S. TANNO
KODAI MATH. J.
13 (1990), 204—209

INTEGRALS OF SOME TRIGONOMETRIC FUNCTIONS

BY SHUKICHI TANNO

1. Introduction

Let a and b be positive numbers, and u, v and m be positive integers such that $u+v=m$, $2\leq m$. We define $I(m)$ and $I(u;v)$ by

$$I(m)=\int_0^\infty \frac{\sin^m at}{t^m}dt, \qquad I(u;v)=\int_0^\infty \frac{\sin^u at\sin^v bt}{t^m}dt.$$

Tables of integrals give values of $I(m)$ and $I(u;v)$ only for some special cases. For example, for $3\leq m\leq 6$, Gradshteyn and Ryzhik [4] (p. 449-p. 450) gives

$$I(3)=3a^2\pi/8, \qquad I(4)=a^3\pi/3,$$
$$I(5)=115a^4\pi/384, \qquad I(6)=11a^5\pi/40.$$

As for $I(u;v)$ with $a<b$, [4] (p. 451-p. 452) gives

$$I(2;2)=(3b-a)a^2\pi/6 \qquad (a<b),$$
$$I(3;1)=a^3\pi/2 \qquad (0<3a\leq b)$$
$$=[24a^3-(3a-b)^3]\pi/48 \qquad (0<a<b\leq 3a),$$
$$I(1;3)=(9b^2-a^2)a\pi/24 \qquad (a<b).$$

In this note we give the general expressions of $I(m)$ and $I(u;v)$. These are special cases of Theorem A below. To state our Theorem A we need the following definition. Let $a_1, a_2, \cdots, a_m$ be positive numbers such that $0<a_1\leq a_2\leq \cdots \leq a_{m-1}\leq a_m$. For a subset $\lambda=\{k_1, k_2, \cdots, k_{m-r}\}$ of $\{1, 2, \cdots, m-1\}$, a polynomial

$$P_r(\lambda)=a_{k_1}+a_{k_2}+\cdots+a_{k_{m-r}}-a_{k_{m-r+1}}-\cdots-a_{k_{m-1}}-a_m$$

is said to be of *r-type*, if $\{a_{k_1}, a_{k_2}, \cdots, a_{k_{m-1}}, a_m\}=\{a_1, a_2, \cdots, a_m\}$ as sets and

$$P_r(\lambda)>0, \quad k_1<k_2<\cdots<k_{m-r}, \quad k_{m-r+1}<\cdots<k_{m-1}.$$

Note that a_m appears with negative sign and r is the number of negative signs contained in a polynomial of r-type. A polynomial of 1-type is unique if it exists.

Received November 28, 1989.

THEOREM A. *For constants $1<a_1\leq a_2\leq \cdots \leq a_{m-1}\leq a_m$, $2\leq m$, the following holds:*

$$\frac{2}{\pi}\int_0^{\infty}\frac{1}{t^m}\Big(\prod_{k=1}^{m}\sin a_k t\Big)dt=\prod_{k=1}^{m-1}a_k-\frac{1}{c'}\sum_{r=1}^{m-2}(-1)^{r-1}\sum_{r\text{-}type}P_r(\lambda)^{m-1}, \tag{1.1}$$

where $c'=(m-1)!\,2^{m-2}$ and $P_r(\lambda)$ denotes a polynomial of r-type.

If $a_m>a_1+a_2+\cdots+a_{m-1}$, then there is no polynomial of r-type $(r\geq 1)$, and so the above integral does not depend on a_m.

For proof of Theorem A we use the volume expression of cube-slicing by Hensley [5] and Ball [1], and actually we give the volume of cube-slicing in terms of numbers a_i (after normalization) using polynomials of r-type by elementary geometric method.

A special case of Theorem A for $m=3$ is given, for example, at p. 79 of Erdélyi [3] and at p. 422 of Gradshteyn and Ryzhik [4]. A special case where there is no polynomial of r-type $(r\geq 1)$ is given at p. 417 of [4].

Three special cases of Theorem A are given as Corollaries C, D and E in § 3. From these one can deduce many analogous formulas. Only several examples are given in Corollary F.

The author is grateful to Professor N. Kurokawa, Professor S. Saitoh and Professor N. Suita for their kind informations.

2. The volume of cube-slicing

By $\{e_j,\ 1\leq j\leq m\}$ we denote the standard base of the Euclidean m-space E^m at the origen O, and by $\{x^j\}$ the standard coordinate system of E^m. Let K^m be the unit cube in E^m, which is expressed by $\{x\,;\,0\leq x^j\leq 1,\ 1\leq j\leq m\}$. Let $b_1, b_2, \cdots, b_m$ be positive numbers such that

$$b_1\geq b_2\geq \cdots \geq b_m>0.$$

Let T be an m-simplex determined by $\{O, p_1, p_2, \cdots, p_m\}$, where p_j is the end point of the vector b_je_j, $1\leq j\leq m$.

LEMMA. *The volume $V(K^m\cap T)$ of the intersection of K^m and T is given by*

$$V(K^m\cap T)=\frac{b_1b_2\cdots b_m}{m!}\sum_{r=0}^{m}(-1)^r\sum_{k_1<k_2<\cdots<k_r}^{*}\Big(1-\frac{1}{b_{k_1}}-\cdots-\frac{1}{b_{k_r}}\Big)^m,$$

where $\sum^*$ denotes the sum over all positive terms $(1-1/b_{k_1}-\cdots-1/b_{k_r}>0)$.

Proof. (i) If $b_1\leq 1$, then the m-simplex T is contained in K^m and $V(K^m\cap T)=V(T)=b_1b_2\cdots b_m/m!$.

(ii) If $b_1>1\geq b_2$, then only one vertex p_1 of T lies outside K^m. The intersection $T\cap(K^m)^C$ of T and the complement $(K^m)^C$ of K^m in E^m defines an m-

simplex T_1, which we call the outer simplex at p_1. $V(K^m \cap T)$ is given by the difference of the volume $V(T)$ and the volume $V(T_1)$ of the outer simplex at p_1, and

$$V(K^m \cap T)=(b_1 b_2 \cdots b_m/m!)[1-(1-1/b_1)^m].$$

(iii) If $b_s>1\geq b_{s+1}$, for $2\leq s\leq m$ (putting $b_{m+1}=0$), then vertices $p_1, p_2, \cdots, p_s$ of T lie outside K^m. In this case the outer simplex T_h at p_h, $1\leq h\leq s$, is defined by $T_h=T\cap\{x; x^h\geq 1\}$.

(iii-1) If the outer simplexes at $p_1, p_2, \cdots, p_s$ are disjoint, then $V(K^m\cap T)=V(T)-V(T_1)-V(T_2)-\cdots-V(T_s)$, where $V(T_h)=(b_1b_2\cdots b_m/m!)(1-1/b_h)^m$, $1\leq h\leq s$.

(iii-2) Two outer simplexes at p_1 and p_2 have a non trivial intersection T_{12}, if and only if $1-1/b_1-1/b_2>0$, which is equivalent to the fact that the vertex $(1, 1, 0, \cdots, 0)$ of K^m lies below the affine hyperplane determined by the face of T opposite to O. The volume $V(T_{12})$ of T_{12} is equal to $(b_1b_2\cdots b_m/m!)(1-1/b_1-1/b_2)^m$. Let $\{T_{12}, T_{13}, T_{23}, \cdots, T_{jh}\}$ be the set of all non trivial intersections of two outer simplexes. If three outer simplexes at p_1, p_2, and p_3 do not have non trivial intersecton T_{123} (or equivalently, $1-1/b_1-1/b_2-1/b_3\leq 0$), then $V(K^m\cap T)$ is given by $V(T)-V(T_1)-V(T_2)-\cdots-V(T_s)+V(T_{12})+\cdots+V(T_{jh})$.

(iii-3) If T_{123} is non trivial, then we need the term $-V(T_{123})=-(b_1b_2\cdots b_m/m!)(1-1/b_1-1/b_2-1/b_3)^m$ in the expression of $V(K^m\cap T)$.

(iii-4) Generally, the intersection $T_{t_1t_2\cdots t_u}$ of $T_{t_1}, T_{t_2}, \cdots, T_{t_u}$ is non trivial, if and only if $1-1/b_{t_1}-1/b_{t_2}-\cdots-1/b_{t_u}>0$, and its volume has sign $(-1)^u$ in the expression of $V(K^m\cap T)$. q. e. d.

Let B^m be the unit cube in E^m centered at the origin: $B^m=\{x; -1/2\leq x^j\leq 1/2, 1\leq j\leq m\}$. Let $a=(a_1, a_2, \cdots, a_m)$ be a unit vector in E^m and $H(a)$ be the hyperplane passing through the origin and orthogonal to a. Since the case where $a_1=0$ reduces to the lower dimensional case, we assume that the components of a satisfy

$$0<a_1\leq a_2\leq \cdots \leq a_{m-1}\leq a_m .$$

Concerning the volume $V_m(a)$ of the slice $B^m\cap H(a)$ corresponding to a, Hensley [5] and Ball [1] gave the best possible inequality; $1\leq V_m(a)\leq\sqrt{2}$, which was verified by using the following expression of $V_m(a)$:

$$V_m(a)=\frac{1}{\pi}\int_{-\infty}^{\infty}\prod_{k=1}^{m}\frac{\sin a_k t}{a_k t}dt . \tag{2.1}$$

Since (1.1) is homothetically invariant with respect to (a_i), to prove Theorem A it suffices to give the value of the left hand side of (2.1). Namely we prove the following.

PROPOSITION B. *The volume $V_m(a)$ of the slice corresponding to a is given by*

$$V_m(a)=\frac{1}{a_m}-\frac{1}{C}\sum_{r=1}^{m-2}(-1)^{r-1}\sum_{r\text{-}type}P_r(\lambda)^{m-1}, \tag{2.2}$$

where $c=(m-1)!\,2^{m-2}a_1a_2\cdots a_m$ *and* $P_r(\lambda)$ *denotes a polynomial of r-type.*

Proof. We define θ by $\cos\theta=\langle e_m, a\rangle=a_m$. By ρ we denote the orthogonal projection of E^m onto E^{m-1} defined by $x_m=0$. First we study the case where $a_m>a_1+a_2+\cdots+a_{m-1}$. The condition $a_m>a_1+a_2+\cdots+a_{m-1}$ is equivalent to the fact that $H(a)$ does not meet the upper face F^{m-1} of B^m defined by $x_m=1/2$. Therefore, $\rho(B^m\cap H(a))=B^{m-1}$. Hence, $V_m(a)=1/\cos\theta=1/a_m$.

Next we assume that $a_m<a_1+a_2+\cdots+a_{m-1}$ holds. Then $H(a)$ meets the upper face F^{m-1}. We denote the part of F^{m-1} which lies below $H(a)$ by $K(a)$. Then the volume $V(K(a))$ of $K(a)$ is given by the preceding Lemma. The relation between (b_k) and (a_k) is given by $b_k=(a_1+a_2+\cdots+a_{m-1}-a_m)/2a_k$, $1\le k\le m-1$. Consequently, we obtain the following:

$$1-1/b_k=(a_1+a_2+\cdots\check{a}_k\cdots+a_{m-1}-a_k-a_m)/A,$$

$$1-1/b_k-1/b_l=(a_1+a_2+\cdots\check{a}_k\cdots\check{a}_l\cdots+a_{m-1}-a_k-a_l-a_m)/A,$$

etc., where $A=a_1+a_2+\cdots+a_{m-1}-a_m$ and $\check{a}_k$ means that a_k is removed. Since $V_m(a)=[1-2V(K(a))]/\cos\theta$, we obtain (2.2). q. e. d.

3. Corollaries

First we give three special cases of Theorem A.

COROLLARY C. *For a positive number a and integer* $m\ge 2$, *the following holds.*

$$\int_0^\infty\frac{\sin^m at}{t^m}dt=\frac{a^{m-1}\pi}{(m-1)!\,2^{m-1}}\Big[(m-1)!\,2^{m-2}-\sum_{r=1}^{[(m-1)/2]}(-1)^{r-1}{}_{m-1}C_{r-1}(m-2r)^{m-1}\Big]. \tag{3.1}$$

COROLLARY D. *For positive numbers* $a<b$ *and positive integers* u, v $(u+v=m)$, *the following holds·*

$$\int_0^\infty\frac{\sin^u at\sin^v bt}{t^m}dt=\frac{\pi}{(m-1)!\,2^{m-1}}\Big[(m-1)!\,2^{m-2}a^ub^{v-1}-\sum_{r=1}^{m-2}(-1)^{r-1}\sum_{p=0}^{*}{}_uC_p\cdot{}_{v-1}C_{r-p-1}\{(u-2p)a+(v-2r+2p)b\}^{m-1}\Big], \tag{3.2}$$

where $\sum^*$ *denotes the sum over all polynomials* $(u-2p)a+(v-2r+2p)b>0$ *and p runs from* $\max\{r-v, 0\}$ *to* $\min\{r-1, u\}$.

COROLLARY E. *For positive numbers* $a<b<c$ *and positive integers* $u, v, w(u+v+w=m)$, *the following holds:*

$$(3.3)\quad \int_0^\infty \frac{\sin^u at \sin^v bt \sin^w ct}{t^m} dt = \frac{\pi}{(m-1)!\, 2^{m-1}} \Big[(m-1)!\, 2^{m-2} a^u b^v c^{w-1} - \sum_{r=1}^{m-2} (-1)^{r-1} \sum^*_{p+q+s=r-1} {}_uC_p \cdot {}_vC_q \cdot {}_{w-1}C_s \cdot P(p, q, s)^{m-1} \Big],$$

where $P(p, q, s) = (u-2p)a + (v-2q)b + (w-2s-2)c$, *and* $\sum^*$ *denotes the sum over all polynomials* $P(p, q, s) > 0$ *and* $0 \leq p \leq \min\{r-1, u\}$, $0 \leq q \leq \min\{r-1, v\}$ *and* $0 \leq s = r-1-p-q \leq w-1$.

Proof of Corollary C. The number of polynomials $a+a+\cdots+a-a$ of 1-type is one for $m \geq 3$. The number of polynomials $a+a+\cdots+a-a-a$ of 2-type is $m-1$ for $m \geq 5$. Similarly, the number of polynomials $a+a+\cdots+a-a-\cdots-a-a$ of r-type is ${}_{m-1}C_{r-1}$. The range of r is from 1 to $[(m-1)/2]$. Therefore, Corollary C follows from Theorem A.

Proof of Corollary D. The number of polynomials of 1-type is at most one; $a+a+\cdots+a+b+\cdots+b-b$ for $m \geq 3$. Each of polynomials of 2-type is one of the following;

$$a+a+\cdots+a+a+b+\cdots+b-b-b \qquad (ua+(v-4)b>0),$$

$$a+a+\cdots+a+b+\cdots+b+b-a-b \qquad ((u-2)a+(v-2)b>0).$$

The numbers of such polynomials are ${}_uC_0 \cdot {}_{v-1}C_1$ and ${}_uC_1 \cdot {}_{v-1}C_0$. By p we denote the number of a with negative sign in the polynomial of r-type. Polynomials of r-type for general r and the number of such polynomials are similarly studied.

Proof of Corollary E is similar.

Corollary C enables us to calculate $I(m)$ for any m, for example, we obtain

$$I(7) = 5887a^6\pi/23040, \qquad I(8) = 151a^7\pi/630,$$

$$I(9) = 259723a^8\pi/1146880, \qquad I(10) = 15619a^9\pi/72576.$$

Also Corollary D enables us to calculate $I(u; v)$ for any u, v, for example, we obtain

$$I(3; 4) = \frac{\pi}{6!\, 2^6}[6!\, 2^5 a^3 b^3 - (3a+2b)^6 + 3(3a)^6 + 3(a+2b)^6 - 3(3a-2b)^6 - 9a^6 - 3(-a+2b)^6] \qquad (2b \leq 3a < 3b)$$

$$= \frac{\pi}{6!\, 2^6}[6!\, 2^5 a^3 b^3 - (3a+2b)^6 + 3(3a)^6 + 3(a+2b)^6 - 9a^6 - 3(-a+2b)^6 + (-3a+2b)^6] \qquad (3a < 2b).$$

The formulas in Corollaries produce many analogous formulas. Here we give some examples. For convenience sake we use the following notations:

$$I_m(u\,;\,a)=\int_0^\infty \frac{\sin^u at}{t^m}dt,$$

$$I_m(u,\,v\,;\,a,\,b)=\int_0^\infty \frac{\sin^u at \sin^v bt}{t^m}dt,$$

$$I_m(u,\,v,\,w\,;\,a,\,b,\,c)=\int_0^\infty \frac{\sin^u at \sin^v bt \sin^w ct}{t^m}dt.$$

Then, for $m\geq 2$, we obtain the following:

COROLLARY F.

$$\int_0^\infty \frac{\sin^{m+2} at}{t^m}dt=I_m(m\,;\,a)-\frac{1}{4}I_m(m-2,\,2\,;\,a,\,2a), \tag{3.4}$$

$$\int_0^\infty \frac{\sin^m at \sin^2 bt}{t^m}dt=\frac{1}{2}I_m(m\,;\,a)-\frac{1}{2}I_m(m-2,\,2\,;\,a,\,a+b)$$
$$+\frac{1}{2}I_m(m-2,\,2\,;\,a,\,b)+\frac{1}{4}I_m(m-2,\,1,\,1\,;\,a,\,2a,\,2b), \tag{3.5}$$

$$\int_0^\infty \frac{\sin^{m+4} at}{t^m}dt=I_m(m+2\,;\,a)-\frac{1}{4}I_m(m,\,2\,;\,a,\,2a), \tag{3.6}$$

$$\int_0^\infty \frac{\sin^m at \cos^2 bt}{t^m}dt=I_m(m\,;\,a)-I_m(m,\,2\,;\,a,\,b), \tag{3.7}$$

$$\int_0^\infty \frac{\sin^m at \cos bt}{t^m}dt=I_m(m-1,\,1\,;\,a,\,a+b)-\frac{1}{2}I_m(m-2,\,1,\,1\,;\,a,\,2a,\,b), \tag{3.8}$$

etc.

In the above, (3.5) is somewhat complicated. If $m\geq 4$, then it has a simpler expression: $I_m(m-2,\ ;\,a,\,b)-(1/4)I_m(m-4,\,2,\,2\,;\,a,\,2a,\,b)$.

REFERENCES

[1] K. BALL, Cube slicing in R^n, Proc. Amer. Math. Soc., 97 (1986), 465-473.

[2] K. BALL, Some remarks on the geometry of convex sets, Lect. Notes in Math. (Springer) No. 1317 (1986-1987), 224-231.

[3] A. ERDÉLYI (Ed.), Tables of integral transforms, McGraw-Hill Book Comp. Inc. 1954.

[4] I.S. GRADSHTEYN AND I.M. RYZHIK, Table of Integrals, Series, and Products. Academic Press, 1980.

[5] D. HENSLEY, Slicing the cube in R^n and probability (Bounds for the measure of a central cube slice in R^n by probability methods), Proc. Amer. Math. Soc., 73 (1979), 95-100.

[6] D. HENSLEY, Slicing convex bodies (Bounds for slice area in terms of the bodies' covariances), Proc. Amer. Math. Soc., 79 (1980), 619-625.

DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY