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Integrality of a ratio of Petersson norms and level-lowering congruences

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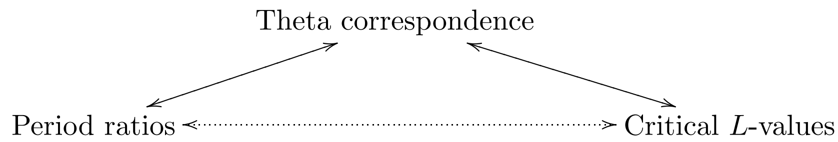
Abstract

We prove integrality of the ratio $\langle f, f\rangle/\langle g, g\rangle$ (outside an explicit finite set of primes), where g is an arithmetically normalized holomorphic newform on a Shimura curve, f is a normalized Hecke eigenform on GL(2) with the same Hecke eigenvalues as g and $\langle , \rangle$ denotes the Petersson inner product. The primes dividing this ratio are shown to be closely related to certain level-lowering congruences satisfied by f and to the central values of a family of Rankin-Selberg L-functions. Finally we give two applications, the first to proving the integrality of a certain triple product L-value and the second to the computation of the Faltings height of Jacobians of Shimura curves.

Introduction

An important problem emphasized in several papers of Shimura is the study of period relations between modular forms on different Shimura varieties. In a series of articles (see for e.g. [34], [35], [36]), he showed that the study of algebraicity of period ratios is intimately related to two other fascinating themes in the theory of automorphic forms, namely the arithmeticity of the theta correspondence and the theory of special values of L-functions. Shimura's work on the theta correspondence was later extended to other situations by Harris-Kudla and Harris, who in certain cases even demonstrate rationality of theta lifts over specified number fields. For instance, the articles [12], [13] study rationality of the theta correspondence for unitary groups and explain its relation, on the one hand, to period relations for automorphic forms on unitary groups of different signature, and on the other to Deligne's conjecture on critical values of L-functions attached to motives that occur in the cohomology of the associated Shimura varieties. To understand these results from a philosophical point of view, it is then useful to picture the three themes mentioned above as the vertices of a triangle, each of which has some bearing on the others.



This article is an attempt to study the picture above in perhaps the simplest possible case, not just up to algebraicity or rationality, but up to p-adic integrality. The period ratio in the case at hand is that of the Petersson norm of a holomorphic newform g of even weight k on a (compact) Shimura curve X associated to an indefinite quaternion algebra D over $\mathbb{Q}$ to the Petersson norm of a normalized Hecke eigenform f on GL(2) with the same Hecke eigenvalues as g. The relevant theta correspondence is from GL(2) to GO(D), the orthogonal similitude group for the norm form on D, as occurs in Shimizu's explicit realization of the Jacquet-Langlands correspondence. The L-values that intervene are the central critical values of Rankin-Selberg products of f and theta functions associated to Grossencharacters of weight k of a certain family of imaginary quadratic fields.

We now explain our results and methods in more detail. Firstly, to formulate the problem precisely, one needs to normalize f and g canonically. Traditionally one normalizes f by requiring that its first Fourier coefficient at the cusp at ∞ be 1. Since compact Shimura curves do not admit cusps, such a normalization is not available for g. However, g corresponds in a natural way to a section of a certain line bundle L on X. The curve X and the line bundle L admit *canonical* models over $\mathbb{Q}$, whence g may be normalized up to an element of K_f, the field generated by the Hecke eigenvalues of f. Let $\langle f, f\rangle$ and $\langle g, g\rangle$ denote the Petersson inner products taken on $X_0(N)$ and X respectively. It was proved by Shimura ([34]) that the ratio $\langle f, f\rangle/\langle g, g\rangle$ lies in $\overline{\mathbb{Q}}$ and by Harris-Kudla ([14]) that it in fact lies in K_f.

Now, let p be a prime not dividing the level of f. For such a p the curve X admits a *canonical* proper smooth model $\mathcal{X}$ over $\mathbb{Z}_p$, and the line bundle L too extends canonically to a line bundle $\mathcal{L}$ over $\mathcal{X}$. The model $\mathcal{X}$ can be constructed as the solution to a certain moduli problem, or one may simply take the minimal regular model of X over $\mathbb{Z}_p$; the line bundle $\mathcal{L}$ is the appropriate power of the relative dualizing sheaf. Let λ be an embedding of $\overline{\mathbb{Q}}$ in $\overline{\mathbb{Q}_p}$, so that λ induces a prime of K_f over p. One may then normalize g up to a λ-adic unit by requiring that the corresponding section of L be λ-adically integral and primitive with respect to the integral structure provided by $\mathcal{L}$. One of our main results (Thm. 2.4) is that with such a normalization, and some restrictions on p, the ratio considered above is in fact a λ-adic integer.

As the reader might expect, our proof of the integrality of $\langle f, f\rangle/\langle g, g\rangle$ builds on the work of Harris-Kudla and Shimura, but requires several new ingredients: an integrality criterion for forms on Shimura curves (§2.3), work of Watson on the explicit Jacquet-Langlands-Shimizu correspondence [43], our computations of ramified zeta integrals related to the Rankin-Selberg L-values mentioned before (§3.4), the use of some constructions (§4.2) analogous to those of Wiles in [40] and an application of Rubin's theorem ([30]) on the *main conjecture* of Iwasawa theory for imaginary quadratic fields (§4.3). Below we describe these ingredients and their role in more detail.

The first main input is Shimizu's realization of the Jacquet-Langlands correspondence (due in this case originally to Eichler and Shimizu) via theta lifts. We however need a more precise result of Watson [43], namely that one can obtain some multiple g' of g by integrating f against a suitable theta function. Crucially, one has precise control over the theta lift; it is not just any form in the representation space of g but a scalar multiple of the newform g. Further one checks easily that $\langle g', g'\rangle = \langle f, f\rangle$. To prove the λ-integrality of $\langle f, f\rangle/\langle g, g\rangle$ is then equivalent to showing the λ-integrality of the form g'.

The next step is to develop an integrality criterion for forms on Shimura curves. While q-expansions are not available, Shimura curves admit CM points, which are known to be algebraic, and in fact defined over suitable class fields of the associated imaginary quadratic field. This fact can be used to identify algebraic modular forms via their values at such points; i.e., their values, suitably defined, should be algebraic. In fact X is a coarse moduli space for abelian surfaces with quaternionic multiplication and level structure. Viewed as points on the moduli space, CM points associated to an imaginary quadratic field K correspond to products of elliptic curves with complex multiplication by K, hence have potentially good reduction. Consequently, the values of an integral modular form at such points (suitably defined, i.e., divided by the appropriate period) must be integral. Conversely, if the form g' has integral values at all or even sufficiently many CM points then it must be integral, since the mod p reductions of CM points are dense in the special fibre of $\mathcal{X}$ at p. In practice, it is hard to evaluate g' at a fixed CM point but easier to evaluate certain *toric integrals* associated to g' and a Hecke character χ of K of the appropriate infinity type. These toric integrals are actually finite sums of the values of g' at all Galois conjugates of the CM point, twisted by the character χ. In the case when the field K has class number prime to p and the CM points are Heegner points, we show (Prop. 2.9) that the integrality of the values of g' is equivalent to the integrality of the toric integrals for all unramified Hecke characters χ.

The toric integrals in question can be computed by a method of Waldspurger as in [14]. In fact, the square of such an integral is equal to the value at the center of the critical strip of a certain global zeta integral which factors into a product of local factors. By results of Jacquet ([20]), at almost all primes, the relevant local factor is equal to the Euler factor $L_q(s, f \otimes \theta_\chi)$ associated to the Rankin-Selberg product of f and $\theta_\chi = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) e^{2\pi i N\mathfrak{a} z}$ (sum over integral ideals in K). For our purposes, knowing all but finitely many factors is not enough, so we need to compute the local zeta integrals at all places, including the ramified ones, the ramification coming from the level of f, the discriminant of K and the Heegner point data. The final result then (Thm. 3.2) is that the square of the toric integral differs from the central critical value $L(k, f \otimes \theta_\chi)$ of the Rankin-Selberg L-function by a p-adic unit.

We now need to prove the integrality of $L(k, f \otimes \theta_\chi)$ (divided by an appropriate period). One sees easily from the Rankin-Selberg method that this follows if one knows the integrality of $\langle f', \theta_\chi \rangle / \Omega'$ for a certain period Ω' and for all integral forms f' of weight $k+1$ and level Nd where N is the level of f and $-d$ is the discriminant of K. In fact $\langle f', \theta_\chi \rangle / \Omega' = \alpha \langle \theta_\chi, \theta_\chi \rangle / \Omega' = \alpha L(k+1, \overline{\chi}\chi^\rho)/\Omega''$ where α is the coefficient of θ_χ in the expansion of f' as a linear combination of orthogonal eigenforms, χ^ρ is the twist of χ by complex conjugation and Ω'' is a suitable period. The crux of the argument is that if α had any denominators these would give congruences between θ_χ and other forms; on the other hand the last L-value is expected to count all congruences satisfied by θ_χ. Thus any possible denominators in α should be cancelled by the numerator of this L-value. The precise mechanism to prove this is quite intricate. Restricting ourselves to the case when p is *split* in K and $p \nmid h_K$ (= the class number of K), we first use analogs of the methods of Wiles ([40], [42]) to construct a certain Galois extension of degree equal to the p-adic valuation of the denominator of α. Next we use results of Rubin ([30]) on the Iwasawa main conjecture for K to bound the size of this Galois group by the p-adic valuation of $L(k+1, \overline{\chi}\chi^\rho)/\Omega''$. The details are worked out in Chapter 4 where the reader may find also a more detailed introduction to these ideas and a more precise statement including some restrictions on the prime p. We should mention at this point that in the case when the base field is a totally real field of even degree over $\mathbb{Q}$, Hida [19] has found a direct proof of the integrality of $\langle f', \theta_\chi \rangle / \Omega'$ under certain conditions and he is able to deduce from it the anticyclotomic main conjecture for CM fields in many cases.

To apply the results of Ch. 4 to the problem at hand, we now need to show that we can find infinitely many Heegner points with p split in K and $p \nmid h_K$. In Section 5.1 we show this using results of Bruinier [3] and Jochnowitz [22], thus finishing the proof of the integrality of the modular form g' (and of the ratio $\langle f, f \rangle / \langle g, g \rangle$). An amazing consequence of the integrality of g' is that we can deduce from it the integrality of the Rankin-Selberg L-values above even if $p \mid h_K$ or p is *inert* in K ! This result, which is also explained in Section 5.1, would undoubtedly be much harder to obtain directly using the Iwasawa-theoretic methods mentioned above.

Having proved the integrality of the ratio $\langle f, f \rangle / \langle g, g \rangle$ we naturally ask for a description of those primes λ for which the λ-adic valuation of this ratio is strictly positive. First we consider the special case in which the weight of f is 2, its Hecke eigenvalues are rational and the prime p is not an Eisenstein prime for f. In this case we show that p divides $\langle f, f\rangle/\langle g, g\rangle$ exactly when for some q dividing the discriminant of the quaternion algebra associated to X, there is a form h of level N/q such that f and h are congruent modulo p. We say in such a situation that p is a level-lowering congruence prime for f at the prime q. In the general case we can only show one direction, namely that the λ-adic valuation is strictly positive for such level-lowering congruence primes. This is accomplished by showing that the λ-adic valuation of the Rankin-Selberg L-value discussed above is strictly positive for such primes. Conversely, one might expect that if the λ-adic valuation of the L-value is strictly positive for infinitely many K and all choices of unramified characters χ, then λ would be a level-lowering congruence prime.

Finally, we give two applications of our results. The first is to prove integrality of a certain triple product L-value. Indeed, the rationality of $\langle f, f\rangle/\langle g, g\rangle$ proved by Harris-Kudla was motivated by an application to prove rationality for the central critical value of the triple product L-function associated to three holomorphic forms of compatible weight. Combining a precise formula proved by Watson [43] with our integrality results we can establish integrality of the central critical value of the same triple product.

The second application is the computation of the Faltings height of Jacobians of Shimura curves over $\mathbb{Q}$. This problem (over totally real fields) was suggested to me by Andrew Wiles and was the main motivation for the results in this article. While we only consider the case of Shimura curves over $\mathbb{Q}$, most of the ingredients of the computation should generalize in principle to the totally real case. Many difficulties remain though, the principal one being that the Iwasawa main conjecture is not yet proven for CM fields. (The reader will note from the proof that we only need the so-called anticyclotomic case of the main conjecture. As mentioned before this has been solved [19] in certain cases but not yet in the full generality needed.) Also one should expect that the computations with the theta correspondence will get increasingly complicated; indeed the best results to date on period relations for totally real fields are due to Harris ([11]) and these are only up to algebraicity.

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1. Notation and conventions

Let $\mathbb{A}$ denote the ring of adeles over $\mathbb{Q}$ and $\mathbb{A}_f$ the finite adeles. We fix an additive character ψ of $\mathbb{Q} \setminus \mathbb{A}$ as follows. Choose ψ so that $\psi_\infty(x) = e^{2\pi\imath x}$ and so that ψ_q for finite primes q is the unique character with kernel $\mathbb{Z}_q$ and such that $\psi_q(x) = e^{-2\pi\imath x}$ for $x \in \mathbb{Z}[\frac{1}{q}]$. Let dx_v be the unique Haar measure on $\mathbb{Q}_v$ such that the Fourier transform $\hat{\varphi}(y_v) = \int_{\mathbb{Q}_v} \varphi(x_v)\psi(x_v y_v)dx_v$ is autodual, i.e., $\hat{\hat{\varphi}}(y) = \varphi(-y)$. On $\mathbb{A}$ we take the product measure $dx = \prod_v dx_v$. On $\mathbb{A}^\times$ we fix the Haar measure $d\xi = \prod_v d^\times x_v$, the local measures being given by $d^\times x_v = \zeta_v(1)\frac{dx_v}{|x_v|}$, where $\zeta_p(s) = (1-p^{-s})^{-1}$ for finite primes p and $\zeta_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s)$.

If D is a quaternion algebra over $\mathbb{Q}$, tr and ν denote the reduced trace and the reduced norm respectively. The canonical involution on D is denoted by i so that $\mathrm{tr}(x) = x + x^i$ and $\nu(x) = xx^i$. Let $\langle , \rangle$ be the quadratic form on D given by $\langle x, y\rangle = \mathrm{tr}(xy^i) = xy^i + yx^i$. We choose a Haar measure dx_v on $D_v = D \otimes \mathbb{Q}_v$ by requiring that the Fourier transform $\hat{\varphi}(y_v) = \int_{D_v} \varphi(x_v)\langle x_v, y_v\rangle dx_v$ be autodual. On $D_v^\times = (D\otimes\mathbb{Q}_v)^\times$ we fix the Haar measure $d^\times x_v = \zeta_v(1)\frac{dx_v}{|\nu(x_v)|}$. These local measures induce a global measure $d^\times x = \prod_v d^\times x_v$ on $D^\times(\mathbb{A})$ (the adelic points of the algebraic group $D^\times$). In the case $D^\times = \mathrm{GL}(2)$, at finite primes p, the volume of the maximal compact $\mathrm{GL}_2(\mathbb{Z}_p)$ with respect to the measure $d^\times x_p$ is easily computed to be $\zeta_p(2)^{-1}$. On the infinite factor $\mathrm{GL}_2(\mathbb{R})$ one sees that

$$d^\times x_\infty = d^\times a_1 d^\times a_2 db d\theta \text{ if } x_\infty = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\kappa_\theta,$$

where $\kappa_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Let $D^{(1)}$ and $PD^\times$ denote the derived and adjoint groups of $D^\times$ respectively. On $D^{(1)}(\mathbb{A})$ we pick the measure $d^{(1)}x = \prod_v dx_{1,v}$ where $dx_{1,v}$ is compatible with the exact sequence $1 \to D_v^{(1)} \to D_v^\times \xrightarrow{\nu} \mathbb{Q}_v^\times \to 1$. Likewise on $PD^\times(\mathbb{A})$ we pick the measure $d^\times x = \prod_v d^\times x_v$ where the local measures $d^\times x_v$ are compatible with the exact sequence $1 \to \mathbb{Q}_v^\times \to D_v^\times \to PD_v^\times \to 1$. It is well known that with respect to these measures, $\mathrm{vol}(D^{(1)}(\mathbb{Q}) \setminus D^{(1)}(\mathbb{A})) = 1$ and $\mathrm{vol}(PD^\times(\mathbb{Q}) \setminus PD^\times(\mathbb{A})) = 2$.

If W is a symplectic space and V an orthogonal space (both over $\mathbb{Q}$), $\mathrm{GSp}(W)$ denotes the group of symplectic similitudes of W and $\mathrm{GO}(V)$ the group of orthogonal similitudes of V, both viewed as algebraic groups. We also denote by $\mathrm{GSp}(W)^{(1)}$ and $\mathrm{GO}(V)^{(1)}$ the subgroups with similitude norm 1 and by $\mathrm{GO}(V)^0$ the identity component of $\mathrm{GO}(V)$. In the text, W will always be two-dimensional and by a choice of basis $\mathrm{GSp}(W)$ and $\mathrm{GSp}^{(1)}(W)$ are identified with GL(2) and SL(2) respectively, the Haar measures on the corresponding adelic groups being as chosen as in the previous paragraph. For $H = \mathrm{GO}(V)$ or $\mathrm{GO}(V)^0$ we pick Haar measures $d^\times h$ on $H(\mathbb{A})$ such that $\int_{\mathbb{A}^\times H(\mathbb{Q})\backslash H(\mathbb{A})} d^\times h = 1$. The similitude norm induces a map $\nu : H(\mathbb{Q})Z_{H,\infty} \setminus H(\mathbb{A}) \to \mathbb{Q}^\times(\mathbb{Q}_\infty^\times)^+ \setminus \mathbb{Q}_\mathbb{A}^\times$ whose kernel is identified with $H^{(1)}(\mathbb{Q}) \setminus H^{(1)}(\mathbb{A})$. As in [15, §5.1], we pick a Haar measure $d^{(1)}h$ on $H^{(1)}(\mathbb{A})$ such that the quotient measures satisfy $d^\times h = d^{(1)}h d\xi$.

Let $\mathfrak{H}$ denote the complex upper half plane. The group $\mathrm{GL}_2(\mathbb{R})^+$ consisting of elements of $\mathrm{GL}_2(\mathbb{R})$ with positive determinant acts on $\mathfrak{H}$ by $\gamma\cdot z = \frac{az+b}{cz+d}$ where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We define also $j(\gamma, z) = (cz+d)\det(\gamma)^{-1}$ and $J(\gamma, z) = (cz+d)$ for any element $\gamma \in \mathrm{GL}_2(\mathbb{R})$ and $z \in \mathfrak{H}$.

As is usual in the theory, we fix once and for all embeddings $i : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\lambda : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. These induce on every number field an infinite and p-adic place.

2. Shimura curves and an integrality criterion

2.1. *Modular forms on quaternion algebras.* Let N be a square-free integer with $N = N^+N^-$ where N^- has an even number of prime factors. Let D be the unique (up to isomorphism) indefinite quaternion algebra over $\mathbb{Q}$ with discriminant N^-. Fix once and for all isomorphisms $\Phi_\infty : D\otimes\mathbb{R} \simeq M_2(\mathbb{R})$ and $\Phi_q : D \otimes \mathbb{Q}_q \simeq M_2(\mathbb{Q}_q)$ for all $q \nmid N^-$. Any order in D gives rise to an order in $D \otimes \mathbb{Q}_q$ for each prime q which for almost all primes q is equal (via Φ_q) to the maximal order $M_2(\mathbb{Z}_q)$. Conversely given local orders R_q in $D\otimes\mathbb{Q}_q$ for all finite q, such that $R_q = M_2(\mathbb{Z}_q)$ for almost all q, they arise from a unique global order R. Let $\mathcal{O}$ be the maximal order in D such that $\Phi_q(\mathcal{O} \otimes \mathbb{Z}_q) = M_2(\mathbb{Z}_q)$ for $q \nmid N^-$ and such that $\mathcal{O} \otimes \mathbb{Z}_q$ is the unique maximal order in $D \otimes \mathbb{Q}_q$ for $q \mid N^-$. It is well known that all maximal orders in D are conjugate to $\mathcal{O}$. Let $\mathcal{O}'$ be the Eichler order of level N^+ given by $\Phi_q(\mathcal{O}' \otimes \mathbb{Z}_q) = \Phi_q(\mathcal{O} \otimes \mathbb{Z}_q)$ for all $q \nmid N^+$, and such that $\Phi_q(\mathcal{O}' \otimes \mathbb{Z}_q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_q), c \equiv 0 \bmod q \right\}$ for all $q \mid N^+$.

2.1.1. *Classical and adelic modular forms.* Let $\Gamma = \Gamma_0^{N^-}(N^+)$ be the group of norm 1 units in $\mathcal{O}'$. (If $N^- = 1$ we will drop the superscript and write Γ simply as $\Gamma_0(N)$.) Via the isomorphism Φ_∞ the group Γ may be viewed as a subgroup of $\mathrm{SL}_2(\mathbb{R})$ and hence acts in the usual way on $\mathfrak{H}$. Let k be an even integer. A (holomorphic) modular form f of weight k and character ω (ω being a Dirichlet character of conductor N_ω dividing N^+) for the group Γ is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ such that $f(\gamma(z))(cz + d)^{-k} = \omega(\gamma)f(z)$, for all $\gamma \in \Gamma$, where we denote also by the symbol ω the character on Γ

associated to ω in the usual way (see [43]). Denote the space of such forms by $M_k(\Gamma,\omega)$. We will usually work with the subspace $S_k(\Gamma,\omega)$ consisting of cusp forms (i.e. those that vanish at all the cusps of Γ). When $N^- > 1$, there are no cusps and $S_k(\Gamma,\omega) = M_k(\Gamma,\omega)$. The space $S_k(\Gamma,\omega)$ is equipped with a Hermitean inner product, the Petersson inner product, defined by $\langle f_1, f_2\rangle = \int_{\Gamma\backslash\mathfrak{H}} f_1(z)\overline{f_2(z)}y^k d\mu$ where $d\mu$ is the invariant measure $\frac{1}{y^2}dxdy$.

To define adelic modular forms, let $\tilde{\omega}$ be the character of $\mathbb{Q}_{\mathbb{A}}^\times$ corresponding to ω via class field theory. Denote by $L^2(D_{\mathbb{Q}}^\times \setminus D_{\mathbb{A}}^\times,\omega)$ the space of functions $F : D_{\mathbb{A}}^\times \to \mathbb{C}$ satisfying $F(\gamma z\beta) = \tilde{\omega}(z)F(\beta)\ \ \forall\gamma \in D_{\mathbb{Q}}^\times$ and $z \in \mathbb{A}^\times$ and having finite norm under the inner product $\langle F_1, F_2\rangle = \frac{1}{2}\int_{\mathbb{Q}_{\mathbb{A}}^\times D_{\mathbb{Q}}^\times\backslash D_{\mathbb{A}}^\times} F_1(\beta)\overline{F_2(\beta)}d^\times\beta$. Also let $L_0^2(D_{\mathbb{Q}}^\times \setminus D_{\mathbb{A}}^\times,\omega) \subseteq L^2(D_{\mathbb{Q}}^\times \setminus D_{\mathbb{A}}^\times,\omega)$ be the closed subspace consisting of cuspidal functions. If $U = \prod_q U_0^{N^-}(N^+)_q$ is the compact subgroup of $D_{\mathbb{A}_f}^\times$ given by $U_0^{N^-}(N^+)_q = (\mathcal{O}' \otimes \mathbb{Z}_q)^\times$ for all finite primes q, one has

$$D^\times(\mathbb{A}) = D^\times(\mathbb{Q}) \cdot (U \times (D_\infty^\times)^+) \tag{1}$$

(by strong approximation) and $(U \times (D_\infty^\times)^+) \cap D^\times(\mathbb{Q}) = \Gamma$. Since $N_\omega \mid N^+$, the character $\tilde{\omega}$ restricted to $\mathbb{Q}_{\mathbb{A}_f}^\times$ can be extended in the usual way to a character of U, also denoted by $\tilde{\omega}$. A (cuspidal) adelic automorphic form of weight k and character ω for U is a smooth (i.e. locally finite in the p-adic variables and C^∞ in the archimedean variables) function $F \in L_0^2(D_{\mathbb{Q}}^\times \setminus D_{\mathbb{A}}^\times,\omega)$ such that $F(\beta\kappa) = \tilde{\omega}(\kappa_{fin})e^{-\imath k\theta}F(\beta)$ if $\kappa = \prod_{q<\infty}\kappa_q \times \kappa_\theta \in U \times \mathrm{SO}_2(\mathbb{R})$. We denote the space of such forms by $S_k(U,\omega)$. The assignment $f \longmapsto F$, $F(\beta) = f(\beta_\infty(\imath))j(\beta_\infty,\imath)^{-k}\tilde{\omega}(\kappa)$, if $\beta = \gamma\kappa\beta_\infty$ is a decomposition of β given by (1), is independent of the choice of decomposition and gives an isomorphism $S_k(\Gamma,\omega) \simeq S_k(U,\omega)$. It is easy to check that if f_i corresponds to F_i under this isomorphism, then $\langle F_1, F_2\rangle = \frac{1}{\mathrm{vol}(\Gamma\backslash\mathfrak{H})}\langle f_1, f_2\rangle$.

If $N_\omega \mid N' \mid N^+$, there is an inclusion $S_k(\Gamma_0^{N^-}(N'),\omega) \hookrightarrow S_k(\Gamma,\omega)$. The subspace of $S_k(\Gamma,\omega)$ generated by the images of all these maps is called the space of oldforms of level N^+ and character ω. The orthogonal complement of the oldspace is called the new subspace and is denoted $S_k(\Gamma)^{\mathrm{new}}$.

We will need to use the language of automorphic representations. (See [8] for details.) If f is a newform in $S_k(\Gamma,\omega)$ then F generates an irreducible automorphic cuspidal representation π_f of (the Hecke algebra of) $D^\times(\mathbb{A})$ that factors as a tensor product of local representations $\pi_f = \otimes\pi_{f,\infty} \otimes \otimes_q\pi_{f,q}$.

2.1.2. *The Jacquet-Langlands correspondence.* We assume now that ω is trivial, and denote the space $S_k(\Gamma,1)$ simply by $S_k(\Gamma)$. This space is equipped with an action of Hecke operators T_q for all primes q (see [32] for instance for a definition). Let $\mathbb{T}_{(N^-,N^+)}$ be the algebra generated over $\mathbb{Z}$ by the Hecke operators T_q for $q \nmid N$. It is well-known that the action of this algebra on the space $S_k(\Gamma)$ is semi-simple. Further, on the new subspace $S_k(\Gamma)^{\mathrm{new}}$, the eigencharacters of $\mathbb{T}_{(N^-,N^+)}$ occur with multiplicity one. In the case when $N^- = 1$ this follows from Atkin-Lehner theory. In the general case it is a consequence of a theorem of Jacquet-Langlands. More precisely one has the following proposition which is an easy consequence of the Jacquet-Langlands correspondence. (We use the symbols λ_f and λ_g to denote the associated characters of the Hecke algebra.)

PROPOSITION 2.1. *Let f be an eigenform of $\mathbb{T}_{(1,N)}$ in $S_k(\Gamma_0(N))^{\mathrm{new}}$ for $N = N^+N^-$. Then there is a unique (up to scaling) $\mathbb{T}_{(N^-,N^+)}$ eigenform g in $S_k(\Gamma)^{\mathrm{new}}$ such that $\lambda_f(T_q) = \lambda_g(T_q)$ for all $q \nmid N$.*

2.1.3. *Shimura curves, canonical models and Heegner points.* Now suppose $N^- > 1$ and denote by X^{an} the compact complex analytic space

$$X^{\mathrm{an}} = D^\times(\mathbb{Q})^+ \setminus \mathfrak{H} \times D^\times(\mathbb{A}_f)/U \simeq \Gamma \setminus \mathfrak{H} \tag{2}$$

and by $X_{\mathbb{C}}$ the corresponding complex algebraic curve. Following Shimura we will define certain special points on $X_{\mathbb{C}}$ called CM points. Let $j : K \hookrightarrow D$ be an embedding of an imaginary quadratic field in D. Then j induces an embedding of $\mathbb{C} = K \otimes \mathbb{R}$ in $D \otimes \mathbb{R}$, hence of $\mathbb{C}^\times$ in $\mathrm{GL}_2(\mathbb{R})^+$. The action of the torus $\mathbb{C}^\times$ on the upper half plane $\mathfrak{H}$ has a unique fixed point z. In fact there are two possible choices of j that fix z. We normalize j so that $J(\Phi_\infty(j(x)), z) = x$ (rather than $\overline{x}$). One refers to such a point z (or even the embedding j itself) as a CM point. Let $\varphi : \mathfrak{H} \to \Gamma \setminus \mathfrak{H}$ be the projection map.

THEOREM 2.2 (Shimura [33]). *The curve $X_{\mathbb{C}}$ admits a unique model over $\mathbb{Q}$ satisfying the following: for any embedding $j : K \hookrightarrow D$ such that $j(\mathcal{O}_K) \subset \mathcal{O}$, and associated* CM *point z, the point $\varphi(z)$ on $X_{\mathbb{C}}$ is defined over K^{ab}, the maximal abelian extension of K in $\overline{\mathbb{Q}}$. If $\sigma \in \mathrm{Gal}(K^{\mathrm{ab}}/K)$ then the action of σ on $\varphi(z)$ is given by $\varphi(z)^\sigma =$ the class of $[z, j_{\mathbb{A}_f}(i(\sigma)_{fin})]$ via the isomorphism* (2), *where $i(\sigma)$ is any element of $K_{\mathbb{A}}^\times$ mapping to σ under the reciprocity map $K_{\mathbb{A}}^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ given by class field theory.*

It is well known that the imaginary quadratic fields that admit embeddings into D are precisely those that are not split at any of the primes dividing N^-. Let $U_j = K_{\mathbb{A}_f}^\times \cap j_{\mathbb{A}_f}^{-1}(U)$. Then it is clear from the above theorem that $\varphi(z)$ is defined over the class field of K corresponding to the subgroup $K^\times U_j K_\infty^\times$. We will be particularly interested in the case when K is unramified at N and $j(\mathcal{O}_K) \subset \mathcal{O}'$, the corresponding CM points being called Heegner points. For any Heegner point it is clear that U_j is the maximal compact subgroup of $K^\times(\mathbb{A}_f)$ and hence such points are defined over the Hilbert class field of K. Heegner points exist if and only if K is split at all the primes dividing N^+ and inert at all the primes dividing N^-. In that case, there are exactly $2^t h_K$ of them ($t =$ the number of primes dividing N, $h_K =$ class number of K), that split up into 2^t conjugacy classes under the action of the class group of K. (See [2] and [39] for more details.)

2.2. *A ratio of Petersson norms.* Let $f \in S_k(\Gamma_0(N))^{\mathrm{new}}$ be a normalized Hecke eigenform (i.e. with first Fourier coefficient $= 1$) and g be the unique (up to a scalar) Hecke eigenform in $S_k(\Gamma)$ with the same Hecke eigenvalues as f. Then $s_g = g(z)(2\pi\imath \cdot dz)^{\otimes k/2}$ is invariant under Γ, hence descends to a section of $\Omega^{\otimes k/2}$ on X^{an} and by GAGA induces a section of $\Omega^{\otimes k/2}$ on $X_{\mathbb{C}}$. In this way, one obtains a natural isomorphism $S_k(\Gamma) \simeq H^0(X_{\mathbb{C}}, \Omega^{\otimes k/2})$. It is well known that the field K_f generated by the Hecke eigenvalues of f is a totally real number field. Suppose for the moment that $K_f = \mathbb{Q}$ and let V be the $\mathbb{Q}$-vector space $H^0(X_{\mathbb{Q}}, \Omega^{\otimes k/2})$. Since λ_g takes values in $\mathbb{Q}$, we can choose g such that s_g lies in V. Let $\mathcal{X}$ be the minimal regular model of $X_{\mathbb{Q}}$ over $\operatorname{spec}\mathbb{Z}$. It is known that $X_{\mathbb{Q}}$ is a semi-stable curve, i.e. that the fibres of $\mathcal{X}$ over any prime q are reduced and have only ordinary double points as singularities. The relative dualizing sheaf $\omega = \omega_{\mathcal{X}/\mathbb{Z}}$ is then an invertible sheaf on $\mathcal{X}$ that agrees with Ω on the generic fibre. Denote by $\mathcal{V}$ the lattice $H^0(\mathcal{X}, \omega^{\otimes k/2})$ and normalize g by requiring that s_g be a primitive element in this lattice. This fixes g up to ± 1, so the Petersson norm $\langle g, g\rangle$ is well defined. Now define $\beta = \frac{\langle f,f\rangle}{\langle g,g\rangle}$.

THEOREM 2.3. (i) (Shimura [34]). *$\beta \in \overline{\mathbb{Q}}$.*

(ii) (Harris-Kudla [14]). *$\beta \in \mathbb{Q}$.*

In this article we study the p-integrality properties of β. In fact we can also prove a corresponding result in the more general case when $K_f \neq \mathbb{Q}$, but, since the class number of K_f need not be 1 we are forced to formulate the result λ-adically. Choosing g such that $s_g \in V \otimes K_f$ and further such that s_g is λ-adically primitive in $\mathcal{V} \otimes \mathcal{O}_{K_f}$, we see that g is well-defined up to a λ-adic unit in K_f. It is known, again due to Shimura, that $\beta = \langle f, f\rangle/\langle g, g\rangle \in \overline{\mathbb{Q}}$ and due to Harris-Kudla that $\beta \in K_f$. We will prove a λ-adic integrality result for β. To motivate our results in the general case, we first study in the next section the special case $k = 2$ and $K_f = \mathbb{Q}$.

2.2.1. *A special case*: *elliptic curves and level-lowering congruences.* In this section we restrict ourselves to the case when $k = 2$ and $K_f = \mathbb{Q}$. Then f corresponds to an isogeny class of elliptic curves over $\mathbb{Q}$, and if E is any curve in this class there exist surjective maps from $J_0(N)$ and J to E (where $J_0(N)$ and J denote the Jacobians of $X_0(N)$ and X respectively). Let E_1 and E_2 be the *strong* elliptic curves corresponding to $J_0(N)$ and J respectively and let $\varphi_1 : J_0(N) \to E_1$ and $\varphi_2 : J \to E_2$ denote the corresponding maps. Also let ω_i be a Neron differential on E_i i.e. a generator of the rank-1 $\mathbb{Z}$-module $H^0(\mathcal{E}_i, \Omega^1)$ where $\mathcal{E}_i$ denotes the Neron model of E_i over $\operatorname{spec}\mathbb{Z}$. If A_2 is the kernel of the map φ_2, we get an exact sequence of abelian varieties

$$0 \to A_2 \to J \to E_2 \to 0.$$

By a theorem of Raynaud ([1, App.]) one then has an exact sequence

$$0 \to \mathrm{Lie}(\mathcal{A}_2) \to \mathrm{Lie}(\mathcal{J}) \to \mathrm{Lie}(\mathcal{E}_2) \to (\mathbb{Z}/2\mathbb{Z})^r \to 0$$

with $r = 0$ or 1, where the script letters denote Neron models. Denote by H the cokernel of the map $\mathrm{Lie}(\mathcal{A}_2) \to \mathrm{Lie}(\mathcal{J})$. Then we have exact sequences

$$0 \to \mathrm{Lie}(\mathcal{A}_2) \to \mathrm{Lie}(\mathcal{J}) \to H \to 0,$$
$$0 \to H \to \mathrm{Lie}(\mathcal{E}_2) \to (\mathbb{Z}/2\mathbb{Z})^r \to 0$$

and hence by taking duals, exact sequences

$$0 \to H^\vee \to H^0(\mathcal{J}, \Omega^1) \to H^0(\mathcal{A}_2, \Omega^1) \to 0,$$
$$0 \to H^0(\mathcal{E}_2, \Omega^1) \to H^\vee \to (\mathbb{Z}/2\mathbb{Z})^r \to 0.$$

Thus the injective map $\varphi_2^* : H^0(\mathcal{E}_2, \Omega^1) \to H^0(\mathcal{J}, \Omega^1)$ remains injective on tensoring with $\mathbb{Z}/q\mathbb{Z}$ for any prime q other than 2. Noting now that one has a canonical isomorphism $H^0(\mathcal{X}, \omega) \simeq H^0(\mathcal{J}, \Omega^1)$, we see that $\varphi_2^*\omega_2$ equals s_g (via this isomorphism) except possibly for a factor of ± 2. Let $\psi_2 : X_{\mathbb{C}} \to J_{\mathbb{C}}$ be the embedding corresponding to the choice of any point in $X(\mathbb{C})$ and let φ_2' denote the composite map $\varphi_2 \circ \psi_2$. Then ${\varphi'}_2^*(\omega_2)$ equals s_g possibly up to a factor of ± 2. Also let $\psi_1 : X_0(N) \to J_0(N)$ be the usual embedding corresponding to the cusp at ∞ and let $\varphi_1' = \varphi_1 \circ \psi_1$. Then it is shown in [1] that ${\varphi'}_1^*(\omega_1) = 2\pi\imath f(z)dz$ possibly up to a factor of 2. Writing $\sim_2$ to mean equality to up to possibly a power of 2, we get $\int_{E_1(\mathbb{C})} \omega_1 \wedge \bar{\omega_1} \sim_2 \frac{1}{\deg \varphi_1'} 4\pi^2 \langle f, f\rangle$ and $\int_{E_2(\mathbb{C})} \omega_2 \wedge \bar{\omega_2} \sim_2 \frac{1}{\deg \varphi_2'} 4\pi^2 \langle g, g\rangle$.

Denote by S the set of Eisenstein primes for the isogeny class containing E_1 and E_2 i.e. S is the set of primes q for which the representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the q-torsion $E_i[q]$ is reducible. Choosing any isogeny $\varphi_3 : E_1 \to E_2$ of minimal degree it is easy to see that the degree of φ_3 can only be divisible by primes in S. Using the symbol $\sim$ to mean equality up to primes in $S \cup \{2\}$, we see then that $\int_{E_1(\mathbb{C})} \omega_1 \wedge \bar{\omega_1} \sim \int_{E_2(\mathbb{C})} \omega_2 \wedge \bar{\omega_2}$ and hence $\frac{\langle f,f\rangle}{\langle g,g\rangle} \sim \frac{\deg \varphi_1'}{\deg \varphi_2'}$.

Now it is known that $\deg \varphi_1'$ measures congruences between f and other forms of level dividing N. Likewise $\deg \varphi_2'$ measures congruences between g and other forms of weight 2 on X. Such forms correspond to forms on $X_0(N)$ which are new at the primes dividing N^-. Thus the ratio $\frac{\deg \varphi_1'}{\deg \varphi_2'}$ should measure congruences between f and other forms on $X_0(N)$ that are old at some prime dividing N^-. Indeed, in [29] it is shown that $\frac{\deg \varphi_1'}{\deg \varphi_2'} \sim \prod_{q|N^-} c_q$, where c_q is the order of the component group of $\mathcal{E}_1$ at q. Also it follows from [28] that a non-Eisenstein prime p divides c_q exactly when p is a level-lowering congruence prime for f at q; i.e., when there exists a form f' of level N/q such that the

Hecke eigenvalues (away from N) of f and f' are congruent to each other modulo some prime of $\overline{\mathbb{Q}}$ above p. As a consequence we see that away from Eisenstein primes, the ratio $\frac{\langle f,f\rangle}{\langle g,g\rangle}$ is integral and is divisible by a prime p exactly when p is a level-lowering congruence prime for f at some prime q dividing N^-.

2.2.2. *The general case*: *statement of the main theorem.* The results of the previous section motivate the following theorem which is the central result of this article. It is proved in Sections 5.1 and 5.2.

THEOREM 2.4. *Suppose g is chosen such that s_g is K_f-rational and λ-adically primitive. Let $\beta = \frac{\langle f,f\rangle}{\langle g,g\rangle}$. Then $\beta \in K_f$. Further, for $p \nmid M := \prod_{q|N} q(q-1)(q+1)$ and $p > k+1$,*

1. $v_\lambda(\beta) \geq 0$.
2. *If there exists a prime $q \mid N^-$ and a newform f' of level dividing N/q and weight k such that $\rho_{f,\lambda} \equiv \rho_{f',\lambda} \bmod \lambda$ then $v_\lambda(\beta) > 0$. Here $\rho_{f,\lambda}$ and $\rho_{f',\lambda}$ denote the two dimensional λ-adic representations associated to f and f'.*

2.3. *An integrality criterion for forms on Shimura curves.* This section is devoted to developing an integrality criterion for forms on Shimura curves using values at CM points. The main result is Proposition 2.9.

2.3.1. *Integral models of Shimura curves.* We now choose an auxiliary integer $N' \geq 4$, prime to N and such that p does not divide N'. Consider the Shimura curve $X' = \Gamma' \setminus \mathfrak{H}$ associated to the subgroup U_1 of U consisting of elements whose component at q for $q \mid N'$ is congruent to $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ mod q. This curve too has a *canonical model* defined over $\mathbb{Q}$ that is the solution to a certain moduli problem (parametrizing abelian varieties of dimension 2 with an action of $\mathcal{O}$ and suitable level structure). The moduli problem can in fact be defined over $\mathbb{Z}[\frac{1}{NN'}]$ and is represented by a fine moduli scheme $\mathcal{X}'$ over $\mathbb{Z}[\frac{1}{NN'}]$, that is geometrically connected, proper and smooth of relative dimension 1. For all these facts, see [6, §§3 and 4].

Let $\mathcal{Y}$ and $\mathcal{Y}'$ denote the base change of $\mathcal{X}$ and $\mathcal{X}'$ to $\mathbb{Z}_{(p)}$ respectively. Then the canonical map from $X'_{\mathbb{C}}$ to $X_{\mathbb{C}}$ is in fact defined over $\mathbb{Q}$ and extends to a map $u : \mathcal{Y}' \to \mathcal{Y}$. Clearly we may choose N' such that $p \nmid \deg(u)$. The following lemma is then evident.

LEMMA 2.5. *Let $s \in H^0(X_L, \Omega^{\otimes l})$ for L a number field and l a positive integer. If L_λ is the completion of L at λ and $\mathcal{O}_\lambda$ the ring of integers of L_λ, then $s \in H^0(\mathcal{X}, \Omega^{\otimes l}) \otimes \mathcal{O}_\lambda = H^0(\mathcal{X}_{\mathcal{O}_\lambda}, \Omega^{\otimes l}) \iff u^*(s) \in H^0(\mathcal{Y}', \Omega^{\otimes l}) \otimes \mathcal{O}_\lambda = H^0(\mathcal{Y}'_{\mathcal{O}_\lambda}, \Omega^{\otimes l})$.*

The field $\mathbb{Q}(\sqrt{-N^-})$ embeds in D. Pick an element $t \in D$ such that $t^2 = -N^-$, and let $*$ denote the involution on D given by $x^* = t^{-1}x^i t$. Let $\pi : \mathcal{A} \to \mathcal{Y}'$ be the universal abelian scheme over $\mathcal{Y}'$. It is known (see [6, Lemma 5]) that there exists a unique principal polarization on $\mathcal{A}/\mathcal{Y}'$ such that on all geometric points x the associated Rosati involution induces the involution $*$ on $\mathcal{O} \hookrightarrow \mathrm{End}(A_x)$. Let $\phi : \mathcal{A} \simeq \mathcal{A}^\vee$ be the isomorphism associated to the principal polarization. Via ϕ, $R^1\pi_*\mathcal{O}_{\mathcal{A}}$ is isomorphic to $R^1\pi_*\mathcal{O}_{\mathcal{A}^\vee}$; hence $R^1\pi_*\mathcal{O}_{\mathcal{A}}$ and $\pi_*\Omega^1_{\mathcal{A}/\mathcal{Y}'}$ are dual to each other so that the adjoint of $\delta \in \mathcal{O}$ is δ^*. The proof of [6, Lemma 7] shows that there is a canonical isomorphism of rank-2 locally free sheaves

$$\varphi : \pi_*\Omega^1_{\mathcal{A}/\mathcal{Y}'} \simeq R^1\pi_*\mathcal{O}_{\mathcal{A}} \otimes \Omega^1_{\mathcal{Y}'/\mathbb{Z}_{(p)}}. \tag{3}$$

Recall that the above map is constructed in the following way. Since $\mathcal{A}/\mathcal{Y}'$ and $\mathcal{Y}'/\mathbb{Z}_{(p)}$ are smooth, the sequence

$$0 \to \pi^*\Omega^1_{\mathcal{Y}'/\mathbb{Z}_{(p)}} \to \Omega^1_{\mathcal{A}/\mathbb{Z}_{(p)}} \to \Omega^1_{\mathcal{A}/\mathcal{Y}'} \to 0$$

is exact. Applying $R^{\cdot}\pi_*$ to this sequence gives a map

$$\pi_*\Omega^1_{\mathcal{A}/\mathcal{Y}'} \to R^1\pi_*\pi^*\Omega^1_{\mathcal{Y}'/\mathbb{Z}_{(p)}} \simeq R^1\pi_*\mathcal{O}_{\mathcal{A}} \otimes \Omega^1_{\mathcal{Y}'/\mathbb{Z}_{(p)}}$$

which is the required one. Taking the second exterior power of (3) we get

$$\wedge^2\pi_*\Omega^1_{\mathcal{A}/\mathcal{Y}'} \simeq \wedge^2(R^1\pi_*\mathcal{O}_{\mathcal{A}} \otimes \Omega^1_{\mathcal{Y}'/\mathbb{Z}_{(p)}}) = \wedge^2 R^1\pi_*\mathcal{O}_{\mathcal{A}} \otimes (\Omega^1_{\mathcal{Y}'/\mathbb{Z}_{(p)}})^{\otimes 2}$$

and by the duality of $R^1\pi_*\mathcal{O}_{\mathcal{A}}$ and $\pi_*\Omega^1_{\mathcal{A}/\mathcal{Y}'}$, a canonical isomorphism

$$\varphi^{(l)} : (\wedge^2\pi_*\Omega^1_{\mathcal{A}/\mathcal{Y}'})^{\otimes l} \simeq (\Omega^1_{\mathcal{Y}'/\mathbb{Z}_{(p)}})^{\otimes l}$$

for every even integer l. Let $x : \mathrm{Spec}\ L \to X'$ be a geometric point of X' defined over a number field L. By the properness of $\mathcal{Y}'$ over $\mathbb{Z}_{(p)}$, x extends to an $\mathcal{O}_\lambda$-valued point of $\mathcal{Y}'$, $x : \mathrm{Spec}\ \mathcal{O}_\lambda \to \mathcal{Y}'$. Denote by $\mathcal{A}_{x,\lambda}, A_{x,\lambda}$ and A_x the Abelian schemes over $\mathcal{O}_\lambda, L_\lambda$ and L, respectively, obtained by pulling back the universal family over $\mathcal{Y}'$ via x. Now suppose $s = G(z)(2\pi\imath \cdot dz)^{\otimes l}$ (with G a form of weight $2l$) descends to an L-rational element of $H^0(X', (\Omega^1)^{\otimes l})$. Via the isomorphism $\varphi^{(l)}$, s gives rise to a section $s_x \in H^0(A_x, (\wedge^2\pi_*\Omega^1_{A_x})^{\otimes l})$. If further s is λ-integral, i.e. if $s \in H^0(\mathcal{Y}', (\Omega^1_{\mathcal{Y}/\mathbb{Z}_{(p)}})^{\otimes l})$ then $s_x \in H^0(\mathcal{A}_{x,\lambda}, (\wedge^2\pi_*\Omega^1_{\mathcal{A}_{x,\lambda}})^{\otimes l})$. Conversely, let R be an infinite set of algebraic points such that the mod λ reductions of $x \in R$ still form an infinite set. If $s_x \in H^0(\mathcal{A}_{x,\lambda}, (\wedge^2\pi_*\Omega^1_{\mathcal{A}_{x,\lambda}})^{\otimes l})$ for all $x \in R$, it is clear that s must be λ-integral (since the reduction of $\mathcal{Y}'$ mod λ is irreducible).

We use the same symbol x to denote also the corresponding point of $X'(\mathbb{C})$ and suppose $\tau \in \mathfrak{H}$ is such that the image of τ in $X' = \Gamma \setminus \mathfrak{H}$ is equal to x. Then one has a canonical identification

$$\frac{\mathbb{C}^2}{\Phi_\infty(\mathcal{O})\begin{bmatrix} \tau \\ 1 \end{bmatrix}} \simeq A_{x,\mathbb{C}}.$$

Via this isomorphism s_x corresponds to some multiple of $(dt_1 \wedge dt_2)^{\otimes l}$ where t_1, t_2 denote the coordinates on $\mathbb{C}^2$.

LEMMA 2.6. $s_x = \frac{(2\pi\imath)^{2l}}{(N^-)^{l/2}} G(\tau)(dt_1 \wedge dt_2)^{\otimes l}$.

Before proving the lemma we note the following consequence. Let ω_x be an element of $H^0(A_x, (\wedge^2 \pi_* \Omega^1_{A_{x,\lambda}}))$ that is λ-integral and λ-adically primitive (with respect to the lattice $H^0(\mathcal{A}_{x,\lambda}, (\wedge^2 \pi_* \Omega^1_{\mathcal{A}_{x,\lambda}}))$, and suppose that via the isomorphism above, ω_x corresponds to $\mu_\tau dt_1 \wedge dt_2$ for some complex period μ_τ. Since $p \nmid N^-$, the following proposition is a corollary of the lemma and the preceding discussion.

PROPOSITION 2.7. *Let R be an infinite set of algebraic points on X' whose reductions* mod λ *still form an infinite set. Then $s := G(z)(2\pi\imath \cdot dz)^{\otimes l}$ is λ-integral if and only if for all $x \in R$ and $\tau \in \mathfrak{H}$ mapping to x, $\frac{(2\pi)^{2l} G(\tau)}{\mu_\tau^l}$ is a λ-adic integer.* (*Note*: *G has weight $2l$.*)

We now prove the lemma. A similar result is proved in [10] (see statement 4.4.3) in the symplectic case, and we only need to adapt the proof to our context. First one needs to note that the map (3) can be defined in a slightly different way using the Gauss-Manin connection on the relative de Rham cohomology of $\mathcal{A}/\mathcal{Y}'$. Let $H^1_{\mathrm{DR}}(\mathcal{A}/\mathcal{Y}')$ denote the first relative de Rham cohomology sheaf. It is an algebraic vector bundle on $\mathcal{Y}'$ equipped with a canonical integrable connection, the Gauss-Manin connection:

$$\nabla : H^1_{\mathrm{DR}}(\mathcal{A}/\mathcal{Y}') \to H^1_{\mathrm{DR}}(\mathcal{A}/\mathcal{Y}') \otimes \Omega^1_{\mathcal{Y}'/\mathbb{Z}_{(p)}}.$$

Also the de Rham cohomology sits in an exact sequence

$$0 \to \pi_* \Omega^1_{\mathcal{A}/\mathcal{Y}'} \xrightarrow{f_1} H^1_{\mathrm{DR}}(\mathcal{A}/\mathcal{Y}') \xrightarrow{f_2} R^1 \pi_* \mathcal{O}_{\mathcal{A}}.$$

The composite map

$$\pi_* \Omega^1_{\mathcal{A}/\mathcal{Y}'} \xrightarrow{f_1} H^1_{\mathrm{DR}}(\mathcal{A}/\mathcal{Y}') \xrightarrow{\nabla} H^1_{\mathrm{DR}}(\mathcal{A}/\mathcal{Y}') \otimes \Omega^1_{\mathcal{Y}'/\mathbb{Z}_{(p)}} \xrightarrow{f_2 \otimes 1} R^1 \pi_* \mathcal{O}_{\mathcal{A}} \otimes \Omega^1_{\mathcal{Y}'/\mathbb{Z}_{(p)}}$$

is then the same as the map given by (3). To prove the lemma we only need to pull back these vector bundles to $\mathfrak{H}$ and compute explicitly the connection ∇. Let $L_\tau = \Phi_\infty(\mathcal{O}) \begin{bmatrix} \tau \\ 1 \end{bmatrix} \subset \mathbb{C}^2$. If $\mathcal{A}_{\mathbb{C}}$ denotes the universal family over $X'_{\mathbb{C}}$, the pull-back of $\mathcal{A}_{\mathbb{C}}$ to $\mathfrak{H}$ is an analytic family of Abelian varieties with fibre $A_\tau = \mathbb{C}^2/L_\tau$ over the point τ. Let $\mathcal{H}^1_{\mathrm{DR}}(\mathcal{A}/\mathfrak{H})$ denote the analytic vector bundle on $\mathfrak{H}$ obtained by pulling back $H^1_{\mathrm{DR}}(\mathcal{A}_{\mathbb{C}}/X'_{\mathbb{C}})$ to $\mathfrak{H}$. The fibre of this vector bundle over $\tau \in \mathfrak{H}$ is naturally interpreted as the de Rham cohomology of A_τ, hence as the complex vector space $\mathrm{Hom}(L_\tau \otimes_{\mathbb{Z}} \mathbb{C}, \mathbb{C})$ (since $L_\tau = H_1(A_\tau, \mathbb{Z})$ and by the isomorphism between de Rham and singular cohomology).

Denote by E_t the nondegenerate skew-symmetric pairing on $\mathcal{O}$ defined by $E_t(a,b) = \frac{1}{N^-}\mathrm{tr}(ab^\iota t)$ so that $E_t(ca,b) = E_t(a,c^*b)$. Via the natural isomorphism $\mathcal{O} \simeq L_\tau$, E_t induces a pairing on L_τ and we extend it $\mathbb{R}$-linearly to a real-valued skew-symmetric pairing on $\mathbb{C}^2$, denoted E_τ. Then E_τ takes integral values on L_τ, and is a nondegenerate Riemann form for A_τ. In fact it is easy to check (for instance using an explicit symplectic basis for $\mathcal{O}$ as constructed in [16]) that the Pfaffian of E_τ restricted to L_τ is 1; hence the associated polarization of A_τ is principal. Since the corresponding Rosati involution is just the involution $*$, we see that the principal polarization associated to the Riemann form E_τ is ϕ_τ. Let $\{e_1,e_2,e_3,e_4\}$ be a symplectic basis for $\mathcal{O}\otimes\mathbb{R}$ with respect to the skew-symmetric form E_t and $e_{i,\tau} = \Phi_\infty(e_i)\begin{bmatrix}\tau\\1\end{bmatrix} \in L_\tau\otimes\mathbb{R}$ so that the $e_{i,\tau}$ form a symplectic basis for $L_\tau\otimes\mathbb{R}$ with respect to the form E_τ. We define $\alpha_1,\alpha_2,\beta_1,\beta_2$ to be the global sections of $\mathcal{H}^1_{\mathrm{DR}}(\mathcal{A}/\mathfrak{H})$ which when restricted to the fibre A_τ give the basis dual to $\{e_{1,\tau},e_{2,\tau},e_{3,\tau},e_{4,\tau}\}$. If H^1 is the complex subspace of $H^0(\mathfrak{H},\mathcal{H}^1_{\mathrm{DR}}(\mathcal{A}/\mathfrak{H}))$ spanned by $\alpha_1,\alpha_2,\beta_1,\beta_2$, one has $\mathcal{H}^1_{\mathrm{DR}}(\mathcal{A}/\mathfrak{H}) = H^1\otimes\mathcal{O}_{\mathfrak{H}}$. The sections $\alpha_1,\alpha_2,\beta_1,\beta_2$ are horizontal for the Gauss-Manin connection and on $\mathcal{H}^1_{\mathrm{DR}}(\mathcal{A}/\mathfrak{H}) = H^1\otimes\mathcal{O}_{\mathfrak{H}}$ the connection ∇ is just $1\otimes d$. The principal polarization ϕ induces a nondegenerate alternating form

$$\langle,\rangle_{\mathrm{DR}} : H^1_{\mathrm{DR}}(\mathcal{A}_{\mathbb{C}}/X'_{\mathbb{C}}) \times H^1_{\mathrm{DR}}(\mathcal{A}_{\mathbb{C}}/X'_{\mathbb{C}}) \to \mathcal{O}_{X'_{\mathbb{C}}}.$$

Pulling back to $\mathfrak{H}$, this form can be computed on the global sections $\alpha_1,\alpha_2,\beta_1,\beta_2$:

$$\begin{aligned}\langle\beta_j,\alpha_k\rangle_{\mathrm{DR}} &= \frac{1}{2\pi\imath}\delta_{j,k} = -\langle\alpha_k,\beta_j\rangle_{\mathrm{DR}},\\ \langle\beta_j,\beta_k\rangle_{\mathrm{DR}} &= 0 = \langle\alpha_j,\alpha_k\rangle_{\mathrm{DR}}.\end{aligned}$$

Let t_1 and t_2 denote the coordinates on $\mathbb{C}^2$ so that $\pi_*\Omega^1_{\mathcal{A}/\mathfrak{H}}$ is generated freely by dt_1 and dt_2. Suppose that $\Phi_\infty(e_i) = \begin{pmatrix} p_i & q_i\\ r_i & s_i\end{pmatrix}$. Clearly,

$$\begin{aligned}dt_1 &= (p_1z+q_1)\alpha_1 + (p_2z+q_2)\alpha_2 + (p_3z+q_3)\beta_1 + (p_4z+q_4)\beta_2,\\ dt_2 &= (r_1z+s_1)\alpha_1 + (r_2z+s_2)\alpha_2 + (r_3z+s_3)\beta_1 + (r_4z+s_4)\beta_2,\end{aligned}$$

whence

$$\begin{aligned}\nabla(dt_1) &= (p_1\alpha_1+p_2\alpha_2+p_3\beta_1+p_4\beta_2)\otimes dz,\\ \nabla(dt_2) &= (r_1\alpha_1+r_2\alpha_2+r_3\beta_1+r_4\beta_2)\otimes dz.\end{aligned}$$

Suppose $\varphi(dt_1) = (x_{11}(dt_1)^\vee + x_{12}(dt_2)^\vee)\otimes dz$ and $\varphi(dt_2) = (x_{21}(dt_1)^\vee + x_{22}(dt_2)^\vee)\otimes dz$. Then $\varphi^{(2)}(dt_1\wedge dt_2)^{\otimes 2} = (x_{11}x_{22}-x_{12}x_{21})dz^{\otimes 2}$ where $x_{ij}dz = \langle\nabla(dt_i),dt_j\rangle_{\mathrm{DR}}$. To compute $\det(x_{ij})$ it is simplest to work with some explicit choice of e_i. Without loss, we may assume that $\Phi_\infty(t) = \begin{pmatrix}0 & -1\\ N^- & 0\end{pmatrix}$ and that Φ_∞ maps e_1, e_2, e_3, e_4 to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ N^- & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, respectively. Thus, $dt_1 = z\alpha_1 + \beta_1, dt_2 = N^- z\alpha_2 + \beta_2$, whence $\nabla(dt_1) = \alpha_1 dz, \nabla(dt_2) = N^-\alpha_2 dz$, $x_{11} = -1/2\pi\imath, x_{12} = 0, x_{21} = 0, x_{22} = -N^-/2\pi\imath$, and finally $\det(x_{ij}) = \frac{N^-}{(2\pi\imath)^2}$. Hence $\varphi^{(l)}(dt_1 \wedge dt_2)^{\otimes l} = \frac{(N^-)^{l/2}}{(2\pi\imath)^l} dz^{\otimes l}$ for any even integer l. It follows then that via the isomorphism $\varphi^{(l)}$ the section $s = (2\pi\imath)^l G(z) dz^{\otimes l}$ corresponds to $(2\pi\imath)^{2l}(N^-)^{-l/2} G(\tau)(dt_1 \wedge dt_2)^{\otimes l}$ on the fibre over τ. This completes the proof of Lemma 2.6.

2.3.2. *Algebraic Hecke characters.* Let K and L be number fields. Let $\underline{\chi} : K_{\mathbb{A}}^\times \to L^\times$ be an algebraic Hecke character and let $\mu : K^\times \to L^\times$ be the restriction of $\underline{\chi}$ to $K^\times$. Thus μ is an algebraic homomorphism of algebraic groups; i.e. there exist integers n_σ such that $\mu(x) = \prod_\sigma (\sigma x)^{n_\sigma}$ where σ ranges over the various embeddings of K into $\overline{\mathbb{Q}}$. The formal sum $\sum_\sigma n_\sigma \sigma$ is called the infinity type of $\underline{\chi}$.

Since μ is algebraic, it induces a continuous homomorphism $\mu_{\mathbb{A}} : K_{\mathbb{A}}^\times \to L_{\mathbb{A}}^\times$. Recall that we have fixed embeddings $i : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\lambda : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. Now $\underline{\chi}$ gives rise to two characters χ and χ_λ which are defined as follows.

(i) Let $\mu_i : K_{\mathbb{A}}^\times \to \mathbb{C}^\times$ be the projection of $\mu_{\mathbb{A}}$ onto the factor corresponding to i. Then $\chi(x) = i(\underline{\chi}(x))/\mu_i(x)$ and χ is a continuous character of $K_{\mathbb{A}}^\times$, trivial on $K^\times$, with values in $\mathbb{C}^\times$ i.e. a Grossencharacter of K.

(ii) Let $\mu_\lambda : K_{\mathbb{A}}^\times \to L_\lambda^\times$ be the projection of $\mu_{\mathbb{A}}$ onto the factor corresponding to λ. Then $\chi_\lambda(x) = (\lambda(\underline{\chi}(x))/\mu_\lambda(x))^{-1}$. Since $L_\lambda^\times$ is a totally disconnected topological group, χ_λ must factor through the group of components of $K_{\mathbb{A}}^\times$, which by class field theory is canonically identified with $\mathrm{Gal}(\overline{K}/K)^{\mathrm{ab}}$. Thus we can think of χ_λ as a character of $\mathrm{Gal}(\overline{K}/K)$ and we shall use the same symbol to denote both the character on the ideles and the Galois group.

For the rest of this article, by a Grossencharacter of K we shall mean a Grossencharacter χ that arises from an algebraic Hecke character $\underline{\chi}$ as in (i) above. By the infinity type of χ we shall mean the infinity type of $\underline{\chi}$. We will also use the same symbol χ to denote the corresponding character on the ideals of K prime to the conductor c_χ. Thus, for $\mathfrak{q}$ an ideal in K coprime to λ and c_χ, we have $\chi(\mathfrak{q}) = \chi_\lambda(\mathrm{Frob}_\mathfrak{q})$ where Frob denotes the arithmetic Frobenius. For any algebraic map $\sigma : K^\times \to K^\times$ we shall denote by χ^σ the Grossencharacter corresponding to the algebraic Hecke character $\underline{\chi}^\sigma$ where $\underline{\chi}^\sigma(x) = \underline{\chi}(\sigma x)$. Especially for K imaginary quadratic, we denote by ρ the nontrivial automorphism of $K/\mathbb{Q}$ and χ^ρ the associated Grossencharacter. Clearly, $\chi_\lambda^\rho(g) = \chi_\lambda(cgc^{-1})$ for any $g \in \mathrm{Gal}(\overline{K}/K)$ where c denotes complex conjugation.

2.3.3. CM *periods.* Let K be an imaginary quadratic field and let p be any prime. We shall define a canonical period Ω associated to the pair (K, p) that will be well defined up to multiplication by a p-adic unit. Let E be any elliptic curve defined over some number field L with CM by $\mathcal{O}_K$ defined also over L. Assume also that E has good reduction over L; we can certainly achieve this by passing to a larger field. Denote by $\mathcal{E}$ the Neron model of E over $\mathcal{O}_L$, the ring of integers of L. Since $M = H^0(\mathcal{E}, \Omega^1)$ is a locally free $\mathcal{O}_L$ module of rank one, we may choose $\omega \in M$ such that the cardinality of $M/\mathcal{O}_L\omega$ is coprime to p. Likewise the module $N = H_1(E(\mathbb{C}), \mathbb{Z})$ is a locally free $\mathcal{O}_K$ module of rank one, so we may choose $\gamma \in N$ such that $N/\mathcal{O}_K\gamma$ has cardinality coprime to p. There is a canonical pairing between M and N given by integration, which we use to define the period $\Omega := \int_\gamma \omega$. If ω_1 and γ_1 also satisfy these conditions, they must differ from ω and γ by p-adic units, so Ω that (up to a p-adic unit) does not depend on the choices of ω and γ. That Ω does not depend on the choice of E is also clear since if E' is another such curve, it must be isogenous to E (over some number field) by an isogeny of degree prime to p.

2.3.4. *The integrality criterion.* Suppose now that z is a Heegner point (see §2.1.3), $j : K^\times \to D^\times$ is the corresponding embedding and p is unramified in K. If $f \in S_k(\Gamma)$ and F is the corresponding adelic automorphic form, define (following [14, App.]) for any Grossencharacter χ of K with weight $(k, 0)$ at infinity,

$$L_\chi(F) = j(\alpha, \imath)^k \int_{K^\times K_\infty^\times \backslash K_\mathbb{A}^\times} F(x\alpha)\chi'(x) d^\times x \tag{4}$$

for any $\alpha \in (D_\infty^\times)^{(1)} = \mathrm{SL}_2(\mathbb{R})$ such that $\alpha(\imath) = z$. Here we pick a Haar measure $d^\times x = \prod_v d^\times x_v$ on $K_\mathbb{A}^\times$ such that for finite v, $d^\times x_v$ gives U_v volume 1 and such that dx_∞ induces on $(\mathbb{R}^\times)^+ \setminus K_\infty^\times$ a measure with volume 1. The quotient measure, also denoted $d^\times x$, on $K^\times K_\infty^\times \setminus K_\mathbb{A}^\times$ has total volume 1. Also $\chi' = \chi^\rho \mathbb{N}^{-k/2}$ and we think of $K_\mathbb{A}^\times$ as a subgroup of $D_\mathbb{A}^\times$ via $j_\mathbb{A}$. It is easy to check that the definition above does not depend on the choice of α. (Indeed, in the notation of [14], $j(\alpha, \imath)^k F(\cdot\alpha)$ is nothing but $\underline{\mathrm{Lift}}(s)$ where s denotes the restriction of the section $f(z)(dz)^{\otimes k/2}$ of the automorphic line bundle $\Omega^{\otimes k/2}$ to the sub-Shimura variety defined by the embedding of the torus $K^\times$ in $D^\times$.) Note that there is some abuse of notation here, since $L_\chi(F)$ depends not only on χ and K but also the specific choice of Heegner point.

We will assume henceforth that $\underline{\chi}$ is an unramified Hecke character of K. We show now that $L_\chi(F)$ is a weighted sum of the values of f at the various Galois conjugates of the CM point z. Pick $y_i \in K_{\mathbb{A}_f}^\times$ such that $K_\mathbb{A}^\times = \bigsqcup_{i=1}^h (K^\times U_K K_\infty^\times) y_i$ where $U_K = \prod_v U_v$, $U_v =$ the units in K_v and $h = h_K$ is the class number of K. Write $j(y_i) = g_i(g_{U,i}\cdot\gamma_i), g_i \in D^\times, g_{U,i} \in U, \gamma_i \in (D_\infty^\times)^+$. One sees immediately that

$$L_\chi(F) = \frac{1}{h} j(\alpha, \imath)^k \sum_{i=1}^h f(\gamma_i \alpha i) j(\gamma_i \alpha, \imath)^{-k} \chi'(y_i) = \frac{1}{h} \sum_{i=1}^h f(\gamma_i z) j(\gamma_i, z)^{-k} \chi'(y_i).$$

Since every element in the class group is $\mathrm{Frob}_{\mathfrak{q}}$ for infinitely many primes $\mathfrak{q}$, we can choose $y_i = (\dots 1, 1, \pi_{\mathfrak{q}_i}, 1, \dots)$ where $\mathfrak{q}_i$ is prime to p and $\pi_{\mathfrak{q}}$ is a uniformiser at $\mathfrak{q}$. Now $y_i^h \in K^\times(U_K \cdot K_\infty^\times)$. Suppose $y_i^h = x(x_U x_\infty)$. Then $x_\infty = x^{-1}$ and it is clear that x is a λ-adic unit (since $xx_{\mathfrak{p}} = 1$ and $x_{\mathfrak{p}}$ is a $\mathfrak{p}$-adic unit for $\mathfrak{p}$ any prime above p). Hence $\chi'(y_i)^h = x_\infty^{k/2}\overline{x}_\infty^{-k/2} = (x/\overline{x})^{-k/2}$ is a λ-adic unit and the same is therefore true of $\chi'(y_i)$.

Let $z_i = \gamma_i z$ and A_{z_i} be the fibre over the image of z_i in X'. Then there is an isomorphism

$$A_{z_i} \simeq \frac{\mathbb{C}^2}{\Phi_\infty(\mathcal{O})\begin{bmatrix} z_i \\ 1 \end{bmatrix}} = \frac{\mathbb{C}^2}{(c_i z + d_i)^{-1}\Phi_\infty(\mathcal{O}g_i^{-1})\begin{bmatrix} z \\ 1 \end{bmatrix}} \simeq \frac{\mathbb{C}^2}{\Phi_\infty(\mathcal{O}g_i^{-1})\begin{bmatrix} z \\ 1 \end{bmatrix}}$$

where $\gamma_i = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. From the choice of y_i we see that $g_i \in (\mathcal{O} \otimes \mathbb{Z}_p)^\times$ and hence the A_{z_i} are all isogenous to A_z by isogenies of degree prime to p. Now it is well known that the A_{z_i} admit models over $\overline{\mathbb{Q}}$. Let A_i be such a model for A_{z_i} over some number field. We will see below that each A_i is isogenous to a product of elliptic curves with CM by $\mathcal{O}_K$ by an isogeny of degree prime to p. Thus by extending scalars to a bigger number field if required we can assume that A_i has good reduction everywhere and in particular at λ. If $\mathcal{A}_i$ is the Neron model of A_i then $H^0(\mathcal{A}_i, \wedge^2\Omega^1)$ is a lattice in the one dimensional vector space $H^0(A_i, \wedge^2\Omega^1)$ and we can pick an element ω_i in this lattice that is λ-adically primitive. Pick an integer n coprime to p such that $ng_i^{-1} \in \mathcal{O}$. Then we have an isogeny

$$\frac{\mathbb{C}^2}{\Phi_\infty(\mathcal{O}g_i^{-1})\begin{bmatrix} z \\ 1 \end{bmatrix}} \simeq \frac{\mathbb{C}^2}{\Phi_\infty(\mathcal{O}ng_i^{-1})\begin{bmatrix} z \\ 1 \end{bmatrix}} \to \frac{\mathbb{C}^2}{\Phi_\infty(\mathcal{O})\begin{bmatrix} z \\ 1 \end{bmatrix}} = A_z$$

given by the inclusion $\mathcal{O}ng_i^{-1} \hookrightarrow \mathcal{O}$. Composing the two maps above, we get an isogeny ψ_i

$$A_{z_i} \simeq \frac{\mathbb{C}^2}{\Phi_\infty(\mathcal{O})\begin{bmatrix} z_i \\ 1 \end{bmatrix}} \to \frac{\mathbb{C}^2}{\Phi_\infty(\mathcal{O})\begin{bmatrix} z \\ 1 \end{bmatrix}} = A_z$$

of degree prime to p. Suppose $\omega_i = \mu_i dt_1 \wedge dt_2$ on $A_{z_i\mathbb{C}}$. Tracing through the above maps we see that $\psi_i^*(\omega) = \psi_i^*(\mu dt_1 \wedge dt_2) = \frac{(c_i z + d_i)^2}{n^2}\mu dt_1 \wedge dt_2$. On the other hand since ψ_i has degree prime to p, $\psi_i^*(\omega)/\omega_i$ must be a λ-adic unit. Hence $\frac{(c_i z+d_i)^2}{n^2}\frac{\mu}{\mu_i}$ is a λ-adic unit. Noting that $\det(\gamma_i) = N(g_i)^{-1}$ and n are λ-adic units we get that $\beta_i = (\mu_i/j(\gamma_i, z)^2\mu)^{k/2}$ is a λ-adic unit. Now $\frac{L_\chi(F)}{\mu^{k/2}} = \frac{1}{h}\sum_{i=1}^h \frac{f(z_i)}{\mu_i^{k/2}}\beta_i \cdot \chi'(y_i)$ and $\sum_\chi \frac{L_\chi(F)}{\mu^{k/2}}\chi'(y_j^{-1}) = \sum_\chi \frac{1}{h}\sum_{i=1}^h \frac{f(z_i)}{\mu_i^{k/2}}\beta_i\chi'(y_i)\chi'(y_j^{-1})$ $= \sum_{i=1}^h \beta_i \frac{f(z_i)}{\mu_i^{k/2}}\frac{1}{h}\sum_\chi \chi'(y_i y_j^{-1}) = \beta_j \frac{f(z_j)}{\mu_j^{k/2}}$. Thus we get the following proposition:

Proposition 2.8. 1. *$\frac{(2\pi)^k L_\chi(F)}{\mu^{k/2}}$ is a λ-adic integer for all unramified χ $\Rightarrow$ $\frac{(2\pi)^k f(z_i)}{\mu_i^{k/2}}$ is a λ-adic integer for all i $\Rightarrow$ $h\frac{(2\pi)^k L_\chi(F)}{\mu^{k/2}}$ is a λ-adic integer for all unramified χ.*

2. *If $p \nmid h$, then $\frac{(2\pi)^k L_\chi(F)}{\mu^{k/2}}$ is a λ-adic integer for all unramified χ $\iff$ $\frac{(2\pi)^k f(z_i)}{\mu_i^{k/2}}$ is a λ-adic integer for all i.*

We now relate μ to the period Ω defined in Section 2.3.3. Write $D = K + K\mathcal{J}$ for some $\mathcal{J} \in N_{K^\times}(D^\times) \setminus K^\times$ so that $\mathcal{J}^i = -\mathcal{J}$ and $x\mathcal{J} = \mathcal{J}\overline{x}$ for all $x \in K$. (This is an orthogonal decomposition for the norm form on D.) Let I be the fractional ideal of $\mathcal{O}_K$ given by $I = \{x \in K : x\mathcal{J} \in \mathcal{O}\}$. Since K and D are both unramified at p, we have $\mathcal{O} \otimes \mathbb{Z}_p = \mathcal{O}_K \otimes \mathbb{Z}_p + (I \otimes \mathbb{Z}_p)\mathcal{J}$. Clearly we may choose $\mathcal{J}$ such that I and $N\mathcal{J}$ are prime to p. The map $K \times K \to D$ given by $(x_1, x_2) \mapsto x_1 + x_2\mathcal{J}$ induces on tensoring with $\mathbb{R}$, maps

$$\mathbb{C} \times \mathbb{C} = (K \otimes \mathbb{R}) \times (K \otimes \mathbb{R}) \to D \otimes \mathbb{R} \xrightarrow{\Phi_\infty(\cdot)\begin{bmatrix} z \\ 1 \end{bmatrix}} \Phi_\infty(D \otimes \mathbb{R}) \begin{bmatrix} z \\ 1 \end{bmatrix} = \mathbb{C}^2.$$

Let α denote the composite map. For all $x \in K$, $x\mathcal{J} = \mathcal{J}\overline{x}$; hence

$$\Phi_\infty(x)\Phi_\infty(\mathcal{J})(z) = \Phi_\infty(\mathcal{J})\Phi_\infty(\overline{x})(z) = \Phi_\infty(\mathcal{J})(z).$$

Thus $\Phi_\infty(\mathcal{J})(z)$ must be either z or $\overline{z}$. Since $N\mathcal{J}$ must be negative, $\Phi_\infty(\mathcal{J})(z) = \overline{z}$. Now, for $(x_1, x_2) \in K \times K$,

$$\begin{aligned} \alpha(x_1, x_2) &= \Phi_\infty(x_1 + x_2\mathcal{J}) \begin{bmatrix} z \\ 1 \end{bmatrix} = x_1 \begin{bmatrix} z \\ 1 \end{bmatrix} + x_2 J(\mathcal{J}, z) \begin{bmatrix} \overline{z} \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} zx_1 + J(\mathcal{J}, z)\overline{z}x_2 \\ x_1 + J(\mathcal{J}, z)x_2 \end{bmatrix} \end{aligned}$$

and so the same formula holds for $(x_1, x_2) \in \mathbb{C} \times \mathbb{C}$. (Note that we are using the fact that j is normalized so that $J(\Phi_\infty(j(x)), z) = x$ for all $x \in K$. Also we write $J(\mathcal{J}, z)$ instead of $J(\Phi_\infty(\mathcal{J}), z)$.) This shows that α is holomorphic and hence it induces an isogeny, also denoted by α,

$$E_1 \times E_2 = \mathbb{C}/\mathcal{O}_K \times \mathbb{C}/I \xrightarrow{\alpha} \frac{\mathbb{C}^2}{\Phi_\infty(\mathcal{O}) \begin{bmatrix} z \\ 1 \end{bmatrix}}$$

from the product of elliptic curves on the left to the abelian variety A_z. Since $(\mathcal{O}_K + I\mathcal{J}) \otimes \mathbb{Z}_p = \mathcal{O} \otimes \mathbb{Z}_p$, the degree of this isogeny is prime to p. If ω_1, ω_2 are holomorphic differentials on E_1, E_2 respectively that are λ-adically primitive, then $\alpha^*(\omega) = \beta\omega_1 \wedge \omega_2$ with β a λ-adic unit. Note that up to λ-adic units $\omega_1 = \Omega dx_1$ and $\omega_2 = \Omega dx_2$ (since I is prime to p), so that $\alpha^*(\omega) = \Omega^2 dx_1 \wedge dx_2$ up to a λ-adic unit. On the other hand $\alpha^*\omega = \alpha^*(\mu dt_1 \wedge dt_2) =$

$\mu\det\begin{pmatrix} z & J(\mathcal{J},z)\overline{z} \\ 1 & J(\mathcal{J},z)\end{pmatrix} dx_1\wedge dx_2 = 2\mu J(\mathcal{J},z)\Im(z)dx_1\wedge dx_2$, which shows that up to a λ-adic unit $\mu = (2J(\mathcal{J},z)\Im(z))^{-1}\Omega^2$. Now combining this computation with Lemma 2.5, Proposition 2.7 (applied to $G = f^2$, $R =$ the image in X' of an appropriate set of Heegner points) and Proposition 2.8 we get

Proposition 2.9. *Let f be an algebraic modular form on Γ. Suppose f is λ-adically integral. Then for all choices of imaginary quadratic fields K with p unramified in K, Heegner points $K \hookrightarrow D$, and unramified Grossencharacters χ of K of type $(k,0)$ at infinity, the algebraic number*

$$(2J(\mathcal{J},z)\Im(z))^k\cdot h_K^2\cdot\frac{(2\pi)^{2k}L_\chi(F)^2}{\Omega^{2k}}$$

is a λ-adic integer. ($F =$ the adelic form associated to f.) Conversely, if there exist infinitely many Heegner points $K\hookrightarrow D$ with K split at p such that for all unramified characters χ of K of weight $(k,0)$ at infinity, the algebraic numbers

$$(2J(\mathcal{J},z)\Im(z))^k\cdot\frac{(2\pi)^{2k}L_\chi(F)^2}{\Omega^{2k}}$$

are λ-adic integers, then f is λ-adically integral.

Note that if the fields K are chosen to be split at p, the mod λ reductions of the corresponding Heegner points give infinitely many distinct points in the mod λ reduction of the Shimura curve X. The reader will notice the appearance of an extra factor h_K^2 in the first half of the proposition. This comes from the fact that $L_\chi(f)$ is of the form $\frac{1}{h_K}$(a sum of values of f at CM points), hence one can only expect the product $h_K\cdot L_\chi(F)/\Omega^k$ (and not $L_\chi(F)/\Omega^k$) to have good integrality properties. Consequently, it will be important for applications to know the existence of infinitely many Heegner points with class number prime to p (see Lemma 5.1).

3. Computations with the theta correspondence

3.1. *The Weil representation.* Let W be a symplectic space of dimension $2n$ and V an orthogonal space of dimension d. Then $W\otimes V$ is naturally a symplectic space with symplectic form $\langle,\rangle = \langle,\rangle_W\otimes\langle,\rangle_V$. The groups $\mathrm{Sp}(W)$ and $\mathrm{O}(V)$ form a dual reductive pair in $\mathrm{Sp}(W\otimes V)$. Recall that we have fixed an additive character ψ of $\mathbb{Q}\setminus\mathbb{A}$. If V is even dimensional the metaplectic cover splits over $\mathrm{Sp}(W)$ and $\mathrm{O}(V)$ and we get a representation ω_ψ of $(\mathrm{Sp}(W)\times\mathrm{O}(V))(\mathbb{A})$ by restricting the Weil representation. Let $W = W_1\oplus W_2$ where W_1 and W_2 are isotropic for the symplectic form. Then the Weil representation (and hence ω_ψ) can be realised on the Schwartz space $\mathcal{S}((W_1\otimes V)(\mathbb{A}))$. The action of the orthogonal group is via its left regular representation

$$L(h)\varphi(\beta) = \varphi(h^{-1}\beta).$$

We now restrict to the case when W is two-dimensional so that the Weil representation is realised on $\mathcal{S}(V(\mathbb{A}))$. Let w_1, w_2 be nonzero elements of W_1 and W_2 respectively and write elements of the symplectic group as matrices with respect to the basis $\{w_1, w_2\}$. The action of $\mathrm{Sp}(W)(\mathbb{A}) = \mathrm{SL}_2(\mathbb{A})$ can be described by giving the action of $\mathrm{Sp}(W)(\mathbb{Q}_v)$ on $\mathcal{S}(V \otimes \mathbb{Q}_v)$ for all primes v and is given by

$$\omega_{\psi_v}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\varphi(\beta) = \psi(\frac{1}{2}\langle u\beta, \beta\rangle_V)\varphi(\beta) \tag{5}$$

$$\omega_{\psi_v}\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\varphi(\beta) = \chi_V(a)|a|^{d/2}\varphi(a\beta)$$

$$\omega_{\psi_v}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\varphi(\beta) = \gamma_V\hat{\varphi}(\beta)$$

where γ_V is an eighth root of unity and χ_V is a certain quadratic character. The values of γ and χ_V in the cases of interest to us can be copied from [21] and are listed below in Section 3.4. Here, $\hat{\varphi}$ is the Fourier transform:

$$\hat{\varphi}(s) = \int_{V \otimes \mathbb{Q}_v} \varphi(t)\psi_v(\langle t, s\rangle)dt$$

the Haar measure on $V \otimes \mathbb{Q}_v$ being chosen to be self-dual with respect to the pairing $\langle , \rangle_V$. (i.e. so that $\hat{\hat{\varphi}}(\beta) = \varphi(-\beta)$).

Following Harris-Kudla [15] we can extend L to an action of $\mathrm{GO}(V)(\mathbb{A})$ and ω_ψ to an action of $\mathrm{G}(\mathrm{Sp}(W) \times \mathrm{O}(V))(\mathbb{A})$ where

$$\mathrm{G}(\mathrm{Sp}(W) \times \mathrm{O}(V)) = \{(x, y) \in \mathrm{GSp}(W) \times \mathrm{GO}(V), \nu_1(x) = \nu_2(y)\}$$

and ν_1 and ν_2 denote the similitude characters on $\mathrm{GSp}(W)$ and $\mathrm{GO}(V)$ respectively. Define $L(h)\varphi(\beta) = |\nu_2(h)|^{-d/4}\varphi(h^{-1}\beta)$ for $h \in \mathrm{GO}(V)(\mathbb{A})$. For $(x, y) \in \mathrm{G}(\mathrm{Sp}(W) \times \mathrm{O}(V))(\mathbb{A})$ let $\delta = \nu_1(x) = \nu_2(y)$, $\alpha = \begin{pmatrix} 1 & \\ & \delta \end{pmatrix}$ and $x^{(1)} = x\alpha^{-1}, x_{(1)} = \alpha^{-1}x$. Then define $\omega_\psi(x, y) = \omega_\psi(x^{(1)})L(y) = L(y)\omega_\psi(x_{(1)})$.

The Weil representation can be used to lift automorphic forms from $\mathrm{GSp}(W)$ to $\mathrm{GO}(V)$ and vice versa. Pick any $\varphi \in \mathcal{S}(V(\mathbb{A}))$. If F is a form on $\mathrm{GSp}(W)(\mathbb{A})$ one defines the theta lift, $\theta_\varphi(F) : \mathrm{GO}(V)(\mathbb{A}) \to \mathbb{C}$, by

$$\theta_\varphi(F)(h) = \int_{\mathrm{GSp}(W)^{(1)}\backslash \mathrm{GSp}(\mathbb{A})^{(1)}} \left[\sum_{x \in V} \omega_\psi(g\tilde{g}, h)\varphi(x)\right] F(g\tilde{g})d^{(1)}g$$

for any $\tilde{g} \in \mathrm{GSp}(\mathbb{A})$ with $\nu_1(\tilde{g}) = \nu_2(h)$. Likewise if G is a form on $\mathrm{GO}(V)(\mathbb{A})$ one defines $\theta^t_\varphi(G) : \mathrm{GSp}(W)^{(V)}(\mathbb{A}) \to \mathbb{C}$, by

$$\theta^t_\varphi(G)(g) = \int_{\mathrm{GO}(V)^{(1)}\backslash \mathrm{GO}(V)(\mathbb{A})^{(1)}} \left[\sum_{x \in V} \omega_\psi(g, h\tilde{h})\varphi(x)\right] G(h\tilde{h})d^{(1)}h$$

for any $\tilde{h} \in \mathrm{GO}(V)(\mathbb{A})$ with $\nu_1(g) = \nu_2(\tilde{h})$ and $\mathrm{GSp}(W)^{(V)}$ is the subgroup of $\mathrm{GSp}(W)(\mathbb{A})$ consisting of those elements g such that $\nu_1(g) = \nu_2(\tilde{h})$ for some $\tilde{h} \in \mathrm{GO}(V)(\mathbb{A})$.

3.2. *Theta correspondence for the dual pair* $\mathrm{GL}(2) \times \mathrm{GO}(D)$. We now consider the case $V = D$ equipped with the quadratic form $\langle \beta_1, \beta_2 \rangle = \beta_1 \beta_2^i + \beta_1^i \beta_2$. Let $\rho : D^\times \times D^\times \to \mathrm{GO}(D)$ be the map $\rho(\beta_1, \beta_2)(\beta) = \beta_1 \beta \beta_2^{-1}$. Then ρ surjects onto H, the identity component of $\mathrm{GO}(D)$. As mentioned above the Weil representation on $\mathcal{S}(D(\mathbb{A}))$ can be used to lift the adelic form F on $\mathrm{GL}_2(\mathbb{A})$ associated to the normalized newform f from Section 2.2 to a form $\theta_\varphi(F)$ on $\mathrm{GO}(D)(\mathbb{A})$. If $\tilde{g}$ is any element of $\mathrm{GL}_2(\mathbb{A})$ such that $\nu_1(\tilde{g}) = \nu_2(h)$,

$$\theta_\varphi(F)(h) := \int_{\mathrm{GL}_2(\mathbb{Q})^{(1)} \backslash \mathrm{GL}_2(\mathbb{A})^{(1)}} \left[\sum_{x \in D} \omega_\psi(g\tilde{g}, h)\varphi(x) \right] F(g\tilde{g}) d^{(1)}g.$$

This lift depends on $\varphi \in \mathcal{S}(D(\mathbb{A}))$, the choice of which will be very important in what follows. We make the following choice for φ: $\varphi = \otimes_q \varphi_q$, where

$$\varphi_q = \frac{1}{\mathrm{vol}((\mathcal{O}' \otimes \mathbb{Z}_q)^\times)} \mathbb{I}_{\mathcal{O}' \otimes \mathbb{Z}_q}, \text{ for finite primes } q,$$
$$\varphi_\infty(\beta) = \frac{1}{\pi} Y(\beta)^k e^{-2\pi(|X(\beta)|^2 + |Y(\beta)|^2)}$$

where for $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) = D \otimes \mathbb{R}$, $X(\beta) = \frac{1}{2}(a+d) + \frac{\imath}{2}(b-c)$ and $Y(\beta) = \frac{1}{2}(a-d) + \frac{\imath}{2}(b+c)$. This is very close to the choice made in [43] except for the place at infinity. It ensures that the theta lift is holomorphic in both variables as opposed to holomorphic in one and antiholomorphic in the other when pulled back to a form on $D^\times(\mathbb{A}) \times D^\times(\mathbb{A})$. We now summarize some results from [43] with the modifications required to account for our different choice of Schwartz function (at infinity).

Let π_g be the automorphic representation on $D^\times(\mathbb{A})$ associated to π_f by the Jacquet-Langlands correspondence (realised as a subspace of $L^2(D_\mathbb{Q}^\times \backslash D_\mathbb{A}^\times, 1)$) and Ψ the adelic form in π_g corresponding to the arithmetically normalized newform g from Section 2.2. Also let $\tilde{\theta_\varphi}(F)$ denote the pull-back of $\theta_\varphi(F)$ to $D^\times(\mathbb{A}) \times D^\times(\mathbb{A})$ and $\varphi' = \omega_\psi(\mathcal{J}_0, (\mathcal{J}_0, 1))\varphi$ where $\mathcal{J}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in D^\times(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})$. Thus $\varphi'_\infty(\beta) = X(\beta)^k e^{-2\pi(|X(\beta)|^2 + |Y(\beta)|^2)}$ and $\varphi'_q = \varphi_q$ for all finite q. By [43, Chap. 2, Thm. 1], $\tilde{\theta_{\varphi'}}(\overline{F}) = \overline{\Psi'} \times \Psi'$ where Ψ' is some scalar multiple of Ψ. Note that this only fixes Ψ' up to a scalar of absolute value 1. However by requiring further that $\overline{\Psi'(\beta\mathcal{J}_0)} = \Psi'(\beta)$, Ψ' is fixed up to

± 1. Then $\tilde{\theta_\varphi}(F)(\beta_1,\beta_2)$ equals

$$\begin{aligned}
&\int_{\mathrm{GL}_2(\mathbb{A})^{(1)}} \sum_{x\in D} \omega_\psi(g\tilde{g},\rho(\beta_1,\beta_2))\varphi(x)F(g\tilde{g})d^{(1)}g \\
&\quad = \int_{\mathrm{GL}_2(\mathbb{A})^{(1)}} \sum_{x\in D} \omega_\psi(g\tilde{g},\rho(\beta_1,\beta_2))\omega_\psi(\mathcal{J}_0,(\mathcal{J}_0,1))\varphi'(x)\overline{F}(g\tilde{g}\mathcal{J}_0)d^{(1)}g \\
&\quad = \int_{\mathrm{GL}_2(\mathbb{A})^{(1)}} \sum_{x\in D} \omega_\psi(g\tilde{g}\mathcal{J}_0,\rho(\beta_1\mathcal{J}_0,\beta_2))\varphi'(x)\overline{F}(g\tilde{g}\mathcal{J}_0)d^{(1)}g \\
&\quad = \tilde{\theta_{\varphi'}}(\overline{F})(\beta_1\mathcal{J}_0,\beta_2) = \overline{\Psi'}(\beta_1\mathcal{J}_0)\Psi'(\beta_2) = \Psi'(\beta_1)\Psi'(\beta_2)
\end{aligned}$$

where $\det(\tilde{g}) = \nu(\beta_1)\nu(\beta_2)^{-1}$. Thus $\tilde{\theta_\varphi}(F) = \Psi' \times \Psi'$. Now define $F' : H(\mathbb{A}) \to \mathbb{C}$ by $F'(\rho(\beta_1,\beta_2)) = \Psi'(\beta_1)\Psi'(\beta_2)$. By see-saw duality (see [23] and [15]),

$$\langle \theta_\varphi(F), F' \rangle = \langle F, \overline{\theta^t_\varphi(\overline{F'})} \rangle \tag{6}$$

where $\theta^t_\varphi(\overline{F'})$ is the theta lift of $\overline{F'}$ to GL(2), given by

$$\theta^t_\varphi(\overline{F'})(g) := \int_{H(\mathbb{Q})^{(1)}\backslash H(\mathbb{A})^{(1)}} \left[\sum_{x\in D} \omega_\psi(g,h\tilde{h})\varphi(x) \right] \overline{F'}(h\tilde{h})d^{(1)}h$$

with $\tilde{h} \in H(\mathbb{A}), \nu_2(\tilde{h}) = \nu_1(g)$. By a computation similar to the one given above, it is easy to check that $\theta^t_\varphi(\overline{F'})(g\mathcal{J}_0) = \theta^t_{\varphi'}(F'')(g)$ where $F''(\beta_1,\beta_2) = \Psi'(\beta_1)\overline{\Psi'(\beta_2)}$. Further, by the computations in [43, §2.2.1], $\theta^t_{\varphi'}(F'')(g) = \langle \Psi', \Psi' \rangle F(g)$, whence $\overline{\theta^t_\varphi(\overline{F'})}(g) = \langle \Psi', \Psi' \rangle \overline{F}(g\mathcal{J}_0) = \langle \Psi', \Psi' \rangle F(g)$. Substituting this in (6) we see that $\langle \Psi', \Psi' \rangle^2 = \langle \Psi', \Psi' \rangle \langle F, F \rangle$, whence $\langle \Psi', \Psi' \rangle = \langle F, F \rangle$. The key point will be to show by Proposition 2.9 that Ψ' is the adelic form associated to a p-adically integral form on $D^\times$.

Let $j : K \hookrightarrow D$ be an embedding of an imaginary quadratic field K in D corresponding to a Heegner point with p unramified in K. Recall that such an embedding gives an algebraic map $K^\times \to D^\times$ and hence a map $j_{\mathbb{A}} : K^\times_{\mathbb{A}} \to D^\times_{\mathbb{A}}$. In what follows we think of $K^\times_{\mathbb{A}}$ as a subgroup of $D^\times_{\mathbb{A}}$ via this embedding. Let $\underline{\chi}$ be an algebraic Hecke character of K of weight $(k,0)$ at infinity and let χ denote the corresponding Grossencharacter at infinity (i.e. corresponding to the identity embedding of K in $\mathbb{C}$). Also, define $\chi' = \chi^\rho \mathbb{N}^{-k/2}$. Recall also (see (4)) that we have defined $L_\chi(\Psi')$ to be the integral

$$L_\chi(\Psi') = j(\alpha,\imath)^k \int_{K^\times K^\times_\infty \backslash K^\times_{\mathbb{A}}} \Psi'(x\alpha)\chi'(x)d^\times x$$

for any $\alpha \in \mathrm{SL}_2(\mathbb{R}) \subset D^\times(\mathbb{R})$ such that $\alpha(\imath) = z$. Note that there is some abuse of notation here, since $L_\chi(\Psi')$ depends not only on χ and K but also on the specific choice of Heegner point. We assume henceforth that $\underline{\chi}$ is an unramified Hecke character of K.

We now compute $L_\chi(\Psi')^2$ by a method of Waldspurger, as in [14]. As in Section 2.3.4 write $D = D_1 + D_2$ where $D_1 = K$ and $D_2 = K\mathcal{J}$ is the orthogonal complement to K for the norm form on D. Then the identity components of $\mathrm{GO}(D_1)$ and $\mathrm{GO}(D_2)$ are both equal to $K^\times$, and the identity component of $\mathrm{G}(\mathrm{O}(D_1) \times \mathrm{O}(D_2))$ is identified with $\mathrm{G}(K^\times \times K^\times)$. The map $\rho : D^\times \times D^\times \to \mathrm{GO}(D)^0 = H$ sends $K^\times \times K^\times$ to $(\mathrm{G}(\mathrm{O}(D_1) \times \mathrm{O}(D_2)))^0 = \mathrm{G}(K^\times \times K^\times)$, and this map is nothing but $(x, y) \mapsto (xy^{-1}, x\overline{y}^{-1})$. Note also that $\chi'(xy) = \chi'(x\overline{y}^{-1})\chi'(y\overline{y}) = \chi'(x\overline{y}^{-1})$ since χ', being unramified, implies $\chi'(y\overline{y}) = 1$. Let T_1 and T_2 denote the tori $\mathrm{GO}(D_1)^0$ and $\mathrm{GO}(D_2)^0$ respectively and T be the torus $G(T_1 \times T_2)$. Now,

$$\begin{aligned} L_\chi(\Psi')^2 &= j(\alpha, \imath)^{2k} \int_{(K^\times K_\infty^\times \backslash K_{\mathbb{A}}^\times)^2} \tilde{\theta_\varphi}(F)(x\alpha, y\alpha)\chi'(xy)d^\times x d^\times y \\ &= \frac{1}{\pi} \frac{j(\alpha, \imath)^{2k}}{\prod_q \mathrm{vol}((\mathcal{O}' \otimes \mathbb{Z}_q)^\times)} \cdot \int_{T(\mathbb{Q})T(\mathbb{R})\backslash T(\mathbb{A})} \theta_{\varphi''}(F)|_{T(\mathbb{A})}\chi'(b)d^\times a d^\times b \end{aligned}$$

where a is the variable on T_1, b that on T_2 and φ'' is given by

$$\begin{aligned} \varphi''_q &= \mathrm{vol}((\mathcal{O}' \otimes \mathbb{Z}_q)^\times)\varphi_q = \mathbb{I}_{\mathcal{O}' \otimes \mathbb{Z}_q} \text{ for finite } q, \\ \varphi''_\infty(x = x_1 + x_2\mathcal{J}) &= \pi\varphi_\infty(\alpha^{-1}x\alpha) = \pi\varphi_\infty(\alpha^{-1}x_1\alpha + (\alpha^{-1}x_2\alpha)(\alpha^{-1}\mathcal{J}\alpha)) \\ &= (Y(\alpha^{-1}x_2\mathcal{J}\alpha))^k e^{-2\pi(N(x_1)+N(x_2)|N(\mathcal{J})|)}. \end{aligned}$$

Let $C = \frac{1}{\pi}\prod_{q|N^+}(q+1)\prod_{q|N^-}(q-1)$. By see-saw duality (see [14, 14.5 and §7.3], this last integral equals

$$C \cdot \zeta(2) j(\alpha, \imath)^{2k} \int_{Z(\mathbb{A})\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})} F(g)\theta^t_{\varphi''}(1, \chi')|_{\mathrm{GL}_2(\mathbb{A})}(g)d^\times g.$$

Note that $\theta^t_{\varphi''}$ is *a priori* defined only on the subgroup $\mathrm{G}(\eta_K) = \{x \in \mathrm{GL}_2(\mathbb{A})$, $\det(x)$ is a norm from $K_{\mathbb{A}}^\times\}$. We extend it to a function on $\mathrm{GL}_2(\mathbb{A})$ by making it left invariant by $\mathrm{GL}_2(\mathbb{Q})$ and extending by zero outside $\mathrm{GL}_2(\mathbb{Q})\mathrm{G}(\eta_K)$ which is a subgroup of index 2 in $\mathrm{GL}_2(\mathbb{A})$.

Since $D = D_1 + D_2$, $\mathcal{S}(D(\mathbb{A}_f)) = \mathcal{S}(D_1(\mathbb{A}_f)) \otimes \mathcal{S}(D_2(\mathbb{A}_f))$. Note that φ''_∞ is of the form $\varphi_{\infty 1} \otimes \varphi_{\infty 2}$. However this is not necessarily the case for finite q. For each finite q write $\varphi''_q = \sum_{i_q \in I_q} \varphi_{q,i_q,1} \otimes \varphi_{q,i_q,2}$. Then

(7)
$$\begin{aligned} L_\chi(\Psi')^2 = C \cdot j(\alpha, i)^{2k} \cdot \zeta(2) \sum_{\prod_q I_q} \int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} & F(g)\theta^t_{\varphi_{\infty 1} \otimes (\otimes_q \varphi_{q,i_q,1})}(1)(g) \\ & \cdot \theta^t_{\varphi_{\infty 2} \otimes (\otimes_q \varphi_{q,i_q,2})}(\chi')(g)d^\times g \end{aligned}$$

where $G = \mathrm{GL}(2)$ and $\theta^t_{\varphi_1}(1)$ and $\theta^t_{\varphi_2}(\chi)$ denote theta lifts from the groups $\mathrm{GO}(D_1)^0(\mathbb{A})$ and $\mathrm{GO}(D_2)^0(\mathbb{A})$ respectively to $\mathrm{GL}(2)$. In the next section we will study the Fourier coefficients of the cusp form $\theta^t_{\varphi_2}(\chi)$ and explicitly identify the form $\theta^t_{\varphi_1}(1)$ as an Eisenstein series for φ_1 and φ_2 in $\mathcal{S}(D_1(\mathbb{A}))$ and $\mathcal{S}(D_2(\mathbb{A}))$ respectively.

3.3. *Theta functions attached to Grossencharacters of K and the Siegel-Weil formula.* We first derive an explicit formula for the Fourier coefficients of $\theta^t_{\varphi_2}(\chi')$ for any $\varphi_2 \in \mathcal{S}(D_2(\mathbb{A}))$. Let $K^{(1)}$ denote the kernel of the norm map $N : K^\times \to \mathbb{Q}^\times$. Recall that we have picked (in Section 2.3.4) a Haar measure $d^\times x$ on $K^\times_{\mathbb{A}}$ such that $\mathrm{vol}(K^\times K^\times_\infty \setminus K^\times_{\mathbb{A}}) = 1$. The norm map induces a map $N : K^\times(\mathbb{R}^\times)^+ \setminus K^\times_{\mathbb{A}} \to \mathbb{Q}^\times(\mathbb{R}^\times)^+ \setminus \mathbb{Q}^\times_{\mathbb{A}}$ whose kernel is identified with $K^{\times(1)} \setminus K^{\times(1)}_{\mathbb{A}}$. We pick $d_1^\times h$ to be the unique measure on $K^{\times(1)}_{\mathbb{A}}$ such that the quotient measure on $K^{\times(1)} \setminus K^{\times(1)}_{\mathbb{A}}$ satisfies $d^\times x = d_1^\times h d\xi$. (Thus the volume of $K^{\times(1)} \setminus K^{\times(1)}_{\mathbb{A}}$ with respect to this measure is 2.) Let $g \in \mathrm{G}(\eta_K)$ and $\tilde{h}$ be any element of $K^\times_{\mathbb{A}}$ with $N(\tilde{h}) = \det(g)$. Then one sees that

$$\theta^t_{\varphi_2}(\chi')(g) = \int_{K^{\times(1)} \setminus K^{\times(1)}_{\mathbb{A}}} \left(\sum_{x \in K} \omega_\psi(g, h\tilde{h}) \varphi_2(x\mathcal{J}) \right) \chi'(h\tilde{h}) d_1^\times h.$$

We split this into two terms, one corresponding to $x = 0$ and one corresponding to all other $x \in K$. Thus $\theta^t_{\varphi_2}(\chi')(g) = I_0 + I$, where

$$I_0 = \int_{K^{\times(1)} \setminus K^{\times(1)}_{\mathbb{A}}} \omega_\psi(g, h\tilde{h}) \varphi_2(0) \chi'(h\tilde{h}) d_1^\times h,$$

$$I = \int_{K^{\times(1)} \setminus K^{\times(1)}_{\mathbb{A}}} \left(\sum_{\tilde{\alpha} \in K^{(1)} \setminus K^\times} \sum_{\gamma \in K^{(1)}} \omega_\psi(g, h\tilde{h}) \varphi_2(\tilde{\alpha}\gamma\mathcal{J}) \right) \chi'(h\tilde{h}) d_1^\times h.$$

Note that $\omega_\psi(g, h\tilde{h})\varphi_2(0) = \omega_\psi(1, h)\omega_\psi(g, \tilde{h})\varphi_2(0) = \omega_\psi(g, \tilde{h})\varphi_2(h^{-1} \cdot 0) = \omega_\psi(g, \tilde{h})\varphi_2(0)$ is independent of h. Then, since $\chi'|_{K^{\times(1)}_{\mathbb{A}}}$ is nontrivial, we must have $I_0 = 0$. Hence $\theta^t_{\varphi_2}(\chi')(g)$ is given by the expression

$$\begin{aligned}
&\int_{K^{\times(1)} \setminus K^{\times(1)}_{\mathbb{A}}} \left(\sum_{\tilde{\alpha} \in K^{(1)} \setminus K^\times} \sum_{\gamma \in K^{(1)}} \omega_\psi(g, h\gamma^{-1}\tilde{h}) \varphi_2(\tilde{\alpha}\gamma\mathcal{J}) \right) \chi'(h\gamma^{-1}\tilde{h}) d_1^\times h \\
&\quad = \int_{K^{\times(1)}_{\mathbb{A}}} \left(\sum_{\tilde{\alpha} \in K^{(1)} \setminus K^\times} \omega_\psi(g, h\tilde{h}) \varphi_2(\tilde{\alpha}\mathcal{J}) \right) \chi'(h\tilde{h}) d_1^\times h \\
&\quad = \sum_{\tilde{\alpha} \in K^{(1)} \setminus K^\times} \int_{K^{\times(1)}_{\mathbb{A}}} \omega_\psi \left(\begin{pmatrix} N(\tilde{\alpha}) & 0 \\ 0 & 1 \end{pmatrix}, \tilde{\alpha}^i \right) \omega_\psi(g, h\tilde{h}) \varphi_2(\mathcal{J}) \chi'(h\tilde{h}) d_1^\times h \\
&\quad = \sum_{\substack{\xi \in \mathbb{Q}^\times \\ \xi \in N(K^\times)}} W\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g \right)
\end{aligned}$$

where

$$W(g)=\int_{K_{\mathbb{A}}^{\times(1)}}\omega_\psi(g,h\tilde{h})\varphi_2(\mathcal{J})\chi'(h\tilde{h})d_1^\times h.$$

Then for any $x\in\mathbb{A}_{\mathbb{Q}}$, by (5),

$$\begin{aligned}W\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)&=\int_{K_{\mathbb{A}}^{\times(1)}}\omega_\psi(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g,h\tilde{h})\varphi_2(\mathcal{J})\chi'(h\tilde{h})d_1^\times h\\ &=\psi(N(\mathcal{J})x)W\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right).\end{aligned}$$

Let $W^\psi_{\theta_{\varphi_2}}(g)=W\left(\begin{pmatrix}N(\mathcal{J})^{-1} & 0\\ 0 & 1\end{pmatrix}g\right)$ so that $W^\psi_{\theta_{\varphi_2}}\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)=\psi(x)W^\psi_{\theta_{\varphi_2}}(g)$. Then we have the following Fourier expansion for $\theta^t_{\varphi_2}(\chi')(g)$:

$$\begin{aligned}\theta^t_{\varphi_2}(\chi')(g)&=\sum_{\substack{\xi\in\mathbb{Q}^\times\\ N(\mathcal{J})^{-1}\xi\in N(K^\times)}}W^\psi_{\theta_{\varphi_2}}\left(\begin{pmatrix}\xi & 0,\\ 0 & 1\end{pmatrix}g\right),\\ W^\psi_{\theta_{\varphi_2}}(g)&=\int_{K_{\mathbb{A}}^{\times(1)}}\omega_\psi\left(\begin{pmatrix}N(\mathcal{J})^{-1} & 0\\ 0 & 1\end{pmatrix}g,h\tilde{h}\right)\varphi_2(\mathcal{J})\chi'(h\tilde{h})d_1^\times h,\end{aligned}$$

where $\tilde{h}$ is chosen such that $N(\tilde{h})=N(\mathcal{J})^{-1}\det(g)$. Now let $U_q^{(1)}$ be the group of norm 1 elements in $(\mathcal{O}_K\otimes\mathbb{Z}_q)^\times$ if q is finite and $U_\infty^{(1)}=K_\infty^{\times(1)}$. Let $d^\times h=\prod_q d^\times h_q$ be the product measure on $K_{\mathbb{A}}^{\times(1)}$ where $d^\times h_q$ is the measure on $K_q^{(1)}$ that gives $U_q^{(1)}$ volume 1 if q is unramified in K or $q=\infty$ and volume 2 otherwise. It is easy to check that the measures $d_1^\times h$ and $d^\times h$ are related by $d_1^\times h=\frac{1}{h_K}d^\times h$ where h_K is the class number of K. Thus, if $\varphi_2=\otimes_q\varphi_{q,2}$, and $g=\prod_q g_q$, then

$$W^\psi_{\theta_{\varphi_2}}(g)=\frac{1}{h_K}\prod_q W^{\psi_q}_{\theta_{\varphi_{2,q}}}(g_q),\qquad\text{where}\tag{8}$$

$$W^{\psi_q}_{\theta_{\varphi_{2,q}}}(g)=\int_{K_q^{(1)}}\omega_{\psi_q}\left(\begin{pmatrix}N(\mathcal{J})^{-1} & 0\\ 0 & 1\end{pmatrix}g,h_q\tilde{h}\right)\varphi_2(\mathcal{J})\chi'_q(h_q\tilde{h})d^\times h_q\tag{9}$$

if $N(\mathcal{J})^{-1}\det(g)=N(\tilde{h})$, for some $\tilde{h}\in K\otimes\mathbb{Q}_q$, and is equal to 0 if $N(\mathcal{J})^{-1}\det(g)$ is not a norm from K_q. Note that while the local Whittaker coefficients depend on the choice of $\mathcal{J}$, the expression in (8) is independent of the choice of $\mathcal{J}$.

On the other hand, by the Siegel-Weil formula (due in this case to Hecke), the theta lift of the character 1 is an Eisenstein series. More precisely, one has ([14, Thm. 13.3])

Proposition 3.1. *Let* $\varphi_1=\varphi_{\infty 1}\otimes(\otimes_q\varphi_{q,i_q,1})$. *Then* $\theta^t_{\varphi_1}(1)(g)=$ *the restriction to* $\mathrm{G}(\eta_K)$ *of* $E(0,\Phi,g)$ *where* $E(s,\Phi,g)$ *is the Eisenstein series cor-*

responding to the function

$$\Phi = \Phi_{\varphi_1} : B(\mathbb{Q}) \setminus \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C},$$
$$\Phi(g) = \omega_\psi(g, g')\varphi_1(0) \ \textit{if } g \in \mathrm{G}(\eta_K)$$

where $g' \in K_{\mathbb{A}}^\times$ *is any element such that* $N(g') = \nu_1(g) = \det(g)$, *and*

$$\Phi\left(\begin{pmatrix} ad & b \\ 0 & d \end{pmatrix} g\right) = \eta_K(d)|a|_{\mathbb{A}}^{1/2}\Phi(g) \ \textit{for } g \in \mathrm{GL}_2(\mathbb{A}).$$

Define $\Phi^s(g) = \|a(g)\|^{s-1/2}\Phi(g)$, where $g = n(g)a(g)k(g)$ in the NAK decomposition, and $\|\begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}\| = |a_1/a_2|$. Using the proposition and unfolding the Eisenstein series in (7), we find (as in [14, §14.6]; see also [20]) that $L_\chi(\Psi')^2$ is the value at $s = 1/2$ of the analytic continuation of

$$\begin{aligned}
&\zeta(2) \cdot C \cdot j(\alpha, i)^{2k} \sum_{\prod I_q} \int_{Z(\mathbb{A})N(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} W_F^\psi(g) W^\psi_{\theta^t_{\varphi_{\infty,2}\otimes(\otimes_q \varphi_{q,i_q,2})}}(\eta g) \\
&\quad \cdot \Phi^s_{\varphi_{\infty,1}\otimes(\otimes_q\varphi_{q,i_q,1})}(g) d^\times g \\
&= C \cdot j(\alpha, i)^k \sum_{\prod I_q} \int_{K(\mathbb{A})} \int_{\mathbb{A}^\times} W_F^\psi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right) W^\psi_{\theta^t_{\varphi_{\infty,2}\otimes(\otimes_q \varphi_{q,i_q,2})}} \\
&\quad \cdot \left(\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} k\right) \Phi^s_{\varphi_{\infty,1}\otimes(\otimes_q\varphi_{q,i_q,1})}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right) |a|^{-1} d^\times a \, dk
\end{aligned}$$

where $\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $K(\mathbb{A}) = \prod_q K_q(\mathbb{A})$ with $K_\infty = \mathrm{SO}_2(\mathbb{R})$, $K_q =$ the maximal compact $\mathrm{GL}_2(\mathbb{Z}_q)$ for finite q and the measure dk is chosen as follows. Let $dk'_\infty = d\theta$ on $\mathrm{SO}_2(\mathbb{R})$ and dk'_q be the restriction of the measure $dg^\times$ to $K_q = \mathrm{GL}_2(\mathbb{Z}_q)$ so that $\mathrm{vol}(\mathrm{GL}_2(\mathbb{Z}_q)$ with respect to this measure is $\zeta_q(2)^{-1}$. If T' is the torus consisting of matrices of the form $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, then $Z(\mathbb{A})N(\mathbb{A})\backslash\mathrm{GL}_2(\mathbb{A})$ is identified above with $T'(\mathbb{A}) \times K(\mathbb{A})$. The quotient measure associated to the Tamagawa measure $dg^\times$ is easily seen to be identified with the measure $|a|^{-1}d^\times a dk'$. Then dk is defined to be the measure on $K(\mathbb{A})$ given by $dk_\infty = dk'_\infty$ and $dk_q = \zeta_q(2)dk'_q$, the product $\prod_q \zeta_q(2) = \zeta(2)$ accounting for the loss of the factor $\zeta(2)$ above. It follows that $L_\chi(\Psi')^2$ is the value at $s = 1/2$ of the analytic continuation of $\frac{C}{h_K} \cdot L_\infty \times \prod_{q<\infty} L_q$ where

$$L_\infty = j(\alpha,i)^{2k}\int_{K_\infty}\int_{\mathbb{R}^\times} W^{\psi_\infty}_{F,\infty}\left(\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}k\right) W^{\psi_\infty}_{\theta^t_{\varphi_{\infty,2}}}\left(\begin{pmatrix} -a & 0\\ 0 & 1\end{pmatrix}k\right)$$
$$\cdot\ \Phi^s_{\varphi_{\infty,1}}\left(\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}k\right)|a|^{-1}d^\times a\, dk_\infty,$$
$$L_q = \int_{K_q}\int_{\mathbb{Q}_q^\times} W^{\psi_q}_{F,q}\left(\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}k\right)\Big[\sum_{i_q\in I_q} W^{\psi_q}_{\theta^t_{\varphi_{q,i_q,2}}}\left(\begin{pmatrix} -a & 0\\ 0 & 1\end{pmatrix}k\right)$$
$$\cdot\ \Phi^s_{\varphi_{q,i_q,1}}\left(\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}k\right)\Big]|a|^{-1}d^\times a\, dk_q,$$

and the local measures dk_∞ and dk_q are such that $\mathrm{vol}(\mathrm{SO}_2(\mathbb{R})) = 2\pi$ and $\mathrm{vol}(\mathrm{GL}_2(\mathbb{Z}_q)) = 1$.

3.4. *Computation of the local zeta integrals L_q and L_∞.* We list below the values of γ and χ_V at the various local places, for the following choices of V: $V = D, K, K\mathcal{J}$. The reader can find these computed in [21, Chap. 1, §1]. Let $-d$ be the discriminant of the field K and η_K the quadratic character associated to the extension $K/\mathbb{Q}$. We assume that d is odd to simplify the local calculations at the prime 2. In the application of interest this can be arranged:

	χ_D	γ_D	χ_K	γ_K	$\chi_{K\mathcal{J}}$	$\gamma_{K\mathcal{J}}$
$q\nmid dN$, split in K	1	1	1	1	1	1
$q\nmid dN$, inert in K	1	1	η_K	1	η_K	1
$q \mid N^+$	1	1	1	1	1	1
$q \mid N^-$	1	-1	η_K	1	η_K	-1
$q \mid d$	1	1	η_K	ζ_1	η_K	ζ_2
$q=\infty$	1	1	$a\mapsto a/\|a\|$	i	$a\mapsto a/\|a\|$	$-i$

where we will only need the fact that $\zeta_1\zeta_2 = 1$.

We now make some reductions in the case that q is unramified in K. In this case $\mathcal{O}'\otimes\mathbb{Z}_q = \mathcal{O}_{K\otimes\mathbb{Q}_q} + I\mathcal{J}$ for some ideal I in $\mathcal{O}_{K_q}$. Thus $\varphi_q = \varphi_1\otimes\varphi_2$, where φ_1 is the characteristic function of $\mathcal{O}_K\otimes\mathbb{Z}_q$ and φ_2 is the characteristic function of $I\mathcal{J}$. Note that K_q is generated by the matrices

$$\begin{pmatrix} 1 & x\\ & 1\end{pmatrix},\quad x\in\mathbb{Z}_q,\qquad \begin{pmatrix} a & \\ & b\end{pmatrix},\quad a,b\in\mathbb{Z}_q^\times \quad\text{and}\quad \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}. \tag{10}$$

We first compute the action of these matrices on φ_1. It is easy to see that

$$\omega_\psi\left(\begin{pmatrix} 1 & x\\ & 1\end{pmatrix},1\right)\varphi_1 = \psi(xN(\cdot))\varphi_1 = \varphi_1$$
$$\omega_\psi\left(\begin{pmatrix} a & \\ & b\end{pmatrix},c\right)\varphi_1 = \varphi_1(a\cdot) = \varphi_1 \quad\text{if}\quad N(c) = ab$$
$$\omega_\psi\left(\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},1\right)\varphi_1 = \hat{\varphi}_1 = \varphi_1.$$

Hence $\Phi^s_{\varphi_1}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right) = |a|^{s-1/2}\Phi_{\varphi_1}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right) = |a|^s\omega_\psi(k,\tilde{h})\varphi_1(0) = |a|^s\varphi_1(0) = |a|^s$ (here $\tilde{h}$ is any element with $N(\tilde{h}) = \det(k)$).

Next we compute the actions of the matrices above on φ_2.

$$\begin{aligned}
\omega_\psi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, 1\right)\varphi_2 &= \psi(xN(\cdot))\varphi_2 = \varphi_2,\\
\omega_\psi\left(\begin{pmatrix} a & \\ & b \end{pmatrix}, c\right)\varphi_2 &= \varphi_2(a\cdot) = \varphi_2 \quad \text{if } \ N(c) = ab,\\
\omega_\psi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1\right)\varphi_2 &= \zeta\hat{\varphi}_2,
\end{aligned}$$

where $\zeta = 1$ unless $q|N^-$ in which case $\zeta = -1$. Let $r = 0$ if $q \nmid N$ and $r = 1$ if $q \mid N$. Since $\mathcal{O}' \otimes \mathbb{Z}_q = \mathcal{O}_{K\otimes\mathbb{Q}_q} + I\mathcal{J}$, taking discriminants shows that we have an equality of ideals $(q)^r = NI \cdot N\mathcal{J}$. Now it is easy to compute that $\hat{\varphi}_2 = q^{-r}\cdot$ (char. function of $q^{-r}I\mathcal{J}$). Thus if $r = 0$, $W^{\psi_q}_{\theta_{\varphi_2}}$ is right-invariant under K_q. If $r = 1$ we show that $W^{\psi_q}_{\theta_{\varphi_2}}$ is right-invariant under the subgroup $\Gamma_0(q)$, where

$$\Gamma_0(q) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_q, \ \ c \equiv 0 \mod q\right\}.$$

To check this note first that $\Gamma_0(q)$ is generated by matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{Z}_q \qquad \begin{pmatrix} a & \\ & b \end{pmatrix}, \quad a, b \in {\mathbb{Z}_q}^\times \qquad \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, \quad y \in q\mathbb{Z}_q.$$

So it will suffice to check that φ_2 is invariant under $\omega_\psi\left(\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, 1\right)$ with $y \in q\mathbb{Z}_q$. Since

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\omega_\psi\left(\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, 1\right)\varphi_2(t\mathcal{J}) = \zeta\omega_\psi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix}, 1\right)\hat{\varphi}_2(-t\mathcal{J}).$$

But

$$\begin{aligned}
\omega_\psi\left(\begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix}, 1\right)\hat{\varphi}_2(-t\mathcal{J}) &= \psi(\frac{1}{2}\langle yt\mathcal{J}, -t\mathcal{J}\rangle)\hat{\varphi}_2(-t\mathcal{J})\\
&= \psi(-yN(t)N\mathcal{J})\hat{\varphi}_2(-t\mathcal{J}) = \hat{\varphi}_2(-t\mathcal{J}),
\end{aligned}$$

since

$$\begin{aligned}
\hat{\varphi}_2(-t\mathcal{J}) \neq 0 \Rightarrow -t\mathcal{J} \in \frac{1}{q}I\mathcal{J} \Rightarrow -yN(t)N\mathcal{J} &\in (q)(1/q^2)NI(N\mathcal{J})\\
&= \mathbb{Z}_q \Rightarrow \psi(-yN(t)N\mathcal{J}) = 1.
\end{aligned}$$

Hence

$$\omega_\psi\left(\left(\begin{array}{cc}1 & 0\\ y & 1\end{array}\right),1\right)\varphi_2(t\mathcal{J}) = \zeta\omega_\psi\left(\left(\begin{array}{cc}0 & 1\\ -1 & 0\end{array}\right),1\right)\hat{\varphi}_2(-t\mathcal{J})$$
$$= \zeta^2\hat{\hat{\varphi}}_2(-t\mathcal{J}) = \varphi_2(t\mathcal{J})$$

which proves our claim that $W^{\psi_q}_{\theta_{\varphi_2}}$ is right invariant by $\Gamma_0(q)$. We now write down a coset decomposition for $\mathrm{GL}_2(\mathbb{Z}_q)/\Gamma_0(q)$:

$$K_q = \mathrm{GL}_2(\mathbb{Z}_q) = \Gamma_0(q) \sqcup \bigsqcup_{z=0}^{q-1}\left(\begin{array}{cc}z & 1\\ -1 & 0\end{array}\right)\Gamma_0(q).$$

Thus, in any case,

$$L_q = \int_{\mathbb{Q}_q^\times}\int_{K_q} W^{\psi_q}_{F,q}\left(\left(\begin{array}{cc}a & 0\\ 0 & 1\end{array}\right)k\right) W^{\psi_q}_{\theta_{\varphi_2}}\left(\left(\begin{array}{cc}-a & 0\\ 0 & 1\end{array}\right)k\right)|a|^{s-1}dk\,d^\times a$$
$$= \frac{1}{q+1}[I_1 + I_2], \text{ where}$$
$$I_1 = \int_{\mathbb{Q}_q^\times} W^{\psi_q}_{F,q}\left(\begin{array}{cc}a & 0\\ 0 & 1\end{array}\right) W^{\psi_q}_{\theta_{\varphi_2}}\left(\begin{array}{cc}-a & 0\\ 0 & 1\end{array}\right)|a|^{s-1}d^\times a$$
$$I_2 = \sum_{z=0}^{q-1}\int_{\mathbb{Q}_q^\times} W^{\psi_q}_{F,q}\left(\left(\begin{array}{cc}a & 0\\ 0 & 1\end{array}\right)\left(\begin{array}{cc}z & 1\\ -1 & 0\end{array}\right)\right)$$
$$\cdot W^{\psi_q}_{\theta_{\varphi_2}}\left(\left(\begin{array}{cc}-a & 0\\ 0 & 1\end{array}\right)\left(\begin{array}{cc}z & 1\\ -1 & 0\end{array}\right)\right)|a|^{s-1}d^\times a$$
$$= q\int_{\mathbb{Q}_q^\times} W^{\psi_q}_{F,q}\left(\begin{array}{cc}0 & a\\ -1 & 0\end{array}\right) W^{\psi_q}_{\theta_{\varphi_2}}\left(\begin{array}{cc}0 & -a\\ -1 & 0\end{array}\right)|a|^{s-1}d^\times a$$

since $\left(\begin{array}{cc}\pm a & 0\\ 0 & 1\end{array}\right)\left(\begin{array}{cc}z & 1\\ -1 & 0\end{array}\right) = \left(\begin{array}{cc}1 & \mp az\\ 0 & 1\end{array}\right)\left(\begin{array}{cc}0 & \pm a\\ -1 & 0\end{array}\right)$. Suppose $\tilde{h} \in (K \otimes \mathbb{Q}_q)^\times$ is such that $N(\tilde{h}) = -(N\mathcal{J})^{-1}a$. Since

$$\omega_\psi\left(\left(\begin{array}{cc}-(N\mathcal{J})^{-1}a & 0\\ 0 & 1\end{array}\right),h\tilde{h}\right)\varphi_2(\mathcal{J}) = |(N\mathcal{J})^{-1}a|^{1/2}\varphi_2(-(N\mathcal{J})^{-1}a(h\tilde{h})^{-1}\mathcal{J})$$

we see by (9) that

$$W^{\psi_q}_{\theta_{\varphi_2}}\left(\begin{array}{cc}-a & 0\\ 0 & 1\end{array}\right) = |(N\mathcal{J})^{-1}a|^{1/2}\int_{(K\otimes\mathbb{Q}_q)^{(1)}}\varphi_2(-(N\mathcal{J})^{-1}a(h\tilde{h})^{-1}\mathcal{J})$$
$$\cdot\,\chi'(h\tilde{h})d^\times h, \tag{11}$$

if $\eta_q(-(N\mathcal{J})^{-1}a) = 1$ and is equal to 0 otherwise. Likewise, since $\begin{pmatrix} 0 & -a \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

(12)
$$W^{\psi_q}_{\theta_{\varphi_2}} \begin{pmatrix} 0 & -a \\ -1 & 0 \end{pmatrix} = \zeta |(N\mathcal{J})^{-1}a|^{1/2} \int_{(K\otimes\mathbb{Q}_q)^{(1)}} \hat{\varphi}_2(-(N\mathcal{J})^{-1}a(h\tilde{h})^{-1}\mathcal{J}) \cdot \chi'(h\tilde{h}) d^\times h$$

if $\eta_q(-(N\mathcal{J})^{-1}a) = 1$ and is equal to 0 otherwise. As for $W^{\psi_q}_{F,q}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ and $W^{\psi_q}_{F,q}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, the reader is referred to [43, §2.5] where the following formulae have been worked out. For $q \nmid N$, $\pi_q(f) \simeq \pi(\mu_1, \mu_2)$,

$$W^{\psi_q}_{F,q} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = |a|^{1/2} \frac{\mu_1(aq) - \mu_2(aq)}{\mu_1(q) - \mu_2(q)} \mathbb{I}_{\mathbb{Z}_q}(a). \tag{13}$$

For $q \mid N$, $\pi_q(f) \simeq \sigma(|\cdot|^{\frac{1}{2}+it_q}, |\cdot|^{-\frac{1}{2}+it_q})$ (with t_q satisfying $q^{2it_q} = 1$),

$$W^{\psi_q}_{F,q} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = |a|^{it_q} |a| \mathbb{I}_{\mathbb{Z}_q}(a), \tag{14}$$

$$W^{\psi_q}_{F,q} \begin{pmatrix} 0 & a \\ -1 & 0 \end{pmatrix} = -|a|^{it_q} |aq| \mathbb{I}_{\mathbb{Z}_q}(qa). \tag{15}$$

For $q = \infty$,

$$W^{\psi_\infty}_{F,\infty} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = a^{k/2} e^{-2\pi a} \mathbb{I}_{\mathbb{R}^+}(a) e^{-ik\theta}. \tag{16}$$

We now evaluate the local integrals L_q. We will often drop the subscript q in the following sections (e.g. we write χ instead of χ_q etc.) Also we will use formulas (11) and (12) repeatedly without comment.

3.4.1. *$q \nmid dN$, q split in K.* Identifying $\mathcal{O}_K \otimes \mathbb{Z}_q$ with $\mathbb{Z}_q \times \mathbb{Z}_q$ suppose that $I = (q^{-n_1}, q^{-n_2})$. Then $(N\mathcal{J}) = (q^{n_1+n_2})$. We have seen that in this case $W^{\psi_q}_{\theta_{\varphi_2}}$ is right invariant by K_q. Identifying $(K \otimes \mathbb{Q}_q)^\times$ with $\mathbb{Q}_q^\times \times \mathbb{Q}_q^\times$, let $h = (t, t^{-1})$ and $\tilde{h} = (1, -(N\mathcal{J})^{-1}a)$. Then $\varphi_2(-(N\mathcal{J})^{-1}a(h\tilde{h})^{-1}\mathcal{J}) \neq 0 \iff (N\mathcal{J})^{-1}a(h\tilde{h})^{-1} \in I \iff -n_2 \leq v_q(t) \leq -n_2 + v_q(a)$. The character χ' restricted to $(K \otimes \mathbb{Q}_q)^\times = \mathbb{Q}_q^\times \times \mathbb{Q}_q^\times$ is of the form (λ, λ^{-1}) since $\chi'|\mathbb{Q}_q^\times = 1$.

Then $W^{\psi_q}_{\theta_{\varphi_2}}\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}$ equals

$$\begin{aligned}
&\mathbb{I}_{\mathbb{Z}_q}(a)|(N\mathcal{J})^{-1}a|^{1/2}\int_{-n_2\le v_q(t)\le -n_2+v_q(a)}\chi'(t,-(N\mathcal{J})^{-1}at^{-1})d^\times t\\
&=\mathbb{I}_{\mathbb{Z}_q}(a)|(N\mathcal{J})^{-1}a|^{1/2}\lambda^{-1}(-(N\mathcal{J})^{-1}a)\sum_{-n_2\le n\le -n_2+v_q(a)}\lambda(q)^{2n}\\
&=\mathbb{I}_{\mathbb{Z}_q}(a)|(N\mathcal{J})^{-1}a|^{1/2}\lambda(q)^{n_1-n_2}\frac{\lambda(aq)-\lambda^{-1}(aq)}{\lambda(q)-\lambda^{-1}(q)}.
\end{aligned}$$

Now, $W^{\psi_q}_{F,q}$ is also right invariant under K_q. If $\pi_q(f)\simeq\pi(\mu_1,\mu_2)$, by (13),

$$\begin{aligned}
L_q=&\int_{\mathbb{Q}_q^\times}|a|^{1/2}\frac{\mu_1(aq)-\mu_2(aq)}{\mu_1(q)-\mu_2(q)}\\
&\cdot|(N\mathcal{J})^{-1}a|^{1/2}\lambda(q)^{n_1-n_2}\frac{\lambda(aq)-\lambda^{-1}(aq)}{\lambda(q)-\lambda^{-1}(q)}\mathbb{I}_{\mathbb{Z}_q}(a)|a|^{s-1}d^\times a\\
=&|(N\mathcal{J})^{-1}|^{1/2}\lambda(q)^{n_1-n_2}L_q(2s,\eta_K)^{-1}L_q(s,\pi_f\otimes\pi_{\chi'})
\end{aligned}$$

where $\pi_{\chi'}$ denotes the automorphic representation of GL(2)) associated to χ' by Langlands functoriality (so that $L_{\mathbb{Q}}(s,\pi_{\chi'})=L_K(s,\chi')$). The last equality above follows from a standard calculation. (See [20] for instance.)

3.4.2. *$q\nmid dN$, q inert in K.* Let $\mathfrak{q}$ denote the unique prime ideal in $\mathcal{O}_K\otimes\mathbb{Z}_q$ and suppose that $I=\mathfrak{q}^{-n}$, so that $(N\mathcal{J})=(q^{2n})$. Note that $W^{\psi_q}_{\theta_{\varphi_2}}\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}=0$ unless $-N(\mathcal{J})^{-1}a$ is a norm from $K\otimes\mathbb{Q}_q$. Since $(N\mathcal{J})=(q^{2n})$ and q is unramified in K, this happens exactly when $v_q(a)$ is even. Now $\varphi_2(-(N\mathcal{J})^{-1}a(h\tilde{h})^{-1}\mathcal{J})\neq 0\iff (N\mathcal{J})^{-1}a(h\tilde{h})^{-1}\in I\iff v_q(a)\ge 0$. Since χ' is unramified and the norm 1 elements in $K\otimes\mathbb{Q}_q$ are all units, χ' factors as $\chi'=\tilde{\chi}\circ N_{(K\otimes\mathbb{Q}_q)^\times/\mathbb{Q}_q^\times}$ for some character $\tilde{\chi}$ of $\mathbb{Q}_q^\times$. Hence

$$\begin{aligned}
W^{\psi_q}_{\theta_{\varphi_2}}\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}&=\int_{(K\otimes\mathbb{Q}_q)^{(1)}}|(N\mathcal{J})^{-1}a|^{1/2}\chi'(h\tilde{h})d^\times h\cdot\mathbb{I}_{\{v_q(\cdot)\ge 0,\equiv 0 \bmod 2\}}(a)\\
&=|(N\mathcal{J})^{-1}a|^{1/2}\tilde{\chi}(-(N\mathcal{J})^{-1}a)\cdot\mathbb{I}_{\{v_q(\cdot)\ge 0,\equiv 0 \bmod 2\}}(a)\\
&=|(N\mathcal{J})^{-1}a|^{1/2}\frac{(\tilde{\chi}\eta_K)(aq)-\tilde{\chi}(aq)}{(\tilde{\chi}\eta_K)(q)-\tilde{\chi}(q)}\mathbb{I}_{\mathbb{Z}_q}(a)
\end{aligned}$$

since χ' unramified implies $\tilde{\chi}(-N\mathcal{J})=1$. Again, $W^{\psi_q}_{F,q}$ is right invariant under K_q. If $\pi_q(f)\simeq\pi(\mu_1,\mu_2)$, by (13),

$$\begin{aligned}
L_q=&\int_{\mathbb{Q}_q^\times}|a|^{1/2}\frac{\mu_1(aq)-\mu_2(aq)}{\mu_1(q)-\mu_2(q)}\\
&\cdot|(N\mathcal{J})^{-1}a|^{1/2}\frac{(\tilde{\chi}\eta_K)(aq)-\tilde{\chi}(aq)}{(\tilde{\chi}\eta_K)(q)-\tilde{\chi}(q)}\mathbb{I}_{\mathbb{Z}_q}(a)|a|^{s-1}d^\times a\\
=&|(N\mathcal{J})^{-1}|^{1/2}L_q(2s,\eta_K)^{-1}L_q(s,\pi_f\otimes\pi_{\chi'}).
\end{aligned}$$

3.4.3. $q \mid N^+$. Writing $\mathcal{O}' \otimes \mathbb{Z}_q = \mathcal{O}_K \otimes \mathbb{Z}_q + I\mathcal{J}$ and taking discriminants, we see that $(q) = (NI)(N\mathcal{J})$. Identifying $\mathcal{O}_K \otimes \mathbb{Z}_q$ with $\mathbb{Z}_q \times \mathbb{Z}_q$, let $I = (q^{-n_1}, q^{-n_2})$, so that $(N\mathcal{J}) = q^{n_1+n_2+1}$. Also we may suppose that the local representation $\pi_{f,q}$ is equivalent to the special representation

$$\sigma(|\cdot|^{\frac{1}{2}+it_q}, |\cdot|^{-\frac{1}{2}+it_q})$$

(with some t_q satisfying $q^{2it_q} = 1$).

We begin by computing I_1. Identifying $(K \otimes \mathbb{Q}_q)^\times$ with $\mathbb{Q}_q^\times \times \mathbb{Q}_q^\times$, let $h = (t, t^{-1})$ and $\tilde{h} = (1, -(N\mathcal{J})^{-1}a)$. Then

$$\begin{aligned}\varphi_2(-(N\mathcal{J})^{-1}a(h\tilde{h})^{-1}\mathcal{J}) \neq 0 &\iff (N\mathcal{J})^{-1}a(h\tilde{h})^{-1} \in I \\ &\iff -n_2 \leq v_q(t) \leq -n_2 - 1 + v_q(a) \Rightarrow v_q(a) \geq 1.\end{aligned}$$

The character χ' restricted to $(K \otimes \mathbb{Q}_q)^\times = \mathbb{Q}_q^\times \times \mathbb{Q}_q^\times$ is of the form (λ, λ^{-1}) since $\chi'|\mathbb{Q}_q^\times = 1$. Hence $W_{\theta_{\varphi_2}}^{\psi_q}\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}$ is equal to

$$\begin{aligned}&\mathbb{I}_{\mathbb{Z}_q}(a/q)|(N\mathcal{J})^{-1}a|^{1/2} \int_{-n_2 \leq v_q(t) \leq -n_2-1+v_q(a)} \chi'(t, -(N\mathcal{J})^{-1}at^{-1})d^\times t \\ =&\mathbb{I}_{\mathbb{Z}_q}(a/q)|(N\mathcal{J})^{-1}a|^{1/2}\lambda^{-1}(-(N\mathcal{J})^{-1}a) \sum_{-n_2 \leq n \leq -n_2-1+v_q(a)} \lambda(q)^{2n} \\ =&\mathbb{I}_{\mathbb{Z}_q}(a/q)|(N\mathcal{J})^{-1}a|^{1/2}\lambda(q)^{n_1-n_2}\frac{\lambda(a) - \lambda^{-1}(a)}{\lambda(q) - \lambda^{-1}(q)}.\end{aligned}$$

By (14), I_1 equals

$$\begin{aligned}&\int_{\mathbb{Q}_q^\times} |a|^{it_q}|a|\mathbb{I}_{\mathbb{Z}_q}(a)|(N\mathcal{J})^{-1}a|^{1/2}\lambda(q)^{n_1-n_2}\frac{\lambda(a) - \lambda^{-1}(a)}{\lambda(q) - \lambda^{-1}(q)}\mathbb{I}_{\mathbb{Z}_q}(a/q)|a|^{s-1}d^\times a \\ =&\int_{\mathbb{Q}_q^\times} |aq|^{it_q}|aq||(N\mathcal{J})^{-1}aq|^{1/2}\lambda(q)^{n_1-n_2}\frac{\lambda(aq) - \lambda^{-1}(aq)}{\lambda(q) - \lambda^{-1}(q)}\mathbb{I}_{\mathbb{Z}_q}(a)|aq|^{s-1}d^\times a \\ =&|q|^{\frac{1}{2}+it_q+s}|(N\mathcal{J})^{-1}|^{1/2}\lambda(q)^{n_1-n_2}\int_{\mathbb{Q}_q^\times} |a|^{\frac{1}{2}+it_q+s}\frac{\lambda(aq) - \lambda^{-1}(aq)}{\lambda(q) - \lambda^{-1}(q)}\mathbb{I}_{\mathbb{Z}_q}(a)d^\times a \\ =&\frac{q^{-\frac{1}{2}-it_q}q^{-s}|(N\mathcal{J})^{-1}|^{1/2}\lambda(q)^{n_1-n_2}}{(1 - (\frac{1}{q})^{\frac{1}{2}+it_q}\lambda(q)q^{-s})(1 - (\frac{1}{q})^{\frac{1}{2}+it_q}\lambda^{-1}(q)q^{-s})}.\end{aligned}$$

We now compute I_2 noting that

$$\begin{aligned}\hat{\varphi}_2(-(N\mathcal{J})^{-1}a(h\tilde{h})^{-1}\mathcal{J}) \neq 0 &\iff (N\mathcal{J})^{-1}a(h\tilde{h})^{-1} \in \frac{1}{q}I \\ &\iff -n_2 - 1 \leq v_q(t) \leq -n_2 + v_q(a) \Rightarrow v_q(a) \geq -1.\end{aligned}$$

Since $\hat{\varphi}_2$ takes the value $\frac{1}{q}$ whenever it is nonzero, $W_{\theta_{\varphi_2}}^{\psi_q}\begin{pmatrix} 0 & -a \\ -1 & 0 \end{pmatrix}$ is equal to

$$\begin{aligned}
&\frac{1}{q}\mathbb{I}_{\mathbb{Z}_q}(aq)|(N\mathcal{J})^{-1}a|^{1/2}\int_{-n_2-1\le v_q(t)\le -n_2+v_q(a)}\chi'(t,-(N\mathcal{J})^{-1}at^{-1})d^\times t\\
&=\frac{1}{q}\mathbb{I}_{\mathbb{Z}_q}(aq)|(N\mathcal{J})^{-1}a|^{1/2}\lambda^{-1}(-(N\mathcal{J})^{-1}a)\sum_{-n_2-1\le n\le -n_2+v_q(a)}\lambda(q)^{2n}\\
&=\frac{1}{q}\mathbb{I}_{\mathbb{Z}_q}(aq)|(N\mathcal{J})^{-1}a|^{1/2}\lambda(q)^{n_1-n_2}\frac{\lambda(aq^2)-\lambda^{-1}(aq^2)}{\lambda(q)-\lambda^{-1}(q)}.
\end{aligned}$$

By (15), I_2 equals

$$\begin{aligned}
&q\int_{\mathbb{Q}_q^\times}-|a|^{it_q}|aq|\mathbb{I}_{\mathbb{Z}_q}(aq)\frac{1}{q}|(N\mathcal{J})^{-1}a|^{1/2}\\
&\qquad\cdot\lambda(q)^{n_1-n_2}\frac{\lambda(aq^2)-\lambda^{-1}(aq^2)}{\lambda(q)-\lambda^{-1}(q)}\mathbb{I}_{\mathbb{Z}_q}(aq)|a|^{s-1}d^\times a\\
&=\int_{\mathbb{Q}_q^\times}-|a/q|^{it_q}|a|\mathbb{I}_{\mathbb{Z}_q}(a)|(N\mathcal{J})^{-1}a/q|^{1/2}\\
&\qquad\cdot\lambda(q)^{n_1-n_2}\frac{\lambda(aq)-\lambda^{-1}(aq)}{\lambda(q)-\lambda^{-1}(q)}\mathbb{I}_{\mathbb{Z}_q}(a)|a/q|^{s-1}d^\times a\\
&=-|q|^{\frac{1}{2}-it_q-s}|(N\mathcal{J})^{-1}|^{1/2}\lambda(q)^{n_1-n_2}\int_{\mathbb{Q}_q^\times}|a|^{\frac{1}{2}+it_q+s}\frac{\lambda(aq)-\lambda^{-1}(aq)}{\lambda(q)-\lambda^{-1}(q)}\mathbb{I}_{\mathbb{Z}_q}(a)d^\times a\\
&=\frac{-q^{-\frac{1}{2}+it_q}q^s|(N\mathcal{J})^{-1}|^{1/2}\lambda(q)^{n_1-n_2}}{(1-(\frac{1}{q})^{\frac{1}{2}+it_q}\lambda(q)q^{-s})(1-(\frac{1}{q})^{\frac{1}{2}+it_q}\lambda^{-1}(q)q^{-s})}.
\end{aligned}$$

Finally, since $L_q=\frac{1}{q+1}[I_1+I_2]$ and $q^{it_q}=q^{-it_q}$, we get

$$\begin{aligned}
L_q&=\frac{1}{q+1}|(N\mathcal{J})^{-1}|^{1/2}\lambda(q)^{n_1-n_2}\frac{-q^{-\frac{1}{2}+it_q}q^s(1-q^{-2s})}{(1-(\frac{1}{q})^{\frac{1}{2}+it_q}\lambda(q)q^{-s})(1-(\frac{1}{q})^{\frac{1}{2}+it_q}\lambda^{-1}(q)q^{-s})}\\
&=-\frac{1}{q+1}|(N\mathcal{J})^{-1}|^{1/2}\lambda(q)^{n_1-n_2}q^{-\frac{1}{2}+it_q+s}L_q(2s,\eta_K)^{-1}L_q(s,\pi_f\otimes\pi_{\chi'}).
\end{aligned}$$

3.4.4. $q\mid N^-$. Again, $\mathcal{O}'\otimes\mathbb{Z}_q=\mathcal{O}_K\otimes\mathbb{Z}_q+I\mathcal{J}$ for I an ideal in $\mathcal{O}_K\otimes\mathbb{Z}_q$. Let $\mathfrak{q}$ denote the unique prime ideal of $\mathcal{O}_K\otimes\mathbb{Z}_q$, and suppose that $I=\mathfrak{q}^{-n}$. Taking discriminants yields $(q)=NI(N\mathcal{J})$ so that $(N\mathcal{J})=q^{2n+1}$.

We begin by computing I_1. We may suppose that the local representation $\pi_{f,q}$ is equivalent to the special representation $\sigma(|\cdot|^{\frac{1}{2}+it_q},|\cdot|^{-\frac{1}{2}+it_q})$ (with some t_q satisfying $q^{2it_q}=1$). Recall that $W_{\theta_{\varphi_2}}^{\psi_q}\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}=0$ unless $-N(\mathcal{J})^{-1}a$ is a norm from $K\otimes\mathbb{Q}_q$. Since $(N\mathcal{J})=(q^{2n+1})$ and q is unramified in K, this happens exactly when $v_q(a)$ is odd. Now $\varphi_2(-(N\mathcal{J})^{-1}a(h\tilde{h})^{-1}\mathcal{J})\neq 0\iff$

$(N\mathcal{J})^{-1}a(h\tilde{h})^{-1} \in I \iff v_q(a) \geq 1$. Since χ' is unramified and the norm-1 elements in $K \otimes \mathbb{Q}_q$ are all units, χ' factors as $\chi' = \tilde{\chi} \circ N_{(K\otimes\mathbb{Q}_q)^\times/\mathbb{Q}_q^\times}$ for some character $\tilde{\chi}$ of $\mathbb{Q}_q^\times$. Hence

$$\begin{aligned} W^{\psi_q}_{\theta_{\varphi_2}}\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} &= \int_{(K\otimes\mathbb{Q}_q)^{(1)}} |(N\mathcal{J})^{-1}a|^{1/2}\chi'(h\tilde{h})d^\times h \cdot \mathbb{I}_{\{v_q(\cdot)\geq 1, \equiv 1 \bmod 2\}}(a) \\ &= |(N\mathcal{J})^{-1}a|^{1/2}\tilde{\chi}(-(N\mathcal{J})^{-1}a) \cdot \mathbb{I}_{\{v_q(\cdot)\geq 1, \equiv 1 \bmod 2\}}(a) \\ &= |(N\mathcal{J})^{-1}a|^{1/2} \cdot \mathbb{I}_{\{v_q(\cdot)\geq 1, \equiv 1 \bmod 2\}}(a) \end{aligned}$$

since $v_q(-N(\mathcal{J})^{-1}a) \equiv 0 \mod 2 \Rightarrow \tilde{\chi}(-(N\mathcal{J})^{-1}a) = 1$. Thus

$$\begin{aligned} I_1 &= \int_{\mathbb{Q}_q^\times} |a|^{it_q}|a|\mathbb{I}_{\mathbb{Z}_q}(a)|(N\mathcal{J})^{-1}a|^{1/2} \cdot \mathbb{I}_{\{v_q(\cdot)\geq 1, \equiv 1 \bmod 2\}}(a)|a|^{s-1}d^\times a \\ &= \int_{\mathbb{Q}_q^\times} |aq|^{it_q}|aq|\mathbb{I}_{\mathbb{Z}_q}(aq)|(N\mathcal{J})^{-1}aq|^{1/2} \cdot \mathbb{I}_{\{v_q(\cdot)\geq 1, \equiv 1 \bmod 2\}}(aq)|aq|^{s-1}d^\times a \\ &= |q|^{\frac{1}{2}+it_q+s}|(N\mathcal{J})^{-1}|^{1/2} \int_{\mathbb{Q}_q^\times} \mathbb{I}_{\{v_q(\cdot)\geq 0, \equiv 0 \bmod 2\}}(a)|a|^{\frac{1}{2}+it_q+s}d^\times a \\ &= |q|^{\frac{1}{2}+it_q+s}|(N\mathcal{J})^{-1}|^{1/2} \sum_{n=0}^{\infty}((\frac{1}{q})^{\frac{1}{2}+it_q+s})^{2n} \\ &= |q|^{\frac{1}{2}+it_q+s}|(N\mathcal{J})^{-1}|^{1/2} \frac{1}{(1-(\frac{1}{q})^{\frac{1}{2}+it_q}q^{-s})(1+(\frac{1}{q})^{\frac{1}{2}+it_q}q^{-s})}. \end{aligned}$$

Next we compute I_2. Now $\hat{\varphi}_2(-(N\mathcal{J})^{-1}a(h\tilde{h})^{-1}\mathcal{J}) \neq 0 \iff (N\mathcal{J})^{-1}a(h\tilde{h})^{-1} \in \mathfrak{q}^{-1}I \iff v_{\mathfrak{q}}((N\mathcal{J})^{-1}a(h\tilde{h})^{-1}) \geq -n-1 \iff v_q(a) \geq -1$. Since $\hat{\varphi}_2$ takes the value $\frac{1}{q}$ whenever it is nonzero, $W^{\psi_q}_{\theta_{\varphi_2}}\begin{pmatrix} 0 & -a \\ -1 & 0 \end{pmatrix}$ equals

$$\begin{aligned} -\frac{1}{q}\int_{(K\otimes\mathbb{Q}_q)^{(1)}} &|N(\mathcal{J})^{-1}a|^{1/2}\mathbb{I}_{\{v_q(\cdot)\geq -1, \equiv 1 \bmod 2\}}(a)\chi'(h\tilde{h})d^\times h \\ &= -\frac{1}{q}|N(\mathcal{J})^{-1}a|^{1/2}\mathbb{I}_{\{v_q(\cdot)\geq -1, \equiv 1 \bmod 2\}}(a) \end{aligned}$$

since $v_q(-N(\mathcal{J})^{-1}a) \equiv 0 \mod 2 \Rightarrow \tilde{\chi}(-(N\mathcal{J})^{-1}a) = 1$. Thus, I_2 is equal to

$$\begin{aligned} q\int_{\mathbb{Q}_q^\times} &\left(-|a|^{it_q}|aq|\mathbb{I}_{\mathbb{Z}_q}(qa)\right)\left(-\frac{1}{q}|N(\mathcal{J})^{-1}a|^{1/2}\mathbb{I}_{\{v_q(\cdot)\geq -1, \equiv 1 \bmod 2\}}(a)\right)|a|^{s-1}d^\times a \\ &= \int_{\mathbb{Q}_q^\times} \left(|\frac{a}{q}|^{it_q}|a|\mathbb{I}_{\mathbb{Z}_q}(a)\right)\left(|N(\mathcal{J})^{-1}\frac{a}{q}|^{1/2}\mathbb{I}_{\{v_q(\cdot)\geq -1, \equiv 1 \bmod 2\}}(\frac{a}{q})\right)|\frac{a}{q}|^{s-1}d^\times a \\ &= |q|^{\frac{1}{2}-it_q-s}|N(\mathcal{J})^{-1}|^{1/2}\int_{\mathbb{Q}_q^\times} |a|^{\frac{1}{2}+it_q+s}\mathbb{I}_{\{v_q(\cdot)\geq 0, \equiv 0 \bmod 2\}}(a)d^\times a \\ &= |q|^{\frac{1}{2}-it_q-s}|(N\mathcal{J})^{-1}|^{1/2}\frac{1}{(1-(\frac{1}{q})^{\frac{1}{2}+it_q}q^{-s})(1+(\frac{1}{q})^{\frac{1}{2}+it_q}q^{-s})}. \end{aligned}$$

Finally, since $L_q = \frac{1}{q+1}[I_1 + I_2]$, we get

$$
\begin{aligned}
L_q &= \frac{1}{q+1}|(N\mathcal{J})^{-1}|^{1/2} \frac{|q|^{\frac{1}{2}+it_q+s} + |q|^{\frac{1}{2}-it_q-s}}{(1-(\frac{1}{q})^{\frac{1}{2}+it_q}q^{-s})(1+(\frac{1}{q})^{\frac{1}{2}+it_q}q^{-s})} \\
&= \frac{1}{q+1}|(N\mathcal{J})^{-1}|^{1/2} \frac{q^{it_q+s-\frac{1}{2}}(1+q^{-2s})}{(1-(\frac{1}{q})^{\frac{1}{2}+it_q}q^{-s})(1+(\frac{1}{q})^{\frac{1}{2}+it_q}q^{-s})} \\
&= \frac{1}{q+1}|(N\mathcal{J})^{-1}|^{1/2} q^{-\frac{1}{2}+it_q+s} L_q(2s,\eta_K)^{-1} L_q(s,\pi_f \otimes \pi_{\chi'}).
\end{aligned}
$$

3.4.5. *q ramified in K, i.e. $q \mid d$.* In this case $D \otimes \mathbb{Q}_q \simeq M_2(\mathbb{Q}_q)$. Since d is odd, $v_q(d) = 1$. Suppose that $K' := K \otimes \mathbb{Q}_q = \mathbb{Q}_q(\sqrt{\epsilon q})$ where ϵ is a unit in $\mathbb{Z}_q$. Then the embedding ϕ of $K \otimes \mathbb{Q}_q$ in $D \otimes \mathbb{Q}_q$ given by

$$
\sqrt{\epsilon q} \mapsto \begin{pmatrix} 0 & 1 \\ \epsilon q & 0 \end{pmatrix}
$$

sends $\mathcal{O}_K \otimes \mathbb{Z}_q$ to $M_2(\mathbb{Z}_q)$. If $j : K \hookrightarrow D$ is the embedding corresponding to the Heegner point chosen, then j and ϕ are conjugate by an element β of $\mathrm{GL}_2(\mathbb{Q}_q)$. We claim that they are in fact conjugate by an element of $\mathrm{GL}_2(\mathbb{Z}_q)$. To prove this, note first that we can assume that $\det(\beta)$ has even valuation by replacing β by $\begin{pmatrix} 0 & 1 \\ \epsilon q & 0 \end{pmatrix}\beta$ if necessary and then that $v_q(\det(\beta)) = 0$ by dividing β by a suitable power of q. Let $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$
\beta^{-1}\begin{pmatrix} 0 & 1 \\ \epsilon q & 0 \end{pmatrix}\beta = (\text{unit})\begin{pmatrix} -ab\epsilon q + cd & -b^2\epsilon q + d^2 \\ a^2\epsilon q - c^2 & ab\epsilon q - cd \end{pmatrix} \in M_2(\mathbb{Z}_q).
$$

Since $-b^2\epsilon q + d^2 \in \mathbb{Z}_q$, both b and d must lie in $\mathbb{Z}_q$, and likewise for a and c, which shows that $\beta \in \mathrm{GL}_2(\mathbb{Z}_q)$ as required.

Note now that we have two different decompositions of $D \otimes \mathbb{Q}_q$: $D \otimes \mathbb{Q}_q = j(K') + j(K')\mathcal{J}$ and $D \otimes \mathbb{Q}_q = \phi(K') + \phi(K')\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, that are conjugate by β. Let $\beta\mathcal{J}\beta^{-1} = \phi(y_0)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for some $y_0 \in K$. Then $N(\mathcal{J}) = -N(y_0)$. Now, if π denotes the uniformiser $\sqrt{\epsilon q}$,

$$
M_2(\mathbb{Z}_q) = \sum_{i=0}^{q-1} \phi\left(\mathcal{O}_{K'} + \frac{i}{\pi}\right) \times \phi\left(\mathcal{O}_{K'} + \frac{i}{\pi}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

Since $\beta \in \mathrm{GL}_2(\mathbb{Z}_q)$, $M_2(\mathbb{Z}_q) = \beta^{-1} M_2(\mathbb{Z}_q)\beta$

$$= \sum_{i=0}^{q-1} (\beta^{-1}\phi\beta)\left(\mathcal{O}_{K'} + \frac{i}{\pi}\right) \times (\beta^{-1}\phi\beta)\left(\mathcal{O}_{K'} + \frac{i}{\pi}\right) \cdot \beta^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\beta$$
$$= \sum_{i=0}^{q-1} j\left(\mathcal{O}_{K'} + \frac{i}{\pi}\right) \times j\left(\mathcal{O}_{K'} + \frac{i}{\pi}\right) j(y_0^{-1})\mathcal{J}.$$

Thus $\varphi_q = \sum_{i=0}^{q-1} \varphi_{1,i} \otimes \varphi_{2,i}$, where $\varphi_{1,i}$ = the characteristic function of $\mathcal{O}_{K'} + \frac{i}{\pi}$ and $\varphi_{2,i}$ the characteristic function of $(\mathcal{O}_{K'} + \frac{i}{\pi})(y_0^{-1}\mathcal{J})$. While the individual terms $W^{\psi_q}_{\theta^t_{\varphi_{2,i}}} \cdot \Phi^s_{\varphi_{1,i}}$ are not invariant under K_q we see that the sum $\sum_i W^{\psi_q}_{\theta^t_{\varphi_{2,i}}} \cdot \Phi^s_{\varphi_{1,i}}$ is in fact invariant under $K_q^{(\eta)} = \{g \in K_q, \eta_K(\det(g)) = 1\}$. It suffices to check that $\sum_i \varphi_{1,i} \otimes \varphi_{2,i}$ is invariant under the action of the matrices $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in K_q^{(\eta)}$. First,

$$\omega_\psi(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1)(\varphi_{1,i} \otimes \varphi_{2,i})(t_1, t_2 y_0^{-1}\mathcal{J})$$
$$= \psi(xN(t_1))\psi(xN(t_2)N(y_0^{-1}\mathcal{J}))(\varphi_{1,i} \otimes \varphi_{2,i})(t_1, t_2 y_0 \mathcal{J}).$$

If $\varphi_{1,i} \otimes \varphi_{2,i}(t_1, t_2 y_0 \mathcal{J}) \neq 0$, then $t_1, t_2 \in \mathcal{O}_{K'} + \frac{i}{\pi}$, hence $N(t_1) - N(t_2) \in \mathcal{O}_{K'}$ and $\psi(xN(t_1))\psi(xN(t_2)N(y_0^{-1}\mathcal{J})) = \psi(x(N(t_1) - N(t_2))) = 1$ which shows that $\omega_\psi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1\right)(\varphi_{1,i} \otimes \varphi_{2,i}) = \varphi_{1,i} \otimes \varphi_{2,i}$. Next, since $\eta_K(a)^2 = 1$,

$$\omega_\psi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, c\right)(\varphi_{1,i} \otimes \varphi_{2,i})(\cdot, \cdot) = (\varphi_{1,i} \otimes \varphi_{2,i})(ac^{-1}\cdot, ac^{-1}\cdot)$$
$$= \varphi_{1,j} \otimes \varphi_{2,j}$$

where $ac^{-1}i \equiv j \bmod \pi$. Hence $\omega_\psi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, c\right)(\sum_i \varphi_{1,i} \otimes \varphi_{2,i}) = \sum_i \varphi_{1,i} \otimes \varphi_{2,i}$. Finally, since $\zeta_1\zeta_2 = 1$,

$$\omega_\psi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1\right)(\varphi_{1,i} \otimes \varphi_{2,i}) = \hat{\varphi}_{1,i} \otimes \hat{\varphi}_{2,i}.$$

We now compute $\hat{\varphi}_{1,i}$ and $\hat{\varphi}_{2,i}$.

$$\hat{\varphi}_{1,i}(y) = \int_{D_1 \otimes \mathbb{Q}_q} \varphi_{1,i}(s)\psi(\langle s, y\rangle)ds = \int_{\mathcal{O}_{K\otimes\mathbb{Q}_q} + \frac{i}{\pi}} \psi(\langle s, y\rangle)ds$$
$$= \psi(\langle \frac{i}{\pi}, y\rangle)\int_{\mathcal{O}_{K\otimes\mathbb{Q}_q}} \psi(\langle s, y\rangle)ds = \psi(\langle \frac{i}{\pi}, y\rangle)\int_{\mathcal{O}_{K\otimes\mathbb{Q}_q}} \psi(Tr(sy^i))ds.$$

This integral is nonzero $\iff$ $\psi(Tr(sy^i)) = 1$ for all s $\iff$ $Tr(sy^i) \in \mathbb{Z}_q$ for all s $\iff$ $y^i \in \pi^{-1}$ $\iff$ $y \in \pi^{-1}$, in which case the integral is equal to $\frac{1}{\sqrt{q}}$. Thus $\hat{\varphi}_{1,i} = \frac{1}{\sqrt{q}}\psi(\langle \frac{i}{\pi}, y\rangle)$char. function of π^{-1}. Likewise, $\hat{\varphi}_{2,i} = \frac{1}{\sqrt{q}}\psi(-\langle \frac{i}{\pi}, \cdot\rangle)$char. function of $\pi^{-1}y_0^{-1}\mathcal{J}$. Then $\sum_i(\hat{\varphi}_{1,i}\otimes\hat{\varphi}_{2,i})(t_1, t_2y_0^{-1}\mathcal{J}) = 0$ unless $t_1, t_2 \in \pi^{-1}$, in which case it is equal to $\frac{1}{q}\sum_i \psi(\langle \frac{i}{\pi}, (t_1 - t_2)\rangle)$. Suppose $t_1 - t_2 = \alpha + \frac{j}{\pi}$ with $\alpha \in \mathcal{O}_{K'}$. Then

$$\frac{1}{q}\sum_i \psi\left(\left\langle \frac{i}{\pi}, (t_1 - t_2)\right\rangle\right) = \frac{1}{q}\sum_i \psi\left(\frac{-2ij}{\epsilon q}\right). = \delta_{0j}.$$

Thus $\sum_i(\hat{\varphi}_{1,i}\otimes\hat{\varphi}_{2,i})(t_1, t_2y_0^{-1}\mathcal{J})$ is zero unless $t_1, t_2 \in \pi^{-1}$ and $t_1 - t_2 \in \mathcal{O}_{K'}$ in which case it is equal to 1. Clearly then, $\sum_i(\hat{\varphi}_{1,i}\otimes\hat{\varphi}_{2,i}) = \sum_i \varphi_{1,i}\otimes\varphi_{2,i}$ as required. This proves our claim that $\sum_i W^{\psi_q}_{\theta^t_{\varphi_{2,i}}}\cdot\Phi^s_{\varphi_{1,i}}$ is invariant under $K_q^{(\eta)}$. Thus

$$L_q = \int_{\mathbb{Q}_q^\times} W^{\psi_q}_{F,q}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\left\{\sum_i W^{\psi_q}_{\theta^t_{\varphi_{2,i}}}\begin{pmatrix} -a & 0 \\ 0 & 1\end{pmatrix}\Phi^s_{\varphi_{1,i}}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\right\}|a|^{-1}d^\times a.$$

But $\Phi^s_{\varphi_{1,i}}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} = |a|^s\delta_{0i}$. Hence

$$L_q = \int_{\mathbb{Q}_q^\times} W^{\psi_q}_{F,q}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}W^{\psi_q}_{\theta^t_{\varphi_{2,0}}}\begin{pmatrix} -a & 0 \\ 0 & 1\end{pmatrix}|a|^{s-1}d^\times a \qquad \text{where}$$

$$W_{\theta_{\varphi_{2,0}}}\left(\begin{pmatrix} -a & 0 \\ 0 & 1\end{pmatrix}\right) = |(N\mathcal{J})^{-1}a|^{1/2}\int_{(K\otimes\mathbb{Q}_q)^{(1)}} \varphi_{2,0}(-(N\mathcal{J})^{-1}a(h\tilde{h})^{-1}\mathcal{J}) \cdot\chi'(h\tilde{h})d^\times h$$

for any $\tilde{h} \in K\otimes\mathbb{Q}_q$ with $N(\tilde{h}) = -aN(\mathcal{J})^{-1}$ if $\eta_K(-aN(\mathcal{J})^{-1}) = 1$ and is equal to zero if $\eta_K(-aN(\mathcal{J})^{-1}) = -1$. Suppose the former holds. Since $\eta_K(-N(\mathcal{J})^{-1}) = 1$, in this case $\eta_K(a) = 1$. Now, $\varphi_{2,0}(-(N\mathcal{J})^{-1}a(h\tilde{h})^{-1}\mathcal{J}) \neq 0 \iff v_{\mathfrak{q}}(-(N\mathcal{J})^{-1}a(h\tilde{h})^{-1}y_0) \geq 0 \iff v_q(a) \geq 0$, in which case it is equal to 1. Since χ' is unramified, $\chi' = \tilde{\chi}\circ N$ for some character $\tilde{\chi}$, and $\chi'(h\tilde{h}) = \chi'(\tilde{h}) = \tilde{\chi}(N(\tilde{h})) = \tilde{\chi}(-aN(\mathcal{J})^{-1})$. Now $\tilde{\chi}$ is *a priori* defined only on the subgroup $N(K^\times)$ of $\mathbb{Q}_q^\times$. But it extends, in exactly two ways to a character of $\mathbb{Q}_q^\times$. Denote the one which is trivial on the units of $\mathbb{Z}_q$ by $\tilde{\chi}$. Then the other extension is $\tilde{\chi}\eta_K$ and in any case,

$$W_{\theta_{\varphi_{2,0}}}\left(\begin{pmatrix} -a & 0 \\ 0 & 1\end{pmatrix}\right) = (1+\eta_K(a))|N(\mathcal{J})^{-1}a|^{1/2}\tilde{\chi}(-aN(\mathcal{J})^{-1})\mathbb{I}_{\mathbb{Z}_q}(a)$$

since the volume of $(K\otimes\mathbb{Q}_q)^{(1)}$ with respect to the measure $d^\times h$ is 2. If $\pi_{f,q}\simeq\pi(\mu_1,\mu_2)$, then

$$\begin{aligned}L_q &= |N(\mathcal{J})^{-1}|^{1/2}\tilde\chi(-N(\mathcal{J})^{-1})\\ &\quad\cdot\int_{\mathbb{Q}_q^\times} W_{F,q}\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}|a|^{1/2}(\tilde\chi(a)+(\eta_K\tilde\chi)(a))|a|^{s-1}\mathbb{I}_{\mathbb{Z}_q}(a)d^\times a\\ &= |N(\mathcal{J})^{-1}|^{1/2}\tilde\chi(-N(\mathcal{J})^{-1})\int_{\mathbb{Q}_q^\times} W_{F,q}\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}|a|^{1/2}\tilde\chi(a)|a|^{s-1}\mathbb{I}_{\mathbb{Z}_q}(a)d^\times a\\ &= |N(\mathcal{J})^{-1}|^{1/2}\tilde\chi(-N(\mathcal{J})^{-1})\int_{\mathbb{Q}_q^\times}\frac{\mu_1(aq)-\mu_2(aq)}{\mu_1(q)-\mu_2(q)}\tilde\chi(a)|a|^s\mathbb{I}_{\mathbb{Z}_q}(a)d^\times a\\ &= |N(\mathcal{J})^{-1}|^{1/2}\tilde\chi(-N(\mathcal{J})^{-1})\frac{1}{(1-\mu_1(q)\chi(\pi)q^{-s})(1-\mu_2(q)\chi(\pi)q^{-s})}\\ &= |N(\mathcal{J})^{-1}|^{1/2}\tilde\chi(-N(\mathcal{J})^{-1})L_q(2s,\eta_K)^{-1}L_q(s,\pi_f\otimes\pi_{\chi'}).\end{aligned}$$

3.4.6. $q=\infty$. In this case it is easy to check that $\omega_\psi(\kappa_\theta)\varphi_1=-e^{-\imath\theta}\varphi_1$ and $\omega_\psi(\kappa_\theta)\varphi_2=-e^{\imath(k+1)\theta}\varphi_2$. Since $W_{F,\infty}^{\psi_\infty}(g\kappa_\theta)=e^{\imath k\theta}W_{F,\infty}^{\psi_\infty}(g)$ we have

$$L_\infty=(2\pi)j(\alpha,\imath)^{2k}\int_{\mathbb{R}^\times}W_{F,\infty}^{\psi}\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}W^{\psi_\infty}_{\theta^t_{\varphi_{\infty,2}}}\begin{pmatrix} -a & 0\\ 0 & 1\end{pmatrix}|a|^{s-1}d^\times a,\quad\text{with}$$

$$\begin{aligned}W^{\psi_\infty}_{\theta^t_{\varphi_{\infty,2}}}\begin{pmatrix} -a & 0\\ 0 & 1\end{pmatrix} &= |(N\mathcal{J})^{-1}a|^{1/2}\int_{(K\otimes\mathbb{R})^{(1)}}\varphi_{\infty,2}(-(N\mathcal{J})^{-1}a(h\tilde h)^{-1}\mathcal{J})\\ &\qquad\cdot\chi'_\infty(h\tilde h)d^\times h,\end{aligned}$$

$\tilde h$ being chosen such that $N(\tilde h)=-(N\mathcal{J})^{-1}a$. Such an $\tilde h$ exists only if $a\geq 0$ in which case we may choose $\tilde h=\sqrt{-N(\mathcal{J})^{-1}a}$. Suppose $h'=h_1+\imath h_2$ where $\alpha^{-1}j(h)\alpha=\begin{pmatrix} h_1 & -h_2\\ h_2 & h_1\end{pmatrix}$, and $y=y_1+\imath y_2$ where $\alpha^{-1}\mathcal{J}\alpha=\begin{pmatrix} y_1 & y_2\\ y_2 & -y_1\end{pmatrix}$. Then

$$\begin{aligned}\varphi_{\infty,2}(-(N\mathcal{J})^{-1}a(h\tilde h)^{-1}\mathcal{J}) &= \varphi_{\infty,2}(\bar h\bar{\tilde h}\mathcal{J})=[Y(\alpha^{-1}j(\bar h)\bar{\tilde h}\mathcal{J}\alpha)]^k e^{-2\pi|N(\mathcal{J})|N(\bar h\bar{\tilde h})}\\ &= y^k(\bar h'\bar{\tilde h})^k e^{-2\pi a}.\end{aligned}$$

Since $\chi'_\infty(h\tilde h)=(h'\tilde h)^{k/2}\cdot(\bar h'\bar{\tilde h})^{-k/2}=h'^k$,

$$\begin{aligned}W^{\psi_\infty}_{\theta^t_{\varphi_{\infty,2}}}\begin{pmatrix} -a & 0\\ 0 & 1\end{pmatrix} &= \mathbb{I}_{\mathbb{R}^+}(a)|(N\mathcal{J})^{-1}a|^{1/2}y^k(-(N\mathcal{J})^{-1}a)^{k/2}e^{-2\pi a}\int d^\times h\\ &= y^k(-(N\mathcal{J})^{-1})^{\frac{k+1}{2}}a^{\frac{k+1}{2}}e^{-2\pi a}\mathbb{I}_{\mathbb{R}^+}(a).\end{aligned}$$

Thus

$$\begin{aligned}L_\infty &= (2\pi)j(\alpha,\imath)^{2k}y^k(-(N\mathcal{J})^{-1})^{\frac{k+1}{2}}\int_0^\infty a^{\frac{k}{2}}e^{-2\pi a}a^{\frac{k+1}{2}}e^{-2\pi a}a^{s-1}d^\times a\\ &= (2\pi)J(\alpha,\imath)^{2k}y^k(-(N\mathcal{J})^{-1})^{\frac{k+1}{2}}\frac{1}{(4\pi)^{k+s-\frac{1}{2}}}\Gamma\left(k+s-\frac{1}{2}\right)\\ &= \left(\frac{\overline{J(\mathcal{J},z)}}{\Im(z)}\right)^k(-(N\mathcal{J})^{-1})^{\frac{k+1}{2}}\frac{2\pi}{(4\pi)^{k+s-\frac{1}{2}}}\Gamma\left(k+s-\frac{1}{2}\right).\end{aligned}$$

The last equality follows from the computation: $-\bar{y} = J(\alpha^{-1}\mathcal{J}\alpha,\imath) = J(\alpha^{-1},\bar{z})J(\mathcal{J},z)J(\alpha,\imath) = \Im(z)J(\mathcal{J},z)J(\alpha,\imath)^2$ since $J(\alpha^{-1},\bar{z}) = J(\alpha^{-1},\alpha\bar{\imath}) = J(\alpha,\bar{\imath})^{-1} = \overline{J(\alpha,\imath)\Im(z)}$. Thus $yJ(\alpha,\imath)^2 = -\Im(z)\overline{J(\mathcal{J},z)}\cdot\overline{J(\alpha,\imath)}^2J(\alpha,\imath)^2 = -\Im(z)^{-1}\overline{J(\mathcal{J},z)}$, as required. Finally, noting that $J(\mathcal{J},z)\overline{J(\mathcal{J},z)} = -N(\mathcal{J})$ (since $-\Im(z) = \Im(\overline{z}) = \Im(\mathcal{J}z) = N(\mathcal{J})|J(\mathcal{J},z)|^{-2}\Im(z)$) we get

$$(\Im(z)J(\mathcal{J},z))^kL_\infty = (-N\mathcal{J})^{k/2}(-N\mathcal{J})^{-1/2}\frac{2\pi}{(4\pi)^{k+s-\frac{1}{2}}}\Gamma\left(k+s-\frac{1}{2}\right).$$

3.4.7. *The final formula for* $L_\chi(\Psi')^2$. Putting together the results at all primes, and evaluating at $s=\frac{1}{2}$, we have:

THEOREM 3.2. *Assume that* K *has odd discriminant. Then, up to a* p*-adic unit,*

$$(4\Im(z)J(\mathcal{J},z))^kL_\chi(\Psi')^2 = \frac{2\cdot\Gamma(k)}{\pi^k\cdot h_K}\prod_{q|N^-}\frac{q-1}{q+1}\cdot L(1,\eta_K)^{-1}L\left(\frac{1}{2},\pi_f\otimes\pi_{\chi'}\right).$$

Let g' be the classical modular form on $\mathfrak{H}/\Gamma$ that corresponds to Ψ'. Then $\langle g',g'\rangle = \mathrm{vol}(\mathfrak{H}/\Gamma)\langle\Psi',\Psi'\rangle = \mathrm{vol}(\mathfrak{H}/\Gamma)\langle F,F\rangle = \frac{\mathrm{vol}(\mathfrak{H}/\Gamma)}{\mathrm{vol}(\mathfrak{H}/\Gamma_0(N))}\langle f,f\rangle$. Letting $g'' = (\frac{\mathrm{vol}(\mathfrak{H}/\Gamma)}{\mathrm{vol}(\mathfrak{H}/\Gamma_0(N))})^{-1/2}g'$, and Ψ'' be the adelic modular form corresponding to g'', we see then that $\langle g'',g''\rangle = \langle f,f\rangle$.

It is shown in [14] that the form $g''\times g''$ on $D^\times\times D^\times$ is K_f-rational. Hence $g''=\delta g$ where $\delta^2\in K_f$. We will be interested in the λ-adic valuation of δ, or what is the same thing, the λ-adic valuation of g''. It follows from Proposition 2.9 that the λ-adic integrality properties of g'' are controlled by the λ-adic valuation of

$$(2\Im(z)J(\mathcal{J},z))^k\frac{(2\pi)^{2k}L_\chi(\Psi'')^2}{\Omega^{2k}} \sim \frac{\pi^{k-1}\cdot L\left(\frac{1}{2},\pi_f\otimes\pi_{\chi'}\right)}{h_K^2\cdot\Omega^{2k}} \tag{17}$$

where $\sim$ denotes equality up to a λ-adic unit. Here we have used that $p>k$, $p\nmid M$ and $L(1,\eta_K) = 2\pi h_K/w\sqrt{d}$. The reader will note that in this section we have not placed any restriction on K except that p be unramified in K. In the next section we will study the λ-adic integrality of (17) under the assumption that p is split in K. While in this chapter we have normalized our L-functions so that the center of the critical strip is $1/2$, in the next chapter we use the classical normalization which is more standard in the algebraic theory.

4. Integrality of the Rankin-Selberg L-value

For the convenience of the reader we recall briefly our notation. f is a newform of even weight k for $\Gamma_0(N)$. Also, N is square-free and $N = N^+N^-$, where N^- has an even number of prime factors and K is an imaginary quadratic field of discriminant $-d$, chosen such that K is split at the primes dividing N^+ and inert at the primes dividing N^-. Further, χ is an unramified Grossencharacter of K of infinity type $(k, 0)$ (see §2.3.2). Denote by $\theta = \theta(\chi)$ the modular form $\sum_{\mathfrak{a}} \chi(\mathfrak{a}) e^{2\pi i N\mathfrak{a} z}$, the sum being taken over integral ideals of K where we think of χ in the usual way as a character on ideals of K. Then θ is a newform of weight $k+1$ and character η_K (η_K = the quadratic character associated to the extension $K/\mathbb{Q}$) on $\Gamma_1(d)$ and its L-function coincides with the L-function of χ over K. The L-value of concern to us in this chapter is the central (critical) value $L(k, f \otimes \theta)$ of the Rankin-Selberg product of f and θ. (In the previous chapter, this same L-value was denoted $L(\frac{1}{2}, \pi_f \otimes \pi_{\chi'})$.)

Let p be a prime number unramified in K. Recall from Section 2.3.3 that associated to the field K and the prime p is a canonical CM period $\Omega = \Omega_p \in \mathbb{C}$ that is well defined up to multiplication by a p-adic unit. It follows from work of Shimura ([31]) that $\pi^{k-1} L(k, f \otimes \theta)/\Omega^{2k}$ is an algebraic number.

THEOREM 4.1. *Suppose* $p > k+1$ *and* $p \nmid M$ *where* $M = \prod_{q|N} q(q+1)(q-1)$. *Then* $\pi^{k-1} L(k, f \otimes \theta)/\Omega^{2k}$ *is a* λ*-adic integer.*

Theorem 4.1 is proved in this chapter under the assumption that p is *split* in K and $p \nmid h_K$ where h_K is the class number of K. A consequence of this special case of the theorem and the results from the previous chapters is a certain result (Theorem 5.2) about the integrality properties of the theta correspondence from the modular curve $X_0(N)$ to the Shimura curve corresponding to an Eichler order of level N^+ in the indefinite quaternion algebra of discriminant N^-. It is interesting that one can now argue in the reverse direction and deduce Theorem 4.1 even when $p \mid h_K$ or p is *inert* in K. This is explained in Section 5.1.

Note. In this chapter we use the normalized Petersson inner product $\langle f_1, f_2 \rangle = \frac{1}{\mathrm{vol}\,(\Gamma_1(dN)\backslash\mathfrak{H})} \int_{\Gamma_1(dN)\backslash\mathfrak{H}} f_1 \overline{f_2}$.

4.1. *Idea of the proof* (*when* p *is split in* K *and* $p \nmid h_K$). Suppose that $\theta = \sum a_n q^n$ and $f = \sum b_n q^n$ are the q-expansions of θ and f, and let $D(s, f, \theta) = \sum \frac{a_n b_n}{n^s}$. It is easy to see that

$$D(s, f, \theta) = L(s, f \otimes \theta) L_N(2s + 1 - 2k, \eta_K)^{-1} \tag{18}$$

where $L(s, f\otimes\theta)$ is the Rankin-Selberg product and L_N denotes the L-function with the Euler factors at N removed. Consider the Eisenstein series

$$E_{1,Nd}(z,s,\eta_K) = \sum_{(m,n)} \eta_K(n)(mNdz+n)^{-1}|mNdz+n|^{-2s}$$

where the sum is over all $(m,n) \in \mathbb{Z}^2, \neq (0,0)$ and we think of η_K as a character mod Nd. The series above converges for $\Re(s) \gg 0$, and it is well known that it admits an analytic continuation to the whole complex plane. Let $E = E_{1,Nd}(z,0,\eta_K)$. By [31, Thm. 2],

$$D(k,f,\theta) = D(k,f,\theta_\rho) = \frac{c}{2}\pi^{k+1}\langle fE,\theta\rangle L_{Nd}(1,\eta_K)^{-1}$$

where $c = \frac{4^k Nd}{3(k-1)!}\prod_{q|dN}(1+\frac{1}{q})$. (Recall that in the notation of [31], $\theta_\rho = \sum \overline{a_n}q^n$. However χ is unramified; hence $\theta_\rho = \theta$.) Thus

$$\pi^{k-1}L(k, f\otimes\theta) = \frac{c}{2}\pi^{2k+1}\left\langle f\cdot\frac{1}{\pi}E,\theta\right\rangle. \tag{19}$$

It is known (due to Hecke) that $\frac{1}{\pi}E$ has Fourier coefficients that are algebraic and p-integral (see [31, (3.3) and (3.4)]). We see then that to prove Theorem 4.1 it suffices to show that $\prod_{q|d}(q+1)\pi^{2k+1}\langle g,\theta\rangle/\Omega^{2k}$ is p-integral for all integral modular forms $g \in S = S_{k+1}(Nd,\eta_K)$.

We now explain informally how this is done, the details being in the next two sections. One notes that $\langle g,\theta\rangle/\Omega^k = \alpha\langle\theta,\theta\rangle/\Omega^k = \alpha L(\overline{\chi}\chi^\rho, k+1)/\Omega''$ where α is the coefficient of θ when g is written as a linear combination of orthogonal eigenforms and Ω'' is a suitable period. The required integrality will follow if one can bound the denominator of α in terms of the L-value $L(\overline{\chi}\chi^\rho, k+1)/\Omega''$.

Let χ^ρ be the Hecke character $\chi\circ\rho$ where ρ is the nontrivial automorphism of $K/\mathbb{Q}$, and denote by χ_λ, χ_λ^ρ the λ-adic characters of $\mathrm{Gal}(\overline{K}/K)$ associated to χ, χ^ρ respectively. Also let K_0 be an abelian extension of K of degree over K prime to p and K_∞ the unique $\mathbb{Z}_p^2$ extension of K_0 abelian over K, the field K_0 being chosen so that χ_λ factors via $\mathrm{Gal}(K_\infty/K)$. First, one constructs L', an abelian p-extension of K_∞ with controlled ramification, such that the action of $\Gamma = \mathrm{Gal}(K_\infty/K_0)$ on $\mathrm{Gal}(L'/K_\infty)$ by conjugation is via the character $\chi_\lambda(\chi_\lambda^\rho)^{-1}$, and such that the degree $[L' : K_\infty]$ is essentially the p-adic valuation of the denominator of α. The main inputs here are the existence and properties of the λ-adic representations $\rho_{h,\lambda}$ associated to holomorphic forms h of weight $k+1$ and congruent to θ modulo λ. While $\rho_{\theta,\lambda}$ restricted to $\mathrm{Gal}(\overline{K}/K)$ is reducible (in fact equal to $\chi_\lambda \oplus \chi_\lambda^\rho$), $\rho_{h,\lambda}$ restricted to $\mathrm{Gal}(\overline{K}/K)$ will be irreducible as long as h is not a theta lift from K. This fact along with the congruence $\rho_{\theta,\lambda} \equiv \rho_{h,\lambda} \bmod \lambda$ can be used to construct a lattice $\mathcal{L}$ in the representation space of $\rho_{\theta,\lambda}$ whose reduction mod λ (as a representation of $\mathrm{Gal}(\overline{K}/K)$) is a nontrivial extension of the character χ_λ^ρ by χ_λ. The splitting field of $\mathcal{L}/\lambda\mathcal{L}$ over K_∞ is then a field extension of the above type and the field L' is essentially the compositum of all fields obtained in this way from different forms h. On the other hand, extensions of the type constructed are controlled precisely by Iwasawa theory. Let $\mathfrak{p}$ be the prime of K induced by λ. The *main conjecture* in this situation (a theorem of Rubin [30]) shows that the degree $[L' : K_\infty]$ is bounded by the p-adic valuation of the $\mathfrak{p}$-adic L-function evaluated at the character $\chi_\lambda(\chi_\lambda^\rho)^{-1}$, which in turn is equal to the p-adic valuation of the special value $L(\overline{\chi}\chi^\rho, k+1)/\Omega''$.

The prototype of the arguments used to construct L' can be found in work of Ribet [27] on the converse to Herbrand's theorem. However we need to employ the analogs in our context of the more refined techniques of Wiles ([40] and [42]) used in his proof of the Iwasawa conjecture for totally real fields. In the work of Ribet and Wiles, the congruence is betweeen an Eisenstein series and a cusp form while we need to study congruences between theta lifts and cusp forms that are not of theta-type. It turns out that it is easier to keep track of such congruences in the case $p \nmid h_K$ since this condition rules out congruences between theta lifts from K, and hence we restrict ourselves to this case in the present chapter.

We briefly mention the reasons for including the hypotheses $p > k+1$ and $p \nmid \prod_{q|N} q(q+1)(q-1)$ in Theorem 4.1. The assumption $p > k+1$ is needed to ensure that χ_λ and χ_λ^ρ remain distinct on reducing modulo p. The condition $p \nmid q(q+1)(q-1)$ for $q \mid N$ is utilized in controlling level-raising congruences satisfied by θ_χ at such primes q. It should however be hoped that by a more careful analysis some of these hypotheses may be removed in future work.

4.2. *Constructing extensions from congruences.* This section closely follows the paper [40], where congruences between Eisenstein series and cusp forms are used to construct nontrivial extensions of certain characters of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Here we use congruences between theta functions from K and cusp forms not of theta type (for K) to construct extensions of the character $\tilde{\chi_\lambda^\rho}$ of $\mathrm{Gal}(\overline{K}/K)$ by the character $\tilde{\chi_\lambda}$ (where the tildes denote reductions mod λ). We will also make use of some results from [42].

4.2.1. *The congruence ideal.* Recall that $S = S_{k+1}(Nd, \eta_K)$. On S one has the actions of the usual Hecke operators T_q for $q \nmid Nd$ and U_q for $q \mid Nd$. For $q \mid N^+$, let α_q, β_q be the roots of the characteristic polynomial of Frob_q acting on the λ-adic representation associated to θ. Write $q = \mathfrak{q}\overline{\mathfrak{q}}$ where $\mathfrak{q}$ is chosen such that $\chi(\mathfrak{q}) = \alpha_q$. Let T be the set of primes q dividing N^+ such that $a_q(\theta)^2 \equiv q^{k-1}(q+1)^2 \mod \lambda$ (i.e. by [5], exactly the set of primes $q \mid N^+$ at which θ admits a level-raising congruence mod λ). Now

$$\begin{aligned} q^{k-1}(\alpha_q/\beta_q - q)(\beta_q/\alpha_q - q) &= q^{k-1}(q+1)^2 - (\alpha_q + \beta_q)^2 \\ &= q^{k-1}(q+1)^2 - a_q(\theta)^2. \end{aligned}$$

Hence if q is in T, one has either $\alpha_q/\beta_q \equiv q \mod \lambda$ or $\beta_q/\alpha_q \equiv q \mod \lambda$. (Both these conditions cannot be satisfied simultaneously for that would imply $q^2 \equiv 1 \mod \lambda$. Also note that $\alpha_q \not\equiv \beta_q \mod \lambda$ since otherwise $q \equiv 1 \mod p$.) Interchanging α_q and β_q if necessary, we can assume that for $q \in T$ the first congruence holds; i.e. $\alpha_q/\beta_q \equiv q \mod \lambda$. Let $\tilde{\mathbb{T}}$ be the subalgebra of $\mathrm{End}_{\mathbb{C}}(S_{k+1}(Nd,\eta))$ generated over $\mathbb{Z}$ by the Hecke operators T_q for $q \nmid Nd$ and U_q for $q \mid d$ or $q \in T$. Then the old-space corresponding to θ is the direct sum of $2^{\#T}$ spaces V_i on each of which $\tilde{\mathbb{T}}$ acts via a character, the different V_i corresponding to the choice of α_q or β_q as the eigenvalue of U_q acting on V_i for each $q \in T$. Let T_i be the set of primes in T such that the eigenvalue of U_q acting on V_i is β_q. Now, let g be any integral form in S and suppose

$$g = \sum_{i \in P(T)} \delta_i g_i + g' \tag{20}$$

where $g_i \in V_i$ is a (p-adically) primitive ($\tilde{\mathbb{T}}$-eigen)form and g' is orthogonal to the old-space spanned by θ. Here $P(T)$ is the power set of T. Then

$$\frac{\langle g,\theta\rangle}{\Omega^{2k}} = \sum_{i \in P(T)} \delta_i \frac{\langle g_i,\theta\rangle}{\Omega^{2k}}. \tag{21}$$

Lemma 4.2.

$$v_p\left(\left(\prod_{q \mid d}(q+1)\right)\frac{\pi^{2k+1}\langle g_i,\theta\rangle}{\Omega^{2k}}\right) \geq \sum_{q \in T_i} v_p(1 - q^{-(k+1)}\alpha_q^2) + v_p\left(\frac{\pi^{k-1}L(k+1,\overline{\chi}\chi^\rho)}{\Omega^{2k}}\right).$$

Proof. Let P be the product of the primes dividing N^+ that are not in T and the primes dividing N^-. Let θ_i denote the eigenform of $\tilde{\mathbb{T}}$ obtained by dropping from the Euler product for θ the Euler factors $(1-\alpha_q q^{-s})^{-1}$ for $q \in T_i$ and the Euler factors $(1-\beta_q q^{-s})^{-1}$ for $q \in T \setminus T_i$. Then θ_i is an integral form in V_i and it is easy to see that a basis for V_i over $\mathbb{C}$ is given by $\{\theta_i(d'z)\}$ where d' runs over the divisors of P. Suppose that

$$g_i = \sum_{d' \mid P} \gamma_{d'}\theta_i(d'z).$$

Now $\gamma_1 = a_1(g_i)$, so that γ_1 is a p-adic integer. For any $d' > 1, d' \mid P$, $a_{d'}(g_i) = \sum_{d'' \mid d'} \gamma_{d''} a_{d'/d''}(\theta_i)$. By induction on the number of prime factors dividing d', one sees then that each $\gamma_{d'}$ is a p-adic integer. By [31, Lemma 3], one has

$$(22)\quad \langle \theta_i(d'z), \theta\rangle = \left(\prod_{q\in T_i} \frac{(q^{k+1}-\alpha_q^2)}{q^k(q+1)} \prod_{q\in T\setminus T_i} \frac{(q^{k+1}-\beta_q^2)}{q^k(q+1)} \prod_{q\mid d'} \frac{a_q(\theta)}{q^k(q+1)}\right) \langle \theta, \theta\rangle.$$

Note that $q^{k+1}-\beta_q^2 = \beta_q(q\alpha_q - \beta_q)$ is a λ-adic unit. Further, by the assumption $p \nmid M$, the denominators on the right-hand side of (22) are coprime to p. Finally, by [31, 2.5],

$$\begin{aligned} \frac{(4\pi)^{k+1}}{k!}\langle \theta,\theta\rangle &= \mathrm{Res}_{s=k+1} D(s,\theta_\rho,\theta) \\ &= \left(\prod_{q\mid d} \frac{q}{q+1}\right) \mathrm{Res}_{s=k+1} \frac{L_K(s,\overline{\chi}\chi^\rho)\zeta_K(s-k)}{\zeta_{\mathbb{Q}}(2s-2k)} \\ &= \frac{6}{\pi^2}\left(\prod_{q\mid d} \frac{q}{q+1}\right) L_K(k+1,\overline{\chi}\chi^\rho) L(1,\eta_K). \end{aligned}$$

Since $L(1,\eta_K) = 2\pi h_K/w\sqrt{d}$ ($w =$ the number of roots of unity in K) and $p > k+1, p \nmid M$, the lemma follows. □

For the rest of this chapter we fix an i and assume that $v_p(\delta_i) < 0$. (It will be clear that there is nothing to prove if $v_p(\delta_i) \geq 0$.) Let F' be a large enough number field, containing all the Hecke eigenvalues of all eigenforms of level dividing Nd. Denote by $\mathcal{O}$ the ring of integers of F' and by π the prime of $\mathcal{O}$ induced by λ. When

$$S = \oplus V_{i'} \oplus W$$

with W orthogonal to $\oplus V_{i'}$, let $\mathbb{T}'$ be the subalgebra of $\mathrm{End}_{\mathbb{C}}(\oplus_{i'\neq i} V_{i'} \oplus W)$ generated by the image of $\tilde{\mathbb{T}}$ and let $\mathbb{T} = \mathbb{T}' \otimes \mathcal{O}$. Suppose that $v_\pi(\delta_i) = -n$. Pick a scalar $\alpha \in F'$ such that $v_\pi(\alpha) = n$. Define $I = \mathrm{Ann}_{\mathbb{T}}(\alpha(\sum_{i'\neq i}\delta_{i'}g_{i'} + g') \bmod \pi^n)$. Then $\mathbb{T}/I \simeq \mathcal{O}/\pi^n$. Also clearly the elements $T' - \lambda_{\theta_i}(T') \in I$ for all $T' \in \mathbb{T}'$.

4.2.2. *Galois representations associated to modular forms.* Let $h \in S$ be a newform of some level N_h dividing Nd and of character η_K (so that $d \mid N_h \mid Nd$), and let $K_{h,\lambda}$ denote the λ-adic completion of K_h. It is a theorem of Shimura and Deligne that one can associate to h a Galois representation $\rho_{h,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K_{h,\lambda})$ which is unramified outside N_h and such that

$$\begin{aligned} \mathrm{trace}\ \rho_{h,\lambda}(\mathrm{Frob}_q) &= a_q(h), \\ \det \rho_{h,\lambda}(\mathrm{Frob}_q) &= \eta_K(q)q^k \end{aligned}$$

for all $q \nmid pN_h$.

Lemma 4.3. *$\rho_{h,\lambda}\,|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$ is unramified at the primes of K dividing d.*

Proof. Let q be a prime dividing d and $\mathfrak{q}$ the prime in K above q. Let π_h be the automorphic representation associated to h by Jacquet-Langlands. The local factor $\pi_{h,q}$ is then a ramified principal series representation $\pi(\epsilon^{-1}, \epsilon\eta_{K,q})$ for some unramified character ϵ of $\mathbb{Q}_q^\times$. The base change of this representation to $\mathrm{GL}_2(K_{\mathfrak{q}})$ is then the representation $\pi(\epsilon^{-1}\circ N_{K_{\mathfrak{q}}/\mathbb{Q}_p}, \epsilon\circ N_{K_{\mathfrak{q}}/\mathbb{Q}_p})$ which is unramified. By the local Langlands correspondence this implies the statement of the lemma. □

We will be especially interested in those h that are congruent to θ. Since θ is ordinary at p, such h are ordinary at p too. The following theorem of Wiles describes in this case the restriction of the representation $\rho_{h,\lambda}$ to a decomposition group at p.

Theorem 4.4 (Wiles [41]). *Let D_p be a decomposition group at p. Then up to equivalence,*

$$\rho_{h,\lambda}\,|_{D_p} \simeq \begin{pmatrix} \epsilon_1 & * \\ 0 & \epsilon_2 \end{pmatrix}$$

for characters ϵ_1 and ϵ_2 of D_p, with ϵ_1 ramified, ϵ_2 unramified and $\epsilon_2(\mathrm{Frob}_p) = \alpha(p,h)$ where $\alpha(p,h)$ is the unit root of $x^2 - a_p(h)x + p^k$.

4.2.3. *The representation space.* Let $[W]$ be a set of representatives for the eigenspaces of $\tilde{\mathbb{T}}$ contained in W and F be the ring $\prod_{i'\neq i} F' \times \prod_{h'\in[W]} F'$ (where by $h' \in [W]$ we mean h' is any normalized eigenform of $\tilde{\mathbb{T}}$ contained in W, i.e. with first Fourier coefficient equal to 1). Then $\mathbb{T}$ is naturally a subring of F via the embedding given by the various characters of $\tilde{\mathbb{T}}$ and $\mathbb{T}\otimes_{\mathcal{O}} F' = F$. Let $V = F \oplus F$ and $L = (\prod_{i'\neq i}\mathcal{O} \times \prod_{h'\in[W]}\mathcal{O})^2 \subset V$. Then L is a sublattice of V that is stable under the action of $\mathbb{T}$. Below we write $K'_{\theta_{i'}}$ or $K'_{h'}$ for the appropriate copy of the field F' in F (and $\mathcal{O}'_{h'}$ for the appropriate copy of $\mathcal{O}$) so that $F = \prod_{i'\neq i} K'_{\theta_{i'}} \times \prod_{h'\in[W]} K'_{h'}$.

Let β be the maximal ideal in $\mathbb{T}$ containing I and consider the completions $I_\beta, \mathbb{T}_\beta, L_\beta$ and V_β. The natural map $L \to L_\beta$ factors through $\prod_{i'\neq i}(\mathcal{O}'_{\theta_{i'},\lambda})^2 \times \prod_{h'\in[W]}(\mathcal{O}'_{h',\lambda})^2$.

Lemma 4.5. (i) *If $(\mathcal{O}'_{h',\lambda})^2$ is not in the kernel of this map, then h' is congruent to θ_i* mod λ.

(ii) *If h' is an eigenform corresponding to a theta lift from K, then $(\mathcal{O}'_{h',\lambda})^2$ is in the kernel of this map.*

(iii) *The terms $(\mathcal{O}'_{\theta_{i'},\lambda})^2$ are in the kernel of this map.*

Proof. Clearly a factor $(\mathcal{O}'_{h',\lambda})^2$ survives in $\mathbb{T}_\beta$ if and only if the extension of β to $\mathcal{O}'_{h',\lambda}$ is not the unit ideal. Since for all $T' \in \mathbb{T}'$, $T' - \lambda_{\theta_i}(T') \in \beta$, we must have $\lambda_{h'}(T') - \lambda_{\theta_i}(T')$ lies in the maximal ideal of $\mathcal{O}'_{h',\lambda}$, which amounts to the congruence claimed in (i). To prove (ii) we only need to show that under our assumptions there are no congruences between θ_i and other θ-lifts from K. This is a special case of [18, Prop. 2.2], since we have assumed that $p \nmid h_K M$ and $p > k+1$. Finally to see (iii) we note that for some $q \in T$ the U_q eigenvalue of $\theta_{i'}$ differs from that of θ_i. However at q we have the congruence $\alpha_q \equiv q\beta_q \mod \lambda$. If we also have $\alpha_q \equiv \beta_q \mod \lambda$, then $q \equiv 1 \mod p$ which is not true. Thus $\lambda_{\theta_{i'}}(U_q)$ is not congruent mod λ to $\lambda_{\theta_i}(U_q)$, whence by the argument used to prove (i), $(\mathcal{O}'_{\theta_{i'},\lambda})^2$ lies in the kernel. □

Let $[\tilde{W}]$ denote the set of newforms h in W such that one of the eigenforms h' of $\tilde{\mathbb{T}}$ corresponding to h is congruent to θ_i mod λ. In fact then, for each such h exactly one of the eigenforms corresponding to h can be congruent to θ_i mod λ. To see this suppose that two different eigenforms h'_1 and h'_2 corresponding to h are congruent to θ_i mod λ. Then there exists a prime $q \in T$ such that the eigenvalues of U_q acting on h'_1 and h'_2 are different. In particular h must be old at q. Since the mod λ representations associated to h and θ are isomorphic, we must have that $a_q(h) \equiv a_q(\theta) \mod \lambda$, these being the traces of Frob_q in these representations. Let $\alpha_q(h), \beta_q(h)$ denote the roots of the characteristic polynomial of Frob_q acting on $\rho_{h,\lambda}$. Now, we have just seen that $\alpha_q(h) + \beta_q(h) \equiv \alpha_q + \beta_q \mod \lambda$. On the other hand, since $q \mid N^+$, $\alpha_q(h)\beta_q(h) \equiv q^k \equiv \alpha_q\beta_q \mod \lambda$. Thus the set $\{\alpha_q(h), \beta_q(h)\}$ equals $\{\alpha_q, \beta_q\}$ when reduced mod λ. Since $\alpha_q \not\equiv \beta_q \mod \lambda$, we have also $\alpha_q(h) \not\equiv \beta_q(h) \mod \lambda$. Now the eigenvalues of U_q corresponding to h'_1 and h'_2 must be $\alpha_q(h), \beta_q(h)$, hence it could not possibly be true that $h'_1 \equiv h'_2 \mod \lambda$. Thus we have the following lemma (where h' is uniquely determined by h):

LEMMA 4.6. (i) $L_\beta \simeq (\prod_{h\in[\tilde{W}]} \mathcal{O}'_{h',\lambda})^2$.

(ii) $\mathbb{T}_\beta \otimes_{\mathcal{O}} F' \simeq \prod_{h\in[\tilde{W}]} K'_{h',\lambda}$.

Since $V_\beta = L_\beta \otimes_{\mathcal{O}} F' \simeq (\prod_{h\in[\tilde{W}]} K'_{h',\lambda})^2 \simeq \prod_{h\in[\tilde{W}]} (K'_{h',\lambda})^2$, and $K_{h,\lambda}$ is contained in $K'_{h',\lambda}$, V_β is naturally a representation space for $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the action on the component $V_{h,\lambda} = (K'_{h',\lambda})^2$ being via $\rho_{h,\lambda}$. The Galois action preserves L_β and thus L_β is a $\mathbb{T}_\beta[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$ module with commuting actions of the Galois group and the Hecke algebra. Henceforth we shall only be concerned with its structure as a $\mathbb{T}_\beta[\mathrm{Gal}(\overline{\mathbb{Q}}/K)]$ module.

We will be interested in the $\mathbb{T}_\beta[\mathrm{Gal}(\overline{\mathbb{Q}}/K)]$ modules $\mathcal{L}/I\mathcal{L}$ where $\mathcal{L}$ is any compact sub-bimodule of $L_\beta \otimes F'$. If $\mathfrak{q}$ is any prime in K not dividing pNd, we have

$$(\mathrm{Frob}_{\mathfrak{q}} - \chi(\mathfrak{q}))(\mathrm{Frob}_{\mathfrak{q}} - \chi^{\rho}(\mathfrak{q})) = 0 \text{ on } \mathcal{L}/I\mathcal{L} \tag{23}$$

where we think of χ as a character on the ideals of K (so $\chi(\mathfrak{q}) = \chi_\lambda(\mathrm{Frob}_{\mathfrak{q}})$). To check this note that for $q \nmid pNd$,

$$(\mathrm{Frob}_q)^2 - T_q \mathrm{Frob}_q + \eta_K(q) q^k = 0 \text{ on } \mathcal{L},$$

this being true in each factor $(K'_{h',\lambda})^2$. If q is split in K, $q = \mathfrak{q}\overline{\mathfrak{q}}$, then $\mathrm{Frob}_{\mathfrak{q}}, \mathrm{Frob}_{\overline{\mathfrak{q}}}$ are conjugate to Frob_q in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, hence satisfy the same polynomial. In this case $T_q - \chi(\mathfrak{q}) - \chi(\overline{\mathfrak{q}}) \in I, \eta_K(q) = 1$ and $\chi(\mathfrak{q})\chi(\overline{\mathfrak{q}}) = q^k$, hence we get (23) for both $\mathfrak{q}$ and $\overline{\mathfrak{q}}$. If q is inert in K, writing $\mathfrak{q}$ for the prime of K above q, we see that $\mathrm{Frob}_{\mathfrak{q}} = \mathrm{Frob}_q^2$, $T_q \in I$, $\eta_K(q) = -1$, and $\chi(\mathfrak{q}) = \chi^{\rho}(\mathfrak{q}) = q^k$. Hence (23) follows for the prime $\mathfrak{q}$. Now, a standard application of the Brauer-Nesbitt theorem gives the following lemma.

Lemma 4.7. *Suppose that U is an irreducible subquotient (as a $\mathbb{T}_\beta[\mathrm{Gal}(\overline{\mathbb{Q}}/K)]$ module) of $\mathcal{L}/\pi^r\mathcal{L}$ for some r. Then U has one of the following two types.*

(i) *$U \simeq \mathbb{T}_\beta/\beta\mathbb{T}_\beta \simeq \mathcal{O}_\pi/\pi\mathcal{O}_\pi$ with $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ acting via $\tilde{\chi_\lambda}$,*

(ii) *$U \simeq \mathbb{T}_\beta/\beta\mathbb{T}_\beta \simeq \mathcal{O}_\pi/\pi\mathcal{O}_\pi$ with $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ acting via $\tilde{\chi_\lambda^{\rho}}$.*

We will refer to these two types of irreducible modules as being of types χ and χ^ρ respectively. (It is easy to check that these types are distinct using the hypothesis $p > k+1$.) A finite $\mathbb{T}_\beta[\mathrm{Gal}(\overline{\mathbb{Q}}/K)]$ module E is said to be of type χ (resp. χ^ρ) if in any Jordan-Holder series for E every irreducible factor is of type χ (resp. χ^ρ).

Proposition 4.8. *For any stable lattice $\mathcal{L}$ there exists a stable lattice $\mathcal{L}' \subseteq \mathcal{L}$ such that $\mathcal{L}/\mathcal{L}'$ is of type χ and such that every sublattice of $\mathcal{L}$ with this property contains $\mathcal{L}'$. A similar result holds with χ replaced by χ^ρ.*

Proof. This is proved exactly as in [40, Prop. 3.2]. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are such that $\mathcal{L}/\mathcal{L}_1$ and $\mathcal{L}/\mathcal{L}_2$ are of type χ then the same holds for $\mathcal{L}_1 \cap \mathcal{L}_2$. If a smallest such lattice $\mathcal{L}'$ did not exist then one could find an infinite descending sequence $\mathcal{L}_1 \supsetneq \mathcal{L}_2 \supsetneq \dots$, each $\mathcal{L}_i$ having the required property. If $M = \cap \mathcal{L}_i$, then M is a sub-bimodule of $\mathcal{L}$ and $\mathcal{L}/M$ is infinite, hence $\mathcal{L}/M \otimes_{\mathcal{O}} F' \neq 0$. Any irreducible constituent of $\mathcal{L}/M \otimes_{\mathcal{O}} F'$ (as $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$-module) must be isomorphic to $\rho_{h,\lambda}$ for some $h \in [\tilde{W}]$ since these are the irreducible constituents of $\mathcal{L} \otimes_{\mathcal{O}} F'$.

Since $\tilde{\chi_\lambda}$ is not isomorphic to $\tilde{\chi_\lambda^{\rho}}$, one can find infinitely many q split in K with $\chi(\mathfrak{q}) \not\equiv \chi^\rho(\mathfrak{q}) \mod \lambda$ for a prime $\mathfrak{q}$ in K over q. Pick such a prime $\mathfrak{q}$ in K not dividing pND. Then $\mathrm{Frob}_{\mathfrak{q}} - \chi^\rho(\mathfrak{q})$ is invertible on $\mathcal{L}/\mathcal{L}_i$ for each i since $\mathcal{L}/\mathcal{L}_i$ is of type χ, and by compactness of $\mathcal{L}$ the same is true on $\mathcal{L}/M$ (and hence on any irreducible constituent of $\mathcal{L}/M \otimes_{\mathcal{O}} F'$). Choose such an irreducible constituent $\simeq \rho_{h,\lambda}$ with $h \in [\tilde{W}]$. On $\rho_{h,\lambda}$ the characteristic polynomial of $\mathrm{Frob}_{\mathfrak{q}}$ has roots congruent to $\chi(\mathfrak{q}), \chi^{\rho}(\mathfrak{q}) \bmod \lambda$ and both these roots occur, so $\mathrm{Frob}_{\mathfrak{q}} - \chi^{\rho}(\mathfrak{q})$ could not possibly act invertibly. This gives the required contradiction. □

The lattice $\mathcal{L}'$ in the proposition is called the type χ deprived sublattice. (It has no type χ quotient.) The next lemma and proposition follow directly from the the corresponding results proved in [42].

LEMMA 4.9. *Suppose E is a $\mathbb{T}_\beta/I\mathbb{T}_\beta[\mathrm{Gal}(\overline{\mathbb{Q}}/K)]$ module of type χ. Then $\sigma = \chi(\sigma)$ on E for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$. A similar result holds with χ replaced by χ^{ρ}.*

Proof. See [42, Lemma 5.3]. □

PROPOSITION 4.10. *Let E be any finite $\mathbb{T}_\beta/I\mathbb{T}_\beta[\mathrm{Gal}(\overline{\mathbb{Q}}/K)]$ module.*

(i) *Suppose that E has no type χ^{ρ} submodule. Let E_χ be the maximal type χ submodule contained in E. Then E/E_χ is of type χ^{ρ}.*

(ii) *Suppose E has no type χ^{ρ} quotient. Let E/E^{χ} be the maximal type χ quotient. Then E^{χ} is of type χ^{ρ}.*

A similar result holds if χ and χ^{ρ} are interchanged.

Proof. See [42, Prop. 5.4] (applied with $R = \mathbb{T}_\beta/I$). □

In case (i), we shall call the exact sequence

$$0 \to E_\chi \to E \to E/E_\chi \to 0$$

the *canonical* sequence associated to E, likewise for case (ii) or with χ and χ^{ρ} interchanged.

4.2.4. *The local representation.* Let $p = \mathfrak{p}\overline{\mathfrak{p}}$ be the prime decomposition of p in K and suppose that λ induces the prime $\mathfrak{p}$. Then χ_λ is ramified at $\mathfrak{p}$ and unramified everywhere else, while χ_λ^{ρ} is ramified at $\overline{\mathfrak{p}}$ and unramified at all other places. Let $D_{\mathfrak{p}}$ be a decomposition group at $\mathfrak{p}$. Then by Theorem 4.4, the representation $\rho_{h,\lambda}$ has a unique $D_{\mathfrak{p}}$-invariant subspace $V_{h,\lambda}^0 \subset V_{h,\lambda}$ on which the action of $D_{\mathfrak{p}}$ is ramified. Also the action of $D_{\mathfrak{p}}$ on the quotient $V_{h,\lambda}/V_{h,\lambda}^0$ is unramified. Set

$$V_\beta^0 = \oplus_{h\in[\tilde{W}]} V_{h,\lambda}^0, \qquad V_\beta^{\text{ét}} = V_\beta/V_\beta^0.$$

It is clear that V_β^0 is preserved by the action of $\mathbb{T}_\beta$, so that V_β^0 and $V_\beta^{\text{ét}}$ are $\mathbb{T}_\beta \otimes_{\mathcal{O}} F'$ modules, free, of rank one (by Lemma 4.6). Define

$$\mathcal{L}^{\text{ét}} = \mathrm{im}(\mathcal{L} \to V_\beta^{\text{ét}}), \qquad \mathcal{L}^0 = \ker(\mathcal{L} \to \mathcal{L}^{\text{ét}}).$$

The Jordan-Holder factors of $\mathcal{L}/\lambda^r\mathcal{L}$ are still of type χ or χ^ρ as $D_\mathfrak{p}$-modules and these types are distinct because $\tilde{\chi_\lambda}$ is ramified at $\mathfrak{p}$ (since $p > k+1$) while $\tilde{\chi}^\rho_\lambda$ is unramified at $\mathfrak{p}$. From the construction of $\mathcal{L}^0$ and $\mathcal{L}^{\text{ét}}$, and using that $\rho_{h,\lambda} \equiv \rho_{\theta,\lambda} \mod \lambda$, one sees that $\mathcal{L}^0/\lambda^r\mathcal{L}^0$ and $\mathcal{L}^{\text{ét}}/\lambda^r\mathcal{L}^{\text{ét}}$ must be of types χ and χ^ρ respectively (as $D_\mathfrak{p}$-modules).

LEMMA 4.11. *There is a stable sublattice $\mathcal{L}$ of L_β such that $\mathcal{L}$ has no type χ quotient and $\mathcal{L}^{\text{ét}}$ is a free $\mathbb{T}_\beta$ module of rank one.*

Proof. This is proved exactly as in [40, Lemma 3.4] to which we refer the reader for details. (All one needs to do is replace the types "1" and "$\mathfrak{r}$" in that proof by χ^ρ and χ respectively.) □

Choose $\mathcal{L}$ as in the lemma. Then $\mathcal{L}/I\mathcal{L}$ has no type χ quotient. If

$$0 \to C \to \mathcal{L}/I\mathcal{L} \to M \to 0 \tag{24}$$

is the canonical sequence associated to it, we must have $M \simeq \mathcal{L}^{\text{ét}}/I\mathcal{L}^{\text{ét}}$ and $C \simeq \mathcal{L}^0/I\mathcal{L}^0$. Also M is a free module of rank one over $\mathbb{T}_\beta/I$.

4.2.5. *A Galois extension of K_∞.* Let K_0 be the Hilbert class field of K, K_∞ the unique $\mathbb{Z}_p^2$ extension of K_0 abelian over K, and L' the splitting field over K_∞ of the representation $\mathcal{L}/I\mathcal{L}$. Denote by G' the Galois group $\mathrm{Gal}(L'/K_\infty)$. We define a pairing

$$G' \times M \to C, \qquad \langle \sigma, m\rangle \mapsto \sigma\tilde{m} - \chi_\lambda(g)\tilde{m} \tag{25}$$

where $\tilde{m}$ is any lift of m to $\mathcal{L}/I\mathcal{L}$.

LEMMA 4.12. *The extension L'/K_∞ is unramified outside the primes lying above the following set of primes in K: the prime $\mathfrak{p}$ and the primes $\mathfrak{q}$ for q in T_i.*

Proof. The representations $\rho_{h,\lambda}$ are unramified outside pNd and restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ are even unramified at the primes of K dividing d (Lemma 4.3). If $h \in [\tilde{W}]$, then h must be old at the primes q dividing N^- and the primes q dividing N^+ but not lying in T_i. To see this note that since $a_q(\theta) = 0$ for $q \mid N^-$, the relation $a_q(\theta)^2 \equiv \eta_K(q)q^{k-1}(q+1)^2 \mod \lambda$ cannot be satisfied for any such q (since $p \nmid q \pm 1$). Hence (see [5]) θ does not admit any level-raising congruences at such q. If $q \mid N^+$ and h is new at q, then $q \in T$ and $a_q(h)^2 = q^{k-1}$. Thus the square of the U_q eigenvalue of θ_i must be congruent to $q^{k-1} \mod \lambda$. Since $\beta_q^2 \equiv q^{k-1} \mod \lambda$, $\alpha_q^2 \equiv q^{k+1} \mod \lambda \not\equiv q^{k-1} \mod \lambda$, we see that h can only be new at q if $q \in T_i$.

Thus the extension L'/K_∞ can only possibly be ramified at the primes above $\mathfrak{p}, \overline{\mathfrak{p}}$ and $\mathfrak{q}, \overline{\mathfrak{q}}$ for $q \in T_i$. We show now that in fact it is unramified over $\overline{\mathfrak{p}}$ and the various $\overline{\mathfrak{q}}$. It suffices to show that the extension (24) splits viewed as a sequence of $D_{\overline{\mathfrak{p}}}$ or $D_{\overline{\mathfrak{q}}}$ modules. We first consider $\overline{\mathfrak{p}}$. Applying Theorem 4.4 to $D_{\overline{\mathfrak{p}}}$ we see that as $D_{\overline{\mathfrak{p}}}$ modules, $\mathcal{L}$ sits in an exact sequence

$$0 \to \mathcal{L}_1 \to \mathcal{L} \to \mathcal{L}_2 \to 0$$

where the action of $D_{\overline{\mathfrak{p}}}$ on $\mathcal{L}_1$ is via a ramified character and that on $\mathcal{L}_2$ is via an unramified character. Then

$$\mathcal{L}_1/I\mathcal{L}_1 \to \mathcal{L}/I\mathcal{L} \to \mathcal{L}_2/I\mathcal{L}_2 \to 0$$

is exact and we must have that as $D_{\overline{\mathfrak{p}}}$ -modules, $\mathcal{L}_1/I\mathcal{L}_1$ is of type χ^ρ and $\mathcal{L}_2/I\mathcal{L}_2$ is of type χ (since $\tilde{\chi_\lambda}$ is unramified and $\tilde{\chi_\lambda^\rho}$ is ramified at $\overline{\mathfrak{p}}$). Comparing with (24), we see that (24) viewed as an extension of $D_{\overline{\mathfrak{p}}}$-modules must split.

We now consider $\overline{\mathfrak{q}}$ for $q \in T_i$. We may assume that h is new at q. Then the restriction of $\rho_{h,\lambda}$ to a decomposition group at q is known by a theorem of Carayol. If D_q denotes a decomposition group at q, one has

$$\rho_{h,\lambda} \simeq \begin{pmatrix} \chi_1\epsilon & * \\ 0 & \chi_1 \end{pmatrix}$$

where χ_1 is an unramified character of D_q and ϵ is the p-adic cyclotomic character. Apply this to the decomposition group $D_{\overline{\mathfrak{q}}}$. Then $\mathcal{L}$ sits in an exact sequence

$$0 \to \mathcal{L}_1 \to \mathcal{L} \to \mathcal{L}_2 \to 0$$

where the action of $D_{\overline{\mathfrak{q}}}$ on $\mathcal{L}_1$ and $\mathcal{L}_2$ is unramified, and if $\alpha = \chi_1(\mathrm{Frob}_{\overline{\mathfrak{q}}})$, we have that $\mathrm{Frob}_{\overline{\mathfrak{q}}}$ acts as $q\alpha$ on $\mathcal{L}_1$ and as α on $\mathcal{L}_2$. Since the eigenvalues of $\mathrm{Frob}_{\overline{\mathfrak{q}}}$ are congruent mod λ to α_q and β_q, and since $\alpha_q \equiv q\beta_q \mod \lambda, \beta_q \not\equiv q\alpha_q$ mod λ, we see that in the exact sequence

$$\mathcal{L}_1/I\mathcal{L}_1 \to \mathcal{L}/I\mathcal{L} \to \mathcal{L}_2/I\mathcal{L}_2 \to 0$$

the ends $\mathcal{L}_1/I\mathcal{L}_1$ and $\mathcal{L}_2/I\mathcal{L}_2$ must be of types χ^ρ and χ respectively as $D_{\overline{\mathfrak{q}}}$ -modules. (Recall that $\chi(\mathfrak{q}) = \alpha_q, \chi^\rho(\mathfrak{q}) = \beta_q$, so that $\chi^\rho(\overline{\mathfrak{q}}) = \alpha_q$, $\chi(\overline{\mathfrak{q}}) = \beta_q$.) Comparing this sequence with (24) we see that the sequence (24) must split when viewed as a sequence of $D_{\overline{\mathfrak{q}}}$ -modules, as required. □

We view the pairing (25) as one of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ modules where $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ acts on G' in the usual way (via conjugation). Let R denote the ring $\mathbb{T}_\beta/I \simeq \mathcal{O}_\pi/\pi^n$. Then we obtain a Galois equivariant injection

$$G' \hookrightarrow \mathrm{Hom}_R(M, C). \tag{26}$$

Let ν be the Grossencharacter $\chi(\chi^\rho)^{-1}$ and R_ν be the ring generated over $\mathbb{Z}_p$ by the values of $\nu_\lambda = \chi_\lambda(\chi_\lambda^\rho)^{-1}$. The image of G' under (26) is easily seen to be stable under R_ν, and this gives G' the structure of an R_ν module. We thus get a map $\phi : G' \otimes_{R_\nu} \mathcal{O}_\pi \to \mathrm{Hom}_{\mathcal{O}_\pi}(M, C) = \mathrm{Hom}_{\mathcal{O}_\pi/\pi^n}(M, C) = C$.

Lemma 4.13. *The map* ϕ *is surjective and* $\mathrm{Fitt}_{\mathcal{O}_\pi}(G' \otimes_{R_\nu} \mathcal{O}_\pi) \subseteq \pi^n$.

Proof. Suppose that ϕ is not surjective. Then its image lands in a proper submodule C_0 of C. Consider the exact sequence

$$0 \to C/C_0 \to X/C_0 \to M \to 0$$

where $X = \mathcal{L}/I\mathcal{L}$. This sequence splits over K_∞ and since K_∞/K is abelian, we can write X/C_0 as

$$X/C_0 = \ker(\mathrm{Frob}_{\mathfrak{q}} - \chi(\mathfrak{q})) \oplus \ker(\mathrm{Frob}_{\mathfrak{q}} - \chi^\rho(\mathfrak{q}))$$

for any $\mathfrak{q}$ with $\chi(\mathfrak{q}) \not\equiv \chi^\rho(\mathfrak{q}) \mod \lambda$. The splitting is in fact one of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ modules. Since C/C_0 is nontrivial by assumption, X would have a type χ quotient, which gives a contradiction.

The surjectivity of the map shows that $\mathrm{Fitt}_{\mathcal{O}_\pi}(G' \otimes_{R_\nu} \mathcal{O}_\pi) \subseteq \mathrm{Fitt}_{\mathcal{O}_\pi}(C)$. Now $C \simeq \mathcal{L}^0/I\mathcal{L}^0$ for the faithful $\mathbb{T}_\beta$ module $\mathcal{L}^0$ and $\mathrm{Fitt}_{\mathbb{T}_\beta}(\mathcal{L}^0) \subseteq \mathrm{Ann}_{\mathbb{T}_\beta}(\mathcal{L}^0) = 0$. Therefore, $\mathrm{Fitt}_{\mathcal{O}_\pi/\pi^n}(C) = \mathrm{Fitt}_{\mathbb{T}_\beta/I}(\mathcal{L}^0/I\mathcal{L}^0) =$ image in $\mathbb{T}_\beta/I$ of $\mathrm{Fitt}_{\mathbb{T}_\beta}(\mathcal{L}^0) = 0$. Thus $\mathrm{Fitt}_{\mathcal{O}_\pi}(C) \subseteq \pi^n$, which gives the second part of the lemma. □

4.3. *Bounding congruences using the Main Conjecture.* In this section we use Rubin's theorem on the main conjecture of Iwasawa theory for imaginary quadratic fields to show the following key proposition.

Proposition 4.14.

$$\mathrm{Fitt}_{R_\nu}(G') \supseteq \left(\prod_{q \in T_i} (1 - q^{-(k+1)}\alpha_q^2)\right) \left(\pi^{k-1} \frac{L(k+1, \overline{\chi}\chi^\rho)}{\Omega^{2k}}\right).$$

Combining this with the results of the previous section, we see that

$$\begin{aligned} \pi^n &\supseteq \mathrm{Fitt}_{\mathcal{O}_\pi}(G' \otimes_{R_\nu} \mathcal{O}_\pi) = \mathrm{Fitt}_{R_\nu}(G') \otimes_{R_\nu} \mathcal{O}_\pi \\ &\supseteq \left(\prod_{q \in T_i} (1 - q^{-(k+1)}\alpha_q^2)\right) \left(\frac{\pi^{k-1} L(k+1, \overline{\chi}\chi^\rho)}{\Omega^{2k}}\right). \end{aligned}$$

Since $n = -v_\pi(\delta_i)$, using Lemma 4.2 we get the following theorem.

Theorem 4.15. *Let* g *and* g_i *be as in Section* 4.2.1 (*see equations* (20) *and* (21)). *Then*

$$v_p\left(\delta_i \prod_{q|d} (q+1) \frac{\pi^{2k+1}\langle g_i, \theta \rangle}{\Omega^{2k}}\right) \geq 0$$

and

$$v_p\left(\prod_{q|d} (q+1) \frac{\pi^{2k+1}\langle g, \theta \rangle}{\Omega^{2k}}\right) \geq 0.$$

Applying the theorem to $g = \frac{1}{\pi} fE$ and using formula (19), we have proved Theorem 4.1 in the case p is split in K and $p \nmid h_K$.

4.3.1. *Statement of the main conjecture.* For every finite extension F of K in K_∞ write $A(F)$ for the p-part of the class group of F, $\mathcal{E}(F)$ for the group of global units of F and $\mathcal{C}(F)$ for the group of elliptic units of F. Also write $U(F)$ for the group of local units of $F \otimes_K K_{\mathfrak{p}}$ which are congruent to 1 modulo the primes above $\mathfrak{p}$ and let $\overline{\mathcal{E}}(F)$ and $\overline{\mathcal{C}}(F)$ denote the closures of $\mathcal{E}(F) \cap U(F)$ and $\mathcal{C}(F) \cap U(F)$, respectively, in $U(F)$. (See [30] for more details.) For every infinite extension F of K in K_∞ define $A(F), \mathcal{E}(F)$ etc. to be the inverse limits over finite extensions L of the groups $A(L), \mathcal{E}(L)$ etc., the limits being taken with respect to the canonical norm maps. If $F \subset K_\infty$ define $M(F)$ to be the maximal abelian p-extension of F which is unramified outside the primes above $\mathfrak{p}$ and let $X(F) = \mathrm{Gal}(M(F)/F)$. We will also write $A_\infty, \mathcal{E}_\infty$ etc. to denote $A(K_\infty), \mathcal{E}(K_\infty)$ etc. Global class field theory gives the following exact sequence

$$0 \to \overline{\mathcal{E}}_\infty/\overline{\mathcal{C}}_\infty \to U_\infty/\overline{\mathcal{C}}_\infty \to X_\infty \to A_\infty \to 0.$$

The Galois group $\mathcal{G} = \mathrm{Gal}(K_\infty/K)$ acts on all the terms in this exact sequence and the "main conjecture" relates the structure of the various terms as $\mathbb{Z}_p[[\mathcal{G}]]$ modules.

Let $\Delta = \mathrm{Gal}(K_0/K), \Gamma = \mathrm{Gal}(K_\infty/K_0) \simeq \mathbb{Z}_p^2$ (noncanonically). Since Δ has order prime to p we have a canonical splitting $\mathcal{G} \simeq \Gamma \times \Delta$, which allows us to view Δ both as a subgroup and a quotient of $\mathcal{G}$. If Y is a $\mathbb{Z}_p[\Delta]$-module and η is a p-adic character of Δ, define $R_\eta = \mathbb{Z}_p[\eta]$ and Y^η to be the η-isotypic component of Y; i.e,

$$Y^\eta = \{y \in Y \otimes_{\mathbb{Z}_p} R_\eta : gy = \eta(g)y \text{ for all } g \in \Delta\}.$$

The functor $Y \to Y^\eta$ is exact on $\mathbb{Z}_p[\Delta]$ modules (since $p \nmid \#(\Delta)$). We denote by e_η the natural projection from Y to Y^η.

Let Λ be the Iwasawa algebra

$$\Lambda = \mathbb{Z}_p[[\mathcal{G}]] = \varprojlim \mathbb{Z}_p[\mathrm{Gal}(F/K)]$$

the limit being taken over all finite extensions F of K contained in K_∞. Then for every p-adic character η of Δ,

$$\Lambda^\eta = \mathbb{Z}_p[[\mathcal{G}]]^\eta \simeq R_\eta[[\Gamma]]$$

is isomorphic (noncanonically) to the power series ring in two variables over R_η.

We recall now some facts about the structure of finitely generated $\Lambda' = R[[\Gamma']]$ modules where $\Gamma' \simeq \mathbb{Z}_p^r$ and R is the ring of integers of a finite extension of $\mathbb{Q}_p$. The ring Λ' is a regular local ring of dimension $r+1$. A Λ'-module N is called torsion if every element of N is annihilated by a nonzero element of Λ'. A finitely generated Λ'-module N is called pseudo-null if for any height 1 prime ideal $\mathfrak{r}$ of Λ', $N_{\mathfrak{r}} = 0$ or equivalently if $\mathrm{Ann}(N)$ contains a height 2 ideal in Λ'. For any two modules N_1 and N_2, N_1 is said to be pseudo-isomorphic to N_2 if there exists an exact sequence

$$0 \to A \to N_1 \to N_2 \to B \to 0$$

for some psuedo-null modules A and B. If N_1 and N_2 are torsion Λ'-modules this is an equivalence relation. The structure theorem for finitely generated torsion Λ'-modules asserts that any such module is pseudo-isomorphic to an *elementary* torsion module i.e. one of the form $\oplus_i \Lambda'/f_i$ for some (nonzero) elements f_i of Λ'. Elementary torsion modules have no nonzero pseudo-null submodules, so in fact one has an injection $\oplus_i \Lambda'/f_i \hookrightarrow N$ with pseudo-null cokernel. The ideal $\prod_i (f_i)$ is an invariant of N. It is called the characteristic ideal of N and denoted by $\mathrm{char}_{\Lambda'}(N)$. This ideal is principal and any generator of it is called a characteristic power series of N. The characteristic power series is well defined up to multiplication by a unit power series. The characteristic ideal is multiplicative in short exact sequences of torsion modules. If one defines the characteristic ideal of a finitely generated nontorsion module to be 0 then the characteristic ideal is multiplicative in short exact sequences of finitely generated modules.

The "main conjecture" of Iwasawa theory in this setting is the following theorem proved by Rubin [30].

THEOREM 4.16 (Rubin). *The Λ^η modules $(\overline{\mathcal{E}}_\infty/\overline{\mathcal{C}}_\infty)^\eta, (U_\infty/\overline{\mathcal{C}}_\infty)^\eta, X_\infty^\eta$ and A_∞^η are finitely generated and torsion. One has equivalently*

$$\mathrm{char}_{\Lambda^\eta}((A_\infty)^\eta) = \mathrm{char}_{\Lambda^\eta}((\overline{\mathcal{E}}_\infty/\overline{\mathcal{C}}_\infty)^\eta),$$
$$\mathrm{char}_{\Lambda^\eta}((X_\infty)^\eta) = \mathrm{char}_{\Lambda^\eta}((U_\infty/\overline{\mathcal{C}}_\infty)^\eta).$$

4.3.2. *The two-variable $\mathfrak{p}$-adic L-function.* We now need to review briefly the existence and properties of the two-variable $\mathfrak{p}$-adic L-function. Let $\mathcal{D}$ denote the ring of integers of the maximal unramified extension of $\mathbb{Q}_p$. The $\mathfrak{p}$-adic L-function is an element of $\mathcal{D}[[\mathcal{G}]]$ that interpolates the special values of L-functions of Grossencharacters of K whose associated λ-adic character factors through $\mathcal{G}$. For ϵ any such Grossencharacter of K let $\eta(\epsilon)$ denote the restriction of ϵ_λ to Δ. Then one has the following theorem due to Katz, Yager and de Shalit (see [4, II.4.14]). ($L_{\overline{\mathfrak{p}}}(\cdot)$ denotes the L-function $L(\cdot)$ with the Euler factor at $\overline{\mathfrak{p}}$ removed.)

THEOREM 4.17. *There exist an element $\mathcal{L}_{\mathfrak{p}}$ in $\mathcal{D}[[\mathcal{G}]]$, a p-adic period $\Omega_p \in \mathcal{D}^\times$ and a complex period Ω such that for any unramified Grossencharacter ϵ of K of infinity type (k', j') with $0 \le -j' < k'$,*

$$\Omega_p^{j'-k'} \epsilon_\lambda(\mathcal{L}_{\mathfrak{p}}{}^{\eta(\epsilon)}) = \Omega^{j'-k'} \Gamma(k') \left(\frac{\sqrt{-d}}{2\pi}\right)^{j'} \left(1 - \frac{\epsilon(\mathfrak{p})}{p}\right) L_{\overline{\mathfrak{p}}}(\epsilon^{-1}, 0). \tag{27}$$

In (27) the left-hand side lies in $\mathbb{C}_p$ while the right-hand side lies in $\mathbb{C}$. The equality should be interpreted as saying that both sides lie in $\overline{\mathbb{Q}}$ and are equal. The period Ω coincides up to a p-adic unit with the period Ω defined in Section 2.3.3 and so we use the same symbol for both.

We will also need the following theorem, first proved by Yager when $h_K = 1$ and extended by de Shalit in general. (See [4, III.1.10, III.1.14 and the discussion on p. 142].)

THEOREM 4.18. *For any character η of Δ,*

$$\mathrm{char}_{\mathcal{D}[[\Gamma]]}((U_\infty/\overline{\mathcal{C}}_\infty)^\eta) = (\mathcal{L}_{\mathfrak{p}}{}^\eta). \tag{28}$$

4.3.3. *The descent argument.* The action of $\mathrm{Gal}(K_\infty/\mathbb{Q})$ on $\mathrm{Gal}(K_\infty/K_0)$ by conjugation induces an action of the group $\mathrm{Gal}(K/\mathbb{Q}) = \langle 1, c\rangle$ on $\mathrm{Gal}(K_\infty/K_0)$. Let G_+ and G_- denote the maximal subgroups of $\mathrm{Gal}(K_\infty/K_0)$ on which the action of c is via -1 and $+1$ respectively. Denote by K_∞^+ and K_∞^- the fixed fields of G_+ and G_- respectively. Let $K_\infty(\mathfrak{p})$ and $K_\infty(\overline{\mathfrak{p}})$ be the maximal subextensions of K_∞ unramified outside $\overline{\mathfrak{p}}$ and $\mathfrak{p}$ respectively. By class field theory there are canonical isomorphisms

$$\kappa_1 : \Gamma_1 = \mathrm{Gal}(K_\infty(\mathfrak{p})/K_0) \simeq 1 + p\mathbb{Z}_p,$$
$$\kappa_2 : \Gamma_2 = \mathrm{Gal}(K_\infty(\overline{\mathfrak{p}})/K_0) \simeq 1 + p\mathbb{Z}_p.$$

Let $u = 1 + p$, and viewing κ_1, κ_2 as characters of $\Gamma = \mathrm{Gal}(K_\infty/K_0)$ define $\gamma_1, \gamma_2, \gamma^+, \gamma^-$ (see [4, III 3.3]) by

$$\kappa_1(\gamma_1) = u, \kappa_1(\gamma_2) = 1,$$
$$\kappa_2(\gamma_1) = 1, \kappa_2(\gamma_2) = u,$$
$$\gamma^+ = \gamma_1\gamma_2, \gamma^- = \gamma_1/\gamma_2.$$

Finally, defining $\kappa^+ = \kappa_1\kappa_2$ and $\kappa^- = \kappa_1\kappa_2^{-1}$, we see that these yield isomorphisms

$$\kappa^+ : \Gamma^+ = \mathrm{Gal}(K_\infty^+/K_0) \simeq 1 + p\mathbb{Z}_p,$$
$$\kappa^- : \Gamma^- = \mathrm{Gal}(K_\infty^-/K_0) \simeq 1 + p\mathbb{Z}_p.$$

Since $p \nmid [K_0 : K]$, we have a canonical isomorphism $\mathcal{G} = \mathrm{Gal}(K_\infty/K) \simeq \Delta \times \Gamma$ where $\Delta = \mathrm{Gal}(K_0/K)$. Let η be the restriction of the character ν_λ to Δ. If R_η is the ring generated over $\mathbb{Z}_p$ by the values of η, one sees easily that $R_\eta = R_\nu$. The isomorphism $\Gamma \simeq \Gamma^+ \times \Gamma^-$ gives rise to an isomorphism

$$\Lambda^\eta = \mathbb{Z}_p[[\mathcal{G}]]^\eta \simeq R_\eta[[\Gamma]] \simeq R_\eta[[S, T]]$$

sending γ^+ to $1 + S$ and γ^- to $1 + T$. Also, let $\mathcal{G}_- = \mathrm{Gal}(K_\infty^-/K) \simeq \Delta \times \Gamma^-$. Since η factors through $\mathcal{G}_-$, one also has an isomorphism

$$\Lambda_-^\eta = \mathbb{Z}_p[[\mathcal{G}_-]]^\eta \simeq R_\eta[[\Gamma^-]] \simeq R_\eta[[T]]$$

sending γ^- to $1 + T$.

LEMMA 4.19. *Suppose X is a finitely generated torsion Λ^η module with no nonzero pseudo-null submodule and such that X/SX is a (finitely generated) torsion Λ^η_- module. Then $\mathrm{char}_{\Lambda^\eta_-}(X/SX) = f(0,T)$, where $f(S,T) = \mathrm{char}_{\Lambda^\eta}(X)$.*

Proof. Since X has no pseudo-null submodule there is an exact sequence

$$0 \to X \to Y \to N \to 0$$

where $Y = \oplus_i \frac{\Lambda^\eta}{f_i(S,T)}$, with $\prod_i f_i(S,T) = f(S,T)$ and N a pseudo-null Λ^η module. This gives an exact sequence

$$Y[S] \to N[S] \to X/SX \to \oplus_i \frac{\Lambda^\eta_-}{f_i(0,T)} \to N/SN \to 0.$$

Since N is a pseudo-null Λ^η module, it is annihilated by a height 2 ideal I in Λ^η. If (S,I) is a height 3 ideal, then its image in Λ^η_- is height 2 and annihilates both $N[S]$ and N/SN. Then $N[S]$ and N/SN are pseudo-null Λ^η_- modules, so $\mathrm{char}_{\Lambda^\eta_-}(X/SX) = \prod_i f_i(0,T) = f(0,T)$. On the other hand, if (S,I) is a height 2 ideal, then $S \in \mathrm{rad}(I)$, so N is a finitely generated torsion A module where A is the ring Λ^η_- thought of as a subring of Λ^η. The exact sequence

$$0 \to N[S] \to N \xrightarrow{S} N \to N/SN \to 0$$

shows that $\mathrm{char}_A(N[S]) = \mathrm{char}_A(N/SN)$. Now, since X/SX is a torsion Λ^η_- module, it follows from the above exact sequence that $f_i(0,T) \neq 0$ for all i. Consequently, $Y[S] = 0$. Since $\mathrm{char}_{\Lambda^\eta_-}(N[S]) = \mathrm{char}_{\Lambda^\eta_-}(N/SN)$, we see that $\mathrm{char}_{\Lambda^\eta_-}(X/SX) = \mathrm{char}_{\Lambda^\eta_-}(Y/SY) = f(0,T)$. □

Recall that in Section 4.2.5, we have constructed an abelian extension L' of K_∞ with Galois group G'. Further, G' is an R_ν module and the action of Λ on G' is via the character $\nu_\lambda = \chi_\lambda(\chi_\lambda^\rho)^{-1}$. Let δ be the idele $(\dots, 1, 1, 1 + p, 1, 1, \dots) \in K_\mathbb{A}^\times$ where all factors except the one at $\mathfrak{p}$ are equal to 1, and let δ' be defined similarly but with $\mathfrak{p}$ replaced by $\overline{\mathfrak{p}}$. Then (see §2.3.2) $\chi_\lambda(\gamma_1) = \chi_\lambda(\delta) = (\lambda(\underline{\chi}(\delta))/\mu_\lambda(\delta))^{-1} = (1+p)^k$ since $\underline{\chi}$ is unramified. Likewise $\chi_\lambda(\gamma_2) = \chi_\lambda(\delta') = (\lambda(\underline{\chi}(\delta'))/\mu_\lambda(\delta'))^{-1} = 1$. Hence $\nu_\lambda(\gamma_1) = (1+p)^k$ and $\nu_\lambda(\gamma_2) = (1+p)^{-k}$, whence $\nu_\lambda(\gamma^+) = 1$ and $\nu_\lambda(\gamma^-) = (1+p)^{2k}$. Thus ν_λ induces a homomorphism of the group algebra $\Lambda^\eta \simeq R_\eta[S,T]$ to R_η, also denoted ν_λ, that sends S to 0 and T to $(1+p)^{2k} - 1$.

Let L be the maximal extension of K_∞ in L' that is unramified outside $\mathfrak{p}$ and set $G = \mathrm{Gal}(L/K_\infty)$. Denote by $\mathfrak{f}$ the ideal $\prod_{q \in T_i} \mathfrak{q}$. Also let $M_{\infty,\mathfrak{f}}$ be the maximal abelian p-extension of K_∞ unramified outside $\mathfrak{p}$ and the primes dividing $\mathfrak{f}$ and set $X_{\infty,\mathfrak{f}} = \mathrm{Gal}(M_{\infty,\mathfrak{f}}/K_\infty)$. Then in the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Y & \longrightarrow & X_{\infty,\mathfrak{f}} & \longrightarrow & X_\infty & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & H & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

the vertical maps are all surjective.

LEMMA 4.20. $\mathrm{Fitt}_{R_\nu}(H) \supseteq \left(\prod_{q\in T_i}(1 - q^{-(k+1)}\alpha_q^2)\right)$.

Proof. The map $\phi : Y \to H$ induces a surjective map from Y^η to H that, by the discussion above, factors via the quotient $Y^\eta/(S,(1+T)-(1+p)^{2k})Y^\eta$. By class field theory one knows that

$$Y \simeq \varprojlim \left(\prod_{q\in T_i} \prod_{\mathfrak{r}_j|\mathfrak{q}} \mu(K_{n,\mathfrak{r}})/\mu(K_n) \right)$$

where $\mathfrak{r}_j$ ranges over the primes of K_n lying over $\mathfrak{q}$ and $\mu(L)$, for any field L, denotes the p-power roots of unity in L. Since $\mu_{p^\infty}(K_\infty) = 1$, as a Λ^η module,

$$Y^\eta \simeq \left(\prod_{q\in T_i} \mathrm{Ind}_{D_\mathfrak{r}}^{\mathrm{Gal}(K_\infty/K)} \mathbb{Z}_p(1) \right)^\eta \simeq \prod_{q\in T_i} \Lambda^\eta/e_\eta(\mathrm{Frob}_\mathfrak{q} - q)$$

where $\mathfrak{r}$ is a prime of K_∞ over $\mathfrak{q}$, $D_\mathfrak{r}$ is the decomposition group of $\mathfrak{r}$ in $\mathrm{Gal}(K_\infty/K)$, and $\mathbb{Z}_p(1) = \varprojlim \mu_{p^n}$. Hence

$$\begin{aligned} Y^\eta/(S,(1+T)-(1+p)^{2k})Y^\eta &\simeq \prod_{q\in T_i} R_\eta/(\nu(\mathrm{Frob}_\mathfrak{q}) - q) \\ &= \prod_{q\in T_i} R_\eta/(\alpha_q/\beta_q - q) = \prod_{q\in T_i} R_\eta/(\alpha_q^2 - q^{k+1}). \end{aligned}$$

Since this last module surjects onto H and $R_\eta = R_\nu$, we get the lemma. □

PROPOSITION 4.21. $\mathrm{Fitt}_{R_\nu}(G) \supseteq \left(\frac{\pi^{k-1}L(k+1,\overline{\chi}\chi^\rho)}{\Omega^{2k}}\right)$.

Proof. We begin by applying (27) with $\epsilon^{-1} = \overline{\chi}\chi^\rho\mathbb{N}^{-(k+1)}$. With this choice of ϵ, $k' = k+1$, $j' = 1-k$, hence (k',j') lies in the range of interpolation. Thus $\pi^{k-1}L(k+1,\overline{\chi}\chi^\rho)/\Omega^{2k} = \pi^{k-1}L(0,\epsilon^{-1})/\Omega^{2k}$ which in turn is equal, up to a λ-adic unit, to $\epsilon_\lambda(\mathcal{L}_\mathfrak{p}{}^{\eta(\epsilon)})$. (It is easy to check that the terms $1 - \frac{\epsilon(\mathfrak{p})}{p}$ and $1 - \epsilon^{-1}(\overline{\mathfrak{p}})$ are λ-adic units.) For any Grossencharacter ϵ' of K denote by $\hat{\epsilon'}$ the Grossencharacter $(\epsilon'^\rho)^{-1}\mathbb{N}^{-1}$. By the functional equation for the p-adic L-function ([4, II.6.4]), $\epsilon_\lambda(\mathcal{L}_\mathfrak{p}{}^{\eta(\epsilon)})$ is equal up to a λ-adic unit to $\epsilon'_\lambda(\mathcal{L}_\mathfrak{p}{}^{\eta(\epsilon')})$ where $(\epsilon')^{-1} = \widehat{\epsilon^{-1}}$. Now we check immediately that $\widehat{\epsilon^{-1}} = (\overline{\chi^\rho}\chi\mathbb{N}^{-(k+1)})^{-1}\mathbb{N}^{-1} = \chi^\rho\chi^{-1} = \nu^{-1}$ since $\chi^\rho\overline{\chi^\rho} = \mathbb{N}^k$. Thus $\epsilon' = \nu$. Recall that we have been denoting $\eta(\nu)$ just by the symbol η. Let $\mathcal{L}_{\mathfrak{p}}{}^{\eta} = f(S,T) \in \Lambda^{\eta}$. Then $\nu_{\lambda}(\mathcal{L}_{\mathfrak{p}}{}^{\eta}) = f(0,\alpha)$ where $\alpha = (1+p)^{2k} - 1$. Also, by Theorem 4.16 and Theorem 4.18, $\mathrm{char}_{\Lambda^{\eta}}(X_{\infty}^{\eta}) = f(S,T)$. Now, G is a quotient of X_{∞}^{η}, in fact of $X_{\infty}^{\eta}/(S, T-\alpha)X_{\infty}^{\eta}$. To compute the size of this last module, first apply Lemma 4.19 with $X = X_{\infty}$. To check that the hypotheses of the lemma apply, note first that X_{∞}^{η} has no nonzero pseudo-null Λ^{η} submodules by a theorem of Perrin-Riou ([25, §II.2, Thm. 23]). Further $X_{\infty}^{\eta}/SX_{\infty}^{\eta}$ injects into $X(K_{\infty}^{-})^{\eta}$ by [30, Thm 5.3(ii)] (applied with $F = K_{\infty}^{-}$) and by the fact that K_{∞} is unramified over K_{∞}^{-}. Since $X(K_{\infty}^{-})^{\eta}$ is a torsion Λ_{-}^{η} module (by [25, §II.2.2, Prop. 20]) the same is true of $X_{\infty}^{\eta}/SX_{\infty}^{\eta}$. Thus the hypotheses of the lemma are verified and we have $\mathrm{char}_{\Lambda_{-}^{\eta}}(X_{\infty}^{\eta}/SX_{\infty}^{\eta}) = f(0,T)$. Further, since $X(K_{\infty}^{-})^{\eta}$ has no nonzero pseudo-null Λ_{-}^{η} submodules ([9, Prop. 5 and the remarks at the end of §4]), the same is true of $X_{\infty}^{\eta}/SX_{\infty}^{\eta}$. Hence there is an exact sequence

$$0 \to X_{\infty}^{\eta}/SX_{\infty}^{\eta} \to \oplus_i \frac{\Lambda_{-}^{\eta}}{f_i(T)} \to N \to 0$$

where $f(0,T) = \prod_i f_i(T)$ and N is a pseudo-null Λ_{-}^{η} submodule. Taking coinvariants of this sequence by $T - \alpha$ we get an exact sequence

$$N[T-\alpha] \to X_{\infty}^{\eta}/(S,T-\alpha)X_{\infty}^{\eta} \to \oplus_i \frac{R_{\eta}}{f_i(\alpha)} \to N/(T-\alpha)N \to 0.$$

Since the two modules at the ends of this sequence have the same size, we get

$$\mathrm{Fitt}_{R_{\eta}}(X_{\infty}^{\eta}/(S,T-\alpha)X_{\infty}^{\eta}) \supseteq \mathrm{Fitt}_{R_{\eta}}\left(\oplus_i \frac{R_{\eta}}{f_i(\alpha)}\right) = \prod_i (f_i(\alpha)) = (f(0,\alpha)).$$

Finally, since $R_{\eta} = R_{\nu}$,

$$\mathrm{Fitt}_{R_{\nu}}(G) \supseteq (f(0,\alpha)) = \left(\frac{\pi^{k-1}L(k+1,\overline{\chi}\chi^{\rho})}{\Omega^{2k}}\right)$$

which completes the proof of the proposition. □

Combining Lemma 4.20 and Proposition 4.21, we have proved Proposition 4.14.

5. Applications

5.1. *Integrality of $\langle f,f\rangle/\langle g,g\rangle$.* Recall that f is a normalized newform on $\Gamma_0(N)$, g its Jacquet-Langlands lift to the group $\Gamma = \Gamma_0^{N^-}(N^+)$. K_f is the field generated by the Hecke eigenvalues of f (or g.) The form g has been chosen to be K_f-rational and also λ-adically integral and primitive with respect to the integral structure provided by an integral model of the Shimura curve associated to Γ (see §2.2.) We are interested in the ratio $\langle f,f\rangle/\langle g,g\rangle$, this being well-defined only up to a λ-adic unit in K_f. Recall also that in Section 3.4.7, using the theta correspondence, we constructed a form g'' for Γ that satisfies

(a) g'' is a scalar multiple of g, indeed $g'' = \delta g$ for a nonzero algebraic number δ such that $\delta^2 \in K_f$.

(b) $\langle f, f\rangle = \langle g'', g''\rangle$.

Now

$$\frac{\langle f, f\rangle}{\langle g, g\rangle} = \frac{\langle g'', g''\rangle}{\langle g, g\rangle} = \delta\overline{\delta} = \pm\delta^2$$

since K_f is totally real. We will show that $v_\lambda(\delta) \geq 0$. The idea is to apply Proposition 2.9 to g'' and its associated adelic form Ψ''. We will need the following lemma which can be obtained by combining results from [3] and [22].

LEMMA 5.1. *Suppose* $p \nmid M = \prod_{q|N} q(q-1)(q+1)$. *Then there exist infinitely many imaginary quadratic fields* K *with odd discriminant*, *such that*

1. K *is split at* p.
2. K *is split at the primes dividing* N^+.
3. K *is inert at the primes dividing* N^-.
4. $p \nmid h_K$.

Proof. It follows from [3, Thm. 7], that under the assumption $p \nmid M$, there exist infinitely many such fields K satisfying $2, 3$ and 4. If infinitely many of these satisfy 1 too, we are done. If not, there exists a field K_0 that satisfies $2, 3$ and 4 but is not split at p. Then by [22, Thm. 7.5] there are infinitely many fields K satisfying $1-4$ of the lemma. □

The lemma provides infinitely many different Heegner points $K \hookrightarrow D$ as needed in Proposition 2.9. Applying the second half of this proposition to g'' and using formula (17) and Theorem 4.1 we see that

THEOREM 5.2. *The form* g'' *is* λ*-integral. Hence* $v_\lambda(\delta) \geq 0$.

This proves the λ-integrality of $\frac{\langle f,f\rangle}{\langle g,g\rangle}$ and thus the first part of Theorem 2.4. Also combining this theorem with the first half of Propostion 2.9 and formula (17) we see that Theorem 4.1 follows even if p is *inert* in K or $p \mid h = h_K$.

5.2. *Level-lowering congruences.* We now assume that λ is a level-lowering congruence prime for f at a prime q dividing N^-; i.e., there exists an eigenform f' of level dividing N/q and weight k such that $\rho_f \equiv \rho'_f(\lambda)$. Recall that this is equivalent to saying that the Hecke eigenvalues of f and f' outside the primes dividing N are congruent mod λ. We will assume that f' is a newform of level N/q, the argument being easily modified if this is not the case. Let $a_{q'}$ and $b_{q'}$

denote the Fourier coefficients of $\theta = \theta_\chi$ and f' at a prime q'. Then it is well known (see [5]) that there is the congruence

$$b_q^2 \equiv q^{k-2}(q+1)^2 \mod \lambda$$

so that $b_q \equiv \pm q^{\frac{k}{2}-1}(q+1) \mod \lambda$. One then checks easily that $f \equiv f' \mp q^{k/2}h \mod \lambda$ where $h(z) = f'(qz)$ and the congruence is one of q-expansions. It follows now from Theorem 4.15 and formulas (18) and (19) that

$$\frac{\pi^k D(k,f,\theta)}{\Omega^{2k}} \equiv \frac{\pi^k D(k,f',\theta)}{\Omega^{2k}} \mp q^{k/2}\frac{\pi^k D(k,h,\theta)}{\Omega^{2k}} \mod \lambda.$$

Now, from [31, Lemma 1], one sees that $D(s,f',\theta)$ and $D(s,h,\theta)$ have Euler products, and the Euler factors are all the same except for the ones at q. Suppose that the Frobenius eigenvalues at q of θ are α, β and those of f' are α', β'. It is also shown in [31, proof of Lemma 1] that the Euler factors at q are

$$D_q(s,f',\theta) = \frac{1-\alpha\beta\alpha'\beta' q^{-2s}}{(1-\alpha\alpha' q^{-s})(1-\alpha\beta' q^{-s})(1-\beta\alpha' q^{-s})(1-\beta\beta' q^{-s})},$$

$$D_q(s,h,\theta) = \frac{(a_q - b_q\alpha\beta q^{-s})q^{-s}}{(1-\alpha\alpha' q^{-s})(1-\alpha\beta' q^{-s})(1-\beta\alpha' q^{-s})(1-\beta\beta' q^{-s})}.$$

Since q is inert in K, $\alpha\beta = -q^k$, $\alpha'\beta' = q^{k-1}$ and $a_q = 0$. Hence the numerator of $D_q(k,f',\theta)$ is equal to $1+\frac{1}{q}$ while that of $D_q(k,h,\theta)$ is $b_q q^{-k}$. Since we have assumed that $p \nmid (q+1)$, we get

$$\begin{aligned}\frac{\pi^k D(k,f',\theta)}{\Omega^{2k}} \mp q^{k/2}\frac{\pi^k D(k,h,\theta)}{\Omega^{2k}} &= (\text{a } \lambda\text{-adic integer })\left(1+\frac{1}{q} \mp q^{k/2}\cdot b_q q^{-k}\right)\\ &= (\text{a } \lambda\text{-adic integer })(q^{\frac{k}{2}-1}(q+1) \mp b_q)\\ &\equiv 0 \mod \lambda.\end{aligned}$$

Now applying again Proposition 2.9 and formulas (17), (18), (19), we see that $v_\lambda(g'') > 0$ in this case and hence $v_\lambda(\frac{\langle f,f\rangle}{\langle g,g\rangle}) = v_\lambda(\delta^2) > 0$. This completes the proof of Theorem 2.4.

5.3. *Integrality of a triple product L-value.* Let g_1, g_2, g_3 be holomorphic newforms on Γ of even weights k_i and let π_i denote the automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ associated to g_i by the Jacquet-Langlands correspondence. Suppose that the g_i are λ-adically normalized and that the weights satisfy $k_1 = k_2 + k_3$. Define

$$P(g_1,g_2,g_3) = \int_{\Gamma\backslash\mathfrak{H}} g_1(z)\overline{g_2(z)g_3(z)}y^{k_1}\frac{dxdy}{y^2}.$$

THEOREM 5.3. *If* $P(g_1, g_2, g_3) \neq 0$ *then* $L^*(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3) \neq 0$ *and*

$$\frac{L^*(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)}{|P(g_1, g_2, g_3)|^2}$$

is a λ*-adic integer*, L^* *being the* L*-function with the gamma factors at infinity included.*

Proof. Let f_i denote the Jacquet-Langlands lift of g_i to $\Gamma_0(N)$, the f_i's being normalized Hecke eigenforms. The main theorem of [43] shows that

$$\frac{|P|^2}{\prod_i \langle g_i, g_i \rangle} = \frac{c}{N^2} \cdot \frac{L^*(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)}{\prod_i L^*(1, \mathrm{ad}^0(\pi_i))}$$

where c is a product of a power of 2 and a product of local terms indexed by the primes dividing N. For each prime $q \mid N$ the local term in question is of the form $1 \pm \epsilon_q$ where $\epsilon_q = \epsilon_{q,1}\epsilon_{q,2}\epsilon_{q,3}$ is the product of the signs $\epsilon_{q,i}$ of the Atkin-Lehner involution W_q acting on f_i, and the $\pm$ sign occurs depending on whether $q \mid N^+$ or N^-. Thus $c \neq 0$ for at most one choice of factorization $N = N^+N^-$, in which case c is a power of 2 and hence a λ-adic unit. One has also (see [43, §4.2, Lemma 5]) that $L^*(1, \mathrm{ad}^0(\pi_i)) = \frac{2^{k_i}}{N}\langle f_i, f_i \rangle$. Our main result (Theorem 2.4) shows that $\langle f_i, f_i \rangle / \langle g_i, g_i \rangle$ is a λ-adic integer. The theorem follows. □

5.4. *The Faltings height of* $J_0^N(1)$. The Faltings height of $J_0(N)$ was computed in [38]. The computation relies crucially on the q-expansion principle and a corresponding duality between the integral Hecke algebra and the space of integral modular forms that is available for modular curves. The main result is that the (exponential) height times an L-value equals the discriminant of the Hecke algebra, which is an integer that measures congruences between all forms of level N and weight 2. As mentioned in the introduction the computation of the Faltings height of Jacobians of Shimura curves was one of the motivating questions behind this article. Let X denote the Shimura curve over $\mathbb{Q}$ associated to $\Gamma_0^N(1)$ (where, as before, N is square-free and a product of an even number of primes) and let $J_0^N(1) = \mathrm{Jac}(X)$. In this section we show two different methods to compute the height of $J := J_0^N(1)$. The first uses the fact that J is isogenous to the new-quotient of $J_0(N)$, and the computation of the height of $J_0(N)^{\mathrm{new}}$ in [38]. Here we can give a result up to Eisenstein primes. The second method uses the main results of this article. Consequently, one can at present only compute the height outside the primes dividing $M = \prod_{q|N} q(q+1)(q-1)$ by this method. In either case, the main result is that the (exponential) height times an L-value equals the discriminant of the Hecke algebra times an integer that measures level-lowering congruences. The second method has the advantage of being generalizable (in principle, at least) to the case of Shimura curves over totally real fields.

Recall the definition of the Faltings height of a semistable Abelian variety A of dimension d defined over a number field L. Let $\omega \in H^0(A, \Omega_A^d)$ be a nonzero element of this one-dimensional L-vector space. Then

$$h_{\mathrm{Fal}}(A) = -\frac{1}{[L:\mathbb{Q}]}\left(\sum_{\sigma \in H_L} \frac{1}{2}\log\left(\int_{A^\sigma(\mathbb{C})} |\omega^\sigma \wedge \overline{\omega^\sigma}|\right) - \sum_{p<\infty}\sum_{\sigma \in H_L} v_p(\omega^\sigma)\log p\right)$$

where H_L denotes the set of distinct embeddings $\sigma : L \hookrightarrow \mathbb{C}$. This definition is independent of the choice of ω and invariant under extension of the base field L. We define the exponential height $H_{\mathrm{Fal}}(A)$ by $H_{\mathrm{Fal}}(A) = \exp(2h_{\mathrm{Fal}}(A))$.

We recall also from [38] the definition of the *Hida constant* attached to an optimal quotient $\phi : J_0(N) \to A$. Since A is optimal we have an injection $\phi^* : H^0(A, \Omega^1) \hookrightarrow H^0(J_0(N), \Omega^1) \simeq H^0(X_0(N), \Omega^1)$. If $\alpha_1, \alpha_2, \ldots, \alpha_d$ is a basis for $H^0(A, \Omega^1)$, define

$$c_{A,\phi} = \left|\frac{\det\left(\int_{X_0(N)} \phi^*(\alpha_i) \wedge \overline{\phi^*(\alpha_j)}\right)}{\int_A \alpha_1 \wedge \cdots \wedge \alpha_d \wedge \overline{\alpha_1} \wedge \cdots \wedge \overline{\alpha_d}}\right|.$$

Clearly this is independent of the choice of the basis $\alpha_1, \ldots, \alpha_d$. If there is no ambiguity in the choice of map ϕ, then this constant is denoted c_A and is called the Hida constant of A. Let $\phi^\vee : A^\vee \to J_0(N)^\vee \simeq J_0(N)$ denote the dual map. Then it is shown in [38] that $c_A = (\deg(\phi \cdot \phi^\vee))^{1/2}$ and that c_A measures congruences between forms corresponding to A and other forms. Further, one has the following formula for the Faltings height of $J_0(N)^{\mathrm{new}}$.

THEOREM 5.4 (Ullmo ([38, 4.10])). *Up to a power of* 2

$$H_{\mathrm{Fal}}(J_0(N)^{\mathrm{new}}) \cdot \prod_{\text{newforms } f \text{ of level } N} 4\pi^2\langle f, f\rangle = (\delta_{\mathbb{T}^{\mathrm{new}}}) \cdot (c_{J_0(N)^{\mathrm{new}}})$$

where for any reduced algebra R, δ_R denotes the discriminant of R. As mentioned above, the term $c_{J_0(N)^{\mathrm{new}}}$ measures congruences between newforms and old forms. The term $\delta_{\mathbb{T}^{\mathrm{new}}}$ measures congruences between newforms of level N. ($\mathbb{T}^{\mathrm{new}}$ denotes the new quotient of the full Hecke algebra on $J_0(N)$.)

Now it is well known that there is a Hecke equivariant isogeny $\varphi : J_0(N)^{\mathrm{new}} \to J_0^N(1)$ (not necessarily canonical) defined over $\mathbb{Q}$. Let K be the kernel of this isogeny. Then $K(\overline{\mathbb{Q}})$ is a $\mathbb{T}[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}]$ module. Any irreducible subquotient of K must be contained in $J_0(N)[\mathfrak{m}^n]$ for some maximal ideal $\mathfrak{m}$ of $\mathbb{T}$ and some integer n. Assume that p is not an Eisenstein prime at level N. (Since Eisenstein primes divide the order of the cuspidal class group of $J_0(N)$, from [37, Thm. 5.1], it is enough to assume that $p \nmid M$.) Then for any maximal ideal $\mathfrak{m}$ of $\mathbb{T}$ above p, $J_0(N)[\mathfrak{m}]$ is an irreducible $\mathbb{T}[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}]$ module and any irreducible subquotient of $J_0(N)[\mathfrak{m}^n]$ is isomorphic to $J_0(N)[\mathfrak{m}]$. Now the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ representation $J_0(N)[\mathfrak{m}]$ is autodual and so we can apply the following lemma.

LEMMA 5.5. *Suppose A is a semistable Abelian variety defined over $\mathbb{Q}$ with good reduction at p and K is a finite subgroup of A defined over $\mathbb{Q}$ of p-power order. Assume that the representation of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *on* $K(\overline{\mathbb{Q}})$ *is autodual and that $p \neq 2$. Then $h_{\mathrm{Fal}}(A) = h_{\mathrm{Fal}}(A/K)$.*

Proof. Let $B = A/K$. The isogeny $\phi : A \to B$ extends to one of Neron models $\mathcal{A} \to \mathcal{B}$. Let $\mathcal{K}_1$ denote the kernel of this map restricted to the connected component of the Neron models. Then $\mathcal{K}_1$ is a quasi-finite flat group scheme over $\mathbb{Z}$ that is finite at p. From [7, Lemma 5],

$$h_{\mathrm{Fal}}(A) = h_{\mathrm{Fal}}(B) + \frac{1}{2}\deg(\phi) - \log(\# s^*(\Omega^1_{\mathcal{K}_{1,\mathbb{Z}_p}/\operatorname{spec}\mathbb{Z}_p}))$$

where s denotes the zero section of $\mathcal{K}_{1,\mathbb{Z}_p}/\operatorname{spec}\mathbb{Z}_p$. Let $\phi^\vee : B^\vee \to A^\vee$ be the dual isogeny. Again $\phi^\vee$ extends to a map of Neron Models and we denote by $\mathcal{K}_2$ the kernel of $\phi^\vee$ restricted to the connected part of the Neron models. Then

$$h_{\mathrm{Fal}}(B^\vee) = h_{\mathrm{Fal}}(A^\vee) + \frac{1}{2}\deg(\phi^\vee) - \log(\# s^*(\Omega^1_{\mathcal{K}_{2,\mathbb{Z}_p}/\operatorname{spec}\mathbb{Z}_p})).$$

Now $\mathcal{K}_1$ and $\mathcal{K}_2$ are finite at p since A and B have good reduction at p. Further, since $p \neq 2$, $\mathcal{K}_{1,\mathbb{Z}_p}$ and $\mathcal{K}_{2,\mathbb{Z}_p}$ are completely determined by the Galois action on their generic fibres. Since K is autodual we see that $\mathcal{K}_{1,\mathbb{Z}_p} \simeq \mathcal{K}_{2,\mathbb{Z}_p}$. Further the height of an Abelian variety equals that of its dual ([26]). Comparing the equations above, we see then that $h_{\mathrm{Fal}}(A) = h_{\mathrm{Fal}}(B)$. □

The lemma and the discussion preceding it show that up to (logarithms of) Eisenstein primes at level N (and therefore up to primes dividing M), $h_{\mathrm{Fal}}(J_0(N)^{\mathrm{new}}) = h_{\mathrm{Fal}}(J_0^N(1))$. Since the (full) Hecke algebra on the Shimura curve, $\overline{\mathbb{T}}$, is isomorphic to $\mathbb{T}^{\mathrm{new}}$, and since (see [43, §4.2, Lemma 5]) $8\pi^3\langle f, f\rangle = L(1, \mathrm{ad}^0(\pi_f)) = L(1, \mathrm{ad}^0(\pi_g))$, we get the following result.

PROPOSITION 5.6. *Up to primes dividing M,*

$$\gamma := H_{\mathrm{Fal}}(J_0^N(1)) \cdot \prod_{\text{eigenforms } g \text{ on } \Gamma = \Gamma_0^N(1)} (2\pi)^{-1} L(1, \mathrm{ad}^0(\pi_g)) = (\delta_{\overline{\mathbb{T}}}) \cdot (c_{J_0(N)^{\mathrm{new}}}).$$

Note that other than the term $c_{J_0(N)^{\mathrm{new}}}$ all the other terms are defined intrinsically in terms of the Shimura curve $X_0^N(1)$. (So rather mysteriously, the Shimura curve "sees" level-lowering congruences !).

We would like to prove a formula as in the previous proposition *intrinsically* on the Shimura curve rather than use the fact that there is an isogeny from $J_0(N)^{\mathrm{new}}$ to $J_0^N(1)$. The advantage of doing so would be that this method might work for Shimura curves over totally real fields, where an isogeny between Jacobians is not available.

Let L be a number field containing all the Hecke eigenvalues of all weight 2 forms on Γ and $\{g_1, \ldots, g_n\}$ be a basis of eigenforms, defined over L; i.e.,

$s_j = 2\pi \imath g_j(z)dz \in H^0(X_L, \Omega^1) \simeq H^0(J_L, \Omega^1)$. Let $\omega = s_1 \wedge \cdots \wedge s_n$. Now,

$$\int_{J^\sigma(\mathbb{C})} |\omega^\sigma \wedge \overline{\omega^\sigma}| = \det\left[\int_{X^\sigma(\mathbb{C})} s_i^\sigma \wedge \overline{s_j^\sigma}\right] = \prod_{i=1}^n \int_{X(\mathbb{C})} s_i^\sigma \wedge \overline{s_i^\sigma} = \prod_{i=1}^n 4\pi^2 \langle g_i, g_i\rangle.$$

If we choose the basis $\{g_i\}$ such that for every $\sigma \in H_L$, g_i^σ also lies in the basis, then it is clear that $\omega \in H^0(J_\mathbb{Q}, \Omega^n)$. Let f_i be the normalised newform on $\Gamma_0(N)$ corresponding to g_i under the Jacquet-Langlands correspondence. Then clearly γ (defined in Prop. 5.6) is given by

$$\gamma = \prod_{i=1}^n \frac{\langle f_i, f_i\rangle}{\langle g_i, g_i\rangle} \prod_{p<\infty} p^{2v_p(\omega)}.$$

By Harris-Kudla [14], the ratio $\frac{\langle f_i,f_i\rangle}{\langle g_i,g_i\rangle}$ lies in L for each i and satisfies $\left(\frac{\langle f_i,f_i\rangle}{\langle g_i,g_i\rangle}\right)^\sigma = \frac{\langle f_i^\sigma,f_i^\sigma\rangle}{\langle g_i^\sigma,g_i^\sigma\rangle}$. This implies the first part of the following theorem. The second part is an integrality result for γ analogous to Proposition 5.6.

Theorem 5.7. (i) $\gamma \in \mathbb{Q}$.

(ii) *Suppose $p \nmid M$. Then $v_p(\gamma) \geq v_p(\delta_{\overline{\mathbb{T}}})$. Also if p is a level-lowering congruence prime for some newform on $\Gamma_0(N)$, then $v_p(\gamma) > v_p(\delta_{\overline{\mathbb{T}}})$.*

The idea of the proof is as follows: one chooses the g_i so that the s_i are p-adically primitive. Then from our previous results the ratio $\frac{\langle f_i,f_i\rangle}{\langle g_i,g_i\rangle}$ is p-adically integral and divisible by p if p is a level-lowering congruence prime for f_i. It remains then to show that with such a choice of g_i, $v_p(\omega) = \frac{1}{2}v_p(\delta_{\overline{\mathbb{T}}})$.

Let $\mathcal{J}$ be the Neron model of J over $\operatorname{spec}\mathbb{Z}_p$. Let us also assume at first that the following holds for all maximal ideals $\mathfrak{m}$ of $\overline{\mathbb{T}}$ lying over p:

(i) The $\overline{\mathbb{T}}_\mathfrak{m}$ module $H^0(\mathcal{J}, \Omega^1)_\mathfrak{m}$ is a free $\overline{\mathbb{T}}_\mathfrak{m}$ module of rank 1.

(ii) The ring $\overline{\mathbb{T}}_\mathfrak{m}$ is Gorenstein; i.e., $\overline{\mathbb{T}}_\mathfrak{m} \simeq \operatorname{Hom}_{\mathbb{Z}_p}(\overline{\mathbb{T}}_\mathfrak{m}, \mathbb{Z}_p)$ as $\overline{\mathbb{T}}_\mathfrak{m}$ modules.

Since $\overline{\mathbb{T}} \otimes \mathbb{Z}_p = \prod_\mathfrak{m} \overline{\mathbb{T}}_\mathfrak{m}$, $H^0(\mathcal{J}, \Omega^1) = \prod_\mathfrak{m} H^0(\mathcal{J}, \Omega^1)_\mathfrak{m}$. Let $x_1, \ldots, x_d$ be primitive eigenvectors of $\overline{\mathbb{T}}_\mathfrak{m}$ in $H^0(\mathcal{J}, \Omega^1)_\mathfrak{m} \otimes_{\mathbb{Z}_p} \mathcal{O}_\lambda$ where λ is some prime of L over p. It will suffice to show that $v_\lambda(x_1 \wedge \cdots \wedge x_d) = \frac{1}{2}v_\lambda(\delta_{\overline{\mathbb{T}}_\mathfrak{m} \otimes_{\mathbb{Z}_p} \mathcal{O}_\lambda})$. Since $H^0(\mathcal{J}, \Omega^1)_\mathfrak{m} \simeq \overline{\mathbb{T}}_\mathfrak{m} \simeq \operatorname{Hom}_{\mathbb{Z}_p}(\overline{\mathbb{T}}_\mathfrak{m}, \mathbb{Z}_p)$ we need to show that $v_\lambda(y_1 \wedge \cdots \wedge y_d) = \frac{1}{2}v_\lambda(\delta_{\overline{\mathbb{T}}_\mathfrak{m} \otimes_{\mathbb{Z}_p} \mathcal{O}_\lambda})$ where $y_1, \ldots, y_d$ are primitive eigenvectors of $\overline{\mathbb{T}}_\mathfrak{m}$ in $\operatorname{Hom}_{\mathbb{Z}_p}(\overline{\mathbb{T}}_\mathfrak{m}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_\lambda = \operatorname{Hom}_{\mathcal{O}_\lambda}(\overline{\mathbb{T}}_\mathfrak{m} \otimes_{\mathbb{Z}_p} \mathcal{O}_\lambda, \mathcal{O}_\lambda)$. If $\overline{\mathbb{T}}_\mathfrak{m} \otimes_{\mathbb{Z}_p} \mathcal{O}_\lambda \hookrightarrow \prod_i \mathcal{O}_\lambda$ is the canonical inclusion given by the various characters of $\overline{\mathbb{T}}_\mathfrak{m}$, then we may choose the y_i to be the projections on the various factors.

Let $\langle,\rangle$ be the bilinear form on $\operatorname{Hom}_{\mathcal{O}_\lambda}(\overline{\mathbb{T}}_\mathfrak{m} \otimes_{\mathbb{Z}_p} \mathcal{O}_\lambda, \mathcal{O}_\lambda) \otimes_{\mathcal{O}_\lambda} L_\lambda$ that is dual to the trace form on $\overline{\mathbb{T}}_\mathfrak{m}$. With respect to this form the volume of $\operatorname{Hom}_{\mathcal{O}_\lambda}(\overline{\mathbb{T}}_\mathfrak{m} \otimes_{\mathbb{Z}_p} \mathcal{O}_\lambda, \mathcal{O}_\lambda)$ is clearly $\delta^{1/2}_{\overline{\mathbb{T}}_\mathfrak{m} \otimes_{\mathbb{Z}_p} \mathcal{O}_\lambda}$. On the other hand it is easy to check that $y_1, \ldots, y_d$ form an orthonormal basis with respect to this form, whence $v_\lambda(y_1 \wedge \cdots \wedge y_d) = \frac{1}{2} v_\lambda(\delta_{\overline{\mathbb{T}}_\mathfrak{m} \otimes_{\mathbb{Z}_p} \mathcal{O}_\lambda})$ as required.

We now only need to check that (i) and (ii) hold for primes $p \nmid M$. It is well known that both would follow if one knew that the multiplicity of the mod p representation $\rho_\mathfrak{m}$ in $J[\mathfrak{m}]$ is 1 for any prime $\mathfrak{m}$ above p. That this is the case follows from the result of Helm [17]. We recall that in the notation of [17], a non-Eisenstein maximal ideal $\mathfrak{m}$ of $\overline{\mathbb{T}}$ is called *controllable* at a prime $q \mid N$ if $\rho_\mathfrak{m}$ is either ramified at q or $\rho_\mathfrak{m}(\mathrm{Frob} q)$ is not a scalar.

THEOREM 5.8 (Helm). *Suppose that* $\mathfrak{m}$ *is controllable at all primes* $q \mid N$. *Then* $\rho_\mathfrak{m}$ *occurs in* $J[\mathfrak{m}]$ *with multiplicity one.*

Note that if $\mathfrak{m}$ is not controllable at q, then $\rho_\mathfrak{m}$ must be unramified at q and $\rho_\mathfrak{m}(\mathrm{Frob} q)$ must be a scalar, which is necessarily ± 1. In this case, $q \equiv 1 \mod p$. But we have assumed that $p \nmid M$, and thus the hypothesis of Theorem 5.8 holds. This completes the proof of Theorem 5.7. However this proof is still not completely satisfactory since the proof of Theorem 5.8 is based on a study of relations between Hecke modules attached to (character groups of) $J_0(N)$ and $J^N(1)$ and ultimately depends on the fact that we know the corresponding mutiplicity-one statement for $J_0(N)$. Ideally, one would like to verify (i) and (ii) working on the Shimura curve without referring to the corresponding modular curve, and thus make the height computation intrinsic to the Shimura curve.

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