

INTEGRAL TRANSFORMS OF THE k -GENERALIZED MITTAG-LEFFLER FUNCTION $E_{k,\alpha,\beta}^{\gamma,\tau}(z)$

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In this paper we derive the Euler transform, Laplace transform, Whittaker transform and fractional Fourier transform of order α , $0 < \alpha \leq 1$, of the generalized k -Mittag-Leffler function $E_{k,\alpha,\beta}^{\gamma}(z)$. The results obtained are of general character and provide extension of the results given by Saxena [9].

1. Introduction

The k -Pochhammer symbol was introduced in [1] in the form:

$$(x)_{n,k} = x(x+k)(x+2k)\dots(x+(n-1)k), \quad (1)$$

$$(x)_{(n+r)q,k} = (x)_{rq,k}(x+qrk)_{nq,k}, \quad (2)$$

where $x \in \mathbb{C}$, $k \in \mathbb{R}$ and $n \in \mathbb{N}$.

Proposition 1.1. *Let $\gamma \in \mathbb{C}$, $k, s \in \mathbb{R}$, then the following identity holds*

$$\Gamma_s(\gamma) = \left(\frac{s}{k}\right)^{\frac{\gamma}{s}-1} \Gamma_k\left(\frac{k\gamma}{s}\right), \quad (3)$$

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and in particular

$$\Gamma_k(\gamma) = (k)^{\frac{\gamma}{k}-1} \Gamma\left(\frac{\gamma}{k}\right), \quad (4)$$

Proposition 1.2. Let $\gamma \in \mathbb{C}$ and $k, s \in \mathbb{R}$ and $n \in \mathbb{N}$, then the following identity holds

$$(\gamma)_{nq,s} = \left(\frac{s}{k}\right)^{nq} \left(\frac{k\gamma}{s}\right)_{nq,k}, \quad (5)$$

and for particular case

$$(\gamma)_{nq,k} = (k)^{nq} \left(\frac{\gamma}{k}\right)_{nq}, \quad (6)$$

Note 1.3. More details of k -Pochhammer symbol, k -special function and fractional Fourier transform one can refer to the papers by Romero et al [7, 8].

Definition 1.4. Let $k \in \mathbb{R}$; $\alpha, \beta, \gamma \in \mathbb{C}$; $Re(\alpha) > 0$, $Re(\beta) > 0$ and $\tau \in \mathbb{C}$, then the generalized k -Mittag-Leffler function, denoted by $E_{k,\alpha,\beta}^{\gamma,\tau}(z)$, is defined as

$$E_{k,\alpha,\beta}^{\gamma,\tau}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k} z^n}{\Gamma_k(n\alpha + \beta)n!}, \quad (7)$$

where $(x)_\tau$, $(x, \tau \in \mathbb{C})$ denotes the Pochhammer symbol with $(1)_n = n!$ for $n \in \mathbb{N} = \mathbb{N} \cup \{0\}$, defined in terms of gamma function as (also see [13, p. 199])

$$(x)_\tau = \frac{\Gamma(x+\tau)}{\Gamma(x)} = \begin{cases} 1 & (\tau = 0; x \in \mathbb{C} \setminus \{0\}) \\ x(x+1)\dots(x+\tau-1) & (\tau = n \in \mathbb{N}; x \in \mathbb{C}). \end{cases}$$

Particular cases of $E_{k,\alpha,\beta}^{\gamma,\tau}(z)$

(i) For $\tau = q$, equation (7) yields generalized k -Mittag-Leffler function, defined by

$$E_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k} z^n}{\Gamma_k(n\alpha + \beta)n!} = E_{k,\alpha,\beta}^{\gamma}(z) \quad (8)$$

(ii) For $k = 1$, equation (8) yields generalized Mittag-Leffler function, defined by [11]

$$E_{1,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq} z^n}{\Gamma(n\alpha + \beta)n!} = E_{\alpha,\beta}^{\gamma,q}(z) \quad (9)$$

(iii) For $q = 1$, equation (8) gives the Mittag-Leffler function defined by Doorego and Cerutti [2]

$$E_{k,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} z^n}{\Gamma_k(n\alpha + \beta)n!} = E_{k,\alpha,\beta}^{\gamma}(z) \quad (10)$$

Note 1.5. A detailed account of Mittag-Leffler function and their applications can be found in the survey paper by Haubold, Mathai and Saxena [4], Saxena [9] and Saxena et al [10].

The following definitions are also needed in the analysis that follows:

Definition 1.6. Euler Transform

The Euler transform of the function $f(z)$ is defined by

$$B\{f(z); a, b\} = \int_0^1 z^{a-1} (1-z)^{b-1} f(z) dz, \quad a, b \in \mathbb{C}, \operatorname{Re}(a) > 0, \operatorname{Re}(b) > 0. \quad (11)$$

Definition 1.7. Laplace Transform

The Laplace transform of the function $f(t)$, denoted by $F(s)$ is defined by the equation

$$F(s) = (Lf)(s) = L\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt, \quad \operatorname{Re}(s) > 0. \quad (12)$$

which may be symbolically written as

$$F(s) = L\{f(t); s\} \text{ or } f(t) = L^{-1}\{F(s); t\}$$

provided that the function $f(t)$ is continuous for $t \geq 0$ and of exponential order as $t \rightarrow \infty$.

Definition 1.8. Let $u = u(t)$ be a function of the space $S(R)$, the Schwartzian space of the function that decay rapidly at infinity together with all derivatives.

The Fourier transform is given by the integral

$$\hat{u}(\omega) = \mathfrak{S}[u](\omega) = \int_R u(t) \exp(i\omega t) dt \quad (13)$$

and the inverse Fourier transform can be defined as

$$\mathfrak{S}^{-1}[\hat{u}](t) = \frac{1}{2\pi} \int_R \hat{u}(\omega) \exp(-i\omega t) d\omega \quad (14)$$

Definition 1.9. Lizorkin space

Let $V(\mathbb{R})$ be the set of functions

$$V(\mathbb{R}) = \left\{ v \in S(\mathbb{R}) : v^{(n)}(0) = 0, n = 0, 1, 2, \dots \right\}. \quad (15)$$

The Lizorkin space of function $\phi(\mathbb{R})$ is defined as

$$\phi(\mathbb{R}) = \{ \varphi \in S(\mathbb{R}) : \mathfrak{S}[\varphi] \in V(\mathbb{R}) \}. \quad (16)$$

Definition 1.10. Let u be a function belonging to $\phi(\mathbb{R})$.

The fractional Fourier transform of the order α , $0 < \alpha \leq 1$ is defined by

$$\hat{u}_\alpha(\omega) = \mathfrak{F}_\alpha[u](\omega) = \int_R e^{i\omega^{1/\alpha}t} u(t) dt. \quad (17)$$

If put $\alpha = 1$, equation (17) reduces to the conventional Fourier transform and for $\omega > 0$, it reduces to the fractional Fourier Transform defined by Luchko et al [5].

Lemma 1.11. Let u be a function of the space $\phi(\mathbb{R})$, let α be a real number, $0 < \alpha \leq 1$, then

$$\mathfrak{F}_\alpha[u](\omega) = \mathfrak{F}[u](x), \quad \text{for } x = \omega^{1/\alpha}. \quad (18)$$

The inversion Fractional Fourier transform of the order α , $0 < \alpha \leq 1$, $u \in \phi(R)$ is defined as

$$\mathfrak{F}_\alpha^{-1}\{\hat{u}_\alpha(\omega)\}(t) = \frac{1}{2\pi\alpha} \int_R e^{-i\omega^{1/\alpha}t} \hat{u}_\alpha(\omega) \omega^{\frac{1-\alpha}{\alpha}} d\omega. \quad (19)$$

The following result will be required in evaluating the integral (30).

$$\int_0^\infty e^{-1/2t} t^{v-1} W_{\lambda,\mu}(t) dt = \frac{\Gamma(1/2 + \mu + v) \Gamma(1/2 - \mu + v)}{\Gamma(1 - \lambda + v)}, \quad \text{Re}(v \pm \mu) > -1/2 \quad (20)$$

where the Whittaker function $W_{\lambda,\mu}(z)$ is defined in [3] (see also Mathai et al [6]).

$$W_{\mu,\nu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2} - \lambda - \mu)} M_{\lambda,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2} - \lambda + \mu)} M_{\lambda,-\mu}(z) \quad (21)$$

where $M_{\lambda,\mu}(z)$ is defined as

$$M_{\lambda,\mu}(z) = z^{1/2+\mu} e^{-1/2z} {}_1F_1\left(\frac{1}{2} + \mu - \lambda; 2\mu + 1; z\right).$$

The generalized Wright hypergeometric function ${}_p\Psi_q(z)$ is defined by Wright [14-16] (also see, [12, p.21]) in the following form :

$${}_p\Psi_q(z) \left[\begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix} ; Z \right] = \sum_{n=0}^{\infty} \left[\frac{\prod_{i=1}^p \Gamma(a_i + A_i n)}{\prod_{j=1}^q \Gamma(b_j + B_j n)} \right] \frac{z^n}{n!} \quad (22)$$

where $a_i, b_j \in C$ and $A_i, B_j \in R (i = 1, \dots, p; j = 1, \dots, q)$ and the defining series (22) converges for

$$\sum_{j=1}^q B_j - \sum_{i=1}^p A_i > -1. \quad (23)$$

The object of this paper is to evaluate the Euler, Laplace, Whittaker and fractional Fourier transforms of the generalized k-Mittag-Leffler function $E_{k,\alpha,\beta}^{\gamma,\tau}(z)$, defined by (7). The results are obtained in a closed form in terms of the generalized Wright hypergeometric function, defined by (22). The results obtained in this paper are the generalization of the results given earlier by Saxena [9].

Theorem 1.12 (Euler Transform). *If $k \in R$; $\alpha, \beta, \gamma, a, b, \sigma \in C$; $Re(\alpha) > 0$, $Re(\beta) > 0$ and $\tau \in \mathbb{C}$, then*

$$\int_0^1 z^{a-1}(1-z)^{b-1} E_{k,\alpha,\beta}^{\gamma,\tau}(xz^\sigma) dz = \frac{k^{1-\frac{\beta}{k}} \Gamma(b)}{\Gamma\left(\frac{\gamma}{k}\right)} {}_2\Psi_2 \left[\begin{matrix} \left(\frac{\gamma}{k}, \tau\right), (a, \sigma) \\ \left(\frac{\beta}{k}, \frac{\alpha}{k}\right), (a+b, \sigma) \end{matrix}; k^{\tau-\frac{\alpha}{k}} x \right] \quad (24)$$

Proof. Using equation (7) and (11) and beta function formula, it gives

$$\begin{aligned} \int_0^1 z^{a-1}(1-z)^{b-1} E_{k,\alpha,\beta}^{\gamma,\tau}(xz^\sigma) dz &= \int_0^1 z^{a-1}(1-z)^{b-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k} (xz^\sigma)^n}{\Gamma_k(n\alpha + \beta) n!} dz \\ &= \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k} x^n}{\Gamma_k(n\alpha + \beta) n!} \int_0^1 z^{a-1}(1-z)^{b-1} z^{\sigma n} dz \\ &= \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k} x^n}{\Gamma_k(n\alpha + \beta) n!} \frac{\Gamma(b)\Gamma(\sigma n + a)}{\Gamma(\sigma n + a + b)} \end{aligned}$$

using equation (4) and (6), it becomes

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{(k)^{n\tau} \left(\frac{\gamma}{k}\right)_{n\tau} \Gamma(b)\Gamma(\sigma n + a)}{(k)^{\frac{n\alpha + \beta}{k} - 1} \Gamma\left(\frac{\alpha}{k} n + \frac{\beta}{k}\right) \Gamma(\sigma n + a + b)} \frac{x^n}{n!} \\ &= \frac{(k)^{1-\frac{\beta}{k}} \Gamma(b)}{\Gamma\left(\frac{\gamma}{k}\right)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\gamma}{k} + n\tau\right) \Gamma(\sigma n + a)}{\Gamma\left(\frac{\alpha}{k} n + \frac{\beta}{k}\right) \Gamma(\sigma n + a + b)} \frac{\left(k^{\tau-\frac{\alpha}{k}} x\right)^n}{n!}. \end{aligned}$$

This completes the proof of the Theorem 1.12. □

Corollary 1.13. *For $\tau = q$, equation (24) reduces in the following form*

$$\int_0^1 z^{a-1}(1-z)^{b-1} E_{k,\alpha,\beta}^{\gamma,q}(xz^\sigma) dz = \frac{k^{1-\frac{\beta}{k}} \Gamma(b)}{\Gamma\left(\frac{\gamma}{k}\right)} {}_2\Psi_2 \left[\begin{matrix} \left(\frac{\gamma}{k}, q\right), (a, \sigma) \\ \left(\frac{\beta}{k}, \frac{\alpha}{k}\right), (a+b, \sigma) \end{matrix}; k^{q-\frac{\alpha}{k}} x \right] \quad (25)$$

Corollary 1.14. *For $k = 1$, equation (25) reduces to*

$$\int_0^1 z^{a-1}(1-z)^{b-1} E_{\alpha,\beta}^{\gamma,q}(xz^\sigma) dz = \frac{\Gamma(b)}{\Gamma(\gamma)} {}_2\Psi_2 \left[\begin{matrix} (\gamma, q), (a, \sigma) \\ (\beta, \alpha), (a+b, \sigma) \end{matrix}; x \right] \quad (26)$$

If we take $q = 1$ in (26), it reduces to one given by Saxena [9].

Theorem 1.15 (Laplace Transform). *If $k \in R$; $\alpha, \beta, \gamma, a, b, \sigma \in C$; $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(s) > 0$ and $\tau \in \mathbb{C}$, $\left|\frac{x}{s^\sigma}\right| < 1$, then*

$$\int_0^\infty z^{a-1} e^{-sz} E_{k,\alpha,\beta}^{\gamma,\tau}(xz^\sigma) dz = \frac{k^{1-\frac{\beta}{k}} s^{-a}}{\Gamma\left(\frac{\gamma}{k}\right)} {}_2\Psi_1 \left[\begin{matrix} \left(\frac{\gamma}{k}, \tau\right), (a, \sigma) \\ \left(\frac{\beta}{k}, \frac{\alpha}{k}\right) \end{matrix}; \frac{k^{\tau-\frac{\alpha}{k}} x}{s^\sigma} \right] \quad (27)$$

Proof. Using equation (7) and (12) and gamma function formula, we obtain

$$\begin{aligned} \int_0^\infty z^{a-1} e^{-sz} E_{k,\alpha,\beta}^{\gamma,\tau}(xz^\sigma) dz &= \int_0^\infty z^{a-1} e^{-sz} \sum_{n=0}^\infty \frac{(\gamma)_{n\tau,k} (xz^\sigma)^n}{\Gamma_k(n\alpha + \beta) n!} dz \\ &= \sum_{n=0}^\infty \frac{(\gamma)_{n\tau,k} x^n}{\Gamma_k(n\alpha + \beta) n!} \frac{\Gamma(\sigma n + a)}{s^{\sigma n + a}} \end{aligned}$$

using equation (4) and (6), it becomes

$$\begin{aligned} &= s^{-a} \sum_{n=0}^\infty \frac{(k)^{n\tau} \left(\frac{\gamma}{k}\right)_{n\tau} \Gamma(\sigma n + a)}{(k)^{\frac{n\alpha + \beta}{k} - 1} \Gamma\left(\frac{\alpha}{k} n + \frac{\beta}{k}\right) n!} \left(\frac{x}{s^\sigma}\right)^n \\ &= \frac{(k)^{1 - \frac{\beta}{k}} s^{-a}}{\Gamma\left(\frac{\gamma}{k}\right)} \sum_{n=0}^\infty \frac{\Gamma\left(\frac{\gamma}{k} + n\tau\right) \Gamma(\sigma n + a)}{\Gamma\left(\frac{\alpha}{k} n + \frac{\beta}{k}\right) n!} \left(\frac{k^{\tau - \frac{\alpha}{k}} x}{s^\sigma}\right)^n. \end{aligned}$$

This completes the proof of the Theorem 1.15. \square

Corollary 1.16. For $\tau = q$, equation (27) reduces in the following form

$$\int_0^\infty z^{a-1} e^{-sz} E_{k,\alpha,\beta}^{\gamma,q}(xz^\sigma) dz = \frac{k^{1 - \frac{\beta}{k}} s^{-a}}{\Gamma\left(\frac{\gamma}{k}\right)} {}_2\Psi_1 \left[\begin{matrix} \left(\frac{\gamma}{k}, q\right), (a, \sigma) \\ \left(\frac{\beta}{k}, \frac{\alpha}{k}\right) \end{matrix}; \frac{k^{q - \frac{\alpha}{k}} x}{s^\sigma} \right] \quad (28)$$

Corollary 1.17. For $k = 1$, equation (28) reduces in the following form

$$\int_0^\infty z^{a-1} e^{-sz} E_{\alpha,\beta}^{\gamma,q}(xz^\sigma) dz = \frac{s^{-a}}{\Gamma(\gamma)} {}_2\Psi_1 \left[\begin{matrix} (\gamma, q), (a, \sigma) \\ (\beta, \alpha) \end{matrix}; \frac{x}{s^\sigma} \right] \quad (29)$$

when $q = 1$ in (29), it yields a result given by Saxena [9].

Theorem 1.18 (Whittaker Transform). If $k \in \mathbb{R}$; $\alpha, \beta, \gamma, \rho, \delta \in \mathbb{C}$; $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(\rho) > 0$, $Re(\rho \pm \mu) > -1/2$ and $\tau \in \mathbb{C}$, then

$$\begin{aligned} \int_0^\infty t^{\rho-1} e^{-pt/2} W_{\lambda,\mu}(pt) E_{k,\alpha,\beta}^{\gamma,\tau}(wt^\delta) dt \\ = \frac{k^{1 - \frac{\beta}{k}} p^{-\rho}}{\Gamma\left(\frac{\gamma}{k}\right)} {}_2\Psi_2 \left[\begin{matrix} \left(\frac{\gamma}{k}, \tau\right), (1/2 \pm \mu + \rho, \delta) \\ \left(\frac{\beta}{k}, \frac{\alpha}{k}\right), (1 - \lambda + \rho, \delta) \end{matrix}; \frac{k^{\tau - \frac{\alpha}{k}} w}{p^\delta} \right] \end{aligned} \quad (30)$$

Proof. By virtue of equation (7) and (20), it yields

$$\int_0^\infty t^{\rho-1} e^{-pt/2} W_{\lambda,\mu}(pt) E_{k,\alpha,\beta}^{\gamma,\tau}(wt^\delta) dt$$

if we set $pt = v$, then the above line is equal to

$$\int_0^\infty e^{-v/2} \left(\frac{v}{p}\right)^{\rho-1} W_{\lambda,\mu}(v) \sum_{n=0}^\infty \frac{(\gamma)_{n\tau,k} w^n}{\Gamma_k(n\alpha + \beta) n!} \left(\frac{v}{p}\right)^{\delta n} \frac{1}{p} dv$$

$$\begin{aligned}
 &= \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k} w^{n\tau}}{\Gamma_k(n\alpha + \beta) n!} \int_0^{\infty} e^{-v/2} \left(\frac{v}{p}\right)^{\rho-1} \left(\frac{v}{p}\right)^{\delta n} W_{\lambda,\mu}(v) \frac{1}{p} dv \\
 &= p^{-\rho} \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha + \beta) n!} \left(\frac{w}{p^\delta}\right)^n \int_0^{\infty} e^{-v/2} v^{\delta n + \rho - 1} W_{\lambda,\mu}(v) dv \\
 &= p^{-\rho} \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k} \Gamma(1/2 + \mu + \delta n + \rho) \Gamma(1/2 - \mu + \delta n + \rho)}{\Gamma_k(n\alpha + \beta) \Gamma(1 - \lambda + \delta n + \rho) n!} \left(\frac{w}{p^\delta}\right)^n
 \end{aligned}$$

using equation (4) and (6), it simplifies to

$$\begin{aligned}
 &p^{-\rho} \sum_{n=0}^{\infty} \frac{(k)^{n\tau} \left(\frac{\gamma}{k}\right)_{n\tau} \Gamma(1/2 + \mu + \delta n + \rho) \Gamma(1/2 - \mu + \delta n + \rho)}{(k)^{\frac{n\alpha + \beta}{k} - 1} \Gamma\left(\frac{\alpha}{k} n + \frac{\beta}{k}\right) \Gamma(1 - \lambda + \delta n + \rho) n!} \left(\frac{w}{p^\delta}\right)^n \\
 &= \frac{(k)^{1 - \frac{\beta}{k}} p^{-\rho}}{\Gamma\left(\frac{\gamma}{k}\right)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\gamma}{k} + n\tau\right) \Gamma(1/2 + \mu + \delta n + \rho) \Gamma(1/2 - \mu + \delta n + \rho)}{\Gamma\left(\frac{\alpha}{k} n + \frac{\beta}{k}\right) \Gamma(1 - \lambda + \delta n + \rho) n!} \left(\frac{k^{\tau - \frac{\alpha}{k}} w}{p^\delta}\right)^n.
 \end{aligned}$$

This completes the proof of the Theorem 1.18. □

Corollary 1.19. For $\tau = q$, equation (30) reduces in the following form

$$\begin{aligned}
 &\int_0^{\infty} t^{\rho-1} e^{-pt/2} W_{\lambda,\mu}(pt) E_{k,\alpha,\beta}^{\gamma,q}(wt^\delta) dt \\
 &= \frac{k^{1 - \frac{\beta}{k}} p^{-\rho}}{\Gamma\left(\frac{\gamma}{k}\right)} {}_2\Psi_2 \left[\begin{matrix} \left(\frac{\gamma}{k}, q\right), (1/2 \pm \mu + \rho, \delta) \\ \left(\frac{\beta}{k}, \frac{\alpha}{k}\right), (1 - \lambda + \rho, \delta) \end{matrix} ; \frac{k^{q - \frac{\alpha}{k}} w}{p^\delta} \right] \quad (31)
 \end{aligned}$$

Corollary 1.20. For $k = 1$, equation (31) reduces in the following form

$$\begin{aligned}
 &\int_0^{\infty} t^{\rho-1} e^{-pt/2} W_{\lambda,\mu}(pt) E_{\alpha,\beta}^{\gamma,q}(wt^\delta) dt \\
 &= \frac{p^{-\rho}}{\Gamma(\gamma)} {}_2\Psi_2 \left[\begin{matrix} (\gamma, q), (1/2 \pm \mu + \rho, \delta) \\ (\beta, \alpha), (1 - \lambda + \rho, \delta) \end{matrix} ; \frac{w}{p^\delta} \right] \quad (32)
 \end{aligned}$$

when $q = 1$, (32) reduces to one given by Saxena [9].

Fractional Fourier Transform (FFT) of generalized k -Mittag-Leffler Function

Theorem 1.21. The FFT of order α of the generalized k -Mittag-Leffler function for $t < 0$ is given by

$$\mathfrak{S}_\alpha \left[E_{k,\alpha,\beta}^{\gamma,\tau}(t) \right] (\omega) = \frac{(k)^{1 - \frac{\beta}{k}}}{\Gamma\left(\frac{\gamma}{k}\right)} \sum_{n=0}^{\infty} \frac{(k)^{\left(\tau - \frac{\beta}{k}\right)n} \Gamma\left(\frac{\gamma}{k} + n\tau\right) (-1)^n (i)^{-n-1} \omega^{-(n+1)/\alpha}}{\Gamma\left(\frac{\alpha}{k} n + \frac{\beta}{k}\right)} \quad (33)$$

Proof. Using equation (7) and (17) and gamma function formula, it gives

$$\begin{aligned}\mathfrak{S}_\alpha \left[E_{k,\alpha,\beta}^{\gamma,\tau}(t) \right] (\omega) &= \int_R e^{i\omega^{1/\alpha}t} E_{k,\alpha,\beta}^{\gamma,\tau}(t) dt \\ &= \int_R e^{i\omega^{1/\alpha}t} \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k} t^n}{\Gamma_k(n\alpha + \beta)n!} dt \\ &= \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha + \beta)n!} \int_R e^{i\omega^{1/\alpha}t} t^n dt\end{aligned}$$

if we set $i\omega^{1/\alpha}t = -\xi$, then

$$\begin{aligned}&= \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha + \beta)n!} \int_{-\infty}^0 e^{-\xi} \left(\frac{-\xi}{i\omega^{1/\alpha}} \right)^n \left(\frac{-d\xi}{i\omega^{1/\alpha}} \right) \\ &= \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha + \beta)(n!)(i)^{n+1}\omega^{(n+1)/\alpha}(-1)^n} \int_0^{\infty} e^{-\xi} \xi^n d\xi \\ &= \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}(i)^{-n-1}\omega^{-(n+1)/\alpha}(-1)^{-n}}{\Gamma_k(n\alpha + \beta)(n!)} \Gamma(n+1)\end{aligned}$$

In view of equation (4) and (6), the above line

$$= \sum_{n=0}^{\infty} \frac{(k)^{n\tau} \left(\frac{\gamma}{k} \right)_{n\tau} (-1)^n (i)^{-n-1} \omega^{-(n+1)/\alpha}}{(k)^{\frac{n\alpha+\beta}{k}-1} \Gamma\left(\frac{\alpha}{k}n + \frac{\beta}{k} \right)}.$$

This completes the proof of the Theorem 1.21. □

Corollary 1.22. When $\tau = q$, equation (33) reduces in the following form

$$\mathfrak{S}_\alpha \left[E_{k,\alpha,\beta}^{\gamma,q}(t) \right] (\omega) = \frac{(k)^{1-\frac{\beta}{k}}}{\Gamma\left(\frac{\gamma}{k}\right)} \sum_{n=0}^{\infty} \frac{(k)^{\left(q-\frac{\beta}{k}\right)n} \Gamma\left(\frac{\gamma}{k} + nq\right) (-1)^n (i)^{-n-1} \omega^{-(n+1)/\alpha}}{\Gamma\left(\frac{\alpha}{k}n + \frac{\beta}{k}\right)} \quad (34)$$

Corollary 1.23. For $k = 1$, equation (34) reduces in the following form

$$\mathfrak{S}_\alpha \left[E_{\alpha,\beta}^{\gamma,q}(t) \right] (\omega) = \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma + nq) (-1)^n (i)^{-n-1} \omega^{-(n+1)/\alpha}}{\Gamma(\alpha n + \beta)}. \quad (35)$$

Note 1.24. Fractional Fourier transform is studied among others, by Luchko et al [5] and Romero et al [7, 8].

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