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INTEGRALS OF SUBHARMONIC FUNCTIONS AND THEIR DIFFERENCES WITH WEIGHT OVER SMALL SETS ON A RAY

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Let E be a measurable subset in a segment $[0,r]$ in the positive part of the real axis in the complex plane, and $U=u-v$ be the difference of subharmonic functions $u\not\equiv-\infty$ and $v\not\equiv-\infty$ on the complex plane. An integral of the maximum on circles centered at zero of $U^+:=\sup\{0,U\}$ or $|u|$ over E with a function-multiplier $g\in L^p(E)$ in the integrand is estimated, respectively, in terms of the characteristic function T_U of U or the maximum of u on circles centered at zero, and also in terms of the linear Lebesgue measure of E and the L^p-norm of g. Our main theorem develops the proof of one of the classical theorems of Rolf Nevanlinna in the case $E=[0,R]$, given in the classical monograph by Anatoly A. Goldberg and Iossif V. Ostrovsky, and also generalizes analogs of the Edrei – Fuchs Lemma on small arcs for small intervals from the works of A. F. Grishin, M. L. Sodin (1988) and A. F. Grishin, T. I. Malyutina (2005). Our estimates are uniform in the sense that the constants in these estimates do not depend on U or u, provided that U has an integral normalization near zero or $u(0)\geqslant 0$, respectively.

1. Introduction.

1.1. Origins and research subject. One of the classical theorems of Rolf Nevanlinna can be considered as the original source of our results in this article [1, pp. 24–27]. Anatolii Asirovich Gol'dberg and Iosif Vladimiriovich Ostrovskii indicate the last reference in their classic monograph [2, Notes, Ch. 1]. The original source [1] remained inaccessible to us. But the mentioned theorem is stated with a complete proof in the monograph of A. A. Gol'dberg and I. V. Ostrovskii [2, Ch. 1, Theorem 7.2]. We give this result exactly in their formulation and notations.

Let f be a meromorphic function on the *complex plane* $\mathbb{C}$ with the *real axis* $\mathbb{R}$,

$$M(r,f):=\max\Bigl\{\bigl|f(z)\bigr|\colon |z|=r\Bigr\},\quad r\in\mathbb{R}^+:=\{x\in\mathbb{R}\colon x\geqslant 0\},\tag{1M}$$

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and with the *Nevanlinna characteristics*

$$T(r,f) \underset{r\in\mathbb{R}^+}{:=} m(r,f)+N(r,f), \tag{1T}$$

$$m(r,f) \underset{r\in\mathbb{R}^+}{:=} \frac{1}{2\pi}\int\limits_0^{2\pi} \ln^+\bigl|f(re^{i\varphi})\bigr|\,\mathrm{d}\varphi, \quad \ln^+ x \underset{x\in\mathbb{R}^+}{:=} \max\{\ln x, 0\}, \tag{1m}$$

$$N(r,f) \underset{r\in\mathbb{R}^+}{:=} \int\limits_0^{r} \frac{n(t,f)-n(0,f)}{t}\,\mathrm{d}t + n(0,f)\ln r, \tag{1N}$$

where $n(r,f)$ is the number of poles of f in the closed disc $\overline{D}(r) := \{z\in\mathbb{C}\colon |z|\leqslant r\}$, taking into account the multiplicity.

Rolf Nevanlinna Theorem ([2, Ch. 1, Theorem 7.2]). *Let $f(z)$ be a meromorphic function, $k>1$ be a real number. Then*

$$\frac{1}{r}\int\limits_0^{r} \ln^+ M(t,f)\,\mathrm{d}t \leqslant C(k)T(kr,f), \tag{2}$$

where the constant $C(k)>1$ depends on k only.

But [2, Ch. 1, proof of Theorem 7.2] uses [2, Chap. 1, Lemma 7.1] for $R' := \sqrt{k}r$, which was proved only for $R'>R>1$. It turns out to be essentially. So, for meromorphic function

$$f(z) \underset{z\in\mathbb{C}}{\equiv} \frac{1}{z}, \qquad M(r,f) \underset{r>0}{\overset{(1M)}{\equiv}} \frac{1}{r}, \tag{3M}$$

$$m(r,f) \underset{r>0}{\overset{(1m)}{\equiv}} \ln^+\frac{1}{r}, \quad N(r,f) \underset{r>0}{\overset{(1N)}{\equiv}} \ln r, \quad T(r,f) \underset{r>0}{\overset{(1T)}{\equiv}} \ln^+ r, \tag{3T}$$

for the left-hand side of (2), we have

$$\frac{1}{r}\int\limits_0^{r} \ln^+ M(t,f)\,\mathrm{d}t \underset{r>0}{\overset{(3M)}{\equiv}} \frac{1}{r}\int\limits_0^{r} \ln^+\frac{1}{t}\,\mathrm{d}t \underset{r>0}{\equiv} 1+\ln^+\frac{1}{r} \underset{r>0}{\geqslant} 1. \tag{3I}$$

Thus, if (2) is satisfied, then $C(k)\ln^+ kr\geqslant 1$. But this is impossible if $0\leqslant r\leqslant 1/k$. For the constant $C(k)$ independent of r, this is possible only if the requirement of the form $r\geqslant r_0>0$ is added to the inequality (2), and the constant $C(k)$ also depends from the choice of a fixed number $r_0>0$. For the sake of fairness, we note that the known cases of application of the Rolf Nevanlinna Theorem are considered, as a rule, only for the cases $r\to+\infty$ or $r\geqslant 1$.

Integrals over small subsets on arc or ray intervals are also widely used in the theory of entire and meromorphic functions. The starting point of these second-type estimates is the A. Edrei and W. H. J. Fuchs Lemma on Small Arcs [3, Sect. 2, Lemma III, Sect. 9], which has found important applications in the theory of meromorphic functions, reflected, in particular, in [2, Chap. 1, Theorems 7.3, 7.4]. A variation on the Edrei – Fuchs Lemma on Small Arcs is a Lemma on Small Intervals by A. F. Grishin and M. L. Sodin. The authors note that its proof repeats verbatim the proof of the Edrei – Fuchs Lemma on Small Arcs, and therefore it is presented in [4, Lemma 3.1] without proof, but with interesting applications [4, Lemma 3.2, Theorem 3.1]. We formulate the Grishin – Sodin Lemma on Small Intervals also in the literal translation of the author's version.

Denote by $\operatorname{mes} E$ the *linear Lebesgue measure* of $E \subset \mathbb{R}$, and $E(R) := E \cap [1, R)$.

Grishin – Sodin Lemma on Small Intervals ([4, Lemma 3.1]). *Let f be a meromorphic function on $\mathbb{C}$, $E \subset [1, +\infty)$. Then*

$$\frac{1}{r}\int\limits_{E(r)} \ln^+ M(t, f)\, \mathrm{d}t \leqslant C\frac{k}{k-1}\Bigl(\frac{\operatorname{mes} E(r)}{r}\ln\frac{2r}{\operatorname{mes} E(r)}\Bigr)T(kr, f), \tag{4}$$

where C is an absolute constant.

A version of the Grishin – Sodin Lemma on Small Intervals for subharmonic functions, but only *of finite order,* is proved in the joint work of A. F. Grishin and T. I. Malyutina [5]. The Grishin – Malyutina Lemma on Small Intervals found several important applications in proofs of key results of [5, Theorems 2, 4] and [6, 4.3, Lemma 4.2]. We do not give this version, because it is embedded in our joint with L. A. Gabdrakhmanova main result from [7, Theorem 1 (on small intervals)], where it is discussed in detail [7, Conclusion of the Grishin – Malyutina Theorem].

In this article, we consider integrals over subsets of intervals on a ray, but without estimates of integrals over small arcs on circles. In particular, we generalize the Rolf Nevanlinna Theorem, as well as the Grishin – Sodin Lemma on Small Intervals. Our Main Theorem on small intervals with integral L^p-norms for functions-multipliers is formulated in Subsection 1.2. Generalizations of the previous results are given in several directions. First, we consider differences of subharmonic functions, i.e., δ-subharmonic functions of arbitrary growth on the plane. We give a simple correction to the summands dictated by counterexample (3). Second, in the integrand, a multiplier function of the class L^p is allowed for $1 < p \leqslant \infty$, and the integrals are estimated using the L^p-norm with corresponding changes in the contribution of small subsets $E \subset \mathbb{R}^+$. Third, the estimates of integrals over intervals are in a certain sense uniform with respect to the class of all (δ-)subharmonic functions with an integral normalization near zero. This will make it possible to translate them in the future into estimates of integrals of plurisubharmonic functions and their differences in a multidimensional complex space both for intervals on rays with common beginnings and for subsets on these intervals.

1.2. Basic definitions and a recent theorem on small intervals. This subsection can be referred to as needed. Our notations may differ from those used above. We write single-point sets without curly braces, if this does not cause confusion. So, $\mathbb{N} := \{1, 2, \dots\}$ is the set of *natural* numbers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$, $\mathbb{Z} := (-\mathbb{N}) \cup \mathbb{N}_0$ is the set of *integers*, and $\overline{\mathbb{R}}$ is the *extended real axis* with $-\infty := \inf \mathbb{R}$ and $+\infty := \sup \mathbb{R}$, $-\infty \leqslant x \leqslant +\infty$ for each $x \in \overline{\mathbb{R}}$, $-(\pm\infty) = \mp\infty$, $\overline{\mathbb{R}}^+ := \mathbb{R}^+ \cup +\infty$, and

$$\begin{gathered} x+(+\infty) = +\infty \text{ for } x \in \overline{\mathbb{R}}\setminus -\infty, x + (-\infty) = -\infty \text{ for } x \in \overline{\mathbb{R}}\setminus +\infty, \\ x\cdot(\pm\infty) := \pm\infty =: (-x)\cdot(\mp\infty) \text{ for } x \in \overline{\mathbb{R}}^+\setminus 0, \\ \frac{\pm x}{0} := \pm\infty \text{ for } x \in \overline{\mathbb{R}}^+\setminus 0,\ \frac{x}{\pm\infty} := 0 \text{ for } x \in \mathbb{R}, \quad \text{but } 0\cdot\pm\infty := 0 \end{gathered} \tag{5}$$

unless otherwise stated; $x^+ := \max\{0, x\} =: (-x)^-$ for $x \in \overline{\mathbb{R}}$; $\inf \varnothing := +\infty$ and $\sup \varnothing := -\infty$ for the *empty set* $\varnothing$. An *interval* $I \subset \overline{\mathbb{R}}$ is a *connected set* with *ends* $\inf I \in \overline{\mathbb{R}}$ and $\sup I \in \overline{\mathbb{R}}$; $(a, b) := \{x \in \overline{\mathbb{R}} \colon a < x < b\}$, $[a, b] := \{x \in \overline{\mathbb{R}} \colon a \leqslant x \leqslant b\}$, $[a, b) := [a, b] \setminus b$, $(a, b] := [a, b] \setminus a$. $D(z, r) := \{z' \in \mathbb{C} \colon |z' - z| < r\}$ is an *open disc,* $\overline{D}(z, r) := \{z' \in \mathbb{C} \colon |z' - z| \leqslant r\}$ is a *closed disc,* $\partial\overline{D}(z, r) := \{z' \in \mathbb{C} \colon |z' - z| = r\}$ is the *circle with center* $z \in \mathbb{C}$ *of radius* $r \in \overline{\mathbb{R}}^+$*;* $D(z, 0) = \varnothing$, $\overline{D}(z, 0) = \partial\overline{D}(z, 0) = z$, $D(z, +\infty) = \mathbb{C}$. Besides, $D(r) := D(0, r), \overline{D}(r) := \overline{D}(0, r)$, $\partial\overline{D}(r) := \partial\overline{D}(0, r)$.

Given a function $f \colon X \to \overline{\mathbb{R}}$, $f^+ := \sup\{0, f\}$ and $f^- := (-f)^+$ are *positive* and *negative parts* of function f, respectively; $|f| := f^+ + f^-$.

Given $S \subset \mathbb{C}$, $\operatorname{sbh}(S)$ is the class of all *subharmonic* on an open neighbourhood of S. The class $\operatorname{sbh}(S)$ contains the *trivial* $(-\infty)$*-function* $\boldsymbol{-\infty}$. We set $\operatorname{sbh}_*(S) := \operatorname{sbh}(S) \setminus \boldsymbol{-\infty}$ for connected subset $S \subset \mathbb{C}$.

By λ we denote the *linear Lebesgue measure* on $\mathbb{R}$ and its restrictions on arbitrary λ*-measurable subsets* $S \subset \overline{\mathbb{R}}$, by setting $\lambda(\pm\infty) := 0$. We also use the notation $\operatorname{mes} S := \lambda(S)$ from Subsection 1.1. The concepts *almost everywhere, measurability* and *integrability* mean λ-almost everywhere, λ-measurability and λ-integrability, respectively.

We denote by

$$\operatorname*{ess\,sup}_S f := \inf\Bigl\{a \in \mathbb{R} \colon \lambda\bigl(\{x \in S \colon f(x) > a\}\bigr) = 0\Bigr\} \tag{6}$$

essential upper bound of measurable function f defined almost everywhere on S, and

$$L^\infty(S) := \Bigl\{f \colon \|f\|_{L^\infty(S)} := \operatorname*{ess\,sup}_S |f| < +\infty\Bigr\}, \tag{7_∞}$$

and for numbers $p > 1$,

$$L^p(S) := \Bigl\{f \colon \|f\|_{L^p(S)} := \Bigl(\int_S |f|^p \,\mathrm{d}\lambda\Bigr)^{1/p} < +\infty\Bigr\}, \quad p \in (1, +\infty), \tag{7p}$$

together with the numbers q associated with p by equality

$$\frac{1}{p} + \frac{1}{q} = 1, \quad 1 < q = \frac{p}{p-1} < +\infty, \quad \text{but } q := 1 \text{ if } p = \infty. \tag{7q}$$

If $S := I$ is an interval with ends $a \leqslant b$, then for *Lebesgue integral* of f over the interval I we will use two forms of notation

$$\int_I f \,\mathrm{d}\lambda =: \int_a^b f(t) \,\mathrm{d}t. \tag{8}$$

For $r \in \mathbb{R}^+$ and an arbitrary function $v \colon \partial\overline{D}(0, r) \to \overline{\mathbb{R}}$, we define

$$\mathsf{M}_v(r) := \sup_{|z|=r} v(z), \quad \mathsf{C}_v(r) := \frac{1}{2\pi}\int_0^{2\pi} v(re^{is}) \,\mathrm{d}s \tag{9}$$

The latter is the *average over* the *circle* $\partial\overline{D}(0, r)$ for v, if the function $s \mapsto re^{is}$ is integrable on $[0, 2\pi]$; $\mathsf{C}_v(0) := \mathsf{M}_v(0) = \mathsf{C}_v(0) = v(0)$. For the properties of the characteristics M_v and

C_v in the case of a subharmonic function v, see [8, 2.6], [9, 2.7]. So, for meromorphic functions f on $\mathbb{C}$, we have

$$M(r,f) \overset{(1\mathrm{M})}{=} \mathsf{M}_{\ln|f|}(r), \quad m(r,f) \overset{(1\mathrm{m})}{=} \mathsf{C}_{\ln^+|f|}(r).$$

Theorem on small intervals with weight ([7, Theorem 1, Remark 1.1])**.** *There is an absolute constant $a \geqslant 1$ such that*

[u] *for any subharmonic function $u \in \mathrm{sbh}_*(\mathbb{C})$, i. e., $v \not\equiv -\infty$ on $\mathbb{C}$,*

[r] *for any numbers $0 \leqslant r_0 \leqslant r < R < +\infty$,*

[E] *for any measurable subset $E \subset [r,R]$,*

[g] *for any measurable function g on E,*

[b] *for any number $b \in (0,1]$,*

the following inequality is fulfilled

$$\int\limits_E \mathsf{M}_{|u|} g \,\mathrm{d}\lambda \leqslant \Bigl(\frac{a}{b}\ln\frac{a}{b}\Bigr)\Bigl(\mathsf{M}_u\bigl((1+b)R\bigr) + 2\mathsf{C}_u^-(r_0)\Bigr)\|g\|_{L^\infty(E)} \times \\ \times \underbrace{\Bigl(\operatorname{mes} E + \min\{\operatorname{mes} E, 3bR\} \ln\frac{3ebR}{\min\{\operatorname{mes} E, 3bR\}}\Bigr)}_{m_\infty(E;R,b)} \tag{10}$$

where $m_\infty(E;R,b) \leqslant 2\operatorname{mes} E$ when $\operatorname{mes} E > 3bR$, and

$$m_\infty(E;R,b) \leqslant 2\operatorname{mes} E \ln\frac{3ebR}{\operatorname{mes} E}, \quad \text{if } \operatorname{mes} E \leqslant 3bR.$$

2. Main Theorem. For a Borel subset $S \subset \mathbb{C}$, the set of all Borel, or Radon, positive measures $\mu \geqslant 0$ on S is denoted by $\mathrm{Meas}^+(S)$, and $\mathrm{Meas}(S) := \mathrm{Meas}^+(S) - \mathrm{Meas}^+(S)$ is the set of all *charges,* or signed measures, on S. For a measure $\mu \in \mathrm{Meas}^+\bigl(\overline{D}(R)\bigr)$, we set

$$\mu^{\mathrm{rad}}(r) := \mu\bigl(\overline{D}(r)\bigr) \in \mathbb{R}^+, \quad 0 \leqslant r \leqslant R, \tag{11m}$$

$$\mathsf{N}_\mu(r,R) := \int\limits_r^R \frac{\mu^{\mathrm{rad}}(t)}{t}\,\mathrm{d}t \in \overline{\mathbb{R}}^+, \quad 0 \leqslant r \leqslant R, \tag{11N}$$

For $0 \leqslant r \leqslant R \in \mathbb{R}^+$ and an arbitrary function $v\colon \partial\overline{D}(r) \cup \partial\overline{D}(R) \to \overline{\mathbb{R}}$, we define

$$\mathsf{C}_v(r,R) \overset{(9)}{:=} \mathsf{C}_v(R) - \mathsf{C}_v(r) = \frac{1}{2\pi}\int\limits_0^{2\pi} \bigl(v(Re^{is}) - v(re^{is})\bigr)\,\mathrm{d}s \tag{12}$$

provided that $\mathsf{C}_v(R)$ and $\mathsf{C}_v(r)$ are well defined.

If $D \subset \mathbb{C}$ is a domain and $u \in \mathrm{sbh}_*(D)$, then there is its *Riesz measure*

$$\varDelta_u := \frac{1}{2\pi}\triangle u \in \mathrm{Meas}^+(D), \tag{13}$$

where $\triangle$ is the *Laplace operator* acting in the sense of the theory of distribution or generalized functions. This definition of the Riesz measures carries over naturally to $u \in \mathrm{sbh}_*(S)$

for connected subsets $S \subset \mathbb{C}$. If $v \in \mathrm{sbh}_*\big(\overline{D}(R)\big)$, then, by the Poisson – Jensen – Privalov formula, we have

$$\mathsf{C}_v(r,R) = \mathsf{N}_{\Delta_v}(r,R) \quad \text{for all } 0<r<R<+\infty. \tag{14}$$

Let $U = u - v$ be a difference of subharmonic functions $u, v \in \mathrm{sbh}_*\big(\overline{D}(0,R)\big)$, i.e., a *$\delta$-subharmonic non-trivial ($\not\equiv \pm\infty$) function* on $\overline{D}(R)$ with the *Riesz charge* $\Delta_U = \Delta_u - \Delta_v$ [10], [11], [12], [13, 3.1]. A representation $U = u_U - v_U$ with $u_U, v_U \in \mathrm{sbh}_*\big(\overline{D}(0,R)\big)$ is *canonical* if the Riesz measure Δ_{u_U} of u_U is the *upper variation* Δ_U^+ of Δ_U and the Riesz measure Δ_{v_U} of v_U is the *lower variation* Δ_U^- of Δ_U. The canonical representation for U is defined up to the harmonic function added simultaneously to each of the representing subharmonic functions u_U and v_U. We define a *characteristic function* of this δ-subharmonic function U as a function of two variables

$$\begin{aligned}\mathsf{T}_U(r,R) &:= \mathsf{C}_{\sup\{u_U,v_U\}}(r,R) = \mathsf{C}_{U^+}(r,R) + \mathsf{C}_{v_U}(r,R) = \\ &\overset{(14)}{=} \mathsf{C}_{U^+}(r,R) + \mathsf{N}_{\Delta_U^+}(r,R), \quad 0<r\leqslant R\in\mathbb{R}^+.\end{aligned} \tag{15}$$

This characteristic function T_U is already uniquely defined for all $0 < r \leqslant R < +\infty$ by positive values in $\mathbb{R}^+$, and is also increasing and convex with respect to ln in the second variable R, but is decreasing in the first variable $r \leqslant R$.

Main Theorem. *Let $0 < r_0 < r < +\infty$, $1 < k \in \mathbb{R}^+$, $E \subset [0, r]$ be measurable, $g \in L^p(E)$, where $1 < p \leqslant \infty$ and $q \in [1, +\infty)$ is from* (7q)*, $U \not\equiv \pm\infty$ be a δ-subharmonic non-trivial functions on $\mathbb{C}$, and $u \not\equiv -\infty$ be a subharmonic function on $\mathbb{C}$. Then*

$$\frac{1}{r}\int_E \mathsf{M}_U^+(t)g(t)\,\mathrm{d}t \leqslant 4q\frac{k}{k-1}\big(\mathsf{T}_U(r_0,kr)+\mathsf{C}_{U^+}(r_0)\big)\|g\|_{L^p(E)}\frac{\sqrt[q]{\operatorname{mes}E}}{r}\ln\frac{4kr}{\operatorname{mes}E}, \tag{16T}$$

$$\frac{1}{r}\int_E \mathsf{M}_{|u|}(t)g(t)\,\mathrm{d}t \leqslant 5q\frac{k}{k-1}\big(\mathsf{M}_{u^+}(kr)+\mathsf{C}_{u^-}(r_0)\big)\|g\|_{L^p(E)}\frac{\sqrt[q]{\operatorname{mes}E}}{r}\ln\frac{4kr}{\operatorname{mes}E}. \tag{16M}$$

3. Lemmata and Proof of Main Theorem.

Lemma 1. *Let $\mu \in \mathrm{Meas}^+\big(\overline{D}(R)\big)$. Then*

$$\mu^{\mathrm{rad}}(r) \leqslant \frac{R}{R-r}\mathsf{N}_\mu(r,R) \quad \textit{for each } 0\leqslant r\leqslant R. \tag{17}$$

Proof. By definitions (11), we have

$$\mu^{\mathrm{rad}}(r) = \int_r^R \frac{\mu^{\mathrm{rad}}(r)}{t}\,\mathrm{d}t \Bigg/ \int_r^R \frac{1}{t}\,\mathrm{d}t \leqslant \mathsf{N}_\mu(r,R) \Bigg/ \int_r^R \frac{1}{R}\,\mathrm{d}t = \frac{R}{R-r}\mathsf{N}_\mu(r,R),$$

and we obtain (17). □

Lemma 2. *Let $0 \leqslant r < R < +\infty$, $E \subset [0, r]$ be measurable, $U = u - v$ be a difference of subharmonic functions $u, v \in \mathrm{sbh}_*\big(\overline{D}(R)\big)$, Δ_v be the Riesz measure of v, and $g \in L^p(E)$. Then*

$$\int_E \mathsf{M}_U^+(t)g(t)\,\mathrm{d}t \leqslant \left(\frac{R+r}{R-r}\mathsf{C}_{U^+}(R)\sqrt[q]{\operatorname{mes}E} + \Delta_v^{\mathrm{rad}}(R)\sup_{0\leqslant x\leqslant R}\Big\|\ln\frac{2R}{|\cdot - x|}\Big\|_{L^q(E)}\right)\|g\|_{L^p(E)}. \tag{18}$$

Proof. For $w \in E \subset \overline{D}(r)$, by the Poisson–Jensen formula ([8, 4.5]), we have

$$U(te^{ia}) = \frac{1}{2\pi}\int_0^{2\pi} U(Re^{is}) \operatorname{Re}\frac{Re^{is}+te^{ia}}{Re^{is}-te^{ia}}\,\mathrm{d}s - \int_{D(R)} \ln\Bigl|\frac{R^2-zte^{-ia}}{R(te^{ia}-z)}\Bigr|\,\mathrm{d}\varDelta_u(z) +$$

$$+\int_{D(R)} \ln\Bigl|\frac{R^2-zte^{-ia}}{R(te^{ia}-z)}\Bigr|\,\mathrm{d}\varDelta_v(z) \leqslant \frac{R+r}{R-r}\mathsf{C}_{U^+}(R) + \int_{D(R)} \ln\frac{2R}{\bigl|t-|z|\bigr|}\,\mathrm{d}\varDelta_v(z)$$

where the right-hand side of the inequality is positive and independent of $a \in [0, 2\pi)$. Hence, by integrating, we get

$$\int_E \mathsf{M}_U^+(t)g(t)\,\mathrm{d}t \leqslant \frac{R+r}{R-r}\mathsf{C}_{U^+}(R)\int_E |g|(t)\,\mathrm{d}t + \int_{D(R)}\int_E \ln\frac{2R}{\bigl|t-|z|\bigr|}|g|(t)\,\mathrm{d}t\,\mathrm{d}\varDelta_v(z).$$

Therefore, by Hölder's inequality, we obtain

$$\int_E \mathsf{M}_U^+(t)g(t)\,\mathrm{d}t \leqslant \frac{R+r}{R-r}\mathsf{C}_{U^+}(R)\|g\|_{L^p(E)}(\operatorname{mes} E)^{1/q} +$$

$$+\varDelta_v\bigl(D(R)\bigr)\|g\|_{L^p(E)} \sup_{z\in D(R)}\Bigl\|\ln\frac{2R}{\bigl|\cdot-|z|\bigr|}\Bigr\|_{L^q(E)}.$$

The latter gives (18). □

Lemma 3. *Let $q \in \mathbb{R}^+$, $0 < A \in \mathbb{R}^+$, and $a \in (0, A/e]$. Then*

$$\int_0^a \ln^q\frac{A}{x}\,\mathrm{d}x \leqslant (1+q^{q+1})a\ln^q\frac{A}{a}. \tag{19}$$

Proof. We denote by $\lfloor q\rfloor := \max\{n \in \mathbb{Z} \colon n \leqslant q\}$ the *integer part* of q. Evidently,

$$\ln\frac{A}{x} \geqslant 1 \quad \text{if } x \in (0, A/e], \tag{20}$$

We integrate the integral from (19) by parts $\lfloor q\rfloor + 1$ times:

$$\int_0^a \ln^q\frac{A}{x}\,\mathrm{d}x = a\ln^q\frac{A}{a} + qa\ln^{q-1}\frac{A}{a} + q(q-1)a\ln^{q-2}\frac{A}{a} + \dots$$

$$\dots + q(q-1)\cdots(q-\lfloor q\rfloor+1)\ln^{q-\lfloor q\rfloor}\frac{A}{a} + q(q-1)\cdots(q-\lfloor q\rfloor)\int_0^a \ln^{q-\lfloor q\rfloor-1}\frac{A}{x}\,\mathrm{d}x \leqslant$$

$$\leqslant \Bigl(a\ln^q\frac{A}{a}\Bigr)\Biggl(1 + \frac{q}{\ln\frac{A}{a}} + \frac{q(q-1)}{\ln^2\frac{A}{a}} + \dots$$

$$\dots + \frac{q(q-1)\cdots(q-\lfloor q\rfloor+1)}{\ln^{\lfloor q\rfloor}\frac{A}{a}} + \frac{q(q-1)\cdots(q-\lfloor q\rfloor)}{a\ln^q\frac{A}{a}}\int_0^a \ln^{q-\lfloor q\rfloor-1}\frac{A}{x}\,\mathrm{d}x\Biggr) \leqslant$$

$$\overset{(20)}{\leqslant} \underbrace{\bigl(1+q+q(q-1)+\dots+q(q-1)\cdots(q-\lfloor q\rfloor+1)+q(q-1)\cdots(q-\lfloor q\rfloor)\bigr)}_{P_q}\, a\ln^q\frac{A}{a}.$$

We have a recurrent formula $P_q = 1 + qP_{q-1}$ for $1 \leqslant q \in \mathbb{R}^+$. Therefore,

$$P_q = \sum_{j=0}^{\lfloor q \rfloor} q^j + q^{\lfloor q \rfloor}\bigl(q - \lfloor q \rfloor\bigr) \leqslant 1 + q^{\lfloor q \rfloor + 1} \leqslant 1 + q^{q+1} \quad \text{for each } p \in \mathbb{R}^+.$$

Thus, we obtain (19). □

Lemma 4. *For $E \subset [0, r] \subset [0, R]$, $q \geqslant 1$, and $0 \leqslant x \leqslant R$, we have*

$$\Bigl\| \ln \frac{2R}{|\cdot - x|} \Bigr\|_{L^q(E)} \leqslant 2q \sqrt[q]{\operatorname{mes} E} \ln \frac{4R}{\operatorname{mes} E}, \tag{21}$$

Proof. We use

Lemma A ([2, Lemma 7.2])**.** *Let $f \colon (-a, a) \to \overline{\mathbb{R}}$ be an even integrable function on $(-a, a)$ decreasing on $(0, a)$, $E \subset (-a, a)$ be a measurable subset. Then*

$$\int_E f \, \mathrm{d}\lambda \leqslant 2 \int_0^{\lambda(E)/2} f(t) \, \mathrm{d}t. \tag{22}$$

By Lemma A we obtain

$$\int_E \ln^q \frac{2R}{|t - x|} \, \mathrm{d}t = \int_{E-x} \ln^q \frac{2R}{|t|} \, \mathrm{d}t \leqslant 2 \int_0^{\lambda(E-x)/2} \ln^q \frac{2R}{t} \, \mathrm{d}t = 2 \int_0^{\lambda(E)/2} \ln^q \frac{2R}{t} \, \mathrm{d}t,$$

where $a := \lambda(E)/2 \leqslant r/2 \leqslant R/2 \leqslant 2R/e =: A/e$. Hence, by Lemma 3, we have

$$\int_E \ln^q \frac{2R}{|t - x|} \, \mathrm{d}t \leqslant 2(1 + q^{q+1}) \frac{\lambda(E)}{2} \ln^q \frac{2R}{\lambda(E)/2} = (1 + q^{q+1})(\operatorname{mes} E) \ln^q \frac{4R}{\operatorname{mes} E}$$

and $(1 + q^{q+1})^{1/q} \leqslant (2q^{q+1})^{1/q} = q\bigl((2q)^{1/2q}\bigr)^2 \leqslant 2q$ for $q \geqslant 1$, since the function $x \underset{x \in \mathbb{R}^+}{\longmapsto} x^x$ is decreasing in $[1/e, +\infty)$. Thus,

$$\Bigl\| \int_E \ln^q \frac{2R}{|\cdot - x|} \Bigr\|_{L^q(E)} \leqslant \sqrt[q]{1 + q^{q+1}} \sqrt[q]{\operatorname{mes} E} \ln \frac{4R}{\operatorname{mes} E} \leqslant 2q \sqrt[q]{\operatorname{mes} E} \ln\Bigl(\frac{4R}{\operatorname{mes} E}\Bigr)$$

which gives (21). □

Main Lemma. *Let $0 < r < +\infty$, $0 < b \in \mathbb{R}^+$, $E \subset [0, r]$ be measurable, $U = u - v$ be a difference of subharmonic functions $u, v \in \operatorname{sbh}_*\Bigl(\overline{D}\bigl((1+b)^2 r\bigr)\Bigr)$, and $g \in L^p(E)$, where $1 < p \leqslant +\infty$ and $q \in [1, +\infty)$ is from (7q). Then*

$$\int_E \mathsf{M}_U^+(t) g(t) \, \mathrm{d}t \leqslant 2q \frac{1+b}{b} \Bigl(\mathsf{C}_{U^+}\bigl((1+b)r\bigr) + \mathsf{N}_{\Delta_v}\bigl((1+b)r, (1+b)^2 r\bigr) \Bigr) \times$$
$$\times \|g\|_{L^p(E)} \sqrt[q]{\operatorname{mes} E} \ln \frac{4(1+b)r}{\operatorname{mes} E}. \tag{23}$$

Proof. We set $R := (1+b)r$. By Lemma 2, Lemma 1 with $(1+b)^2r$ instead of R and R instead of r, and Lemma 4, we have

$$\int_E \mathsf{M}_U^+(t)g(t)\,\mathrm{d}t \leqslant \Bigl(\frac{2+b}{b}\mathsf{C}_{U^+}\bigl((1+b)r\bigr)\sqrt[q]{\operatorname{mes} E}+$$
$$+\frac{1+b}{b}\mathsf{N}_{\Delta_v}\bigl((1+b)r,(1+b)^2r\bigr)2q\sqrt[q]{\operatorname{mes} E}\ln\frac{4(1+b)r}{\operatorname{mes} E}\Bigr)\|g\|_{L^p(E)} \leqslant$$
$$\leqslant 2\frac{1+b}{b}\Bigl(\mathsf{C}_{U^+}\bigl((1+b)r\bigr)+\mathsf{N}_{\Delta_v}\bigl((1+b)r,(1+b)^2r\bigr)\Bigr)\|g\|_{L^p(E)}q\sqrt[q]{\operatorname{mes} E}\ln\frac{4(1+b)r}{\operatorname{mes} E},$$

which gives (23). □

Proof of the Main Theorem. We can assume that $U=u-v$ is the canonical representation of U. Consider a number $b>0$ such that $(1+b)^2=k$. By the Main Lemma, we have

$$\int_E \mathsf{M}_U^+(t)g(t)\,\mathrm{d}t \leqslant 2q\frac{\sqrt{k}}{\sqrt{k}-1}\Bigl(\mathsf{C}_{U^+}(\sqrt{k}r)+\mathsf{N}_{\Delta_v}(\sqrt{k}r,kr)\Bigr)\|g\|_{L^p(E)}\sqrt[q]{\operatorname{mes} E}\ln\frac{4\sqrt{k}r}{\operatorname{mes} E}$$
$$\leqslant 2q\frac{2k}{k-1}\Bigl(\underbrace{\mathsf{C}_{U^+}(r_0,kr)+\mathsf{N}_{\Delta_v}(r_0,kr)}_{\mathsf{T}_U(r_0,kr)}+\mathsf{C}_{U^+}(r_0)\Bigr)\|g\|_{L^p(E)}\sqrt[q]{\operatorname{mes} E}\ln\frac{4kr}{\operatorname{mes} E},$$

and, by definition (15), obtain (16T).

Evidently, for any function u with values in $\overline{\mathbb{R}}$, we have

$$\mathsf{M}_u^+ = \mathsf{M}_{u^+},\quad \mathsf{M}_{|u|}\leqslant \mathsf{M}_{u^+}+\mathsf{M}_{u^-}=\mathsf{M}_{u^+}+\mathsf{M}_{(-u)^+}, \tag{24}$$

If $u\in \operatorname{sbh}_*(\mathbb{C})$, then, under conditions of the Main Theorem, the function M_u^+ is increasing, and, by Hölder's inequality,

$$\int_E \mathsf{M}_{u^+}(t)g(t)\,\mathrm{d}t \leqslant \mathsf{M}_{u^+}(r)\|g\|_{L^p(E)}\sqrt[q]{\operatorname{mes} E}. \tag{25}$$

For $U_u := 0-u$, the difference $0-u$ is the canonical representation of δ-subharmonic non-trivial function U_u and we have

$$\mathsf{T}_{U_u}(r,R) \overset{(15)}{=} \mathsf{C}_{\sup\{0,u\}}(r,R)=\mathsf{C}_{u^+}(r,R)\leqslant \mathsf{C}_{u^+}(R)\leqslant \mathsf{M}_{u^+}(R). \tag{26}$$

Hence, by the Main Theorem in part (16T) for U_u in the role of U, we obtain

$$\frac{1}{r}\int_E \mathsf{M}_{(-u)^+}(t)g(t)\,\mathrm{d}t \overset{(24)}{=} \frac{1}{r}\int_E \mathsf{M}_{U_u}^+(t)g(t)\,\mathrm{d}t \leqslant$$
$$\overset{(16T)}{\leqslant} \frac{4kq}{k-1}\bigl(\mathsf{T}_{U_u}(r_0,kr)+\mathsf{C}_{U_u^+}(r_0)\bigr)\|g\|_{L^p(E)}\frac{\sqrt[q]{\operatorname{mes} E}}{r}\ln\frac{4kr}{\operatorname{mes} E}\leqslant$$
$$\overset{(26)}{\leqslant} \frac{4kq}{k-1}\bigl(\mathsf{M}_{u^+}(kr)+\mathsf{C}_{(-u)^+}(r_0)\bigr)\|g\|_{L^p(E)}\frac{\sqrt[q]{\operatorname{mes} E}}{r}\ln\frac{4kr}{\operatorname{mes} E}.$$

The latter together with (25) gives (16M). □

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