proof. We pick $\phi \in \widehat{G}$. If $T(f)(\phi)=0$ for every $f \in L^{1}(G)$, then we define $\theta_{\mathcal{F}}(\phi):=0$ so that the equality (3) holds. Suppose then that $\phi$ is such that $T(f)(\phi) \neq 0$ for some $f \in L^{1}(G)$. We consider a nonzero, linear functional $T_{\phi}: L^{1}(G) \longrightarrow \mathbb{C}$ given by $T_{\phi}(f):=T(f)(\phi)$. We pick $g_{*} \in L^{1}(G)$ such that $T_{\phi}\left(g_{*}\right)=1$ and define a function $\chi_{\phi}: G \longrightarrow \mathbb{C}$ by the formula

$$
\chi_{\phi}(x):=\overline{T_{\phi}\left(L_{x} g_{*}\right)} .
$$

Let us remark, that the choice of $g_{*}$ need not be unique, so there might be many functions $\chi_{\phi}$ corresponding to $\phi$.

We will now focus on proving various properties of the function $\chi_{\phi}$. To begin with, we observe that $\forall_{x, y \in G} T_{\phi}\left(L_{x} g_{*}\right) T_{\phi}\left(L_{y} g_{*}\right)=T_{\phi}\left(L_{x} g_{* \star \mathcal{F}} L_{y} g_{*}\right) \stackrel{\text { Lemma }}{=} T_{\phi}\left(g_{* \star}{ }_{\mathcal{F}} L_{x+y} g_{*}\right)=T_{\phi}\left(g_{*}\right) T_{\phi}\left(L_{x+y} g_{*}\right)=T_{\phi}\left(L_{x+y} g_{*}\right)$.
Taking the complex conjugate reveals the equation

$$
\forall_{x, y \in G} \chi_{\phi}(x) \chi_{\phi}(y)=\chi_{\phi}(x+y) .
$$

Furthermore, since $T_{\phi}$ is a continuous linear functional, then Lemma 1.4.2 in [4], p. 18 implies that $\chi_{\phi}$ is continuous. It is also nonzero (as $\chi_{\phi}(0)=1$ ) and bounded, since

$$
\begin{equation*}
\forall_{x \in G}\left|\chi_{\phi}(x)\right| \leqslant\left|T_{\phi}\left(L_{x} g_{*}\right)\right| \leqslant\left\|T_{\phi}\right\| \cdot\left\|L_{x} g_{*}\right\|=\left\|T_{\phi}\right\| \cdot\left\|g_{*}\right\| \tag{4}
\end{equation*}
$$

Finally we argue that $\left|\chi_{\phi}(x)\right|=1$ for every $x \in G$. Indeed, suppose there exists $\bar{x} \in G$ such that $\left|\chi_{\phi}(\bar{x})\right| \neq 1$. Since

$$
1=\left|\chi_{\phi}(0)\right|=\left|\chi_{\phi}(\bar{x}-\bar{x})\right|=\left|\chi_{\phi}(\bar{x})\right|\left|\chi_{\phi}(-\bar{x})\right|
$$

then either $\left|\chi_{\phi}(\bar{x})\right|>1$ or $\left|\chi_{\phi}(-\bar{x})\right|>1$. Without loss of generality, we may assume that the former is true. Consequently, we have

$$
\lim _{n \rightarrow \infty}\left|\chi_{\phi}(n \bar{x})\right|=\left|\chi_{\phi}(\bar{x})\right|^{n}=\infty
$$

which contradicts boundedness of $\chi_{\phi}$. Hence, we conclude that $\left|\chi_{\phi}(x)\right|=1$ for every $x \in G$. This means that $\chi_{\phi} \in \widehat{G}$.

Finally, using Lemma 11.45 in [2], p. 427 (or Proposition 7 in [5], p. 123) we compute

$$
\begin{aligned}
\int_{G} f(x) \overline{\chi_{\phi}(x)} d x & =\int_{G} f(x) T_{\phi}\left(L_{x} g_{*}\right) d x=T_{\phi}\left(\int_{G} f(x) L_{x} g_{*} d x\right) \\
& =T_{\phi}\left(f \star_{\mathcal{F}} g_{*}\right)=T_{\phi}(f) T_{\phi}\left(g_{*}\right)=T_{\phi}(f)
\end{aligned}
$$

which concludes the proof if we put $\theta_{\mathcal{F}}(\phi):=\chi_{\phi}$.
We move on to show that the cosine transform admits a similar convolution characterization. The cosine transform is a map $\mathcal{C}: L^{1}(G) \longrightarrow C_{0}(\widehat{G})$ given by the formula

$$
\begin{equation*}
\underset{\substack{f, g \in L^{1}(G) \\ \chi \in \mathcal{C O S}(G)}}{ } \mathcal{C}(f)(\chi):=\int_{G} f(x) \chi(x) d x \tag{5}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{C O S}(G):=\left\{\chi \in C^{b}(G): \chi \neq 0, \forall_{x, y \in G} \chi(x) \chi(y)=\frac{\chi(x+y)+\chi(x-y)}{2}\right\} . \tag{6}
\end{equation*}
$$

Functions in $\mathcal{C O S}(G)$ are called cosine functions on group $G$, since for $G=\mathbb{R}$ we have

$$
\mathcal{C O S}(G)=\{x \mapsto \cos (y x): y \in \mathbb{R}\}
$$

The cosine convolution $\star_{\mathcal{C}}: L^{1}(G) \times L^{1}(G) \longrightarrow L^{1}(G)$ is given by

$$
\begin{equation*}
\forall_{x \in G} f \star_{\mathcal{C}} g(x):=\int_{G} f(u) \cdot \frac{g(x+u)+g(x-u)}{2} d u \tag{7}
\end{equation*}
$$

Let us recall a property of the cosine convolution which will be crucial in the sequel (see Theorem 2 in [8]):

Lemma 3. Let $x, y \in G$. If $g \in L^{1}(G)$ is an even function, then

$$
\begin{equation*}
L_{y} g \star_{\mathcal{C}} L_{x} g=g \star_{\mathcal{C}} \frac{L_{x+y} g+L_{x-y} g}{2} \tag{8}
\end{equation*}
$$

Before we present a characterization of the cosine transform we need one more technical result:
Lemma 4. Let $T: L^{1}(G) \longrightarrow L^{\infty}(\mathcal{C O S}(G))$ be a linear and bounded operator, which satisfies the cosine convolution property

$$
\begin{equation*}
\forall_{f, g \in L^{1}(G)} T\left(f \star_{\mathcal{C}} g\right)=T(f) T(g) \tag{9}
\end{equation*}
$$

If $\phi \in \mathcal{C O S}(G)$ is such that $T\left(f_{*}\right)(\phi) \neq 0$ for some $f_{*} \in L^{1}(G)$, then there exists an even function $g_{*} \in L^{1}(G)$ such that $T\left(g_{*}\right)(\phi) \neq 0$.

Proof. Let $\iota: G \longrightarrow G$ be the inverse function $\iota(x):=-x$ and let $T_{\phi}: L^{1}(G) \longrightarrow \mathbb{C}$ be a nonzero, linear functional given by $T_{\phi}(f):=T(f)(\phi)$. Then

$$
\begin{aligned}
& \forall_{f \in L^{1}(G)}(f \circ \iota) \star_{\mathcal{C}} f(x)=\int_{G} f \circ \iota(u) \cdot \frac{f(x+u)+f(x-u)}{2} d u \\
& \stackrel{u \mapsto-u}{=} \int_{G} f(u) \cdot \frac{f(x-u)+f(x+u)}{2} d u=f \star_{\mathcal{C}} f
\end{aligned}
$$

which leads to

$$
\forall_{f \in L^{1}(G)} T_{\phi}(f \circ \iota) T_{\phi}(f)=T_{\phi}\left((f \circ \iota) \star_{\mathcal{C}} f\right)=T_{\phi}\left(f \star_{\mathcal{C}} f\right)=T_{\phi}(f)^{2}
$$

Consequently, we have $T_{\phi}\left(f_{*} \circ \iota\right)=T_{\phi}\left(f_{*}\right)$. Finally, we put $g_{*}:=f_{*}+f_{*} \circ \iota$ and observe that

$$
T_{\phi}\left(g_{*}\right)=T_{\phi}\left(f_{*}+f_{*} \circ \iota\right)=T_{\phi}\left(f_{*}\right)+T_{\phi}\left(f_{*} \circ \iota\right)=2 T_{\phi}\left(f_{*}\right) \neq 0
$$

which concludes the proof.
We are now ready to demonstrate a counterpart of Theorem 2 for the cosine transform:

Theorem 5. Let $T: L^{1}(G) \longrightarrow L^{\infty}(\mathcal{C O S}(G))$ be a linear and bounded operator. If it satisfies the cosine convolution property (19), then there exists a function $\theta_{\mathcal{C}}: \mathcal{C O S}(G) \longrightarrow \mathcal{C O S}(G) \cup\{0\}$ such that

$$
\begin{equation*}
\forall_{f \in L^{1}(G)} T(f)=\mathcal{C}(f) \circ \theta_{\mathcal{C}} \tag{10}
\end{equation*}
$$

where $\theta_{\mathcal{C}}(\phi)=0$ if and only if $T(f)(\phi)=0$ for every $f \in L^{1}(G)$.
Proof. The proof follows the same lines as the proof of Theorem 2, We pick $\phi \in \mathcal{C O S}(G)$ and check if $T(f)(\phi)=0$ for every $f \in L^{1}(G)$. If so, we define $\theta_{\mathcal{C}}(\phi):=0$. Otherwise, if $T(f)(\phi) \neq 0$ for some $f \in L^{1}(G)$, then we consider a nonzero, linear functional $T_{\phi}: L^{1}(G) \longrightarrow \mathbb{C}$ given by $T_{\phi}(f):=T(f)(\phi)$. By Lemma 4 there exists an even function $g_{*} \in L^{1}(G)$ such that $T_{\phi}\left(g_{*}\right)=1$. We may thus define a function $\chi_{\phi}: G \longrightarrow \mathbb{C}$ by the formula

$$
\chi_{\phi}(x):=T_{\phi}\left(L_{x} g_{*}\right)
$$

As in Theorem 2 we study the properties of $\chi_{\phi}$ and see that

- it is nonzero due to $\chi_{\phi}(0)=1$,
- it is bounded due to (4),
- it is continuous due to Lemma 1.4.2 in [4], p. 18.

Furthermore, we have

$$
\begin{aligned}
\forall_{x, y \in G} T_{\phi}\left(L_{x} g_{*}\right) T_{\phi}\left(L_{y} g_{*}\right) & =T_{\phi}\left(L_{x} g_{*} \star_{C} L_{y} g_{*}\right) \stackrel{\mathrm{Lemma}}{=} T_{\phi}\left(g_{*} \star_{\mathcal{C}} \frac{L_{x+y} g_{*}+L_{x-y} g_{*}}{2}\right) \\
& =T_{\phi}\left(g_{*}\right) \cdot \frac{T_{\phi}\left(L_{x+y} g_{*}\right)+T_{\phi}\left(L_{x-y} g_{*}\right)}{2}=\frac{T_{\phi}\left(L_{x+y} g_{*}\right)+T_{\phi}\left(L_{x-y} g_{*}\right)}{2}
\end{aligned}
$$

which can be written as

$$
\forall_{x, y \in G} \chi_{\phi}(x) \chi_{\phi}(y)=\frac{\chi_{\phi}(x+y)+\chi_{\phi}(x-y)}{2} .
$$

We conclude that $\chi_{\phi} \in \mathcal{C O S}(G)$, which in particular means that

$$
\begin{equation*}
\forall_{y \in G} \chi_{\phi}(y)=\frac{\chi_{\phi}(y)+\chi_{\phi}(-y)}{2} \tag{11}
\end{equation*}
$$

Finally, using Lemma 11.45 in [2], p. 427 (or Proposition 7 in [5], p. 123) we compute

$$
\begin{aligned}
\int_{G} f(x) \chi_{\phi}(x) d x & \stackrel{(11)}{=} \int_{G} f(x) \cdot \frac{\chi_{\phi}(x)+\chi_{\phi}(-x)}{2} d x=\int_{G} f(x) \cdot \frac{T_{\phi}\left(L_{x} g_{*}\right)+T_{\phi}\left(L_{-x} g_{*}\right)}{2} d x \\
& =T_{\phi}\left(\int_{G} f(x) \cdot \frac{L_{x} g_{*}+L_{-x} g_{*}}{2} d x\right)=T_{\phi}\left(f \star_{\mathcal{C}} g_{*}\right)=T_{\phi}(f) T_{\phi}\left(g_{*}\right)=T_{\phi}(f)
\end{aligned}
$$

which concludes the proof if we put $\theta_{\mathcal{C}}(\phi):=\chi_{\phi}$.

## 3 Laplace transform

In the previous section we focused on Fourier and cosine transforms and characterized them via suitable convolution properties. The purpose of the current section is to demonstrate that the Laplace transform enjoys a similar characterization. Let us recall that the Laplace transform is a map $\mathcal{L}: L^{1}\left(\mathbb{R}_{+}\right) \longrightarrow C_{0}\left(\mathbb{R}_{+}\right)$ given by

$$
\forall_{\substack{f \in L^{1}\left(\mathbb{R}_{+}\right) \\ y \in \mathbb{R}_{+}}} \mathcal{L}(f)(y):=\int_{0}^{\infty} e^{-y x} f(x) d x
$$

whereas the Laplace convolution $\star_{\mathcal{L}}: L^{1}\left(\mathbb{R}_{+}\right) \times L^{1}\left(\mathbb{R}_{+}\right) \longrightarrow L^{1}\left(\mathbb{R}_{+}\right)$is given by

$$
\begin{equation*}
\forall_{f, g \in L^{1}\left(\mathbb{R}_{+}\right)}^{x \in \mathbb{R}_{+}} \boldsymbol{f} \star_{\mathcal{L}} g(x):=\int_{0}^{x} f(u) g(x-u) d u \tag{12}
\end{equation*}
$$

It is well-known that the Laplace transform satisfies the equality (see Theorem 2.39 in [13], p. 92):

$$
\begin{equation*}
\forall_{f, g \in L^{1}\left(\mathbb{R}_{+}\right)} \mathcal{L}\left(f \star_{\mathcal{L}} g\right)=\mathcal{L}(f) \mathcal{L}(g) \tag{13}
\end{equation*}
$$

Similarly to the previous section, our goal is to "reverse the implication" and ask whether there are any other operators satisfying (13). A (almost) negative answer will constitute a characterization of the Laplace transform in terms of the Laplace convolution (12). To this end, we need the following auxilary lemma:

Lemma 6. Let $(\Omega, \Sigma, \mu)$ be a measure space, $g \in L^{\infty}(\Omega)$ and let $D$ be a dense subset in $L^{1}(\Omega)$. If

$$
\forall_{f \in D} \int_{\Omega} f(x) g(x) d x=0
$$

then $g=0$.
Proof. We define a linear and bounded functional $T: L^{1}(\Omega) \longrightarrow \mathbb{C}$ with the formula

$$
T(f):=\int_{\Omega} f(x) g(x) d x
$$

Since $D$ is dense in $L^{1}(\Omega)$ and $\left.T\right|_{D}=0$ then by Theorem 1.7 in [11], p. 9 we conclude that $T(f)=0$ for every $f \in L^{1}(\Omega)$. In particular, $T(g)=0$, which implies that $g=0$.

At last, we are ready to characterize the Laplace transform in terms of the Laplace convolution property:
Theorem 7. Let $T: L^{1}\left(\mathbb{R}_{+}\right) \longrightarrow L^{\infty}\left(\mathbb{R}_{+}\right)$be a linear and bounded operator. If it satisfies the Laplace convolution property

$$
\begin{equation*}
\forall_{f, g \in L^{1}(G)} T\left(f \star_{\mathcal{L}} g\right)=T(f) T(g) \tag{14}
\end{equation*}
$$

then there exists a function $\theta_{\mathcal{L}}: \mathbb{R}_{+} \longrightarrow \mathbb{C}$ such that

$$
\forall_{f \in L^{1}\left(\mathbb{R}_{+}\right)} T(f)=\mathcal{L}(f) \circ \theta_{\mathcal{L}}
$$

where $\theta_{\mathcal{L}}(y)=0$ if and only if $T(f)(y)=0$ for every $f \in L^{1}\left(\mathbb{R}_{+}\right)$.

Proof. We pick $y \in \mathbb{R}_{+}$and check whether $T(f)(y)=0$ for every $f \in L^{1}\left(\mathbb{R}_{+}\right)$. If so, we define $\theta_{\mathcal{L}}(y):=0$. Otherwise, if $T(f)(y) \neq 0$ for some $f \in L^{1}\left(\mathbb{R}_{+}\right)$, then we consider a nonzero, linear functional $T_{y}$ : $L^{1}\left(\mathbb{R}_{+}\right) \longrightarrow \mathbb{C}$ given by $T_{y}(f):=T(f)(y)$. By Theorem 5.2 .16 in [3], p. 161 there exists a function $\chi_{y} \in L^{\infty}\left(\mathbb{R}_{+}\right)$such that

$$
\forall_{f \in L^{1}(G)} T_{y}(f)=\int_{0}^{\infty} f(x) \chi_{y}(x) d x
$$

By Fubini's theorem (see Theorem B.3.1 in [4], p. 287 or Theorem 8.8 in [12], p. 164) we have

$$
\begin{aligned}
\forall_{f, g \in L^{1}\left(\mathbb{R}_{+}\right)} T_{y}\left(f \star_{\mathcal{L}} g\right) & =\int_{0}^{\infty} f \star_{\mathcal{L}} g(u) \chi_{y}(u) d u=\int_{0}^{\infty} \int_{0}^{u} f(v) g(u-v) \chi_{y}(u) d v d u \\
& =\int_{0}^{\infty} \int_{v}^{\infty} f(v) g(u-v) \chi_{y}(u) d u d v \stackrel{u \mapsto \underline{u}+v}{=} \int_{0}^{\infty} \int_{0}^{\infty} f(v) g(u) \chi_{y}(u+v) d u d v
\end{aligned}
$$

Since $T_{y}$ is nonzero we pick $g_{*} \in C_{c}\left(\mathbb{R}_{+}\right)$such that $T_{y}\left(g_{*}\right)=1$. By the Laplace convolution property (14) we have

$$
\forall_{f \in L^{1}\left(\mathbb{R}_{+}\right)} T_{y}\left(f \star_{\mathcal{L}} g_{*}\right)=T_{y}(f) T_{y}\left(g_{*}\right)=T_{y}(f)
$$

so

$$
\forall_{f \in L^{1}\left(\mathbb{R}_{+}\right)} \int_{0}^{\infty} \int_{0}^{\infty} f(v) g_{*}(u) \chi_{y}(u+v) d u d v=\int_{0}^{\infty} f(v) \chi_{y}(v) d v
$$

which we rewrite as

$$
\forall_{f \in L^{1}\left(\mathbb{R}_{+}\right)} \int_{0}^{\infty} f(v)\left(\int_{0}^{\infty} g_{*}(u) \chi_{y}(u+v) d u-\chi_{y}(v)\right) d v=0
$$

Since $v \mapsto \int_{0}^{\infty} g_{*}(u) \chi_{y}(u+v) d u-\chi_{y}(v)$ is a $L^{\infty}$-function then by Lemma we conclude that

$$
\forall_{v \in \mathbb{R}_{+}} \chi_{y}(v)=\int_{0}^{\infty} g_{*}(u) \chi_{y}(u+v) d u
$$

We use this integral functional equation to establish continuity of $\chi_{y}$. For a fixed $v_{*} \in \mathbb{R}_{+}$and $h \in\left(-v_{*}, v_{*}\right)$ we have

$$
\begin{aligned}
\left|\chi_{y}\left(v_{*}+h\right)-\chi_{y}\left(v_{*}\right)\right| & =\left|\int_{0}^{\infty} g_{*}(u) \chi_{y}\left(u+v_{*}+h\right) d u-\int_{0}^{\infty} g_{*}(u) \chi_{y}\left(u+v_{*}\right) d u\right| \\
& =\left|\int_{v_{*}+h}^{\infty} g_{*}\left(u-v_{*}-h\right) \chi_{y}(u) d u-\int_{v_{*}}^{\infty} g_{*}\left(u-v_{*}\right) \chi_{y}(u) d u\right| \\
& \leqslant \int_{v_{*}+h}^{\infty}\left|g_{*}\left(u-v_{*}-h\right)-g_{*}\left(u-v_{*}\right)\left\|\chi_{y}(u)\left|d u+\int_{v_{*}}^{v_{*}+h}\right| g_{*}\left(u-v_{*}\right)\right\| \chi_{y}(u)\right| d u \\
& \leqslant\left\|\chi_{y}\right\|_{\infty}\left(\int_{v_{*}+h}^{\infty}\left|g_{*}\left(u-v_{*}-h\right)-g_{*}\left(u-v_{*}\right)\right| d u+\left\|g_{*}\right\|_{\infty} h\right)
\end{aligned}
$$

Since

$$
\lim _{h \rightarrow 0} \int_{v_{*}+h}^{\infty}\left|g_{*}\left(u-v_{*}-h\right)-g_{*}\left(u-v_{*}\right)\right| d u=0
$$

due to the dominated convergence theorem, then

$$
\lim _{h \rightarrow 0}\left|\chi_{y}\left(v_{*}+h\right)-\chi_{y}\left(v_{*}\right)\right|=0
$$

establishing the continuity of $\chi_{y}$.
For the second time we use the Laplace convolution property (14) to obtain

$$
\forall_{f, g \in L^{1}\left(\mathbb{R}_{+}\right)} \int_{0}^{\infty} \int_{0}^{\infty} f(v) g(u) \chi_{y}(u+v) d u d v=\int_{0}^{\infty} \int_{0}^{\infty} f(v) g(u) \chi_{y}(v) \chi_{y}(u) d u d v
$$

which we rewrite as

$$
\forall_{f, g \in L^{1}\left(\mathbb{R}_{+}\right)} \int_{0}^{\infty} \int_{0}^{\infty} f(v) g(u)\left(\chi_{y}(u+v)-\chi_{y}(v) \chi_{y}(u)\right) d u d v=0
$$

Since $(u, v) \mapsto \chi_{y}(u+v)-\chi_{y}(v) \chi_{y}(u)$ is a $L^{\infty}$-function, then by Lemma we have

$$
\forall_{u, v \in \mathbb{R}_{+}} \chi_{y}(u+v)=\chi_{y}(v) \chi_{y}(u)
$$

This functional equation proves that if $\chi_{y}(\bar{u})=0$ for some $\bar{u} \in \mathbb{R}_{+}$then $\chi_{y}=0$ and consequently, $T_{y}=0$. We have already dealt with this case at the beginning of the proof, so we assume that $\chi_{y}(u) \neq 0$ for every $u \in \mathbb{R}_{+}$. By Theorem 1.6 .11 in [3], p. 36 (and boundedness of $\chi_{y}$ ) we obtain $\chi_{y}(u)=e^{-z u}$ for some $z \in \mathbb{R}_{+}$ (which is dependent on $y$ ). This concludes the proof if we put $\theta_{\mathcal{L}}(y):=z$.

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