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INTEGRALS OF POLYLOGARITHMIC FUNCTIONS, RECURRENCE RELATIONS, AND ASSOCIATED EULER SUMS

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ABSTRACT. We show that integrals of the form

$$\int_0^1 x^m \mathrm{Li}_p(x)\mathrm{Li}_q(x)dx \quad (m \geq -2, p, q \geq 1)$$

and

$$\int_0^1 \frac{\log^r(x)\mathrm{Li}_p(x)\mathrm{Li}_q(x)}{x}dx \quad (p, q, r \geq 1)$$

satisfy certain recurrence relations which allow us to write them in terms of Euler sums. From this we prove that, in the first case for all m, p, q and in the second case when $p+q+r$ is even, these integrals are reducible to zeta values. In the case of odd $p+q+r$, we combine the known results for Euler sums with the information obtained from the problem in this form to give an estimate on the number of *new* constants which are needed to express the above integrals for a given weight $p+q+r$.

The proofs are constructive, giving a method for the evaluation of these and other similar integrals, and we present a selection of explicit evaluations in the last section.

1. INTRODUCTION

The evaluation of series of the form

$$S_{r^p,q} := \sum_{n=1}^{\infty} \frac{\left[H_n^{(r)}\right]^p}{n^q}, \text{ where } H_n^{(r)} := \sum_{k=1}^{n} \frac{1}{k^r}, \quad H_n \equiv H_n^{(1)},$$

dates back to Euler (see the introductory note in [BBG]) and has recently been the object of active research [BA, BB, BBG, C, D, FS]. Apart from the actual evaluation of the series, one of the main questions that one sets out to solve is whether or not a given series can be expressed in terms of a linear rational combination of known constants. When this is the case, we say that the series is reducible to these values.

The purpose of this paper is to carry out a similar analysis in the case of integrals involving polylogarithmic functions and which can be related to Euler sums of the above type. We then use these relations to establish whether or not these integrals are reducible to known constants and to evaluate them.

Within this scope, a typical result that can be obtained by using the techniques in this paper is the following theorem.

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Theorem 1. *Let p, q and m be integers with $p, q \geq 1$ and $m \geq -2$. Then the integrals*

$$J(m,p,q) := \int_0^1 x^m \mathrm{Li}_p(x)\mathrm{Li}_q(x)dx$$

are reducible to a rational constant and zeta values at positive integers.

As usual, we have denoted by Li_p the polylogarithmic function defined for $|x| < 1$ by

$$\mathrm{Li}_p(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^p}.$$

A second result is

Theorem 2. *Let p, q and r be integers such that $p, q \geq 0$ with at least one of them positive, $r \geq 1$ and $w = p + q + r$ even. Then the integrals*

$$K(r,p,q) := \int_0^1 \frac{\log^r(x)\mathrm{Li}_p(x)\mathrm{Li}_q(x)}{x}dx$$

are reducible to zeta values at positive integers.

In the case of odd weight w, these integrals are reducible to zeta values at positive integers and at most $\lfloor (w-1)/6 \rfloor$ other constants which can be taken from the set

$$\mathcal{S}_w = \left\{ \kappa_{2j-1,p+q+r-2j+1}, j = 1, \ldots, \lfloor (w+1)/4 \rfloor \right\},$$

where $\kappa_{rq} := K(r,0,q)/r!$.

Remark 1. For low odd weights ($w < 7$), it is still possible to reduce the integrals only to zeta values, since the integrals mentioned above will also be reducible. See the discussion following Theorem 3.1 and also the examples in Section 4, where the cases of weight 3 and 5 are given. There we also include the case of weight 7 where all integrals can be written in terms of $K(1,0,6)$.

Remark 2. We actually prove that if one considers the set of zeta values at positive integers together with the constants in the set $\mathcal{S}_w$ above, then a specific integral $K(r,p,q)$ can be reduced to zeta values and at most $\lfloor (r+1)/2 \rfloor$ elements of the set $\mathcal{S}_w$ in the case of r smaller than or equal to $(w-1)/2$ and at most $\lfloor (w-r)/2 \rfloor$ elements of $\mathcal{S}_w$ otherwise.

Although in a completely different spirit from the rest of the paper, we also give a very simple approximate expression for the elements of the set $\mathcal{S}_w$. More precisely, we have

Theorem 3. *The constants κ_{rq} satisfy the following estimate*

$$\left|\kappa_{rq} - (-1)^r \zeta(q) \left[\zeta(r+1) - 1\right]\right| \leq \frac{\zeta(q-1) - \zeta(q)}{2^{r+1}}, \quad r \geq 1\,, q \geq 3.$$

Integrals of the type considered here have been studied in the literature either in connection with the general study of polylogarithms [L] or due to the role they play in calculations involving higher order Feynman diagrams, relativistic phase space integrals for scattering processes under some restrictions, etc., [DD, GT]. In particular, the connection between some of these integrals and Euler series had

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already been explored in [GT], although there the authors restricted their study to integrals containing only logarithms such as

$$\int_0^1 \frac{\log^m(x)\log^n(1-x)\log^p(1+x)}{a(x)}dx,$$

where $a(x) = x, 1-x$, or $1+x$, and m, n and p are integers. On the other hand, this line of research had also been suggested in [SC], where the authors point out the relation between linear Euler sums and integrals of the type of $K(r,0,q)$ above; see Remark 3 on page 157 of [SC] and Lemma 2.1 below.

In the present paper, and although we do give explicit evaluations (see Theorems 3.1, 3.2 and 3.3, Corollary 3.4 and Lemma 3.8), the emphasis is on the determination of a set of recurrence relations that is satisfied by the integrals. We believe that this point of view has certain advantages, as in many cases of this type the expressions for the integrals quickly get quite large, and while getting a general closed expression may be quite cumbersome, it is quite simple to write down a set of rules that can then be run on a software package such as Maple or Mathematica and which will then make it possible to evaluate any specific integral that is needed. Of course in order to proceed in this way, one then needs to carry out the explicit evaluation of the *initial conditions* for these relations. One of the ingredients in this last step in some of the cases considered is the relations already mentioned between the integrals and certain Euler sums.

Proceeding in a similar fashion to the evaluation of the integrals K and J, it is possible to evaluate other integrals involving polylogarithms, and we shall now mention some further examples. These include, for instance, certain multiple integrals related to the sums $S_{1^p,q}$ and integrals of the form

$$L(m,r,p) := \int_0^1 x^m \log^r(x)\mathrm{Li}_p(x)dx, \quad m \geq -1,$$

for which it is quite straightforward to obtain that

$$L(-1,r,p) = (-1)^r r!\zeta(p+r+1)$$

and then to use recurrence relations in a similar fashion to what was done for J to show that the remaining cases can also be reduced to zeta values at positive integers plus a rational constant.

The plan of the paper is as follows. In Section 2 we introduce the relations between the integrals and Euler sums. These are then used to evaluate some integrals directly in Section 3, where we also establish the recurrence relations which are needed to prove Theorems 1 and 2. This is done in Section 3.3, and the estimates in Theorem 3 are proven in Section 3.4. In the last section, we present a collection of some explicit evaluations. Finally, in the Appendix we give an example where the procedure is reversed and where we use the corresponding integral to evaluate the Euler sum $S_{1^2,2}$.

2. Identities between integrals of polylogarithms and Euler sums

We begin by establishing some relations between Euler sums and certain integrals. Although it is possible to obtain more general identities by proceeding in the same fashion, for simplicity we have chosen to highlight only some possibilities. The first is a known relation between linear Euler sums and $K(r-1,0,q)$ which can be found, for instance, in [SC, Proposition 3.7]. For completeness, and since

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this will play an important role in what follows, we provide a (different) proof of this result.

Lemma 2.1. *For $q, r \geq 2$ we have that*

$$S_{r,q} = \zeta(r)\zeta(q) - \frac{(-1)^{r-1}}{(r-1)!}\int_0^1 \frac{\log^{r-1}(x)\mathrm{Li}_q(x)}{1-x}dx.$$

Proof. From the fact that

$$\int_0^1 x^{k-1}\log^{r-1}(x)dx = (-1)^{r-1}\frac{(r-1)!}{k^r} \quad (k \geq 1, r \geq 2), \tag{1}$$

we have that

$$\sum_{k=1}^{n} \frac{1}{k^r} = \frac{(-1)^{r-1}}{(r-1)!}\int_0^1 \log^{r-1}(x)\frac{1-x^n}{1-x}dx,$$

and thus

$$\begin{aligned} S_{r,q} &= \frac{(-1)^{r-1}}{(r-1)!}\int_0^1 \frac{\log^{r-1}(x)}{1-x}\sum_{n=1}^{\infty}\frac{1-x^n}{n^q}dx \\ &= \frac{(-1)^{r-1}}{(r-1)!}\int_0^1 \frac{\log^{r-1}(x)}{1-x}\left[\zeta(q) - \mathrm{Li}_q(x)\right]dx \\ &= \zeta(r)\zeta(q) - \frac{(-1)^{r-1}}{(r-1)!}\int_0^1 \frac{\log^{r-1}(x)}{1-x}\mathrm{Li}_q(x)dx, \end{aligned}$$

where the last step follows from (see identity 4.271 in [GR], for instance)

$$\int_0^1 \frac{\log^{r-1}(x)}{1-x}dx = (-1)^{r-1}\zeta(r)\Gamma(r). \tag{2}$$ □

For the higher-order Euler sums $S_{1^p,q}$ we have the following relation.

Lemma 2.2. *For $p \geq 1$ and $q \geq 2$ we have that*

$$S_{1^p,q} = (-1)^p \int_{Q_p} \frac{\mathrm{Li}_{q-p}(x_1\cdots x_p)\prod_{j=1}^{p}\log(1-x_j)}{x_1\cdots x_p}dx,$$

where $Q_p = (0,1)^p$.

Proof. Since $H_n = -n\int_0^1 y^{n-1}\log(1-y)dy$, we have that

$$\begin{aligned} S_{1^p,q} &= \sum_{n=1}^{\infty}\frac{(-1)^p n^p\left[\int_0^1 y^{n-1}\log(1-y)dy\right]^p}{n^q} \\ &= (-1)^p\int_{Q_p}\sum_{n=1}^{\infty}\frac{(x_1\cdots x_p)^{n-1}}{n^{q-p}}\log(1-x_1)\cdots\log(1-x_p)dx \\ &= (-1)^p\int_{Q_p}\frac{\mathrm{Li}_{q-p}(x_1\cdots x_p)\log(1-x_1)\cdots\log(1-x_p)}{x_1\cdots x_p}dx \end{aligned}$$

as desired. □

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Although this result will be used mainly to evaluate the integrals based on the knowledge of the value of the series, as was mentioned in the Introduction, in the Appendix we show how this can be yet another way of evaluating sums such as $S_{1^2,2}$.

3. Reduction of integrals involving polylogarithms

3.1. **Integrals related directly to $S_{r^p,q}$.** We begin by presenting a simple example which is a direct consequence of Lemma 2.1 and the result in [BBG] for the case of $S_{r,q}$.

Theorem 3.1. *For even weight $w = q + r$ $(q \geq 2, r \geq 1)$ the integral*

$$K(r,0,q) = \int_0^1 \frac{\log^r(x)\mathrm{Li}_q(x)}{1-x}dx$$

is reducible to zeta values. More precisely,

$$\begin{aligned} K(r,0,q) = r!\Bigg\{&(-1)^{r+1}\zeta(w+1)\left[\frac{1}{2} - \frac{(-1)^{r+1}}{2}\binom{w}{r+1} - \frac{(-1)^{r+1}}{2}\binom{w}{q}\right] \\ &+ \frac{(-1)^r - 1}{2}\zeta(r+1)\zeta(q) \\ &+ \sum_{k=1}^{\lfloor (r+1)/2 \rfloor}\binom{w-2k}{q-1}\zeta(2k)\zeta(w-2k+1) \\ &+ \sum_{k=1}^{\lfloor q/2 \rfloor}\binom{w-2k}{r}\zeta(2k)\zeta(w-2k+1)\Bigg\}, \end{aligned}$$

where $\zeta(1)$ should be interpreted as 0 whenever it occurs.

Proof. This follows directly from Lemma 2.1 and the fact that $S_{r,q}$ is reducible to zeta values for $q + r$ even [BBG]. Here we used the expression for $S_{r,q}$ given in [FS]. $\square$

When w is odd, it will no longer be possible in general to reduce the integrals above to zeta values. This is in line with the discussion on page 17 of [FS], for instance, which states that there are *general* and *exceptional* classes of evaluations, in the sense that although there are families of sums which are not reducible to zeta values, some low weights are still reducible. Thus, in the case of low odd weights we may still obtain some explicit expressions for the above integrals such as

$$\begin{aligned} \int_0^1 \frac{\log(x)\mathrm{Li}_2(x)}{1-x}dx &= -\frac{3}{4}\zeta(4), \\ \int_0^1 \frac{\log^2(x)\mathrm{Li}_3(x)}{1-x}dx &= \zeta^2(3) - \zeta(6), \\ \int_0^1 \frac{\log(x)\mathrm{Li}_4(x)}{1-x}dx &= \zeta^2(3) - \frac{25}{12}\zeta(6). \end{aligned}$$

These may be derived from the expressions for the linear sums $S_{2,2}$, $S_{3,3}$ and $S_{2,4}$ given in [FS].

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For other odd weights, and as is mentioned in [FS], it is possible to obtain linear relations between different integrals. Here we give one example obtained using the corresponding relation on page 23 of [FS] (see also Table 9):

$$\int_0^1 \frac{\log^2(x)\mathrm{Li}_5(x)}{1-x}dx - 5\int_0^1 \frac{\log(x)\mathrm{Li}_6(x)}{1-x}dx = \frac{163}{12}\zeta(8) - 8\zeta(3)\zeta(5).$$

Another example is obtained by taking p equal to 2 in Lemma 2.2, leading to a double integral which can then be reduced to a quadratic Euler sum and thus to linear sums and zeta values. This last step follows from the result in [BBG] regarding the reducibility of the sum $S_{1^2,q}$ to linear sums and polynomials in zeta values for all weights. In the formulation of the result below, we used the expression for $S_{1^2,q}$ given by Theorem 4.1 in [FS]. Note that there is a misprint in the statement of Theorem 4.1 in [FS] where the residue term is missing.

Theorem 3.2. *For any $q \geq 0$ we have that*

$$\int_{Q_2} \frac{\mathrm{Li}_q(xy)\log(1-x)\log(1-y)}{xy}dxdy$$
$$= S_{2,q+2} + (q+2)S_{1,q+3} - \frac{(q+2)(q+3)}{6}\zeta(q+2) + \zeta(2)\zeta(q+2) - \frac{1}{3}R(q),$$

where

$$R(q) = \mathrm{Res}_{z=0}\left[\frac{(\psi(-z)+\gamma)^3}{z^{q+2}}\right].$$

Here ψ is the logarithmic derivative of the Gamma function, and γ is Euler's constant.

This gives, for instance,

$$\int_{Q_2} \frac{\mathrm{Li}_3(xy)\log(1-x)\log(1-y)}{xy}dxdy = -\frac{5}{2}\zeta(3)\zeta(4) - \zeta(2)\zeta(5) + 6\zeta(7).$$

We shall now consider the case of $J(-1,p,q)$, which may also be related directly to an Euler sum and which will be used in the proof of Theorem 1 for the remaining values of m.

Theorem 3.3. *For $p, q \geq 1$ the integral $J(-1,p,q)$ is reducible to zeta values. More precisely, for $p \geq q$, we have*

$$J(-1,p,q) = (-1)^{q+1}\left(1+\frac{p+q}{2}\right)\zeta(p+q+1) + 2(-1)^q$$
$$\times \sum_{j=1}^{\lfloor q/2 \rfloor} \zeta(2j)\zeta(p+q-2j+1) + \frac{(-1)^q}{2}\sum_{j=1}^{p-q}\zeta(j+q)\zeta(p-j+1).$$

A similar expression can be found for $p < q$.

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Proof. Since $J(-1,p,q) = J(-1,q,p)$, we may, without loss of generality, assume that $p \geq q$. Writing $m = p - q \geq 0$ and integrating by parts $q-1$ times gives that

$$\begin{aligned}\int_0^1 \frac{\mathrm{Li}_{q+m}(x)\mathrm{Li}_q(x)}{x}dx &= \sum_{j=1}^{q-1}(-1)^{(j+1)}\zeta(q+m+j)\zeta(q-j+1)\\ &\qquad +(-1)^{q+1}\int_0^1 \frac{\mathrm{Li}_{2q+m-1}(x)\mathrm{Li}_1(x)}{x}dx\\ &= \sum_{j=1}^{q-1}(-1)^{(j+1)}\zeta(q+m+j)\zeta(q-j+1)\\ &\qquad +(-1)^{q+1}S_{1,2q+m},\end{aligned}$$

where the last step follows from Lemma 2.2. Using the expression for $S_{1,2q+m}$ (already known to Euler), we get

$$\begin{aligned}\int_0^1 \frac{\mathrm{Li}_{q+m}(x)\mathrm{Li}_q(x)}{x}dx &= \sum_{j=1}^{q-1}(-1)^{(j+1)}\zeta(q+m+j)\zeta(q-j+1)\\ &\quad + (-1)^{q+1}\Bigg[(1+\frac{2q+m}{2})\zeta(2q+m+1)\\ &\qquad - \frac{1}{2}\sum_{j=1}^{2q+m-2}\zeta(j+1)\zeta(2q+m-j)\Bigg]\\ &= (-1)^{q+1}(q+\frac{m}{2}+1)\zeta(2q+m+1)\\ &\quad + 2(-1)^q\sum_{j=1}^{\lfloor q/2\rfloor}\zeta(2j)\zeta(2(q-j)+m+1)\\ &\quad + \frac{(-1)^q}{2}\sum_{j=1}^{m}\zeta(j+q)\zeta(q+m-j+1). \qquad \square\end{aligned}$$

Letting $p = q$ gives the following

Corollary 3.4.

$$\int_0^1 \frac{\mathrm{Li}_p^2(x)}{x}dx = (-1)^{p+1}(p+1)\zeta(2p+1) + 2(-1)^p\sum_{j=1}^{\lfloor p/2\rfloor}\zeta(2j)\zeta[2(p-j)+1].$$

3.2. **Recurrence relations.** We shall now develop the recurrence relations that will allow us to reduce the integrals $J(m,p,q)$ and $K(r,p,q)$.

Lemma 3.5. *For $p, q \geq 2$ and $m \geq 0$, the following recurrence relations hold:*

(i) $J(m,p,q) = \dfrac{\zeta(p)\zeta(q)}{m+1} - \dfrac{1}{m+1}\left[J(m,p-1,q) + J(m,p,q-1)\right].$

(ii)

$$\left.\begin{aligned}J(m,1,q) =& \frac{\zeta(q)}{m+1} - \frac{1}{m+1}\left[mJ_0(m,q) + J_0(m,q-1)\right]\\ &+ \frac{1}{m+1}[mJ(m-1,1,q)\\ &\qquad + J(m-1,1,q-1) - J(m,1,q-1)],\end{aligned}\right\}$$

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where $J_0(m,q) := \int_0^1 x^m \mathrm{Li}_q(x)dx$.

(iii) $J_0(m,q) = \dfrac{\zeta(q)}{m+1} - \dfrac{1}{m+1} J_0(m,q-1)$.

Proof. The proof of the three relations follows by integration by parts, in the first and third cases using the integral of x^m and in the second case using the integral of $\log(1-x)$. □

Remark 3. Note that the first relation in the above lemma also holds in the case of $m=-2$. Also, in the case where $m=0$, (ii) simplifies to

$$J(0,1,q) = \zeta(q) + \sum_{k=2}^{q-1} (-1)^{q+k}\zeta(k) - (-1)^q - J(0,1,q-1) + J(-1,1,q-1),$$

where we have used the fact that

$$\int_0^1 \mathrm{Li}_q(x)dx = (-1)^{q-1} + \sum_{k=2}^{q} (-1)^{k+q}\zeta(k).$$

Successive application of these relations in the case of nonnegative m leads to expressions containing only the integrals $J(-1,1,q)$, $J(m,1,1)$ and $J_0(m,1)$; note that $J(0,p,q) = J(0,q,p)$. For $m=-2$, we cannot apply the second recurrence in the lemma, and it becomes necessary to be able to evaluate $J(-2,1,q)$ separately. Thus, in order for it to be possible to actually use the relations in Lemma 3.5 to evaluate $J(m,p,q)$, we have to determine the values of the above four integrals.

The integral $J(-1,1,q)$ is given by Theorem 3.3.

In the case of $J_0(m,1)$ we have from the proof of Lemma 2.2 that

$$-\int_0^1 x^m \log(1-x)dx = \frac{1}{m+1} H_{m+1},$$

while for $J(m,1,1)$ identity (4.2.5) in [DD] gives that

$$\int_0^1 x^m \log^2(1-x)dx = \frac{2}{m+1}\left[H_{m+1}^{(2)} + \sum_{k=1}^{m+1} \frac{H_k}{k+1}\right].$$

It remains to determine $J(-2,1,q)$. This can again be done by means of a recurrence relation, which relates these integrals to $J(-1,1,q-1)$.

Lemma 3.6. *For $q \geq 2$ the following recurrence relation holds:*

$$J(-2,1,q) = \zeta(q+1) + J(-2,1,q-1) - J(-1,1,q-1).$$

Furthermore, $J(-2,1,1) = 2\zeta(2)$.

Proof. One again proceeds by integration by parts, but now starting from $J(-1,1,q)$ and using the integral of $\log(1-x)$. The terms in $J(-1,1,q)$ cancel out, and the result follows. The value of $J(-2,1,1)$ can be found in [DD], for instance. □

For the case of K we need two recurrence relations. The first is similar to those used in the previous case and is again based on integration by parts.

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Lemma 3.7. *For positive integers p, q and r we have*

$$K(r,p,q) = -\frac{1}{r+1}\left[K(r+1,p-1,q) + K(r+1,p,q-1)\right].$$

Proof. Integrate by parts, using the integral of $\log^r(x)/x$. □

The second recursion is based on the following symmetry relation for linear sums:

$$S_{r,q} + S_{q,r} = \zeta(r)\zeta(q) + \zeta(r+q). \tag{3}$$

Lemma 3.8. *For $r \geq 1, q \geq 2$ we have that*

$$K(r,0,q) = (-1)^{r+q}\frac{r!}{(q-1)!}K(q-1,0,r+1) + (-1)^r r!\left[\zeta(r+1)\zeta(q) - \zeta(r+q+1)\right].$$

In particular.

$$K(r,0,r+1) = \int_0^1 \frac{\log^r(x)\mathrm{Li}_{r+1}(x)}{1-x}dx = (-1)^{r+1}\frac{r!}{2}\left[\zeta(2r+2) - \zeta^2(r+1)\right].$$

Proof. The first expression is just a *translation* of the symmetry relations (3) into the case of integrals using Lemma 2.1. Applying this expression to the case of $q = r+1$ gives the second statement. □

3.3. **Proof of Theorems 1 and 2.** By combining the above results, it is now possible to prove Theorems 1 and 2.

Proof of Theorem 1. As was mentioned above, the application of the recurrence relations in Lemma 3.5 allows us to reduce $J(m,p,q)$ to rational combinations of zeta values and integrals of the type $J(m,1,1), J_0(m,1)$ and $J(0,1,q)$. The first two of these reduce to rational numbers (see identities (4.2.4) and (4.2.5) in [DD], for instance), while the third can, by Lemma 3.6, be written in terms of an integer, zeta values, and $J(-1,1,q)$, which is, in turn, reducible to zeta values by Theorem 3.3. □

Proof of Theorem 2. By Lemma 3.7 we have that the weight $w = p+q+r$ does not change as we successively apply the recurrence relation. This, together with the fact that $K(r,p,q) = K(r,q,p)$, allows us to reduce the integral $K(r,p,q)$ to integrals of the form $K(r',0,q')$, with $r'+q' = p+q+r$. By Theorem 3.1, we have that these integrals are reducible if $r'+q'$ is even and $q' \geq 2$. On the other hand, the case $K(r,0,1)$ is reducible for all r (see [K], for instance, and the references therein), and so the result follows.

In the case of odd weights, by applying directly the results from [BBG, Z], we get the $\lfloor (w-1)/6 \rfloor$ bound.

We now use the following version of Lemma 3.7

$$K(r,p,q) = -rK(r-1,p,q+1) - K(r,p-1,q+1).$$

By applying this recursion repeatedly, we see that it is possible to reduce the original integral to integrals of the form $K(0,p',q') = J(-1,p',q')$ (reducible by Theorem 1) and integrals $K(i,0,p+q+r-i)$ with $i = 1,\ldots,r$.

We shall now prove that when i is even, $K(i,0,q')$ can be reduced to integrals where the first entry is $i-1$. To see this, note that for even i we have $K(i,0,q')$ with q' odd. If we apply the above recurrence to $K(i,q',0)$ repeatedly, we get

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a linear combination of integrals with the first entry $i-1$, plus one of the form $K(i,p'+1,p') = K(i,p',p'+1)$. Aplying the recursion once more to this (or by letting $p=q=p'+1$ in Lemma 3.7), we get

$$K(i,p',p'+1) = -\frac{i}{2}K(i-1,p'+1,p'+1)$$

which enables us to reduce this last term to one with odd first entry as desired.

Finally, the relations in Lemma 3.8 allow us to reduce any $K(r,0,q)$ with $q>r$ to one with $q<r-2$. $\square$

3.4. **Estimates for the integrals** $K(r,0,q)$**.** We begin by estimating the Euler sums S_{rq}, and then we obtain the result for κ_{rq} from these. We have

$$\begin{aligned} S_{rq} &= \sum_{n=1}^{\infty} \frac{1}{n^q}\left(1+\frac{1}{2^r}+\cdots+\frac{1}{n^r}\right) \\ &\leq \zeta(q) + \frac{1}{2^r}\sum_{n=1}^{\infty}\frac{n-1}{n^q} \\ &\leq \zeta(q) + \frac{1}{2^r}\left[\zeta(q-1)-\zeta(q)\right], \end{aligned}$$

and so

$$0 \leq S_{rq} - \zeta(q) \leq \frac{1}{2^r}\left[\zeta(q-1)-\zeta(q)\right].$$

Theorem 3 now follows by combining this with Lemma 2.1.

4. Some examples

We begin by considering the integrals $J(m,p,q)$, for $-2 \leq m \leq 1$ and $2 \leq p$, $q \leq 4$.

Table 1. $J(-2,p,q)$, $2 \leq p,q \leq 4$.

$$\begin{aligned}
\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x^2}dx &= 4\zeta(2) - 2\zeta(3) - \frac{5}{2}\zeta(4), \\
\int_0^1 \frac{\mathrm{Li}_2(x)\mathrm{Li}_3(x)}{x^2}dx &= 6\zeta(2) - 3\zeta(3) - \frac{11}{4}\zeta(4) - \zeta(2)\zeta(3), \\
\int_0^1 \frac{\mathrm{Li}_2(x)\mathrm{Li}_4(x)}{x^2}dx &= 8\zeta(2) - 4\zeta(3) - 3\zeta(4) - 2\zeta(5) - \frac{7}{4}\zeta(6), \\
\int_0^1 \frac{\mathrm{Li}_3^2(x)}{x^2}dx &= 12\zeta(2) - 6\zeta(3) - \frac{11}{2}\zeta(4) - 2\zeta(2)\zeta(3) - \zeta^2(3), \\
\int_0^1 \frac{\mathrm{Li}_3(x)\mathrm{Li}_4(x)}{x^2}dx &= 20\zeta(2) - 10\zeta(3) - \frac{17}{2}\zeta(4) - 2\zeta(5) - 2\zeta(2)\zeta(3) \\
&\quad - \frac{7}{4}\zeta(6) - \zeta^2(3) - \zeta(3)\zeta(4), \\
\int_0^1 \frac{\mathrm{Li}_4^2(x)}{x^2}dx &= 40\zeta(2) - 20\zeta(3) - 17\zeta(4) - 4\zeta(5) - 4\zeta(2)\zeta(3) \\
&\quad - \frac{7}{2}\zeta(6) - 2\zeta^2(3) - 2\zeta(3)\zeta(4) - \frac{7}{6}\zeta(8).
\end{aligned}$$

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TABLE 2. $J(-1,p,q)$, $2 \le p, q \le 4$.

$$\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}dx = 2\zeta(2)\zeta(3) - 3\zeta(5),$$

$$\int_0^1 \frac{\mathrm{Li}_2(x)\mathrm{Li}_3(x)}{x}dx = \frac{1}{2}\zeta^2(3),$$

$$\int_0^1 \frac{\mathrm{Li}_2(x)\mathrm{Li}_4(x)}{x}dx = 2\zeta(2)\zeta(5) + \zeta(3)\zeta(4) - 4\zeta(7),$$

$$\int_0^1 \frac{\mathrm{Li}_3^2(x)}{x}dx = -2\zeta(2)\zeta(5) + 4\zeta(7),$$

$$\int_0^1 \frac{\mathrm{Li}_3(x)\mathrm{Li}_4(x)}{x}dx = \frac{7}{12}\zeta(8),$$

$$\int_0^1 \frac{\mathrm{Li}_4^2(x)}{x}dx = 2\zeta(4)\zeta(5) + 2\zeta(2)\zeta(7) - 5\zeta(9).$$

TABLE 3. $J(0,p,q)$, $2 \le p, q \le 4$.

$$\int_0^1 \mathrm{Li}_2^2(x)dx = 6 - 2\zeta(2) - 4\zeta(3) + \frac{5}{2}\zeta(4),$$

$$\int_0^1 \mathrm{Li}_2(x)\mathrm{Li}_3(x)dx = -10 + 4\zeta(2) + 5\zeta(3) - \frac{15}{4}\zeta(4) + \zeta(2)\zeta(3),$$

$$\int_0^1 \mathrm{Li}_2(x)\mathrm{Li}_4(x)dx = 15 - 7\zeta(2) - 5\zeta(3) + 4\zeta(4) - 3\zeta(5) + \frac{7}{4}\zeta(6),$$

$$\int_0^1 \mathrm{Li}_3^2(x)dx = 20 - 8\zeta(2) - 10\zeta(3) - \frac{15}{2}\zeta(4) - 2\zeta(2)\zeta(3) + \zeta^2(3),$$

$$\begin{aligned}\int_0^1 \mathrm{Li}_3(x)\mathrm{Li}_4(x)dx = {}& -35 + 15\zeta(2) + 15\zeta(3) - \frac{23}{2}\zeta(4) + 3\zeta(5) + 2\zeta(2)\zeta(3) \\ & - \zeta^2(3) - \frac{7}{4}\zeta(6) + \zeta(3)\zeta(4),\end{aligned}$$

$$\begin{aligned}\int_0^1 \mathrm{Li}_4^2(x)dx = {}& 70 - 30\zeta(2) - 30\zeta(3) + 23\zeta(4) - 6\zeta(5) - 4\zeta(2)\zeta(3) \\ & + \frac{7}{2}\zeta(6) + 2\zeta^2(3) - 2\zeta(3)\zeta(4) + \frac{7}{6}\zeta(8).\end{aligned}$$

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TABLE 4. $J(1,p,q)$, $2 \le p,q \le 4$.

$$\int_0^1 x\mathrm{Li}_2^2(x)dx = \frac{25}{16} - \frac{3}{4}\zeta(2) - \zeta(3) + \frac{5}{4}\zeta(4),$$

$$\int_0^1 x\mathrm{Li}_2(x)\mathrm{Li}_3(x)dx = -\frac{47}{32} + \frac{7}{8}\zeta(2) + \frac{3}{8}\zeta(3) - \frac{15}{16}\zeta(4) + \frac{1}{2}\zeta(2)\zeta(3),$$

$$\int_0^1 x\mathrm{Li}_2(x)\mathrm{Li}_4(x)dx = \frac{173}{128} - \frac{31}{32}\zeta(2) + \frac{3}{16}\zeta(3) + \frac{1}{4}\zeta(4) - \frac{3}{4}\zeta(5) + \frac{7}{8}\zeta(6),$$

$$\int_0^1 x\mathrm{Li}_3^2(x)dx = \frac{47}{32} - \frac{7}{8}\zeta(2) - \frac{3}{8}\zeta(3) + \frac{15}{16}\zeta(4) - \frac{1}{2}\zeta(2)\zeta(3) + \frac{1}{2}\zeta^2(3),$$

$$\begin{aligned}\int_0^1 x\mathrm{Li}_3(x)\mathrm{Li}_4(x)dx = &-\frac{361}{256} + \frac{59}{64}\zeta(2) + \frac{3}{32}\zeta(3) - \frac{19}{32}\zeta(4) + \frac{1}{4}\zeta(2)\zeta(3)\\ &+ \frac{3}{8}\zeta(5) - \frac{7}{16}\zeta(6) - \frac{1}{4}\zeta^2(3) + \frac{1}{2}\zeta(3)\zeta(4),\end{aligned}$$

$$\begin{aligned}\int_0^1 x\mathrm{Li}_4^2(x)dx = &\frac{361}{256} - \frac{59}{64}\zeta(2) - \frac{3}{32}\zeta(3) + \frac{19}{32}\zeta(4) - \frac{1}{4}\zeta(2)\zeta(3)\\ &- \frac{3}{8}\zeta(5) + \frac{7}{16}\zeta(6) + \frac{1}{4}\zeta^2(3) - \frac{1}{2}\zeta(3)\zeta(4) + \frac{7}{12}\zeta(8).\end{aligned}$$

We now give some values of the integrals $K(r,p,q)$ for $3 \le p+q+r \le 7$.

TABLE 5. $K(r,p,q)$, $p+q+r=3$.

$$\int_0^1 \frac{\log(x)\log^2(1-x)}{x}dx = -\frac{1}{2}\zeta(4),$$

$$\int_0^1 \frac{\log(x)\mathrm{Li}_2(x)}{1-x}dx = -\frac{3}{4}\zeta(4),$$

$$\int_0^1 \frac{\log^2(x)\log(1-x)}{1-x}dx = -\frac{1}{2}\zeta(4).$$

TABLE 6. $K(r,p,q)$, $p+q+r=4$.

$$\int_0^1 \frac{\log(x)\mathrm{Li}_3(x)}{1-x}dx = 2\zeta(2)\zeta(3) - \frac{9}{2}\zeta(5),$$

$$\int_0^1 \frac{\log(x)\log(1-x)\mathrm{Li}_2(x)}{x}dx = \zeta(2)\zeta(3) - \frac{9}{6}\zeta(5),$$

$$\int_0^1 \frac{\log^2(x)\mathrm{Li}_2(x)}{1-x}dx = 6\zeta(2)\zeta(3) - 11\zeta(5),$$

$$\int_0^1 \frac{\log^2(x)\log^2(1-x)}{x}dx = -4\zeta(2)\zeta(3) + 8\zeta(5),$$

$$\int_0^1 \frac{\log^3(x)\log(1-x)}{1-x}dx = -6\zeta(2)\zeta(3) + 12\zeta(5).$$

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TABLE 7. $K(r,p,q)$, $p+q+r=5$.

$$\int_0^1 \frac{\log(x)\mathrm{Li}_4(x)}{1-x}dx = -\frac{25}{12}\zeta(6)+\zeta^2(3),$$
$$\int_0^1 \frac{\log(x)\log(1-x)\mathrm{Li}_3(x)}{x}dx = -\frac{1}{3}\zeta(6)+\frac{1}{2}\zeta^2(3),$$
$$\int_0^1 \frac{\log(x)\mathrm{Li}_2^2(x)}{x}dx = -\frac{1}{3}\zeta(6),$$
$$\int_0^1 \frac{\log^2(x)\mathrm{Li}_3(x)}{1-x}dx = -\zeta(6)+\zeta^2(3),$$
$$\int_0^1 \frac{\log^2(x)\log(1-x)\mathrm{Li}_2(x)}{x}dx = -\frac{1}{3}\zeta(6),$$
$$\int_0^1 \frac{\log^3(x)\mathrm{Li}_2(x)}{1-x}dx = 8\zeta(6)-6\zeta^2(3),$$
$$\int_0^1 \frac{\log^3(x)\log^2(1-x)}{x}dx = -9\zeta(6)+6\zeta^2(3),$$
$$\int_0^1 \frac{\log^4(x)\log(1-x)}{1-x}dx = -18\zeta(6)+12\zeta^2(3).$$

TABLE 8. $K(r,p,q)$, $p+q+r=6$.

$$\int_0^1 \frac{\log(x)\mathrm{Li}_5(x)}{1-x}dx = 2\zeta(3)\zeta(4)+4\zeta(2)\zeta(5)-10\zeta(7),$$
$$\int_0^1 \frac{\log(x)\log(1-x)\mathrm{Li}_4(x)}{x}dx = \zeta(3)\zeta(4)+3\zeta(2)\zeta(5)-6\zeta(7),$$
$$\int_0^1 \frac{\log(x)\mathrm{Li}_2(x)\mathrm{Li}_3(x)}{x}dx = \zeta(2)\zeta(5)-2\zeta(7),$$
$$\int_0^1 \frac{\log^2(x)\mathrm{Li}_4(x)}{1-x}dx = 2\zeta(3)\zeta(4)+20\zeta(2)\zeta(5)-36\zeta(7),$$
$$\int_0^1 \frac{\log^2(x)\log(1-x)\mathrm{Li}_3(x)}{x}dx = 14\zeta(2)\zeta(5)-24\zeta(7),$$
$$\int_0^1 \frac{\log^2(x)\mathrm{Li}_2^2(x)}{x}dx = 12\zeta(2)\zeta(5)-20\zeta(7),$$
$$\int_0^1 \frac{\log^3(x)\mathrm{Li}_3(x)}{1-x}dx = 60\zeta(2)\zeta(5)-102\zeta(7),$$
$$\int_0^1 \frac{\log^3(x)\log(1-x)\mathrm{Li}_2(x)}{x}dx = -18\zeta(2)\zeta(5)+30\zeta(7),$$
$$\int_0^1 \frac{\log^4(x)\mathrm{Li}_2(x)}{1-x}dx = 120\zeta(2)\zeta(5)+48\zeta(3)\zeta(4)-264\zeta(7),$$

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TABLE 8. $K(r,p,q)$, $p+q+r=6$ (continued).

$$\int_0^1 \frac{\log^4(x)\log^2(1-x)}{x}dx = -48\zeta(2)\zeta(5) - 48\zeta(3)\zeta(4) + 144\zeta(7),$$
$$\int_0^1 \frac{\log^5(x)\log(1-x)}{1-x}dx = -120\zeta(2)\zeta(5) - 120\zeta(3)\zeta(4) + 360\zeta(7).$$

Table 9 presents the values of the integrals $K(r,0,q)$, with $r+q=7$. In this case, all values can be reduced to zeta values at integers plus a new constant, which we took to be $\kappa_{16} = K(1,0,6) \approx -0.651565$. Theorem 3 yields

$$|\kappa_{16} - \zeta(6)\,[1-\zeta(2)]| \leq \frac{\zeta(5)-\zeta(6)}{4} \approx 0.0049.$$

TABLE 9. $K(r,0,q)$, $q+r=7$.

$$\int_0^1 \frac{\log(x)\mathrm{Li}_6(x)}{1-x} = \kappa_{16},$$
$$\int_0^1 \frac{\log^2(x)\mathrm{Li}_5(x)}{1-x} = \frac{163}{12}\zeta(8) + 5\kappa_{16} - 8\zeta(3)\zeta(5),$$
$$\int_0^1 \frac{\log^3(x)\mathrm{Li}_4(x)}{1-x} = -\frac{\zeta(8)}{2},$$
$$\int_0^1 \frac{\log^4(x)\mathrm{Li}_3(x)}{1-x} = -187\zeta(8) - 60\kappa_{16} + 120\zeta(3)\zeta(5),$$
$$\int_0^1 \frac{\log^5(x)\mathrm{Li}_2(x)}{1-x} = -80\zeta(8) - 120\kappa_{16},$$
$$\int_0^1 \frac{\log^6(x)\log(1-x)}{1-x} = 720\zeta(3)\zeta(5) - 900\zeta(8).$$

APPENDIX A. THE EULER SUM $S_{1^2,2}$ AND A RELATED DOUBLE INTEGRAL

The Euler sum
$$\sum_{n=1}^{\infty} \frac{H_n^2}{(n+1)^2}$$
was shown in [BB] to be equal to the integral
$$\frac{1}{\pi}\int_0^{\pi} t^2 \log^2\left[2\cos(\frac{t}{2})\right] dt,$$
by using an appropriate Fourier series and then Parseval's identity; for other ways of evaluating this series, see [BA, C, D, FS]. The above integral was then evaluated via contour integration (see also Section 7.9 in [L], especially 7.9.10), and was used to establish the value of $S_{1^2,2}$. Here we use Lemma 2.2 to transform the series $S_{1^2,2}$ into a double integral which we then evaluate directly.

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Proposition A.1.
$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2=\frac{17\zeta(4)}{4}.$$

Proof. By Lemma 2.2 we have that
$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2=\int_0^1\int_0^1\frac{\log(1-x)\log(1-y)}{1-xy}dxdy.$$
On the other hand,
$$\begin{aligned}\int_0^1\int_0^1\frac{\log(1-x)\log(1-y)}{1-xy}dxdy &= \int_0^1\log(1-y)\int_0^1\frac{\log(1-x)}{1-xy}dxdy\\ =-\int_0^1\log(1-y)&\left[\frac{\log(1-x)\log\left(\frac{xy-1}{y-1}\right)+\mathrm{Li}_2\left[\frac{y(1-x)}{y-1}\right]}{y}\right]_0^1 dy\\ =\int_0^1\frac{\log(1-y)}{y}&\mathrm{Li}_2\left(\frac{y}{y-1}\right)dy.\end{aligned}$$
Since, from [L], we have that
$$\mathrm{Li}_2\left(\frac{y}{y-1}\right)=-\frac{1}{2}\log^2(1-y)-\mathrm{Li}_2(y),$$
the above integral is equal to
$$\begin{aligned}&-\int_0^1\frac{\log(1-y)}{y}\left[\frac{1}{2}\log^2(1-y)+\mathrm{Li}_2(y)\right]\\ &\quad=-\frac{1}{2}\int_0^1\frac{\log^3(1-y)}{y}-\int_0^1\frac{\log(1-y)}{y}\mathrm{Li}_2(y)dy.\end{aligned}$$
From (2) it follows that the first of these integrals is equal to $-6\zeta(4)$. In the case of the second integral, the integrand is the derivative of $-\mathrm{Li}_2^2(y)/2$, and so we get $-10\zeta(4)/4$. Substituting this back, we obtain the desired result. □

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