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RONALD OWEN FULP AND JOE ALTON MARLIN

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Let M be a C^∞ manifold of dimension m and E an integrable subbundle (foliation) of the tangent bundle TM . We are interested in structures on the set of all local integrals of E . For example, if M is a symplectic manifold then the Poisson brackets operation on the set C_{loc}^∞ of all local functions of M defines an algebraic structure on C_{loc}^∞ . Earlier authors have called such structures "function groups." In particular, if X_H is a nonvanishing Hamiltonian vector field, then X_H defines a foliation E of M and the set of all local integrals of E is also a function group.

The Poisson brackets operation can be defined on manifolds with somewhat less restrictive requirements than that of being symplectic. Other authors such as S. Lie and C. Carathéodory [4] have studied this more general notion of Poisson brackets in the classical local setting. Hermann [9, p. 31] has indicated how to extend the definition of Poisson brackets to functions on manifolds having a closed 2-form ω of constant rank (Recall that M is called symplectic if ω_p has rank m for each $p \in M$).

The paper is largely self-contained, but does require the use of the following basic identities:

$$L_X Y = [X, Y], \quad L_X = i_X d + di_X, \quad L_X i_Y - i_Y L_X = i_{[X, Y]}.$$

The proofs of these identities may be found in Chapter IV of the first volume of [7]. Other undefined terms appear either in [1] or [7].

1. Generalized symplectic structures on manifolds. Let M be a C^∞ manifold of dimension m and let ω be a closed 2-form on M . Recall that the kernel of a 2-form ω can be defined at each point $p \in M$ by

$$\begin{aligned} \ker \omega_p &= \{v \in M_p \mid \omega(v, M_p) = 0\} \\ &= \{v \in M_p \mid \omega(M_p, v) = 0\}. \end{aligned}$$

The rank of ω at p is defined to be the rank of the bilinear map $\omega_p: M_p \times M_p \rightarrow R$. Of course, since ω_p is a skew-symmetric bilinear map its rank is the even integer $m - \dim(\ker \omega_p)$.

Let Γ denote the set of sections of TM and Γ^* the set of sections of T^*M . Define $\alpha: \Gamma \rightarrow \Gamma^*$ by

$$\alpha_x = i_x \omega .$$

Let $\Gamma_\omega = \{X \in \Gamma \mid i_x \omega = 0\} = \ker \alpha$.

If we fix $p \in M$ then we may regard $\alpha = \alpha_p$ as a map from $T_p M$ into $T_p^* M$. Since $T_p M$ is finite dimensional, $T_p M \cong T_p^{**} M$ and we may apply the standard duality theorems of linear algebra. Thus, if we use the usual pairing between $T_p M$ and $T_p^* M$ we have, for $x, y \in T_p M$,

$$\langle \alpha(y), x \rangle = \alpha(y)(x) = \omega_p(y, x) = -\omega_p(x, y) = \langle y^{**}, -\alpha(x) \rangle .$$

Thus α is skew adjoint: $\alpha^* = -\alpha$, and

$$\text{im}(\alpha^*) = \text{im}(\alpha) = \ker(\alpha)^\perp$$

where $\ker(\alpha)^\perp$ is the annihilator of $\ker(\alpha)$ in $T_p^* M$.

From this we see that if $\Gamma_\omega^* \equiv \{\beta \in \Gamma^* \mid \beta(\Gamma_\omega) = 0\}$, then $\Gamma_\omega^* = \ker(\alpha)^\perp \subseteq \Gamma^*$. From these remarks it follows that $\Gamma_\omega^* = \text{im}(\alpha)$.

If $\text{inv}(\Gamma)$ is defined by $\text{inv}(\Gamma) = \{X \in \Gamma \mid L_X \Gamma_\omega \subseteq \Gamma_\omega\}$ then $\text{inv}(\Gamma)$ is the normalizer of Γ_ω in Γ and thus is a Lie subalgebra of Γ . Moreover, it is immediate from the definitions any subalgebra of a Lie algebra is always an ideal in its normalizer, thus Γ_ω is an ideal in $\text{inv}(\Gamma)$. We summarize all these remarks as a proposition.

PROPOSITION 1.1. *The image of the map $\alpha: \Gamma \rightarrow \Gamma^*$ is precisely*

$$\Gamma_\omega^* \equiv \{\beta \in \Gamma^* \mid \beta(\Gamma_\omega) = 0\} .$$

Moreover, $\text{inv}(\Gamma) \equiv \{X \in \Gamma \mid L_X \Gamma_\omega \subseteq \Gamma_\omega\}$ is a Lie subalgebra of Γ which contains Γ_ω as an ideal.

We now want to show that $\alpha|_{\text{inv}(\Gamma)}$ is a Lie algebra anti-homomorphism from $\text{inv}(\Gamma)$ onto the set $\text{inv}(\Gamma_\omega^*) \subseteq \Gamma^*$ where $\text{inv}(\Gamma_\omega^*)$ is defined by

$$\text{inv}(\Gamma_\omega^*) \equiv \{\beta \in \Gamma_\omega^* \mid L_Z \beta = 0 \text{ for all } Z \in \Gamma_\omega\} .$$

Before doing this we need to define a Lie algebra structure on $\text{inv}(\Gamma_\omega^*)$. For this we need a lemma.

LEMMA 1.2. *If $Z \in \Gamma_\omega$, then $L_Z \Gamma_\omega^* \subseteq \Gamma_\omega^*$. In fact, $L_Z \alpha_x = \alpha_{L_Z x}$ for each $X \in \Gamma$.*

Proof. Since $L_Z \omega = (i_Z d + di_Z)\omega = 0$, $L_Z \alpha_x = L_Z i_x \omega = i_x L_Z \omega + i_{[x, Z]} \omega = \alpha_{[x, Z]} = \alpha_{L_Z x}$.

COROLLARY 1.3. $\alpha(\text{inv} \Gamma) = \text{inv}(\Gamma_\omega^*)$.

Proof. From Proposition 1.1, we know that $\text{inv}(\Gamma_\omega^*)$ is contained

in $\text{im}(\alpha)$. By the lemma above, for $Z \in \Gamma_\omega$, $L_Z\alpha_X = \alpha_{L_ZX} = -\alpha_{L_XZ}$; thus $\alpha_X \in \text{inv}(\Gamma_\omega^*)$ iff $L_XZ \in \Gamma_\omega$ for all $Z \in \Gamma_\omega$. It follows that $\alpha(\text{inv } \Gamma) = \text{inv}(\Gamma_\omega^*)$.

The map α is a linear transformation from $\text{inv}(\Gamma)$ onto $\text{inv}(\Gamma_\omega^*)$ with kernel Γ_ω . Thus $\text{inv}(\Gamma_\omega^*) \cong \text{inv}(\Gamma)/\Gamma_\omega$ as vector spaces. Since Γ_ω is a Lie ideal in $\text{inv}(\Gamma)$, the quotient $\text{inv}(\Gamma)/\Gamma_\omega$ is a Lie algebra. We impose this Lie structure on $\text{inv}(\Gamma_\omega^*)$ via the vector space isomorphism induced by α .

PROPOSITION 1.4. *The set $\text{inv}(\Gamma_\omega^*)$ of all invariant elements of Γ_ω^* is a Lie algebra under $\{, \}$ where $\{, \}$ is defined by*

$$\{\alpha_X, \alpha_Y\} = -\alpha_{[X, Y]}.$$

The map $\alpha: \text{inv}(\Gamma) \rightarrow \text{inv}(\Gamma_\omega^*)$ is a Lie algebra antihomomorphism with kernel Γ_ω , thus the sequence

$$0 \longrightarrow \Gamma_\omega \longrightarrow \text{inv}(\Gamma) \xrightarrow{\alpha} \text{inv}(\Gamma_\omega^*) \longrightarrow 0,$$

is an exact sequence of Lie algebras.

REMARK. It is easy to see that for $\alpha, \beta \in \text{inv}(\Gamma_\omega^*)$ one has

$$\{\alpha, \beta\}|_U = \{\alpha|_U, \beta|_U\}$$

for open subsets U of M .

REMARK. We now call attention to certain identities which have proven useful in our work. If β and γ are closed 1-forms in Γ_ω^* and X and Y are vector fields such that $\beta = \alpha_X$, $\gamma = \alpha_Y$, then

$$\{\beta, \gamma\} = -i_{[X, Y]}\omega = -L_XY = L_Y\beta = d(2\omega(X, Y)).$$

Note, in particular, that $\{\beta, \gamma\}$ is exact.

To see that the above identities hold, observe that

$$\begin{aligned} \{\beta, \gamma\} &= \{\alpha_X, \alpha_Y\} = -\alpha_{[X, Y]} \\ &= -i_{[X, Y]}\omega = -L_Xi_Y\omega + i_YL_X\omega \\ &= -L_X\alpha_Y + i_Y(di_X + i_Xd)\omega = -L_X\alpha_Y + i_Y(d\alpha_X) \\ &= -L_X\gamma = -(di_X + i_Xd)\gamma = -d(i_X\gamma) \\ &= 2d(\omega(X, Y)). \end{aligned}$$

Let $C^\infty(\omega)$ denote the set of all invariant functions of $\ker \omega$, i.e.

$$C^\infty(\omega) = \{f \mid L_Zf = df(Z) = 0 \text{ for all } Z \in \Gamma_\omega\}.$$

We now define the Poisson bracket $\{, \}$ for pairs of invariant functions of $\ker \omega$:

$$\{f, g\} = 2\omega(X_f, X_g)$$

where X_f and X_g are any two vector fields such that

$$dh = i_{X_h}\omega$$

for $h = f, g$. Clearly $\{, \}$ is well-defined.

PROPOSITION 1.5. *If $f, g \in C^\infty(\omega)$ the following statements are true:*

$$(1) \quad \{f, g\} = -L_{X_f}(g) = L_{X_g}(f)$$

$$(2) \quad d\{f, g\} = \{df, dg\}.$$

Moreover, $C^\infty(\omega)$ is a Lie algebra with respect to $\{, \}$ and

$$(3) \quad X_{\{f, g\}} + [X_f, X_g] \in \Gamma_\omega.$$

Proof. If $f, g \in C^\infty(\omega)$ then (1) follows from $\{f, g\} = 2\omega(X_f, X_g) = df(X_g) = L_{X_g}(f)$. By the above remark we have $d\{f, g\} = d(2\omega(X_f, X_g)) = \{df, dg\}$ and thus (2) follows. The statement (3) is immediate from definitions.

PROPOSITION 1.6. *If $f, g \in C^\infty(\omega)$ and $dg = i_{X_g}\omega$ then f is constant on integral curves of X_g iff $\{f, g\} = 0$.*

Proof. $X_g(f) = L_{X_g}(f) = \{f, g\} = 0$.

2. Function groups. Let M be a connected C^∞ -manifold of dimension m with a 2-form ω of constant rank $\rho \leq m$. In this case $\ker \omega$ is locally trivial, i.e., $\ker \omega$ is a subbundle of TM . Moreover, $\ker \omega$ is actually an integrable subbundle of TM and thus is a foliation of M . To see this observe that for $X \in \Gamma_\omega$,

$$L_X\omega = i_X(d\omega) + d(i_X\omega) = 0.$$

Thus for X, Y in Γ_ω ,

$$i_{[X, Y]}\omega = L_X(i_Y\omega) - i_Y(L_X\omega) = 0.$$

A function f is called a local C^∞ function on M iff the domain $U = \text{dom}(f)$ of f is an open subset of M and $f \in C^\infty(U)$. Let $C_{\text{loc}}^\infty = C_{\text{loc}}^\infty(M)$ denote the set of all local C^∞ functions of M . Let $C_{\text{loc}}^\infty(\omega)$ denote the set of all local integrals of the foliation $\ker \omega$, i.e.,

$$C_{\text{loc}}^\infty(\omega) = \{f \in C_{\text{loc}}^\infty \mid df(\ker(\omega_p)) = 0 \text{ for all } p \in \text{dom } f\}.$$

Note that in the symplectic case $C_{\text{loc}}^\infty(\omega) = C_{\text{loc}}^\infty$.

Recall that a function $f \in C_{\text{loc}}^\infty$ is said to be C^∞ -dependent on $f_1, f_2, \dots, f_r \in C_{\text{loc}}^\infty$ at $p \in M$ provided that there is a neighborhood U

of p and a function $F \in C_{10c}^\infty(\mathbf{R}^r)$ such that

- (1) the functions f, f_1, f_2, \dots, f_r are all defined on U , and
- (2) $f(x) = F(f_1(x), f_2(x), \dots, f_r(x))$ for each $x \in U$.

If $f, g \in C_{10c}^\infty(\omega)$ and $U = \text{dom } f \cap \text{dom } g \neq \emptyset$, then U can be regarded as a manifold with $\omega|_U$ a 2-form of constant rank on U . Thus $\{f, g\} = \{f|U, g|U\}$ is a well-defined element of $C^\infty(\omega|U)$. It follows that X_f and X_g have domains $\text{dom } f$ and $\text{dom } g$ respectively and thus $[X_f, X_g]$ and $X_{\{f, g\}}$ are well-defined vector fields on U . Similarly, $\{df, dg\}$ is a well-defined 1-form on U .

DEFINITION 2.1. A nonvoid subset \mathcal{S} of $C_{10c}^\infty(\omega)$ is called a *function group* iff the following conditions hold:

- (1) $M = \bigcup_{f \in \mathcal{S}} \text{dom } (f)$,
- (2) if $f \in \mathcal{S}$ and U is an open subset of $\text{dom } f$ then $f|U \in \mathcal{S}$,
- (3) if $f, g \in \mathcal{S}$ and $\text{dom } (f) \cap \text{dom } (g) \neq \emptyset$, then $\{f, g\} \in \mathcal{S}$,
- (4) if f_1, f_2, \dots, f_k are elements of \mathcal{S} and f is C^∞ -dependent on f_1, f_2, \dots, f_k then $f \in \mathcal{S}$,
- (5) Let $U = \bigcup_j U_j$ where U_j is an open subset of M for each j . If $f \in C^\infty(U)$ and $f|U_j \in \mathcal{S}$, for each j , then $f \in \mathcal{S}$.

A function group is said to be of rank r at a point $p \in M$ provided that there are r functions f_1, f_2, \dots, f_r in \mathcal{S} such that

- (1) there is a neighborhood U of p contained in the domain of each of the functions f_1, f_2, \dots, f_r such that for each $q \in U$

$$df_{1q}, df_{2q}, \dots, df_{rq}$$

are independent elements of M_q^* , and

- (2) for each $f \in \mathcal{S}$, with $p \in \text{dom } f$, f is C^∞ -dependent on f_1, f_2, \dots, f_r on some neighborhood of p .

In case f_1, f_2, \dots, f_r satisfy (1) and (2) we say that f_1, f_2, \dots, f_r *generate* \mathcal{S} at p .

REMARK. If f_1, f_2, \dots, f_r generate \mathcal{S} at p and g_1, g_2, \dots, g_s generate \mathcal{S} at p , then $r = s$. To see this observe that the definition implies that there exists functions $F_i \in C_{10c}^\infty(\mathbf{R}^r)$, $G_j \in C_{10c}^\infty(\mathbf{R}^r)$ such that for $i = 1, 2, \dots, s$ and $j = 1, 2, \dots, r$

$$g_i = F_i(f_1, \dots, f_r) \quad \text{and} \quad f_j = G_j(g_1, \dots, g_s).$$

Then the chain rule applied to the equalities

$$\begin{aligned} g_i &= F_i(G_1(g_1, \dots, g_s), \dots, G_r(g_1, \dots, g_s)) \\ f_j &= G_j(F_1(f_1, \dots, f_r), \dots, F_s(f_1, \dots, f_r)) \end{aligned}$$

implies that $(\partial F_i / \partial f_j)$ and $(\partial G_k / \partial g_l)$ are inverse matrices. Hence $r = s$

REMARK. If \mathcal{S} is a function group of rank r at $p \in M$, then one can easily show that if h_1, h_2, \dots, h_r are elements of \mathcal{S} such that $dh_{1p}, dh_{2p}, \dots, dh_{rp}$ are independent in M_p^* then they generate \mathcal{S} at p .

A function group is said to be of rank r iff it is of rank r at each point of M .

The following is an example to show that a function group may not have the same rank at each point of M . Let $M = \mathbf{R}^2$ and $\omega = dx \wedge dy$. Let $f \in C^\infty(\mathbf{R})$ such that

$$f(x) = 0, x \leq 0 \quad \text{and} \quad f(x) > 0, x > 0.$$

Define functions F and G on \mathbf{R}^2 by $F(x, y) = x$ and $G(x, y) = f(x)y$. Let \mathcal{S} denote the set of all functions of the form

$$(x, y) \longrightarrow \Phi(F(x, y), G(x, y))$$

where Φ is any element of $C_{\text{loc}}^\infty(\mathbf{R}^2)$. Then \mathcal{S} is a function group which has rank 2 at points (x, y) where $x > 0$ and rank 1 at points (x, y) where $x < 0$.

We describe the relation between function groups of rank r and foliations.

THEOREM 2.2. *Let \mathcal{S} be a function group of rank r and let $E_p = \{X_p \mid 2\omega_p(X_p, \cdot) = df(\cdot) \text{ for } f \in \mathcal{S}\}$ for each $p \in M$. Then $E = \bigcup_{p \in M} E_p \subseteq TM$ is an integrable subbundle of TM which contains $\ker(\omega)$.*

Proof. We show E is locally trivial. Choose $p \in M$, U a neighborhood of p , and f_1, \dots, f_r in \mathcal{S} as in the definition of a generating set for \mathcal{S} at p . Let $X_i = X_{f_i}$. If $q \in U$ and $v \in E_q$ then $v = (X_h)_q$ for some $h \in \mathcal{S}$. Since df_{1q}, \dots, df_{rq} are independent we know that there exists $F \in C_{\text{loc}}^\infty(\mathbf{R}^r)$ such that

$$h = F(f_1, \dots, f_r)$$

on a neighborhood V of q . One sees that

$$X_h - \sum_1^r \frac{\partial F}{\partial x_i} X_i \in \Gamma(\ker(\omega|V))$$

and thus $v = (X_h)_q \in \langle X_{1q}, \dots, X_{rq} \rangle + \ker(\omega_q)$. Therefore E is a subbundle of TM .

We show E is integrable. Let X, Y belong to $\Gamma(E)$ and let $p \in M$. On a neighborhood U of p both X and Y are of the form

$$\sum \lambda_i X_i + Z$$

for $\lambda_i \in C^\infty(U)$, $Z \in \Gamma(\omega|_U)$, and $X_i = X_{f_i}$. Then $[X, Y]$ will be in $\Gamma(E)$ provided that for $1 \leq i, j \leq r$, $[X_i, X_j] \in \Gamma(E)$ and for $Z \in \Gamma(\omega|_U)$, $[X_i, Z] \in \Gamma(E)$. Since \mathcal{S} is a function group, $\{f_i, f_j\} \in \mathcal{S}$ and $X_{(f_i, f_j)} \in \Gamma(E|U)$. By (3) of Proposition 1.5 it follows that $[X_i, X_j] \in \Gamma(E|U)$. Moreover, $2\omega([Z, X_j], Y) = (i_{[Z, X_j]}\omega)(Y) = L_Z(i_{X_j}\omega)(Y) = L_Z(df_j)(Y) = d(i_Z df_j)(Y) = 0$ for all $Y \in \Gamma$. Thus $[Z, X_j] \in \Gamma_\omega$ for each $Z \in \Gamma_\omega$ and consequently E is integrable.

Hereafter the foliation E described above will be called the foliation determined by \mathcal{S} .

If \mathcal{S} is a function group then the reciprocal of \mathcal{S} is defined to be the set of all $g \in C^\infty_{loc}(\omega)$ such that $\{f, g\} = 0$ for all $f \in \mathcal{S}$ such that $\text{dom } f \cap \text{dom } g \neq \emptyset$. We denote the reciprocal of \mathcal{S} by \mathcal{S}' . The fact that \mathcal{S}' is a function group is somewhat trivial. To see that \mathcal{S}' is closed under $\{, \}$ one uses the Jacobi identity. To see that (4) of Definition 2.1 holds we need an identity which is useful in subsequent sections of our paper: for arbitrary $h_1, h_2, \dots, h_n \in C^\infty_{loc}(\omega)$ and $F \in C^\infty_{loc}(\mathbf{R}^n)$, then

$$(2.4) \quad \{f, F(h_1, h_2, \dots, h_n)\} = \sum_i \frac{\partial F}{\partial h_i}(h_1, h_2, \dots, h_n)\{f, h_i\}.$$

Part (4) follows immediately from this identity. To prove 2.4 observe that

$$\begin{aligned} \{f, F(h_1, h_2, \dots, h_n)\} &= -2\omega(X_F, X_f) = -dF(X_f) \\ &= -\sum_i \frac{\partial F}{\partial h_i}\{h_i, f\} = \sum_i \frac{\partial F}{\partial h_i}\{f, h_i\}. \end{aligned}$$

REMARK. It is obvious that $\mathcal{S} \subseteq \mathcal{S}''$ for any function group \mathcal{S} . Observe that if \mathcal{S} has rank r , then $\mathcal{S} = \mathcal{S}''$.

If \mathcal{S} is a function group then \mathcal{T} is a subgroup of \mathcal{S} iff \mathcal{T} is a function group such that $\mathcal{T} \subseteq \mathcal{S}$.

Observe that every function group is a subgroup of the function group $C^\infty_{loc}(\omega)$. Also the intersection of two subgroups is a subgroup. In particular $\mathcal{S} \cap \mathcal{S}'$ is a subgroup of both \mathcal{S} and \mathcal{S}' .

PROPOSITION 2.6. *Let \mathcal{S} be a function group of rank r at p . Then its reciprocal has rank $\rho - r$ at p .*

Proof. Let $p \in M$ and let f_1, \dots, f_r be generators of \mathcal{S} at p . Choose coordinates x_1, \dots, x_m at p such that $X_i = X_{f_i} = \partial/\partial x_i$ for $1 \leq i \leq r$ and such that $\{\partial/\partial x_{r+j}\} 1 \leq j \leq m - r$ generate Γ_ω near p . Then any integral of the integrable system $X_1, \dots, X_r, \partial/\partial x_{r+1}, \dots, \partial/\partial x_{r+m-r}$ depends only on the last coordinates. Since each

$f \in \mathcal{S}'$ is an integral of this system it follows that $x_{m+r-\rho+1}, \dots, x_m$ generates \mathcal{S}' at p .

Using arguments similar to those above we obtain the following corollary.

COROLLARY 2.7. *Let \mathcal{S} be a function group of rank r , \mathcal{S}' the reciprocal of \mathcal{S} , and E the foliation determined by \mathcal{S} . Then*

(1) $E_p = \cap \{\ker dg_p \mid g \in \mathcal{S}'\}$, for each $p \in M$,

(2) *if $g_1, g_2, \dots, g_{\rho-r}$ generate \mathcal{S}' at $p \in M$, then there is a neighborhood U of p such that the map $x \rightarrow (g_1(x), g_2(x), \dots, g_{\rho-r}(x))$ is constant on each leaf of the foliation $E|U$ of U .*

We say that a subbundle E of TM is locally Hamiltonian iff $\ker(\omega) \subseteq E$ and for each $p \in M$ there is a neighborhood U of p such that $\Gamma(E|U)$ is spanned by vector fields X which satisfy $df = i_X \omega$ for some $f \in C_{\text{loc}}^\infty(\omega)$.

PROPOSITION 2.8. *An integrable subbundle E is the foliation determined by some function group iff E is locally Hamiltonian. Moreover, the function group which determines such an E is unique.*

Proof. Clearly if E is determined by some function group, then E is locally Hamiltonian.

Conversely, suppose that E is locally Hamiltonian and consider the set \mathcal{S} of all local integrals of E . We now show that \mathcal{S} is a function group and that E is determined by the reciprocal, \mathcal{S}' , of \mathcal{S} . Let $f, g \in \mathcal{S}$, $p \in M$, and $X \in \Gamma(E)$. There is no loss of generality in assuming that there is an $H \in C_{\text{loc}}^\infty(\omega)$ such that $2\omega(X, \cdot) = dH(\cdot)$ in a neighborhood of p . It follows that

$$\begin{aligned} d\{f, g\}(X) &= L_{X_H}(\{f, g\}) = \{f, \{g, H\}\} + \{g, \{H, f\}\} \\ &= \{L_X g, f\} + \{L_X f, g\} = 0 \end{aligned}$$

by Proposition 1.5, the Jacobi identity, and the fact that $X \in \Gamma(E)$. Thus $\{f, g\} \in \mathcal{S}$ and it follows that \mathcal{S} is a function group with constant rank. Since $\mathcal{S} = \mathcal{S}''$ it follows from Corollary 2.7 that

$$E = \cap \{\ker df \mid f \in \mathcal{S}'' = \mathcal{S}\}.$$

REMARK. If \mathcal{S} is any function group then \mathcal{S} determines a unique integrable locally Hamiltonian subbundle E of TM and conversely. If E is determined by \mathcal{S} then the reciprocal of \mathcal{S} is precisely the set of all local integrals of E . If E is an integrable locally Hamiltonian subbundle of TM then the set of all local inte-

grals of E is a function group. The foliation determined by the reciprocal of this function group is precisely E .

Let \mathcal{S} be a function group of rank r . We say that a set $S \subseteq C^\infty(M)$ globally generates \mathcal{S} provided that for each $p \in M$ there exist functions $f_1, f_2, \dots, f_r \in S$ and a neighborhood U of p such that $\{f_1|U, f_2|U, \dots, f_r|U\}$ generates \mathcal{S} at p . We say that a set $T \subseteq \Gamma^*$ of closed 1-forms globally generates \mathcal{S} provided that for each $p \in M$ there exist forms $\beta_1, \dots, \beta_r \in T$, a neighborhood U of p , functions f_1, f_2, \dots, f_r satisfying $df_i = \beta_i$ on U for $i = 1, 2, \dots, r$ such that $\{f_1|U, f_2|U, \dots, f_r|U\}$ generates \mathcal{S} at p .

PROPOSITION 2.9. *Suppose that there exist closed 1-forms $\beta_1, \beta_2, \dots, \beta_n$ in Γ_ω^* and $r > 0$ such that*

- (i) $\beta_1(p), \beta_2(p), \dots, \beta_n(p)$ span an r -dimensional subspace of M_p^* for each $p \in M$,
- (ii) *there exist functions $a_{ijk} \in C^\infty(M)$ such that*

$$\{\beta_i, \beta_j\} = \sum_{k=1}^n a_{ijk} \beta_k .$$

Then there exists a unique function group \mathcal{S} of rank r which is globally generated by $\{\beta_1, \beta_2, \dots, \beta_n\}$. Conversely, if \mathcal{S} is a function group of rank r which is globally generated by $\beta_1, \beta_2, \dots, \beta_n$ then conditions (i) and (ii) are satisfied.

Proof. The details of this proof are much like those of Theorem 2.2 and are left to the reader.

Recall that $\text{inv}(\Gamma_\omega^*)$ is a Lie algebra under $\{ , \}$. Observe that if $\alpha_1, \alpha_2, \dots, \alpha_n$ are elements of $\text{inv}(\Gamma_\omega^*)$ they span a finite dimensional subalgebra of $\text{inv}(\Gamma_\omega^*)$ iff

$$\{\alpha_i, \alpha_j\} = \sum_k c_{ijk} \alpha_k$$

for constants $c_{ijk} \in \mathbf{R}$.

We now give an application of function groups which is a slight generalization of certain well-known theorems.

THEOREMS 2.10. *Let M be a symplectic manifold ($\rho = m = 2N$) and \mathcal{S} a function group of rank r on M . Suppose that the closed 1-forms $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ globally generate \mathcal{S} and that they span an n -dimensional subalgebra \mathcal{L} of $\text{inv}(\Gamma_\omega^*) = \Gamma(T^*M)$. If the vector field X_{α_i} is complete for each $i = 1, 2, \dots, n$, then each leaf of the foliation determined by \mathcal{S} is diffeomorphic to a homogeneous space*

G/H where G is the unique simply connected Lie group with Lie algebra \mathcal{L} and H is a closed subalgebra of G .

Proof. This is a consequence of a well-known theorem due to Palais [11] (see also Loos [10]). The details of the proof of Theorem 2.10 are similar to those of Theorem 1 of [2].

REMARK. Note that if we take $r = 2N - 1$ we obtain a part of Theorem 1 of Andrié and Simms [2]. Note that if we take $r = N$ and assume that \mathcal{L} is commutative we obtain a part of a theorem of Arnold [1] in which the leaves of the foliation turn out to be cylinders or tori (see, for example, Abraham [1, page 113]).

3. Invariant metrics and transverse structures. Let M be a connected C^∞ -manifold of dimension m and let E be an integrable subbundle of TM of dimension r . The normal bundle TM/E of E will be denoted by Q and its dual Q^* will be identified with the bundle E^0 where, for each $x \in M$, E_x^0 is the annihilator of E_x in T_x^*M , i.e.,

$$E_x^0 = \{\beta \in T_x^*M \mid \beta(E_x) = 0\}.$$

Define a connection ∇^* on $\Gamma(E^0)$ along the leaves of E by $\nabla_X^* \beta = L_X \beta$ for $\beta \in \Gamma(E^0)$ and $X \in \Gamma(E)$.

Observe that if f is any local integral of E then $\nabla_X^*(df) = L_X(df) = df(X) = 0$ and thus df is covariant constant along leaves of E . Also, if f_1, f_2, \dots, f_{m-r} are independent local integrals of E defined on an open set $U \subseteq M$, then $df_1, df_2, \dots, df_{m-r}$ span E^0 on U .

LEMMA 3.1. *If $\beta \in \Gamma(E^0)$ is closed, then β is parallel along the leaves of E , i.e., $\nabla_X^* \beta = 0$ for all $X \in \Gamma(E)$.*

Proof. $\nabla_X^* \beta = L_X \beta = (i_X d)\beta + (di_X)\beta = 0$ for all $X \in \Gamma(E)$ and $\beta \in \Gamma(E^0)$.

COROLLARY 3.2. *If $\beta_1, \beta_2, \dots, \beta_{m-r}$ are global, independent, closed elements of $\Gamma(E^0)$, then E^0 is parallelizable, i.e., it has $m-r$ global, independent, parallel sections.*

If σ is a Riemannian metric on M , then Q may be identified with the orthogonal complement of E in TM . Let $\sigma_Q = \sigma|_{(Q \times Q)}$ be the induced metric on Q . If $\beta \in \Gamma(E^0)$, then $\text{grad } \beta$ is that unique vector field in $\Gamma(Q)$ such that

$$\sigma(\text{grad } \beta, \cdot) = \beta$$

and, for $\xi \in \Gamma(Q)$, β_ξ is that element of $\Gamma(E^0)$ defined by

$$\beta_\xi = \sigma(\xi, \cdot)$$

We define the dual connection ∇ of ∇^* to be that connection on $\Gamma(Q)$ along leaves of E such that

$$\nabla_X(\xi) = \text{grad}(\nabla_X^* \beta_\xi)$$

for $X \in \Gamma(E)$ and $\xi \in \Gamma(Q)$. Another connection $\tilde{\nabla}$ for $\Gamma(Q)$ along the leaves of E is defined by

$$\tilde{\nabla}_X(\xi) = [L_X \xi]_Q$$

where $X \in \Gamma(E)$, $\xi \in \Gamma(Q)$ and where $[Y]_Q$ denotes the component of Y in Q .

LEMMA 3.3. *If σ_Q is invariant with respect to $\tilde{\nabla}$ then $\tilde{\nabla} = \nabla$.*

Proof. For $\xi, \eta \in \Gamma(Q)$ we have: $(\nabla_X^* \beta_\xi)(\eta) = (L_X \beta_\xi)(\eta) = i_\eta(L_X \beta_\xi) = L_X(i_\eta \beta_\xi) - i_{[X, \eta]}(\beta_\xi) = L_X(\sigma_Q(\xi, \eta)) - \sigma(\xi, [X, \eta]) = [\sigma_Q(\tilde{\nabla}_X \xi, \eta) + \sigma_Q(\xi, \tilde{\nabla}_X \eta)] - \sigma(\xi, [X, \eta])_Q = \sigma_Q(\tilde{\nabla}_X \xi, \eta)$. Thus $\tilde{\nabla}_X \xi = \text{grad}(\nabla_X^* \beta_\xi) = \nabla_X \xi$.

We say that σ is invariant when σ_Q is invariant with respect to the connection $\tilde{\nabla}$ in which case $\nabla = \tilde{\nabla}$. Observe that a metric σ satisfies this property iff it is "bundle-like" in the sense of Reinhart [12]. Also the connection $\tilde{\nabla}_X$ can be defined for all $X \in \Gamma(TM)$ in such a way that $\tilde{\nabla}$ is a "basic connection" (see Conlon [5]). Moreover the last result is a reflection of the fact that restrictions of basic connections to $\Gamma(E)$ are unique.

LEMMA 3.4. *If σ is an invariant metric, then β is parallel with respect to ∇^* iff $\text{grad} \beta$ is parallel with respect to $\tilde{\nabla}$.*

Proof. It is a standard result that β is ∇^* -parallel iff $\text{grad} \beta$ is parallel relative to the dual connection ∇ (see [7], Vol. II, page 342). Since $\nabla = \tilde{\nabla}$ the result follows.

REMARK. If σ is an invariant metric the usual one-to-one correspondence between $\Gamma(Q)$ and $\Gamma(E^0)$ induces a one-to-one correspondence between $\tilde{\nabla}$ -parallel sections of Q and ∇^* -parallel sections to E^0 .

REMARK. If ξ and η are $\tilde{\nabla}$ -parallel along leaves of E then the invariance of σ implies that $\sigma(\xi, \eta)$ is an integral of E . Thus if β is a closed element of $\Gamma^*(E)$ we conclude that $\sigma(\text{grad} \beta, \text{grad} \beta)$ is constant on leaves of E . If σ is complete as well as invariant then the vector field

$$\frac{1}{\sigma(\text{grad } \beta, \text{grad } \beta)} \cdot \text{grad } \beta$$

is a complete vector field for nonvanishing closed β in $\Gamma(E^0)$.

The foliation E is *transversally* parallelizable iff there exist $m-r$ independent elements of ΓQ each of which is $\tilde{\nu}$ -parallel along the leaves of E .

THEOREM 3.5. *Suppose there exist $m-r$ everywhere independent closed 1-forms $\beta_1, \beta_2, \dots, \beta_{m-r}$ such that*

$$\beta_i(\Gamma(E)) = 0 \quad \text{for } i = 1, 2, \dots, m-r.$$

Then E is transversally parallelizable.

Proof. If we show that there exists an invariant metric on E , then the theorem will be a consequence of Lemmas 3.1 and 3.4. Let Q be the orthogonal complement of E in TM relative to an arbitrary Riemannian τ on TM . Define σ on TM by

$$\sigma = \tau | (E \times E) \oplus \sum_{i=1}^{m-r} (\beta_i \otimes \beta_i).$$

Clearly σ is a Riemannian on TM . We show that σ is invariant. First observe that for $\xi, \eta \in \Gamma(Q)$ and $X \in \Gamma(E)$,

$$L_X(\sigma_Q(\xi, \eta)) = \sum_{i=1}^{m-r} L_X(\beta_i(\xi)\beta_i(\eta)) = \sum_{i=1}^{m-r} [\beta_i(\xi)L_X(\beta_i(\eta)) + \beta_i(\eta)L_X(\beta_i(\xi))].$$

But

$$\begin{aligned} L_X(\beta_i(\eta)) &= L_X(i_\eta \beta_i) = i_{[X, \eta]} \beta_i + i_\eta(L_X \beta_i) \\ &= \beta_i([X, \eta]) + i_\eta([i_X d + di_X](\beta_i)) = \beta_i([X, \eta]_Q) = \beta_i(\tilde{\nu}_X(\eta)). \end{aligned}$$

Thus

$$\begin{aligned} L_X(\sigma_Q(\xi, \eta)) &= \Sigma[\beta_i(\xi)\beta_i(\tilde{\nu}_X(\eta)) + \beta_i(\eta)\beta_i(\tilde{\nu}_X(\xi))] \\ &= \sigma_Q(\xi, \tilde{\nu}_X \eta) + \sigma_Q(\tilde{\nu}_X \xi, \eta) \end{aligned}$$

as required. The theorem follows.

REMARK. In the proof of the preceding theorem we have introduced a new metric $\sigma = \tau |_E \oplus \sum_{i=1}^{m-r} (\beta_i \otimes \beta_i)$. Observe that the orthogonal complement of E relative to σ is the same as for τ , namely Q . The gradient vector fields of the 1-forms $\beta_1, \beta_2, \dots, \beta_{m-r}$ with respect to this metric are parallel along the leaves of E . In the following we will use these vector fields without specific refe-

rences to the metric σ . Thus $\text{grad } \beta_i$ is the unique section of Q satisfying

$$(3.6) \quad \sum_{j=1}^{m-r} \beta_j(\text{grad } \beta_i)\beta_j(Y) = \beta_i(Y)$$

for all $Y \in \Gamma(Q)$.

We make a few remarks regarding completeness. First note that if the metric τ is complete then the metric σ will also be complete if there exist numbers l and L such that

$$l\tau_p(X_p, X_p) \leq \sum_{i=1}^{m-r} \beta_i(X_p)^2 \leq L\tau_p(X_p, X_p)$$

for all $p \in M$ and $X \in \Gamma(Q)$. If this is the case then the vector fields $[1/\beta_i(\text{grad } \beta_i)] \text{grad } \beta_i$ are complete vector fields. In any case (assuming τ is complete) the vector fields $\text{grad } \beta_i$ will be complete if they are bounded in the metric τ . Moreover, in this case, every linear combination in the $\text{grad } \beta_i$ is complete.

COROLLARY 3.7. *If in addition to the hypothesis of Theorem 3.5 we require that every linear combination of the vector fields $\text{grad } \beta_i$ (see 3.6) be complete, then*

- (1) *any two leaves of E are diffeomorphic and if any leaf of E is closed in M they all are,*
- (2) *if E admits a closed leaf then there is a fibre bundle $p: M \rightarrow N$ where N is parallelizable and E is the foliation of M whose leaves are the fibres of p .*

Proof. The corollary follows immediately from Theorem 3.5 above and Propositions 4.3 and 4.4 of Conlon [5].

We now apply the results of this section to function groups.

As an example consider the case where M is symplectic and suppose there is a Hamiltonian function $H \in C^\infty(M)$ such that $dH(p) \neq 0$ for each $p \in M$. Clearly $\{H\}$ globally generates a function group \mathcal{H} of rank 1. This leads to a foliation E which is generated by the unique Hamiltonian vector field $X_H = X_{dH}$. The reciprocal function group \mathcal{H}' , which consists of all local integrals of E , also determines a foliation E' . Thus by Theorem 3.5, E' is transversally parallelizable. Indeed, if the vector field $\text{grad}(dH)$ is complete then each two leaves of E' are diffeomorphic. Also since the leaves are the components of the level surfaces of H , they are closed and hence, by Corollary 3.7, they fibre M over a parallelizable manifold. The following theorem generalizes this example where

M is not necessarily symplectic and the grad β_i are as defined by 3.6.

THEOREM 3.8. *Let ω be a closed 2-form of constant rank ρ on M . Let $\beta_1, \beta_2, \dots, \beta_r$ be closed 1-forms which globally generate a function group \mathcal{S} of rank r . Then the foliation E' determined by the reciprocal function group of \mathcal{S} is transversally parallelizable. Moreover, if every linear combination of the vector fields grad β_i , $i = 1, 2, \dots, r$ is complete then E' is a complete transversally parallelizable foliation and each two leaves of E' are diffeomorphic. Furthermore, if one of the leaves of E' is closed then they all are and M is a fibre bundle over a parallelizable manifold in which the fibres are the leaves of E' .*

Proof. The theorem is an immediate consequence of what it means for $\{\beta_1, \beta_2, \dots, \beta_r\}$ to globally generate \mathcal{S} , Theorem 3.5 and Corollary 3.7.

REMARK. Suppose that in the above theorem we have $r = \rho$. In this case $E' = \ker \omega$. Moreover, if some leaf L of the foliation E' is closed then the manifold M is fibered by $\pi: M \rightarrow N$ where $\pi^{-1}(x) \cong L$, for each x , and N , the manifold of leaves of $\ker \omega$, is a symplectic manifold. This is true since $N_p \cong \mathbb{Q}_p$ and $\omega|_{(\mathbb{Q} \times \mathbb{Q})}$ is nondegenerate.

REMARK. If in the above theorem $r = 1$, then E' is a foliation of codimension 1 and thus by [5, Proposition 5.1] we conclude that either every leaf of E' is closed or else every leaf of E' is dense in M .

REMARK. If in addition to the hypothesis of the above theorem we assume that the 1-forms $\beta_1, \beta_2, \dots, \beta_r$ are exact, then there exist functions H_1, H_2, \dots, H_r such that $dH_i = \beta_i$ and the leaves of E' , being components of level surfaces of $H_i = h_i$, are necessarily closed. Thus we see that if the functions $\{H_1, H_2, \dots, H_r\}$ globally generate a function group of rank r and every linear combination of the grad (H_i) is complete then each two components of the level surfaces $H_i = h_i$ are diffeomorphic.

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