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Integrals of a Lotka-Volterra System of Odd Number of Variables

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A Lotka-Volterra system with $s+1$ conserved quantities of $2s+1$ variables is introduced. In the system each species interacts with the other $2s$ species. The conserved quantities are explicitly represented by polynomials in a simple way.

We introduce a Lotka-Volterra system with $s+1$ conserved quantities of $2s+1$ variables. It is known that nonlinear lattice can provide a Lotka-Volterra system[1)~4)] of n variables,

$$\frac{d}{dt}P_i = P_i(P_{i-1} - P_{i+1}) \tag{1}$$

with $P_{i+n} = P_i$ for each integer i, which has soliton solutions.[2),5)] In our system[6)~10)] of $2s+1$ variables each species interacts with the other $2s$ species as

$$\frac{d}{dt}P_i = P_i\left(\sum_{j=1}^{s} P_{i-j} - \sum_{j=1}^{s} P_{i+j}\right) \tag{2}$$

with $P_{i+2s+1} = P_i$ for each integer i. The conserved quantities for the case $s=2$ are

$$P_1+P_2+P_3+P_4+P_5 = C(1),$$

$$P_1P_2P_4+P_2P_3P_5+P_3P_4P_1+P_4P_5P_2+P_5P_1P_3 = C(2)$$

and

$$P_1P_2P_3P_4P_5 = C(3). \tag{3}$$

Henon[11)] and Flaschka[12)] independently showed analytically m conserved quantities for the general cyclic Toda lattice[13)] of $2m$ variables. Henon's proof is based on combinatorial enumeration. Flaschka proved it by making use of Lax formalism. Sawada and Kotera[14)] showed the theorem starting with the general equation of motion of Poisson brackets. They applied the method also to Calogero's system.[15)]

Our Lotka-Volterra equation is a deterministic version of a stochastic model[6),9)] and hence our present result is a deterministic version of the previous result.[9)] Here we give the $s+1$ conserved quantities explicitly for the $2s+1$ variables by a combinatorial method.

We assume:

1) $I_r = \{0, 1, \cdots, 2r\}$.
2) $f_r(n)$ is a function for every integer n which is defined by

   i) $f_r(n) \equiv n \pmod{2r+1}$

ii) $0\leqq f_r(n)\leqq 2r$.

3) For $i, j\in I_s$

$$a_{ij}=\begin{cases}1 & \text{iff } 1\leqq f_s(i-j)\leqq s\\ -1 & \text{iff } s+1\leqq f_s(i-j)\leqq 2s\\ 0 & \text{iff } f_s(i-j)=0.\end{cases}$$

4)

$$S(r)=\{(k_0, k_1, \cdots, k_{2r})|a_{k_ik_j}=\begin{cases}1 & \text{iff } 1\leqq f_r(i-j)\leqq r\\ -1 & \text{iff } r+1\leqq f_r(i-j)\leqq 2r\\ 0 & \text{iff } f_r(i-j)=0\end{cases}$$

for $i, j\in I_r$ where $0\leqq k_0<k_1<\cdots<k_{2r}\leqq 2s\}$.

Under the above assumption, we have the following theorem.

THEOREM

$$\text{Let } \frac{d}{dt}P_i(t)=P_i(t)\sum_{j=0}^{2s}a_{ij}P_j(t)$$

and

$$P_i(t)>0 \text{ for } i=0, 1, \cdots, 2s, \tag{4}$$

then

$$\sum_{(k_0,k_1,k_2,\cdots,k_{2r})\in S(r)}\prod_{m=0}^{2r}P_{k_m}(t)=C(r) \text{ for } r=0, 1, \cdots, s, \tag{5}$$

where $C(r)$ are time independent constant.

We prove the lemmas.

Lemma 1.

Let $(k_0, k_1, k_2, \cdots, k_{2r})\in S(r)$ and $k_0=0$, then $k_r\leqq s<k_{r+1}$.

Proof Obvious from the definition of $S(r)$.

Lemma 2.

Let $(k_0, k_1, k_2, \cdots, k_{2r})\in S(r)$, $k_0=0$ and $k_r<l<k_{r+1}$,

then $(k_0', k_1', k_2', \cdots, k_{2r}')\in S(r)$, which satisfies

$k_m'\in\{k_0, k_1, k_2, \cdots, k_{2r}, l\}$ for $m=0, 1, 2, \cdots, 2r$, and

$l\in\{k_0', k_1', k_2', \cdots, k_{2r}'\}$, is uniquely determined in the following way:

i) iff $k_r<l\leqq s$, $k_r'=l$ and for $m\neq r$ $k_m'=k_m$,

ii) iff $s<l<k_{r+1}$, $k_{r+1}=l$ and for $m\neq r+1$ $k_m'=k_m$.

Proof From Lemma 1, we see the classification by i) and ii) is reasonable. We see the following two from the conditions.

(1) l is included in $(k_0', k_1', k_2', \cdots, k_{2r}')$.

(2) For every fixed m_0', $1 \leqq f_s(k_{m_0}' - k_m') \leqq s$ holds just for r of k_m' where $m = 0, 1, 2, \cdots, 2r$.

The problem is which k_m must be replaced by l. We consider Case i). Since $1 \leqq f_s(l - k_m) \leqq s$ holds for $r+1$ of k_m, that is to say, for $m = 0, 1, 2, \cdots, r$, one of them must be eliminated (because of (2)). If one of k_m, in which $0 \leqq m \leqq r-1$, is eliminated, $1 \leqq f_s(k_r - k_m) \leqq s$ holds just for $r-1$ of k_m. This does not satisfy (2). So k_r must be replaced by l. l plays the same role as k_r, that is,

$$1 \leqq f_s(l - k_m) \leqq s \text{ for } m = 0, 1, 2, \cdots, r-1,$$

$$s+1 \leqq f_s(l - k_m) \leqq 2s \text{ for } m = r+1, r+2, \cdots, 2r.$$

Thus we see $(k_0', k_1', \cdots, k_{2r}')$, in which $k_r' = l$ and for $m \neq r$ $k_m' = k_m$, belongs to $S(r)$. We can prove Case ii) in the same way. Q. E. D.

For the following Lemma 3 we define k_β for every integer β as $k_\beta = k_m$ iff $m - \beta \equiv 0 \pmod{2r+1}$, where $0 \leqq m \leqq 2r$.

Lemma 3.

Let $1 \leqq f_s(l - k_\alpha) \leqq s$ and $1 \leqq f_s(k_{\alpha+1} - l) \leqq s$, then $(k_0', k_1', \cdots, k_{2r}') \in S(r)$, which satisfies

$$k_0', k_1', k_2', \cdots, k_{2r}' \in \{k_0, k_1, k_2, \cdots, k_{2r}, l\},$$

$$l \in \{k_0', k_1', \cdots, k_{2r}'\},$$

is uniquely determined in the way:

i) iff $1 \leqq f_s(k_{\alpha-r} + s - l) \leqq s$, $k_\alpha' = l$ and for $m \neq \alpha$ $k_m' = k_m$,

ii) iff $1 \leqq f_s(l - s - k_{\alpha-r}) \leqq s$, $k_{\alpha+1}' = l$ and for $m \neq \alpha+1$ $k_m' = k_m$.

Proof This is proved by a little modification of the proof of Lemma 2.

Proof of the theorem

$$\frac{d}{dt} \sum_{(k_0, k_1, \cdots, k_{2r}) \in S(r)} \prod_{m=0}^{2r} P_{k_m}(t)$$

$$= \sum_{(k_0, k_1, \cdots, k_{2r}) \in S(r)} \left(\prod_{m=0}^{2r} P_{k_m}(t)\right) \sum_{m=0}^{2r} \sum_{l=0}^{2s} a_{k_m l} P_l(t).$$

We see each term of the right-hand side of the above equation has the form $P_{k_0} P_{k_1} \cdots P_{k_{2r}} P_l$, in which $(k_0, k_1, \cdots, k_{2r}) \in S(r)$, $l \in \{0, 1, \cdots, 2s\}$. We calculate the coefficient of these terms.

Since $(k_0, k_1, \cdots, k_{2r}) \in S(r)$, for $l \in \{k_0, k_1, \cdots, k_{2r}\}$ $\varepsilon_{k_0 k_1 \cdots k_{2r}}(l) = \sum_{m=0}^{2r} a_{k_m l} = 0$.

Next we prove for the case $l \notin \{k_0, k_1, \cdots, k_{2r}\}$.

From Lemma 3 we see the coefficient of the term $P_{k_0} P_{k_1} \cdots P_{k_{2r}} P_l$ is, in Case i) of Lemma 3,

$$\varepsilon_{k_0 k_1 \cdots k_{2r}}(l) + \varepsilon_{k_0 k_1 \cdots k_{r-1} l k_{r+1} \cdots k_{2r}}(k_r),$$

in Case ii) of Lemma 3,

$$\varepsilon_{k_0k_1\cdots k_{2r}}(l)+\varepsilon_{k_0k_1\cdots k_rlk_{r+2}\cdots k_{2r}}(k_{r+1}) .$$

We consider Case i) of Lemma 3,

$$\varepsilon_{k_0k_1\cdots k_{2r}}(l)=\sum_{m=0}^{2r}a_{k_ml}=-1 ,$$

since

$$a_{k_ml}=\begin{matrix}-1 & \text{for } m=0,1,2,\cdots,r \\ +1 & \text{for } m=r+1,r+2,\cdots,2r .\end{matrix}$$

$$\varepsilon_{k_0k_1\cdots k_{r-1}lk_{r+1}\cdots k_{2r}}(k_r)=1 ,$$

since

$$a_{k_m'k_r}=\begin{matrix}-1 & \text{for } m=0,1,2,\cdots,r-1 \\ +1 & \text{for } m=r,r+1,\cdots,2r\end{matrix}$$

in which k_m' is defined in Lemma 3. Thus we see

$$\varepsilon_{k_0k_1\cdots k_{2r}}(l)+\varepsilon_{k_0'k_1'\cdots k_{2r}'}(k_r)=0 .$$

For Case ii) we can discuss in the same way.

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