Integrated OU Processes and Non-Gaussian OU-based Stochastic Volatility Models

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ABSTRACT. In this paper, we study the detailed distributional properties of integrated non-Gaussian Ornstein–Uhlenbeck (intOU) processes. Both exact and approximate results are given. We emphasize the study of the tail behaviour of the intOU process. Our results have many potential applications in financial economics, as OU processes are used as models of instantaneous variance in stochastic volatility (SV) models. In this case, an intOU process can be regarded as a model of integrated variance. Hence, the tail behaviour of the intOU process will determine the tail behaviour of returns generated by SV models.

Key words: background driving Lévy process, chronometer, co-break, econometrics, integrated variance, Kumulant function, Lévy density, Lévy process, option pricing, OU processes, stochastic volatility

1. Introduction

In the stochastic volatility (SV) model for log-prices of stocks and for log exchange rates, a basic Brownian motion is generalized to allow the volatility term to vary over time. Then, the log-price $y^*(t)$ follows the solution to the stochastic differential equation (SDE),

$$\mathrm{d}y^*(t) = \{\mu + \beta\tau(t)\}\mathrm{d}t + \tau^{1/2}(t)\mathrm{d}w(t), \tag{1}$$

where $\tau(t)$, the *instantaneous* or *spot variance*, is going to be assumed to (almost surely) have locally square integrable sample paths, while being positive, stationary and stochastically independent of the standard Brownian motion $w(t)$. Over an interval of time of length $\hbar > 0$, returns are defined as

$$y_n = y^*(\hbar n) - y^*((n-1)\hbar), \quad n = 1, 2, \ldots. \tag{2}$$

which implies that whatever the model for τ, it follows that

$$y_n|\tau_n \sim N(\mu\hbar + \beta\tau_n, \tau_n), \tag{3}$$

where

$$\tau_n = \tau^*(n\hbar) - \tau^*\{(n-1)\hbar\} \quad \text{and} \quad \tau^*(t) = \int_0^t \tau(u)\,\mathrm{d}u.$$

The terms $\tau^*(t)$ and τ_n are called *integrated variance* and *actual variance*, respectively. Both definitions play a central role in the probabilistic and statistical analysis of SV models. Of course, $\tau^*(t)$ can be thought of as a generalized subordinator, or 'chronometer', for Brownian motion with drift; more specifically, $y^*(t)$ is representable as $\mu t + b^\beta(\tau^*(t))$ where b^β denotes Brownian motion with drift β and is independent of τ^* (cf. Barndorff-Nielsen & Shephard, 2001). Reviews of the literature on SV models are given in Taylor (1994), Shephard, (1996) and Ghysels *et al.* (1996), while statistical and probabilistic aspects are studied in detail in Barndorff-Nielsen & Shephard (2001).

As a result of (3), SV models can deliver returns which are stationary, serially dependent so long as the τ_n are serially dependent, while the marginal distribution of returns will be thicker tailed than normal due to the mixing over the random τ_n. However, when $\hbar$ is large the dependence is mild, while the distribution of returns is close to normality. The latter result holds so long as $\tau(u)$ is ergodic for then, as $t \to \infty$,

$$t^{-1}\tau^*(t) = t^{-1}\int_0^t \tau(u)\,\mathrm{d}u \overset{\text{a.s.}}{\to} \xi = \mathrm{E}(\tau(1)),$$

implying, for the SV model, that (e.g. Barndorff-Nielsen and Shephard, 2001)

$$\hbar^{-1/2}\{y_n - \mu\hbar - \beta\tau_n\} \overset{\mathscr{L}}{\to} N(0, \xi)$$

as $\hbar \to \infty$. This result is called 'aggregational Gaussianity.'

Aggregational Gaussianity has been much discussed in the econometric literature (e.g. in the ARCH literature it goes back to Diebold, 1988, pp. 12–16). Here, we use an exchange rate dataset kindly made available to us by Olsen and Associates to empirically verify this. It records every 5 min the most recent quote to appear on the Reuters screen from 1 December 1986 until 30 November 1996. These data, together with various adjustments we have made to it, are discussed extensively in the Appendix. Here, we focus on the dollar/Deutschmark series. We report in Fig. 1 non-parametric estimates of the log-density of returns recorded over intervals of length 5 min ($\hbar = 1$, $T = 705,313$), 20 min ($\hbar = 4$, $T = 176,325$), 1 h ($\hbar = 12$,

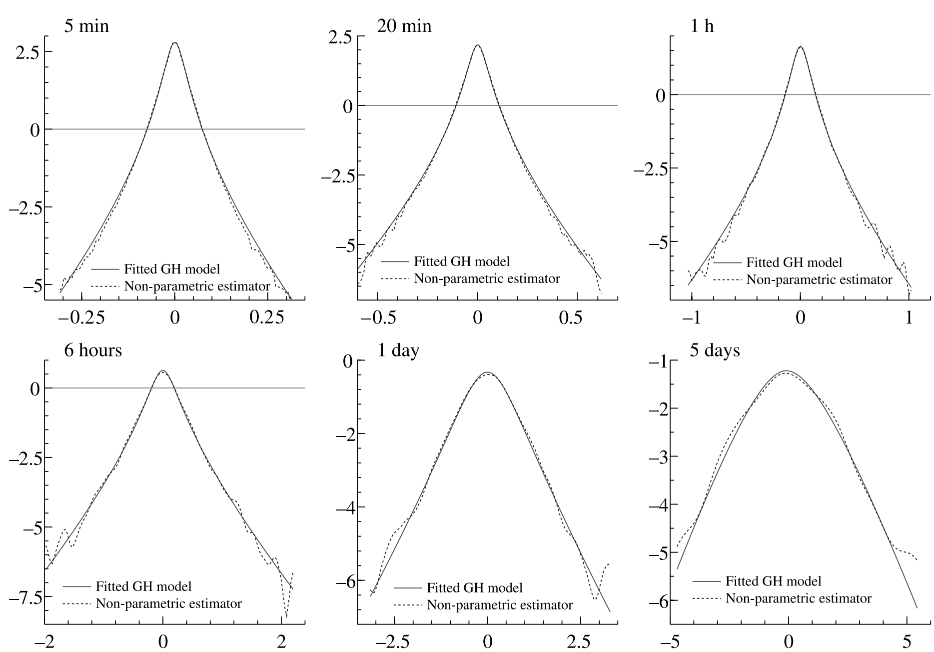

Fig. 1. Log-densities based on the 5 min Olsen group data. Movements on the US dollar against German DM from December 1986 to November 1996 over various intervals of time. Drawn log-densities are computed using a non-parametric estimator as well as the ML estimation of a generalized hyperbolic model.

$T = 58,775$), 6 h ($\hbar = 72$, $T = 9775$), 1 day ($\hbar = 288$, $T = 2440$) and 1 week ($\hbar = 1440$, $T = 488$); here, T denotes the sample size. Also plotted is the fit of the generalized hyperbolic (GH) distribution, which is a five parameter flexible mixture of normals model discussed by, for instance, Barndorff-Nielsen (1977, 1997) and Eberlein & Prause (2002). Its parameters were chosen using maximum likelihood methods. The non-parametric estimator of the log-density is constructed by using the log of Gaussian kernel estimator coded in Applied Statistics Algorithm AS 176 by Bernard Silverman, which is available at `StatLib` and `NAG`, as well as in many statistical software environments such as `Ox` (Doornik 2001). The bandwidth is chosen to be $1.06\hat{\sigma}T^{-1/5}$, where T is the sample size and $\hat{\sigma}$ is the empirical standard deviation of the returns (this is an optimal choice against a mean squared error loss for Gaussian data).

The figures show some interesting features. At low levels of aggregation, the 'pine tree' feature of the log of the density of price changes in the exchange rate holds. This can be seen even at 6-h returns. However, for daily data the log-density is more linear in the tails. At the weekly level, the tails seem lighter than linear and the quadratic approximation of the Gaussian seems to be closer to the mark. These observations also appear if we study the moments of the changes data. Table 1 shows the first four moments for the changes at different levels of aggregation. The important feature of the table is that as the level of aggregation increases the kurtosis falls. At the weekly level the kurtosis is still above three, indeed this value is statistically significant; however, it is not massively so. An interesting feature of the table is the skewness statistics, which are all positive. However, these statistics do not really yield a consistent pattern, which suggests the changes are mildly positively skewed but this is not a prominent feature of the series. Closely similar features to those discussed here are observed in other areas of study, particularly in turbulence (see, for instance, Barndorff-Nielsen, 1979).

As we saw above, if spot variance is ergodic then SV models imply aggregational Gaussianity. In this paper, we try to refine this result. We study the situation where spot variance follows a non-Gaussian Ornstein–Uhlenbeck (OU) process, which is the solution to the SDE

$$d\tau(t) = -\lambda\tau(t)dt + dz(\lambda t), \quad \lambda > 0,$$

where $z(t)$ is a subordinator – that is a process with non-negative, independent and stationary increments (see, e.g. Bertoin, 1996; Sato, 1999). The unusual timing $z(\lambda t)$ is deliberately chosen so that the marginal distribution of τ is independent of the choice of λ. Such models have been introduced in this context by Barndorff-Nielsen & Shephard (2001), while we call the corresponding $\tau^*(t)$ integrated OU or intOU processes. We will study the distribution of τ_n for these models with $\hbar$ fixed, which will imply the distribution of returns y_n. In particular, we will derive the behaviour of the tails of actual variance and so of returns from these SV models. This is of considerable practical importance and of quite some interest in the recent literature. Andersen *et al.* (2000, 2001a) have recently used realized variance estimators of τ_n to claim that actual variance is typically close to being lognormal for a wide range of $\hbar$ (see also

Table 1. *Raw mean, variance and standardized (by the standard deviation) third and fourth moments of the aggregated versions of the Olsen exchange rate data*

	Mean	Variance	Skewness	Kurtosis
5 min	−0.0000256	0.001847	0.146	44.2
20 min	−0.000102	0.006803	0.0628	27.6
1 h	−0.000307	0.01929	0.263	21.3
6 h	−0.00184	0.1162	0.0959	9.47
1 day	−0.00738	0.4903	0.00328	5.27
1 week	−0.0369	2.427	0.144	3.77

Barndorff-Nielsen & Shephard, 2002). This would imply returns are normal lognormal (NLN). Can such a claim hold if variance is of OU type? Barndorff-Nielsen & Shephard (2001) have assumed $\tau(t)$ is distributed as an inverse Gaussian variable and then claimed that actual variance is close to being inverse Gaussian for all $\hbar$, implying returns would be approximately normal inverse Gaussian. Can such claims be rationalized?

The outline of the paper is as follows. In section 2, we introduce the intOU processes, which are integrals of OU processes. The theory for these processes will be developed in the first case for general OU processes – not constraining ourselves to the non-negative case needed for volatility. The section continues with a discussion of the properties of predictions from such models, as well as the behaviour of increments from intOU process. Section 3 gives more concrete results in the special case of non-negative OU processes. This section will contain the answers to the above questions. Section 4 looks at superposition extensions of our basic models, while section 5 concludes. Section 6 contains a discussion of the data used in this paper.

2. intOU processes

2.1. Basic model structure

This paper discusses analytic results on the distributional behaviour of the stochastic process $x^*(t)$ defined by

$$x^*(t) = \int_0^t x(s)\,\mathrm{d}s,$$

where $x(t)$ is a strictly stationary process on the real line which satisfies a SDE of the form

$$\mathrm{d}x(t) = -\lambda x(t)\mathrm{d}t + \mathrm{d}z(\lambda t).$$

Here, the rate parameter λ is arbitrary positive and $z(t)$ is a homogeneous background driving Lévy process (BDLP) – that is, it is a process with independent and stationary increments. The $x(t)$ process is said to be of Ornstein–Uhlenbeck type or an OU process (and is familiar in the Gaussian case where the Lévy process is Brownian motion). Correspondingly, we say that $x^*(t)$ is an intOU process. The OU process is representable (in law) as

$$x(t) = \mathrm{e}^{-\lambda t}x(0) + \mathrm{e}^{-\lambda t}\int_0^t \mathrm{e}^{\lambda s}\,\mathrm{d}z(\lambda s).$$

As indicated in section 1, our main interest is where $x(t)$ is a purely non-negative process. In such cases, we will often switch notation from $x(t)$ to $\tau(t)$ in order to make this clear.

Barndorff-Nielsen and Shephard (2001) have studied some of the stochastic properties of $x(t)$ and the reader is referred there for a discussion of the associated literature. They established the notation that if $x(t)$ is an OU process with a marginal law D, then we say $x(t)$ is a D-OU process. Further, if $z(1)$ has law D, then we say $x(t)$ is an OU-D process. Typical choices of D are the inverse Gaussian (IG) and gamma distributions.

A major feature of the intOU process $x^*(t)$ is that (see Barndorff-Nielsen, 1998)

$$\begin{aligned} x^*(t) &= \lambda^{-1}\{z(\lambda t) - x(t) + x(0)\} \\ &= \lambda^{-1}(1-\mathrm{e}^{-\lambda t})x(0) + \lambda^{-1}\int_0^t \left\{1-\mathrm{e}^{-\lambda(t-s)}\right\}\mathrm{d}z(\lambda s), \end{aligned} \quad (4)$$

which has a simple structure. An interesting characteristic of this expression is that $x^*(t)$ has continuous sample paths when $\lambda > 0$, while $z(\lambda t)$ and $x(t)$ have identical jumps (or breaks). Hence, we have produced a process with a continuous sample path by taking linear

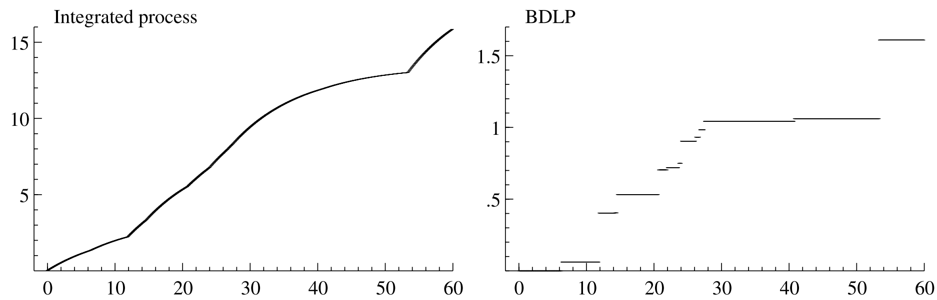

Fig. 2. Γ-OU process with $\nu = 3$, $\alpha = 8.5$, $\lambda = 0.02$. Left: plot of $x^*(t)$ against t. Right: plot of $z(\lambda t)$, the BDLP, against t.

combinations of two upward jumping processes. As a result, $z(\lambda t)$ and $x(t)$ co-break (Clements & Hendry, 1999, Chapter 9 introduced the concept of a co-break, where components of a multivariate series exhibit breaks but a linear combination of that series does not). Figure 2 shows this feature for a Γ-OU process, which is a process with a gamma distributed marginal law $\Gamma(\nu, \alpha)$ with probability density

$$\frac{\alpha^\lambda}{\Gamma(\nu)} x^{\nu-1} \mathrm{e}^{-\alpha x}.$$

For the simulated process we plot $x^*(t)$ and $z(t\lambda)$ against t. The intOU process has no jumps, although the gradient of the process clearly changes over time. The BDLP has its familiar upward jumps.

Further, in the case of a non-negative process, $\tau^*(t)$ has a lower bound made up of $\lambda^{-1}(1 - \mathrm{e}^{-\lambda t})\tau(0)$.

The fact that $z(\lambda t)$ and $x(t)$ co-break has deep implications for the use of this model. Suppose we focus for a moment on the case where $x^*(t)$ is an increasing positive process. We can then use it as a chronometer of Brownian motion with drift, implying the resulting process $y^*(t)$ has continuous sample paths. This contrasts with the usual case of subordination in the probability literature where the Brownian motion plus drift is subordinated by a Lévy process, $z(t)$. In that case, the resulting $y^*(t)$ process must have jumps.

Although $z(\lambda t)$ and $x(t)$ co-break, they do not co-integrate (Engle & Granger, 1987, introduced the concept of a co-integration, where components of a multivariate series exhibit non-stationarity but linear combinations of that series do not). Instead, the long-run behaviour of $x^*(t)$ is dominated by $z(\lambda t)$. This is clear from rewriting (4) as

$$\lambda x^*(t) - z(\lambda t) = x(0) - x(t),$$

which means $x^*(t)$ and $z(\lambda t)$ [rather than $x(t)$ and $z(\lambda t)$] co-integrate. So roughly, for large λt, $\lambda x^*(t)$ will have the same distribution as $z(\lambda t)$ – the error in this approximation is a stationary process. The distribution of the error, for large t and $x(t)$ being a D-OU process, is approximately the difference of two independent random variables drawn from the distribution D.

2.2. *The problem of prediction*

In this section, we will calculate the cumulants of $x^*(t)$, both unconditionally and conditionally on $x(0)$. The latter result is of fundamental importance in option pricing where analytically calculating the conditional cumulant function is enough to be able to compute European style options very rapidly. The former result will allow us to think about the unconditional distribution of returns.

The attractive feature of (4) is that the density of the future intOU process $x^*(t)|x(0)$ is determined by just

$$z(\lambda t) - x(t)|x(0).$$

As both $z(\lambda t)$ and $x(t)$ are linear, we can see that this will be mathematically tractable. In particular, (4) implies we have only to study the stochastic properties of the innovations for the intOU process

$$\lambda^{-1}\int_0^t \left\{1 - \mathrm{e}^{-\lambda(t-s)}\right\} \mathrm{d}z(\lambda s) = \int_0^t \varepsilon(t-s;\lambda)\, \mathrm{d}z(\lambda s) \tag{5}$$

$$\overset{\mathscr{L}}{=} \lambda^{-1}\int_0^{\lambda t} (1 - \mathrm{e}^{-s})\, \mathrm{d}z(s), \tag{6}$$

where

$$\varepsilon(t;\lambda) = \lambda^{-1}(1 - \mathrm{e}^{-\lambda t}). \tag{7}$$

This allows us to compactly write

$$x^*(t) = \int_0^t \varepsilon(t-s;\lambda)\mathrm{d}z(\lambda s) + \varepsilon(t;\lambda)x(0) \tag{8}$$

$$= \int_0^t \varepsilon(t-s;\lambda)\, \mathrm{d}z(\lambda s) + \varepsilon(t;\lambda)\int_{-\infty}^0 \mathrm{e}^s \mathrm{d}z(s), \tag{9}$$

the latter being entirely in terms of the BDLP z, here extended to be defined on the whole real line. In this section, we will show how to compute the cumulants of the $x^*(t)$ process.

2.3. Notation

It will be helpful later to define here various pieces of notation which will be central to our results. First, we note that

$$\varepsilon(t;\lambda) = t\varepsilon(1;\lambda t)$$

and

$$\varepsilon(t;\lambda) = \lambda^{-1}\left\{1 - \frac{\partial\varepsilon(t;\lambda)}{\partial t}\right\} \tag{10}$$

or, equivalently,

$$\frac{\partial\varepsilon(t;\lambda)}{\partial t} = 1 - \lambda\varepsilon(t;\lambda). \tag{11}$$

We shall use the following notation for cumulant and log Laplace transforms of a random variate x:

$$\mathrm{C}\{\zeta \ddagger x\} = \log \mathrm{E}\{\mathrm{e}^{\mathrm{i}\zeta x}\} \quad \text{and} \quad \bar{\mathrm{K}}\{\theta \ddagger x\} = \log \mathrm{E}\{\mathrm{e}^{-\theta x}\},$$

where the latter notation is primarily used for positive variates x. Further, in the context of OU processes we write

$$\acute{\kappa}(\zeta) = \mathrm{C}\{\zeta \ddagger x(t)\} \quad \text{and} \quad \kappa(\zeta) = \mathrm{C}\{\zeta \ddagger z(1)\}, \tag{12}$$

$$\acute{k}(\theta) = \bar{\mathrm{K}}\{\theta \ddagger x(t)\} \quad \text{and} \quad k(\theta) = \mathrm{K}\{\theta \ddagger z(1)\}, \tag{13}$$

and note that (Barndorff-Nielsen, 2001, Barndorff-Nielsen & Shephard, 2001)

$$\acute{\kappa}(\zeta) = \int_0^\infty \kappa(\mathrm{e}^{-s}\zeta)\,\mathrm{d}s \quad \text{and} \quad \kappa(\zeta) = \zeta\acute{\kappa}'(\zeta), \tag{14}$$

while

$$\acute{k}(\theta) = \int_0^\infty k(\mathrm{e}^{-s}\theta)\,\mathrm{d}s \quad \text{and} \quad k(\theta) = \theta\acute{k}'(\theta). \tag{15}$$

It then follows that if we write the cumulants of $x(t)$ and $z(1)$ (when they exist) as, respectively, $\acute{\kappa}_m$ and κ_m $(m = 1, 2, \ldots)$ we have that

$$\kappa_m = m\acute{\kappa}_m, \quad \text{for } m = 1, 2, \ldots$$

A special case here is that the means of $x(t)$ and $z(1)$ are identical, while the variance of the former is twice that of the latter. Finally, we introduce the notation

$$\acute{k}^*(\theta) = \bar{\mathrm{K}}\{\theta \ddagger x^*(t)\},$$

for the integrated process.

Henceforth, for clarity, we shall refer to the quantities $\bar{\mathrm{K}}\{\theta \ddagger x\}$, $k(\theta)$, $\acute{k}(\theta)$ and $\acute{k}^*(\theta)$ as *kumulant* functions, to distinguish them from the other *cumulant* functions $\mathrm{C}\{\zeta \ddagger x\}, \kappa(\zeta)$ and $\acute{\kappa}(\zeta)$. Some examples of the structure of such kumulants are given in Table 2. The only troublesome derivation is the OU-Γ case where we know that $k(\theta) = \nu \log(1 + \theta/\alpha)$ which implies

$$\begin{aligned}
\acute{k}(\theta) &= \nu \int_0^\infty \log\left(1 + \frac{\theta}{\alpha}\mathrm{e}^{-s}\right)\mathrm{d}s \\
&= \nu \int_0^{\theta/\alpha} \frac{1}{t}\log(1+t)\,\mathrm{d}t \quad \text{where } t = \frac{\theta}{\alpha}\mathrm{e}^{-s} \\
&= \nu \sum_{j=1}^\infty (-1)^j \frac{(\theta/\alpha)^j}{j^2} \quad \text{for } 0 \le \theta/\alpha < 1.
\end{aligned}$$

In Table 2, $P(\psi)$ denotes a Poisson distribution with parameter ψ, $\mathrm{IG}(\delta, \gamma)$ is an inverse Gaussian variable which has the density

$$\frac{\delta}{\sqrt{2\pi}}\mathrm{e}^{\delta\gamma}x^{-3/2}\exp\left\{-\frac{1}{2}\left(\delta^2 x^{-1} + \gamma^2 x\right)\right\}, \quad \gamma \ge 0, \;\; \delta, x > 0, \tag{16}$$

and $\mathrm{TS}(\kappa, \delta, \gamma)$ is the tempered stable law. The tempered stable derives by exponential tilting from the positive κ-stable law $S(\kappa, \delta)$ which has the kumulant transform $-(2\delta^2\theta)^\kappa$, $0 < \kappa < 1$ and density $p(x; \kappa, \delta)$. Then, the density of the tempered stable is defined by

$$\mathrm{e}^{(\delta\gamma)^{2\kappa}} p(x; \kappa, \delta)\mathrm{e}^{-1/2\gamma^2 x}, \qquad \kappa \in (0, 1), \;\; \delta > 0, \;\; \gamma \ge 0, \tag{17}$$

Table 2. *Kumulant functions for common OU models. In the OU-P case,* γ *is Euler's constant and* $\mathrm{E}_1(x) = \int_x^\infty y^{-1}\mathrm{e}^{-y}\,\mathrm{d}y$*, the exponential integral*

Model	$k(\theta) = \log \mathrm{E}\{\mathrm{e}^{-\theta z(1)}\}$	$\acute{k}(\theta) = \log \mathrm{E}\{\mathrm{e}^{-\theta x(t)}\}$
OU-$\Gamma(\nu, \alpha)$	$-\nu\log(1 + \theta\alpha^{-1})$	$\nu\sum_{j=1}^\infty (-1)^j(\theta/\alpha)^j j^{-2}$
OU-IG(δ, γ)	$\delta\gamma - \delta\gamma(1 + 2\gamma^{-2}\theta)^{1/2}$	Not known
OU-P(ψ)	$-\psi\left(1 - \mathrm{e}^{-\theta}\right)$	$-\psi\{\mathrm{E}_1(\theta) + \log\theta + \gamma\}$
OU-TS(κ, δ, γ)	$(\delta\gamma)^{2\kappa} - \delta^{2\kappa}(\gamma^2 + 2\theta)^\kappa$	Not known
$\Gamma(\nu, \alpha)$-OU	$-\nu\theta(\alpha + \theta)^{-1}$	$-\nu\,\log\left(1 + \theta\alpha^{-1}\right)$
IG(δ, γ)-OU	$-\theta\delta\gamma^{-1}\left(1 + 2\theta\gamma^{-2}\right)^{-1/2}$	$\delta\gamma - \delta\gamma\left(1 + 2\theta\gamma^{-2}\right)^{1/2}$
TS(κ, δ, γ)-OU	$-2\delta^{2\kappa}\kappa\theta(\gamma^2 + 2\theta)^{\kappa-1}$	$(\delta\gamma)^{2\kappa} - \delta^{2\kappa}(\gamma^2 + 2\theta)^\kappa$

the $\mathrm{IG}(\delta, \gamma)$ being the special case of $\mathrm{TS}(\kappa, \delta, \gamma)$ determined by $\kappa = 1/2$. The class of TS laws was introduced by Hougaard (1986), for applications to survival modelling. However, the same laws had been considered earlier, in an exponential family setting, by Tweedie (1984).

2.4. *Conditional first two moments*

Recalling we wrote the first two cumulants of $z(1)$ as κ_1 [which also equals $\mathrm{E}(x(t))$] and κ_2 [which equals $2\mathrm{Var}(x(t))$], then

$$\begin{aligned}\mathrm{E}\left\{\lambda^{-1}\int_0^{\lambda t}(1-\mathrm{e}^{-s})\,\mathrm{d}z(s)\right\} &= \lambda^{-1}\kappa_1\int_0^{\lambda t}(1-\mathrm{e}^{-s})\,\mathrm{d}s \\ &= \lambda^{-1}\kappa_1\left(\lambda t - 1 + \mathrm{e}^{-\lambda t}\right).\end{aligned}$$

The implication is that

$$\begin{aligned}\mathrm{E}\{x^*(t)|x(0)\} &= \lambda^{-1}(1-\mathrm{e}^{-\lambda t})x(0) + \lambda^{-1}\kappa_1\left(\lambda t - 1 + \mathrm{e}^{-\lambda t}\right) \quad (18)\\ &= \varepsilon(t;\lambda)\{x(0)-\kappa_1\} + \kappa_1 t, \quad (19)\end{aligned}$$

which implies, of course, $\mathrm{E}\{x^*(t)\} = \kappa_1 t$. The corresponding result for the conditional variance is

$$\begin{aligned}\mathrm{Var}\{x^*(t)|x(0)\} &= \lambda^{-2}\kappa_2\int_0^{\lambda t}(1-\mathrm{e}^{-s})^2\,\mathrm{d}s \\ &= \lambda^{-2}\kappa_2\left(\lambda t - 2 + 2\mathrm{e}^{-\lambda t} + \frac{1}{2} - \frac{1}{2}\mathrm{e}^{-2\lambda t}\right), \quad (20)\end{aligned}$$

while

$$\mathrm{Var}\{x^*(t)\} = \kappa_2\lambda^{-2}\left\{\mathrm{e}^{-\lambda t} - 1 + \lambda t\right\}.$$

2.5. *Cumulant functions for $x^*(t)|x(0)$ and $x^*(t)$*

One of the main advantages of the OU process is that we are able to derive the conditional cumulant function of $x^*t)|x(0)$. From (8), it follows that

$$\mathrm{C}\{\zeta \ddagger x^*(t)|x(0)\} = \mathrm{C}\left\{\zeta \ddagger \int_0^t \varepsilon(t-s;\lambda)\,\mathrm{d}z(\lambda s)\right\} + \mathrm{i}\zeta\varepsilon(t;\lambda)x(0), \quad (21)$$

where

$$\begin{aligned}\mathrm{C}\left\{\zeta \ddagger \int_0^t \varepsilon(t-s;\lambda)\,\mathrm{d}z(\lambda s)\right\} &= \mathrm{C}\left\{\zeta \ddagger \lambda^{-1}\int_0^{\lambda t}(1-\mathrm{e}^{-\lambda t+u})\,\mathrm{d}z(u)\right\} \\ &= \int_0^{\lambda t}\mathrm{C}\{\zeta\lambda^{-1}(1-\mathrm{e}^{-\lambda t+u}) \ddagger z(1)\}\,\mathrm{d}u \\ &= \lambda\int_0^t \mathrm{C}\{\zeta\varepsilon(t-s;\lambda) \ddagger z(1)\}\,\mathrm{d}s \\ &= \lambda\int_0^t \kappa(\zeta\varepsilon(t-s;\lambda))\,\mathrm{d}s \\ &= \lambda\int_0^t \kappa(\zeta\varepsilon(s;\lambda))\,\mathrm{d}s. \quad (22)\end{aligned}$$

The result for the conditional cumulant function allows us to easily calculate the unconditional cumulant function. From (8) and (22), it follows that

$$\begin{aligned} \mathrm{C}\{\zeta \ddagger x^*(t)\} &= \mathrm{C}\left\{\zeta \ddagger \int_0^t \varepsilon(t-s;\lambda)\,\mathrm{d}z(\lambda s)\right\} + \acute{\kappa}(\zeta\varepsilon(t;\lambda)) \\ &= \lambda \int_0^t \kappa(\zeta\varepsilon(s;\lambda))\,\mathrm{d}s + \acute{\kappa}(\zeta\varepsilon(t;\lambda)). \end{aligned} \tag{23}$$

It is sometimes helpful to express

$$\int_0^t \kappa(\zeta\varepsilon(s;\lambda))\,\mathrm{d}s = \int_0^{\varepsilon(t;\lambda)} (1-\lambda r)^{-1}\kappa(\zeta r)\,\mathrm{d}r, \tag{24}$$

which implies (21) and (23) become

$$\mathrm{C}\{\zeta \ddagger x^*(t)|x(0)\} = \lambda \int_0^{\varepsilon(t;\lambda)} (1-\lambda r)^{-1}\kappa(\zeta r)\,\mathrm{d}r + \mathrm{i}\zeta\varepsilon(t;\lambda)x(0)$$

and

$$\mathrm{C}\{\zeta \ddagger x^*(t)\} = \lambda \int_0^{\varepsilon(t;\lambda)} (1-\lambda r)^{-1}\kappa(\zeta r)\,\mathrm{d}r + \acute{\kappa}(\zeta\varepsilon(t;\lambda)).$$

2.6. *Cumulant functionals*

At a more abstract level, it is sometimes helpful to have generic results for the cumulant functions of the x^* process. A convenient way of doing this is via

$$f \bullet x^* = \int_0^\infty f(t)\,\mathrm{d}x^*(t),$$

where f denotes an 'arbitrary' function. We find that

$$\begin{aligned} f \bullet x^* &= \int_0^\infty f(t)x(t)\,\mathrm{d}t \\ &= \int_0^\infty f(t)\left\{\mathrm{e}^{-\lambda t}\int_0^t \mathrm{e}^{\lambda s}\,\mathrm{d}z(\lambda s) + \mathrm{e}^{-\lambda t}x(0)\right\}\mathrm{d}t \\ &= \int_0^\infty f(t)\mathrm{e}^{-\lambda t}\int_0^t \mathrm{e}^{\lambda s}\,\mathrm{d}z(\lambda s)\,\mathrm{d}t + \int_0^\infty f(t)\mathrm{e}^{-\lambda t}\,\mathrm{d}tx(0) \\ &= \int_0^\infty \int_s^\infty f(t)\mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}t\,\mathrm{d}z(\lambda s) + \int_0^\infty f(t)\mathrm{e}^{-\lambda t}\,\mathrm{d}tx(0) \\ &= \int_0^\infty \int_0^\infty f(t+s)\mathrm{e}^{-\lambda t}\,\mathrm{d}t\,\mathrm{d}z(\lambda s) + \int_0^\infty f(t)\mathrm{e}^{-\lambda t}\,\mathrm{d}tx(0) \end{aligned}$$

and hence

$$\mathrm{C}\{\zeta \ddagger f \bullet x^*|x(0)\} = \lambda \int_0^\infty \kappa\left(\zeta\int_0^\infty f(t+s)\mathrm{e}^{-\lambda t}\mathrm{d}t\right)\mathrm{d}s + \mathrm{i}\zeta\int_0^\infty f(t)\mathrm{e}^{-\lambda t}\mathrm{d}tx(0).$$

The corresponding unconditional cumulant functional is

$$\mathrm{C}\{\zeta \ddagger f \bullet x^*\} = \lambda \int_0^\infty \kappa\left(\zeta\int_0^\infty f(t+s)e^{-\lambda t}\mathrm{d}t\right)\mathrm{d}s + \acute{\kappa}\left(\zeta\int_0^\infty f(t)e^{-\lambda t}\mathrm{d}t\right). \tag{25}$$

We note, in passing, that using (14), (25) may be given the alternative forms

$$\mathrm{C}\{\zeta \ddagger f \bullet x^*\} = \lambda \int_{-\infty}^{\infty} \kappa\left(\zeta \int_0^{\infty} f(t+s)\mathrm{e}^{-\lambda t}\,\mathrm{d}t\right)\mathrm{d}s \tag{26}$$

$$= \zeta \int_0^{\infty} f(s)\acute{\kappa}'\left(\zeta \int_0^{\infty} f(t+s)\mathrm{e}^{-\lambda t}\,\mathrm{d}t\right)\mathrm{d}s, \tag{27}$$

see Barndorff-Nielsen (2001).

Formula (23) is recovered from (25) by choosing $f(s) = \mathbf{1}_{[0,t]}(s)$. The joint cumulant function of $x^*(s)$ and $x^*(t)$, for $0 < s < t$, may be obtained by letting $f = \phi\mathbf{1}_{[0,s]} + \psi\mathbf{1}_{[0,t]}$, etc.

3. Positive processes

3.1. Background

From now on, we suppose that the OU process is positive and, correspondingly, we switch notation from x and x^* to τ and τ^*. We wish to investigate the nature of the intOU process τ^* somewhat more closely as this is of main concern in connection with the models introduced in Barndorff-Nielsen & Shephard (2001). In particular, we are interested in establishing whether the marginal distribution of τ, which we are free to choose as modellers, is maintained (to a close approximation) when we look at the implied τ^*. This is important in the context of SV models, for if this result is true then we would be in a position to specify models for τ directly knowing the implication for the marginal distribution of returns y_n. At the moment the link between the distribution for τ and that for y_n is unclear.

3.2. Lévy densities

In order to study the tail behaviour of $\tau^*(t)$, we will study the tail behaviour of the Lévy density in detail. We recall that knowledge of the Lévy density is enough to produce the characteristic function via the Lévy–Khintchine formula, which obviously characterizes the density of τ^*. However, the connection between the probability density of τ^* and the associated Lévy density is more intimate than this indicates, particular when we are interested in explicitly studying the tail behaviour of the density of $\tau^*(t)$. This is important, for the right-hand tail will determine an essential part of the behaviour of returns when the intOU models are used for integrated variance in stochastic volatility models. More specifically, for the infinitely divisible laws there are many useful relations and points of similarity between the probability measures and probability densities of the laws on the one hand and their associated Lévy measures and Lévy densities on the other. There are also very intriguing differences, so that simple-minded guessing about similarities will often not work. See Sato (1999, Corollary 25.8, Theorems 28.4, 53.6 and 53.8) and Bingham *et al.* (1989, p. 341) and references given there; compare also Embrechts and Goldie (1981), Sato & Steutel (1998) and Barndorff-Nielsen & Hubalek (2003).

By (23), the kumulant function of τ^* is

$$\bar{\mathrm{K}}\{\theta \ddagger \tau^*(t)\} = \lambda \int_0^t k(\theta\varepsilon(s;\lambda))\,\mathrm{d}s + \acute{k}(\theta\varepsilon(t;\lambda)). \tag{28}$$

Let u and $\acute{u}$ denote the Lévy densities of $z(1)$ and $\tau(t)$, respectively. They are related by

$$u(x) = -\acute{u}(x) - x\acute{u}'(x) \quad \text{and} \quad \acute{u}(x) = x^{-1}U^+(x),$$

where

$$U^{+}(x) = \int_{x}^{\infty} u(y)\,\mathrm{d}y$$

compare Barndorff-Nielsen and Shephard (2001). From (28), we may determine an expression for the Lévy density $v(y;t;\lambda)$ of $\tau^{*}(t)$. We have

$$\begin{aligned}\bar{\mathrm{K}}\{\theta \ddagger \tau^{*}(t)\} &= -\lambda \int_{0}^{t}\int_{0}^{\infty}(1-\mathrm{e}^{-\theta\varepsilon(s;\lambda)x})u(x)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad - \int_{0}^{\infty}(1-\mathrm{e}^{-\theta\varepsilon(t;\lambda)x})\acute{u}(x)\,\mathrm{d}x\end{aligned}$$

and by the substitutions $r = \varepsilon(s;\lambda)$ and $y = rx$, and using (11), this gives

$$\bar{\mathrm{K}}\{\theta \ddagger \tau^{*}(t)\} = -\int_{0}^{\infty}(1-\mathrm{e}^{-\theta y})v(y;t;\lambda)\,\mathrm{d}y,$$

where

$$v(y;t;\lambda) = \lambda \int_{0}^{\varepsilon(t;\lambda)} \{r(1-\lambda r)\}^{-1}u(r^{-1}y)\,\mathrm{d}r + \varepsilon(t;\lambda)^{-1}\acute{u}(\varepsilon(t;\lambda)^{-1}y). \tag{29}$$

Recalling that $\acute{u}(x) = x^{-1}U^{+}(x)$, the latter expression may be written as

$$v(y;t;\lambda) = \lambda \int_{0}^{\varepsilon(t;\lambda)} \{r(1-\lambda r)\}^{-1}u(r^{-1}y)\,\mathrm{d}r + y^{-1}U^{+}(\varepsilon(t;\lambda)^{-1}y). \tag{30}$$

Letting $\delta(t;\lambda) = \varepsilon(t;\lambda)^{-1}$ and using the substitution $q = r^{-1}$ we then obtain

$$\begin{aligned}v(y;t;\lambda) &= \lambda \int_{\delta(t;\lambda)}^{\infty} (q-\lambda)^{-1}u(qy)\,\mathrm{d}q + y^{-1}U^{+}(\varepsilon(t;\lambda)^{-1}y) &(31)\\ &= \lambda \int_{0}^{\infty} (w+\delta(t;\lambda)-\lambda)^{-1}u((w+\delta(t;\lambda))y)\mathrm{d}w + y^{-1}U^{+}(\delta(t;\lambda)y). &(32)\end{aligned}$$

Thus, in particular, if $u(x) \sim x^{-1-a}$ for an $a \in (0,1)$ and $x \downarrow 0$ then

$$v(y;t;\lambda) \sim \left\{\lambda \int_{\delta(t;\lambda)}^{\infty} q^{-1-a}(q-\lambda)^{-1}\,\mathrm{d}q + a^{-1}\varepsilon(t;\lambda)^{a}\right\} y^{-1-a}$$

for $y \downarrow 0$.

More detailed calculations can be carried out in particular cases, as for the TS and Γ settings that we discuss next.

Example 1 (OU-TS case)
In this model u is of the TS form

$u(x) = x^{-1-a}\mathrm{e}^{-x}$,

with $0 < a < 1$. Hence,

$$\begin{aligned}U^{+}(x) &= \int_{x}^{\infty} y^{-1-a}\mathrm{e}^{-y}\,\mathrm{d}y \\ &= a^{-1}x^{-a}\mathrm{e}^{-x} - a^{-1}\int_{x}^{\infty} y^{-a}\mathrm{e}^{-y}\,\mathrm{d}y &(33)\\ &= \Gamma(-a,x) &(34)\end{aligned}$$

and

$$\acute{u}(x) = x^{-1}\Gamma(-a, x). \tag{35}$$

Here,

$$\Gamma(\alpha, x) = \int_x^\infty \xi^{\alpha-1} \mathrm{e}^{-\xi}\,\mathrm{d}\xi$$

is the incomplete gamma function. We recall that

$$\Gamma(\alpha + 1, x) = \alpha\Gamma(\alpha, x) + x^\alpha \mathrm{e}^{-x} \tag{36}$$

and, for $\alpha > 0$ and $x \to \infty$,

$$\Gamma(\alpha, x) \sim x^{\alpha-1}\mathrm{e}^{-x}\{1 + (\alpha - 1)x^{-1} + (\alpha - 1)(\alpha - 2)x^{-2} + \cdots\} \tag{37}$$

(cf. Abramowitz & Stegun, 1970, formula 6.5.32). This implies, in particular, that

$$\Gamma(-a, x) \sim x^{-a-1}\mathrm{e}^{-x} \tag{38}$$

for $x \to \infty$.

It follows, from (35) and (37), that

$$\acute{u}(x) \sim a^{-1}x^{-1-a} \quad \text{for } x \downarrow 0,$$

while

$$\acute{u}(x) \sim x^{-5/2}\mathrm{e}^{-x} \quad \text{for } x \to \infty.$$

Combining (32) and (34) we moreover find

$$v(y; t; \lambda) = \lambda y^{-1-a}\mathrm{e}^{-\delta(t;\lambda)y}\int_0^\infty (w + \delta(t; \lambda))^{-1-a}(w + \delta(t; \lambda) - \lambda)^{-1}\mathrm{e}^{-wy}\,\mathrm{d}w + y^{-1}\Gamma(-a, \delta(t; \lambda)y). \tag{39}$$

Here, for $y \to \infty$

$$\int_0^\infty (w + \delta(t; \lambda))^{-1-a}(w + \delta(t; \lambda) - \lambda)^{-1}\mathrm{e}^{-wy}\mathrm{d}w \sim \varepsilon(t; \lambda)^{2+a}\mathrm{e}^{\lambda t}y^{-1}, \tag{40}$$

while, by (38),

$$y^{-1}\Gamma(-a, \delta(t; \lambda)y) \sim \varepsilon(t; \lambda)^{1+a}y^{-2-a}\mathrm{e}^{-\delta(t;\lambda)y}.$$

Thus, for $y \to \infty$

$$v(y; t; \lambda) \sim c_\infty(t; \lambda, a)y^{-2-a}\mathrm{e}^{-\delta(t;\lambda)y} \tag{41}$$

and for $y \downarrow 0$, by (29) and (33),

$$v(y; t; \lambda) \sim c_0(t; \lambda, a)y^{-1-a}, \tag{42}$$

where

$$c_\infty(t; \lambda, a) = \mathrm{e}^{\lambda t}\varepsilon(t; \lambda)^{1+a}$$

$$c_0(t; \lambda, a) = \lambda\int_0^{\varepsilon(t;\lambda)} r^a(1 - \lambda r)^{-1}\,\mathrm{d}r + a^{-1}\varepsilon(t; \lambda)^a.$$

Example 2 (TS-OU case)
In this case,

$$\acute{u}(x) = x^{-1-a}\mathrm{e}^{-x} \tag{43}$$

$(0 < a < 1)$. Thus we have

$$u(x) = -\acute{u}(x) - x\acute{u}'(x) = ax^{-1-a}\mathrm{e}^{-x} + x^{-a}\mathrm{e}^{-x}$$

and, again with $\delta(t;\lambda) = \varepsilon(t;\lambda)^{-1}$,

$$\delta(t;\lambda)\acute{u}(\delta(t;\lambda)y) = \varepsilon(t;\lambda)^{a}y^{-1-a}\exp\{-\delta(t;\lambda)y\}.$$

Consequently,

$$\begin{aligned} v(y;t;\lambda) = {} & a\lambda y^{-1-a}\int_0^\infty (w+\delta(t;\lambda))^{-1-a}(w+\delta(t;\lambda)-\lambda)^{-1}\mathrm{e}^{-wy}\,\mathrm{d}w\mathrm{e}^{-\delta(t;\lambda)y} \\ & + \lambda y^{-a}\int_0^\infty (w+\delta(t;\lambda))^{-a}(w+\delta(t;\lambda)-\lambda)^{-1}\mathrm{e}^{-wy}\,\mathrm{d}w\mathrm{e}^{-\delta(t;\lambda)y} \\ & + \varepsilon(t;\lambda)^{a}y^{-1-a}\exp\{-\delta(t;\lambda)y\}. \end{aligned} \tag{44}$$

It follows from (40) that for $y \to \infty$

$$v(y;t;\lambda) \sim c_\infty(t;\lambda,a)y^{-1-a}\mathrm{e}^{-\delta(t;\lambda)y}, \tag{45}$$

whereas for $y \downarrow 0$

$$v(y;t;\lambda) \sim c_0(t;\lambda,a)y^{-1-a} \tag{46}$$

and here

$$c_\infty(t;\lambda,a) = \mathrm{e}^{\lambda t}\varepsilon(t;\lambda)^{a}$$

$$c_0(t;\lambda,a) = a\lambda\int_0^\infty (w+\delta(t;\lambda))^{-1-a}(w+\delta(t;\lambda)-\lambda)^{-1}\,\mathrm{d}w + \varepsilon(t;\lambda)^{a}.$$

Thus, in particular, for the IG-OU process τ, for any $t > 0$ the Lévy density of $\tau^*(t)$ has asymptotically the same upper and lower tail behaviour as for IG laws, so that the law of $\tau^*(t)$ is close to being IG. This follows as the Lévy density of an $\mathrm{IG}(\delta,\gamma)$ is

$$\frac{\delta}{\sqrt{2\pi}}x^{-3/2}\exp\left(-\frac{\gamma^2}{2}x\right),$$

which implies $\acute{u}(x)$ in (43) results from the IG special case $\mathrm{IG}(\sqrt{2\pi},\sqrt{2})$. The implication is that the distribution of $\tau^*(t)$ is close to an $\mathrm{IG}(\sqrt{2\pi}c_\infty,\sqrt{2\delta(t,\lambda)})$ in the upper tail and to $\mathrm{IG}(\sqrt{2\pi}c_0(t;\lambda,a),0)$ in the lower tail. A similar conclusion holds in general for the TS-OU processes.

Example 3 OU-Γ case
For this model u is of the form $u(x) = x^{-1}e^{-x}$ and proceeding as in Example 1 we find $U^+(x) = E_1(x)$, where $E_1(x)$ is the exponential integral. Further,

$$\begin{aligned} v(y;t;\lambda) = {} & \lambda y^{-1}\mathrm{e}^{-\delta(t;\lambda)y}\int_0^\infty \{(w+\delta(t;\lambda))(w+\delta(t;\lambda)-\lambda)\}^{-1}\mathrm{e}^{-wy}\,\mathrm{d}w \\ & + y^{-1}E_1(\delta(t;\lambda)y). \end{aligned} \tag{47}$$

Since

$$E_1(x) \sim \begin{cases} x^{-1}\mathrm{e}^{-x} & \text{for } x \downarrow 0, \\ -\log x & \text{for } x \to \infty, \end{cases} \tag{48}$$

(Abramowitz & Stegun, 1970, pp. 229 and 231) it follows that

$$v(y;t;\lambda) \sim \begin{cases} \mathrm{e}^{\lambda t} y^{-1} \mathrm{e}^{-\delta(t;\lambda)y} & \text{for } y \to \infty, \\ y^{-1} \log y^{-1} & \text{for } y \downarrow 0. \end{cases}$$

Example 4 Γ-*OU case*
With $\acute{u}(x) = x^{-1}\mathrm{e}^{-x}$ we have $u(x) = U^{+}(x) = \mathrm{e}^{-x}$ and

$$v(y;t;\lambda) = \lambda \int_{\delta(t;\lambda)}^{\infty} (q-\lambda)^{-1} \mathrm{e}^{-qy} \mathrm{d}q + y^{-1} \exp\{-\delta(t;\lambda)y\}.$$

Here, by (48),

$$\begin{aligned} \int_{\delta(t;\lambda)}^{\infty} (q-\lambda)^{-1} \mathrm{e}^{-qy} \mathrm{d}q &= \mathrm{e}^{-\lambda y} \int_{\delta(t;\lambda)-\lambda}^{\infty} s^{-1} \mathrm{e}^{-s} \mathrm{d}s \\ &= \mathrm{e}^{-\lambda y} E_1((\delta(t;\lambda)-\lambda)y) \\ &\sim \begin{cases} \mathrm{e}^{\lambda t} \mathrm{e}^{-\delta(t;\lambda)y} & \text{for } y \to \infty \\ \mathrm{e}^{-\lambda y} \log y^{-1} & \text{for } y \downarrow 0 \end{cases} \end{aligned}$$

implying

$$v(y;t;\lambda) \sim \begin{cases} \mathrm{e}^{\lambda t} y^{-1} \mathrm{e}^{-\delta(t;\lambda)y} & \text{for } y \to \infty \\ y^{-1} & \text{for } y \downarrow 0. \end{cases}$$

Thus, the conclusion is similar to that for the TS-OU processes.

Example 5 LN-*OU case* The lognormal (LN) distribution is known to be self-decomposable (Bondesson, 1992, p. 18, Theorem 3.1.1, p. 48: Notes). However, an explicit expression for the Lévy density of LN is not known. Bondesson (2002) discusses this open problem and notes that, based among other things on results in Embrechts *et al.* (1979), the Lévy density u corresponding to the standard lognormal LN(0,1) must, in all likelihood, satisfy

$$xu(x) \sim \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-1/2 \, \log^2 x} \tag{49}$$

for $x \to \infty$. Based on this assumption we find for the intOU process, derived from the LN-OU process with standard LN marginal, that for $y \to \infty$

$$v(y;t;\lambda) \sim \frac{1}{\sqrt{2\pi}} (y \log y)^{-1} \mathrm{e}^{-1/2 \, \log^2(\delta(t;\lambda)y)} \sim \delta(t;\lambda)(\log y)^{-1} u(\delta(t;\lambda)y). \tag{50}$$

In particular, then, the upper tails of the marginal distributions of the int(LN-OU) process do not behave as do LN laws.

The verification of (50) is as follows. By the substitution $r = \log(w + \delta(t;\lambda))$, for the integral in (32) we obtain, as $y \to \infty$,

$$
\begin{aligned}
y\int_{\log\delta(t;\lambda)}^{\infty}(\mathrm{e}^{r}-\lambda)^{-1}\mathrm{e}^{r}u(y\mathrm{e}^{r})\,\mathrm{d}r &\sim \frac{1}{\sqrt{2\pi}}\int_{\log\delta(t;\lambda)}^{\infty}\mathrm{e}^{-r}\mathrm{e}^{-1/2(r+\log y)^{2}}\mathrm{d}r\\
&\sim \frac{\sqrt{e}}{\sqrt{2\pi}}\mathrm{e}^{\log y}\int_{\log\delta(t;\lambda)}^{\infty}\mathrm{e}^{-1/2(r+\log y+1)^{2}}\mathrm{d}r\\
&= \sqrt{e}\mathrm{e}^{\log y}\{1-\Phi(\log(\delta(t;\lambda)y)+1)\}\\
&\sim \frac{\sqrt{e}}{\sqrt{2\pi}}\mathrm{e}^{\log y}(\log y)^{-1}\mathrm{e}^{-1/2(\log y+\log\delta(t;\lambda)+1)^{2}}\\
&= \frac{\varepsilon(t;\lambda)}{\sqrt{2\pi}}\mathrm{e}^{-1/2\log^{2}\delta(t;\lambda)}(\log y)^{-1}\mathrm{e}^{-1/2\log^{2}y-(\log(\delta(t;\lambda)+1)\log y}\\
&= o((\log y)^{-1}\mathrm{e}^{-1/2\log^{2}y}).
\end{aligned}
\tag{51}
$$

Furthermore, (49) implies

$$
U^{+}(x)\sim\frac{1}{\sqrt{2\pi}}(\log x)^{-1}\mathrm{e}^{-1/2\log^{2}x} \tag{52}
$$

and combining (32), (51) and (52), we find (50). This is the end of this example.

We summarize these results in Table 3.

3.3 *Assessing the accuracy of results in* IG-*OU case*

The above results are asymptotic and so it is important to assess their predictive power in realistic situations. Here, we carry this out by looking at the important IG-OU case.

We use the expression we have already seen, that

$$
\begin{aligned}
\bar{\mathrm{K}}\{\theta\ddagger\tau^{*}(t)|\tau(0)\} &= -\theta\varepsilon(t;\lambda)\tau(0)+\lambda\int_{0}^{t}k(\theta\varepsilon(s;\lambda))\,\mathrm{d}s\\
&= -\theta\varepsilon(t;\lambda)\tau(0)+\int_{0}^{1-\mathrm{e}^{-\lambda t}}(1-u)^{-1}k(\lambda^{-1}\theta u)\,\mathrm{d}u,
\end{aligned}
\tag{53}
$$

recalling that the kumulant function $k(\theta)=\log\mathrm{E}[\exp\{-\theta z(1)\}]$. Determining the expression for $\bar{\mathrm{K}}\{\theta\ddagger\tau^{*}(t)|\tau(0)\}$ in particular cases has been carried out in the context of option pricing based on OU volatility by Barndorff-Nielsen & Shephard (2001) and subsequently by Nicolato & Venardos (2001) and Tompkins & Hubalek (2000). Here, we discuss only the IG-OU case, which was independently derived by Nicolato & Venardos (2001) and Tompkins & Hubalek (2000). From Table 2 we have that

$$
k(\theta)=-\frac{\theta\delta}{\gamma}\left(1+2\theta\gamma^{-2}\right)^{-1/2}.
$$

Table 3. *The Lévy densities and their tail behaviour for some familiar models. Note the form of the lognormal case is the conjectured asymptotic form*

		Lévy density $v(y;t;\lambda)\propto$	
	Lévy density u or $\acute{u}$	$y\downarrow 0$	$y\uparrow\infty$
OU-TS	$x^{-1-a}\mathrm{e}^{-x}$	y^{-1-a}	$y^{-2-a}\mathrm{e}^{-\delta(t;\lambda)y}$
TS-OU	$x^{-1-a}\mathrm{e}^{-x}$	y^{-1-a}	$y^{-1-a}\mathrm{e}^{-\delta(t;\lambda)y}$
OU-Γ	$x^{-1}\mathrm{e}^{-x}$	$y^{-1}\log y^{-1}$	$y^{-1}\mathrm{e}^{-\delta(t;\lambda)y}$
Γ-OU	$x^{-1}\mathrm{e}^{-x}$	y^{-1}	$y^{-1}\mathrm{e}^{-\delta(t;\lambda)y}$
LN-*OU*	$\sim\frac{1}{\sqrt{2\pi}}x^{-1}\mathrm{e}^{-1/2\log^{2}x}$ as $x\to\infty$	Not known	$(y\log y)^{-1}\mathrm{e}^{-1/2\log^{2}(\delta(t;\lambda)y)}$

Then,

$$\int_0^{1-e^{-\lambda t}} (1-u)^{-1} k(\lambda^{-1}\theta u)\,du = -\frac{\delta\theta}{\gamma\lambda}\int_0^{1-e^{-\lambda t}} (1-u)^{-1} u(1+\varkappa u)^{-1/2}\,du, \tag{54}$$

where $\varkappa = 2\gamma^{-2}\lambda^{-1}\theta$. Now,

$$\int (1-u)^{-1} u(1+\varkappa u)^{-1/2}\,du = -\frac{2\sqrt{1+\varkappa u}}{\varkappa} + \frac{2\,\mathrm{arctanh}\{\sqrt{1+\varkappa u}/\sqrt{1+\varkappa}\}}{\sqrt{1+\varkappa}}.$$

Having derived $\bar{\mathrm{K}}\{\theta \ddagger \tau^*(t)|\tau(0)\}$, it is straightforward to calculate $\mathrm{C}\{\zeta \ddagger \tau^*(t)\}$ via (23). This cumulant function can be inverted to give the exact density of $\tau^*(t)$. Recall that for a random variable Y the distribution function can be obtained via the characteristic function using the Fourier inversion:

$$\Pr(\tau^*(t) > y) = \frac{1}{2} + \frac{1}{2\pi}\int_0^{\infty} \mathrm{Im}\ \exp[-\mathrm{i}\zeta y + \mathrm{C}\{\zeta \ddagger \tau^*(t)\}]\zeta^{-1}\,d\zeta.$$

Here, we use this result to compute the density of $\tau^*(t)$ for the IG-OU case, written $p^*(x;\delta,\lambda)$. This is given, for three different choices of λ, in Fig. 3. Together with this we have also plotted a right-hand tail approximation $\mathrm{IG}(\sqrt{2\delta(t,\lambda)}\mathrm{E}(\tau^*(t)), \sqrt{2\delta(t,\lambda)})$, which makes the mean of the process correct as well as the right-hand tail. We also plot these densities on the log scale. The tail approximation works very well for small values of λ and less well when λ is large.

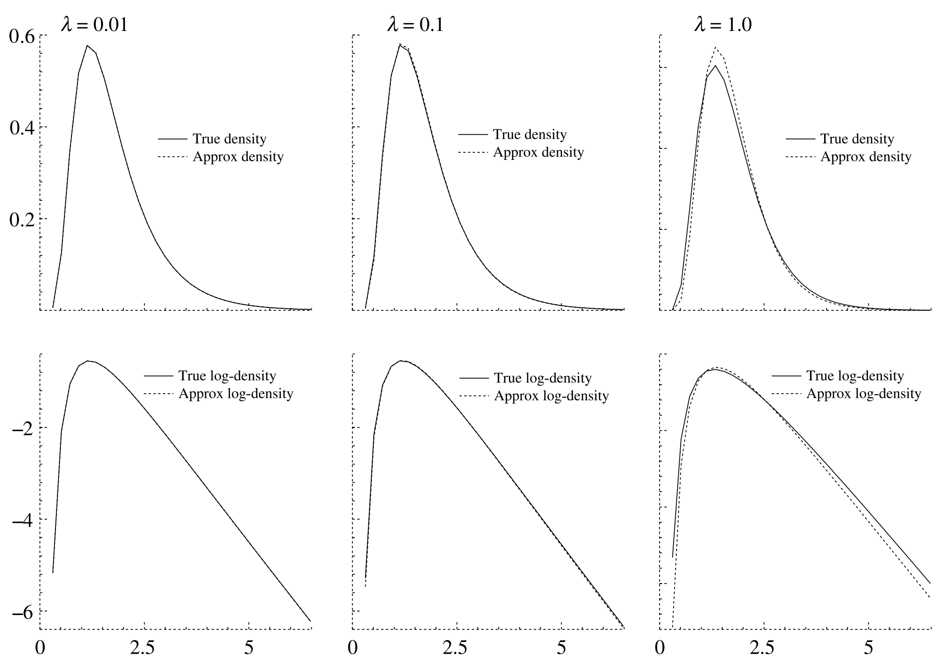

Fig. 3. The density $p(x)$ (top graphs) and $\log p(x)$ (bottom graphs) of $\tau^*(1)$, where $\tau(t)$ of the OU process is distributed as $\mathrm{IG}(\sqrt{2\pi}, \sqrt{2})$. The dotted line is the upper tail approximation $IG(\sqrt{2\delta(t,\lambda)}\mathrm{E}(\tau^*(1)), \sqrt{2\delta(t,\lambda)})$. The left-hand side graphs have $\lambda = 0.01$, the middle 0.1 and the right-hand side graphs have $\lambda = 1$.

4. Superposition of intOU processes

In practical application, it is often helpful to allow for more flexible dynamic structures. A simple and mathematically tractable way of doing this is by adding together two (or more) independent OU processes (Barndorff-Nielsen & Shephard, 2001).

It is helpful in thinking about this issue to work with the IG(δ, γ)-OU case, with

$$\tau(t) = \sum_{j=1}^{2} \tau_j(t), \quad \text{where } \tau_j(t) \sim \text{IG}(\delta w_j, \gamma)\text{-OU},$$

where the weights $\{w_j\}$ are strictly positive and sum to one, while the corresponding damping values are $\{\lambda_j\}$. Again, the tail behaviour of $\tau^*(t)$ will be as for the IG laws, as is simple to show.

5. Conclusions

In this paper, we have carefully studied some of the properties of integrated OU processes. The main focus has been on studying cumulant functions of $x^*(t)$ unconditionally and conditionally on $x(0)$. The results have important implications for their use in, for example, option pricing models. In particular, we asked in section 1 whether the modeller's control over the choice of τ, the spot volatility, would provide knowledge of the behaviour of τ^* and consequently the distribution of returns y_n. Our main analytic conclusion is that if $x(t)$ is TS-OU or Γ-OU then while $x^*(t)$ is not distributed exactly as TS its tails do have this behaviour. A special case of this analysis are the important inverse Gaussian-based models. These results are potentially important, for it means that stochastic volatility models built out of OU processes with gamma or inverse Gaussian marginals will have tails which behave like normal gamma or normal inverse Gaussian distributions.

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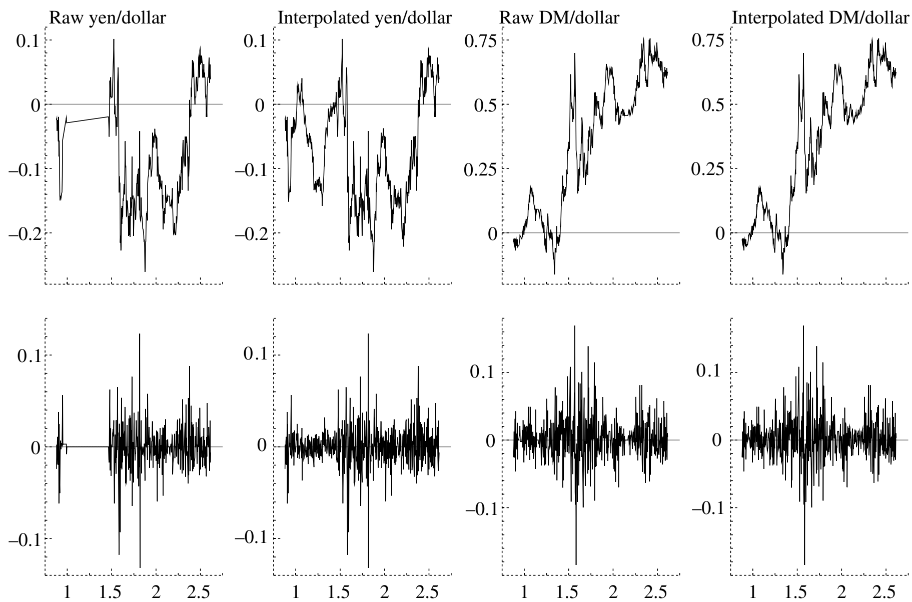

Fig. A1. Top line of graphs are the raw and interpolated data using a Brownian bridge interpolator. Bottom line of graphs are the corresponding returns. The x-axes are marked off in days.

Data appendix

The Olsen group have kindly made available to us a dataset that records every 5 min the most recent quote to appear on the Reuters screen from 1 December 1986 until 30 November 1996. When prices are missing they have interpolated them. Details of this processing are given in Dacorogna *et al.* (2001). The same dataset was analysed by Andersen *et al.* (2001b). We follow the extensive work of Torben Andersen and Tim Bollerslev on this dataset, who remove much of the times when the market is basically closed. This includes almost all of the weekend, while they have taken out most US holidays. The result is what we will regard as a single time series of length 705,313 observations. Although many of the breaks in the series have been removed, sometimes there are sequences of very small price changes caused by, for example, unmodelled non-US holidays or data feed breakdowns. We deal with this by adding a Brownian bridge simulation to sequences of data where at each time point the absolute change in a 5-min period is below 0.01 percent. That is, when this happens, we interpolate prices stochastically, adding a Brownian bridge with a standard deviation of 0.01 for each time period. By using a bridge process we are not effecting the long run trajectory of prices. Code to carry out this interpolation is in `mult_data.ox`. It is illustrated in Fig. A1, which shows the first 500 observations in the dollar/DM series together with another series on the yen/dollar. Later stretches of the data have less breaks in them; however, this graph illustrates the effects of our intervention. Clearly our approach is ad hoc. However, a proper statistical modelling of these breaks is very complicated due to their many causes and the fact that our dataset is enormous. And a more refined approach is unlikely to change the conclusions.