Integrated Volatility Measuring from Unevenly Sampled Observations

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Introduction

- DGP:

$$dp(t) = \mu(t)dt + \sigma(t)dW(t), \quad 0 \le t \le T$$

- unevenly sampled: $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$
- unevenly sampled observations $\{p(t_i)\}_{i=0}^{N} \Rightarrow$ integrated volatility $\int_0^T \sigma^2(t)dt$
- $\max_i\{t_i - t_{i-1}\} \to 0$ as $N \to 0$

Linear Interpolation

■ $\{p(t_i)\}_{i=0}^{N} \rightarrow$ Linear interpolation $\rightarrow \{q(iT/M)\}_{i=0}^{M}$

$$q\left(\frac{iT}{M}\right) = (1-\rho_i)p(t_i^-) + \rho_i p(t_i^+)$$

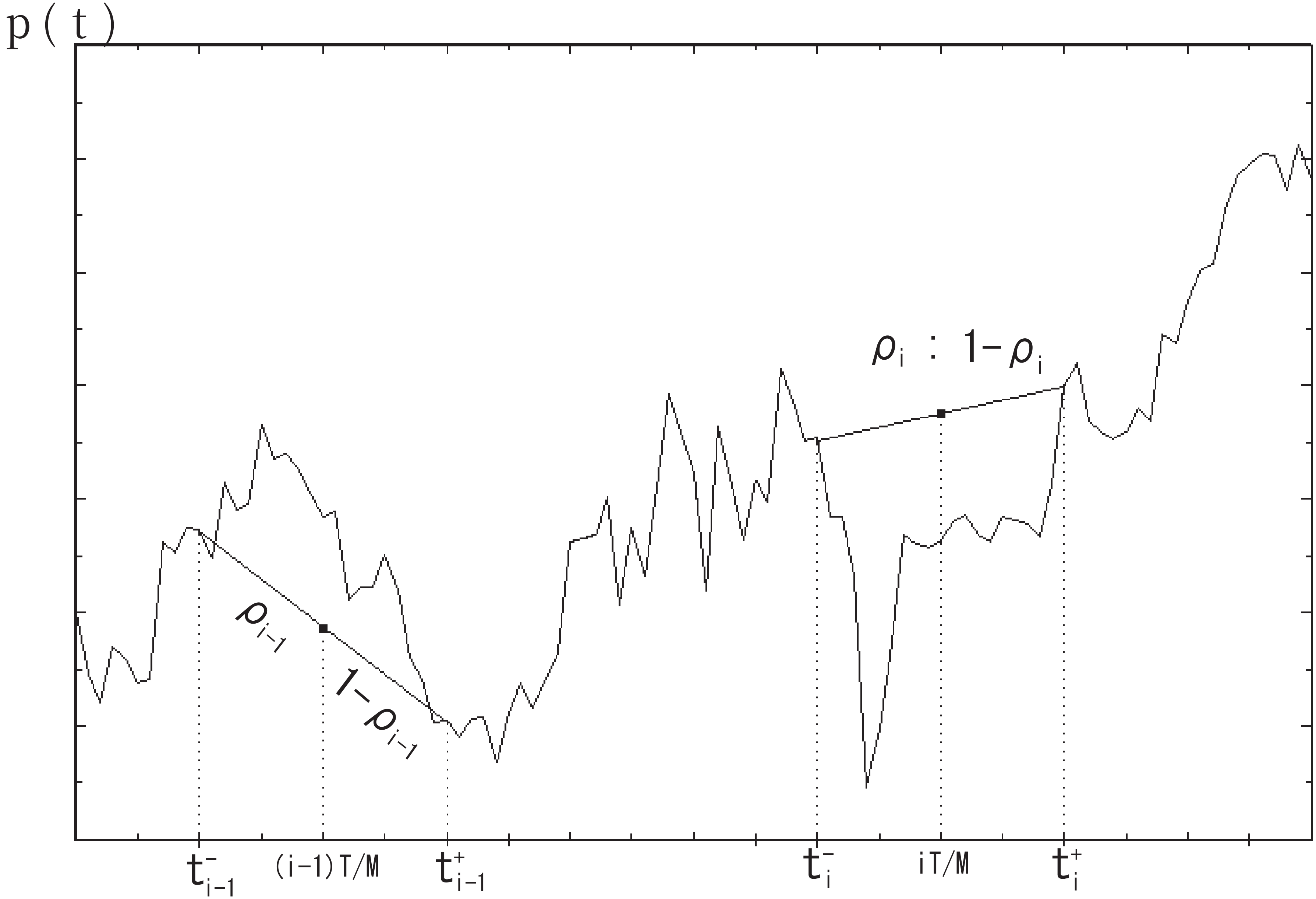
where $\rho_i = ((iT/M) - t_i^-)/(t_i^+ - t_i^-)$.

■ Linear interpolation Realized Volatility

$$\hat{\sigma}^2(M) = \sum_{i=1}^{M} \left\{ q\left(\frac{iT}{M}\right) - q\left(\frac{(i-1)T}{M}\right) \right\}^2$$

■ $bias = E[\hat{\sigma}^2(M)] - \int_0^T \sigma^2(t)dt$:

$$\sum_{i=0}^{M} E\left[\left\{ q\left(\frac{iT}{M}\right) - q\left(\frac{(i-1)T}{M}\right) \right\}^2 \right] - \int_{(i-1)T/M}^{iT/M} \sigma^2(t)dt$$


p (t)
ρ_i : 1−ρ_i
ρ_{i−1}
1−ρ_{i−1}
t⁻_{i−1}
(i−1)T/M
t⁺_{i−1}
t⁻_i
iT/M
t⁺_i

Downward Bias

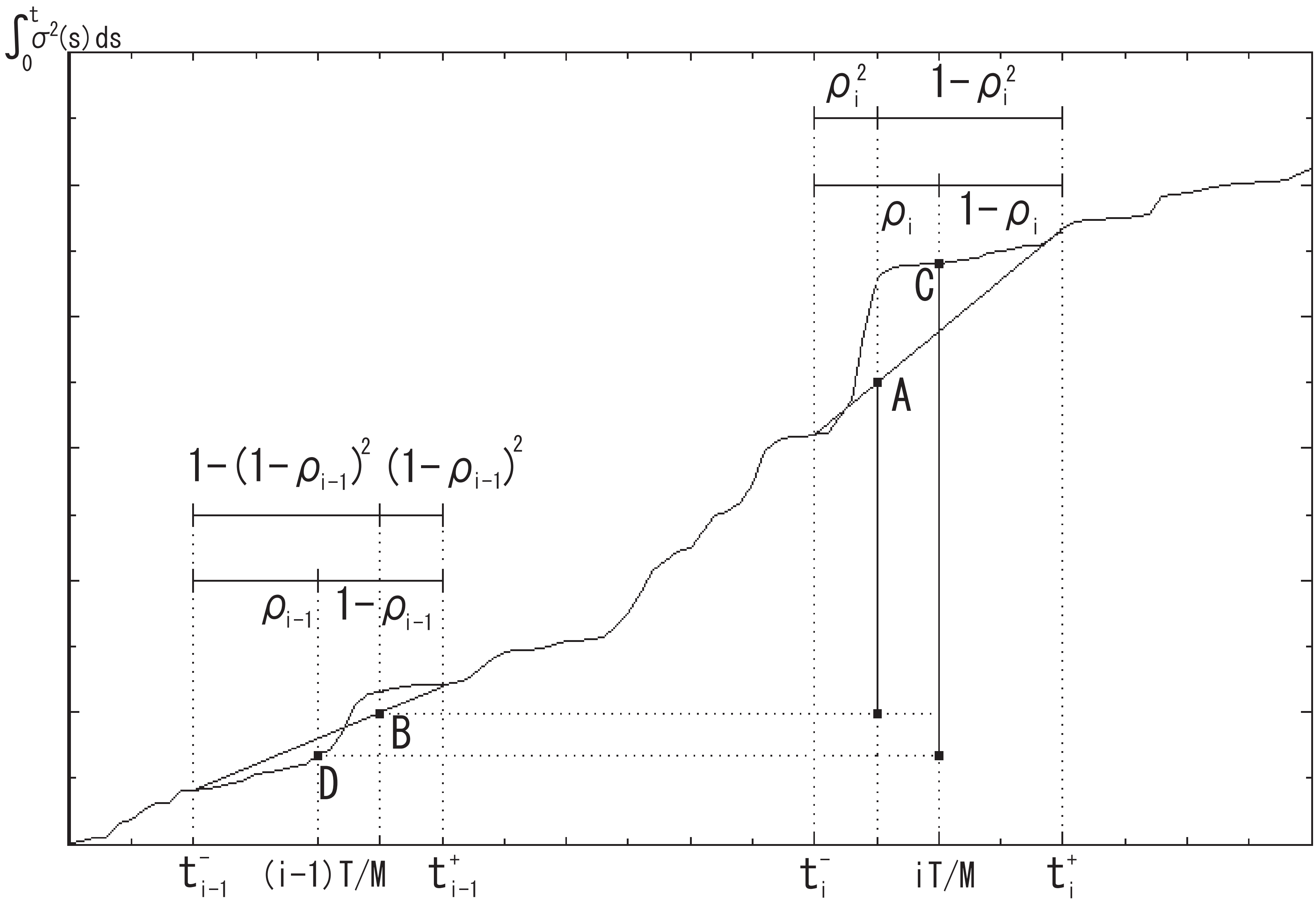
- ith term of the bias $E[\hat{\sigma}^2(M)] - \int_0^T \sigma^2(t)dt$

$$E\left[\left\{q\left(\frac{iT}{M}\right) - q\left(\frac{(i-1)T}{M}\right)\right\}^2\right] - \int_{(i-1)T/M}^{iT/M} \sigma^2(t)dt$$

$$= \overbrace{\left[\rho_i^2 \int_0^{t_i^+} \sigma^2(t)dt + (1-\rho_i^2)\int_0^{t_i^-} \sigma^2(t)dt\right]}^{A}$$

$$- \overbrace{\left[(1-(1-\rho_{i-1})^2)\int_0^{t_{i-1}^+} \sigma^2(t)dt + (1-\rho_{i-1})^2 \int_0^{t_{i-1}^-} \sigma^2(t)dt\right]}^{B}$$

$$- \overbrace{\int_0^{iT/M} \sigma^2(t)dt}^{C} + \overbrace{\int_0^{(i-1)T/M} \sigma^2(t)dt}^{D}$$


$\int_0^t \sigma^2(s)\,ds$
ρ_i^2
$1-\rho_i^2$
ρ_i
$1-\rho_i$
C
A
$1-(1-\rho_{i-1})^2$
$(1-\rho_{i-1})^2$
ρ_{i-1}
$1-\rho_{i-1}$
B
D
t_{i-1}^-
$(i-1)T/M$
t_{i-1}^+
t_i^-
iT/M
t_i^+

Downward Bias

■ $E[\hat{\sigma}^2(M)] - \int_0^T \sigma^2(t)dt$:

(1) $$-\sum_{i=1}^{M}\{\rho_{i-1}(1-\rho_{i-1})(t_{i-1}^{+} - t_{i-1}^{-}) + \rho_i(1-\rho_i)(t_i^{+} - t_i^{-})\} \times \frac{M}{T}\int_{(i-1)T/M}^{iT/M}\sigma^2(t)dt$$

Fourier Series Estimator of Malliavin and Mancino (2002)

■ Fourier representation of $\sigma^2(t)$:

$$\sigma^2(t) = \sum_{k=0}^{\infty} a_k(\sigma^2)\cos(kt) + b_k(\sigma^2)\sin(kt)$$

where $a_k(\sigma^2)$ and $b_k(\sigma^2)$ are Fourier coefficients of $\sigma^2(t)$.

■ $\int_0^{2\pi} \sigma^2(t)dt = 2\pi a_0(\sigma^2)$

■ Malliavin and Mancino (2002)

$$a_0(\sigma^2) = \lim_{K\to\infty} \frac{\pi}{2(K+1-k_0)} \sum_{k=k_0}^{K} a_k^2(dp) + b_k^2(dp)$$

where $a_k(dp)$ and $b_k(dp)$ are Fourier coefficients of $dp(t)$.

■ $\hat{\sigma}_F^2 \equiv \{\pi^2/(K+1-k_0)\} \sum_{k=k_0}^{K} a_k^2(dp) + b_k^2(dp)$.

Realized Volatility

- $$\hat{\sigma}^2 = \sum_{i=1}^{N} \left(p(t_i) - p(t_{i-1})\right)^2$$

Conjecture 1.

$$\hat{\sigma}_F^2 - \hat{\sigma}^2 = O_p(N^{-1})$$

Conjecture 2.

$$V(\hat{\sigma}_F^2) = O_p(N^{-1}), \quad V(\hat{\sigma}^2) = O_p(N^{-1})$$

- $\hat{\sigma}_F^2 - \int \sigma^2(t)dt = O_p(N^{-1/2})$, $\hat{\sigma}^2 - \int \sigma^2(t)dt = O_p(N^{-1/2})$
  1. $\hat{\sigma}_F^2 - \hat{\sigma}^2 \to 0$
  2. $\hat{\sigma}_F^2 \to \int \sigma^2(t)dt$, $\hat{\sigma}^2 \to \int \sigma^2(t)dt$

Monte Carlo Study

■ DGP:

$$d \log \sigma^2(t) = -\kappa \log \sigma^2(t) dt + \gamma dW(t)$$
$$dp(t) = \sigma(t) dW(t), \quad 0 \le t \le 60 \times 60 \times 24 \text{ seconds}$$

where $\kappa = 0.01, \gamma = 0.1$.

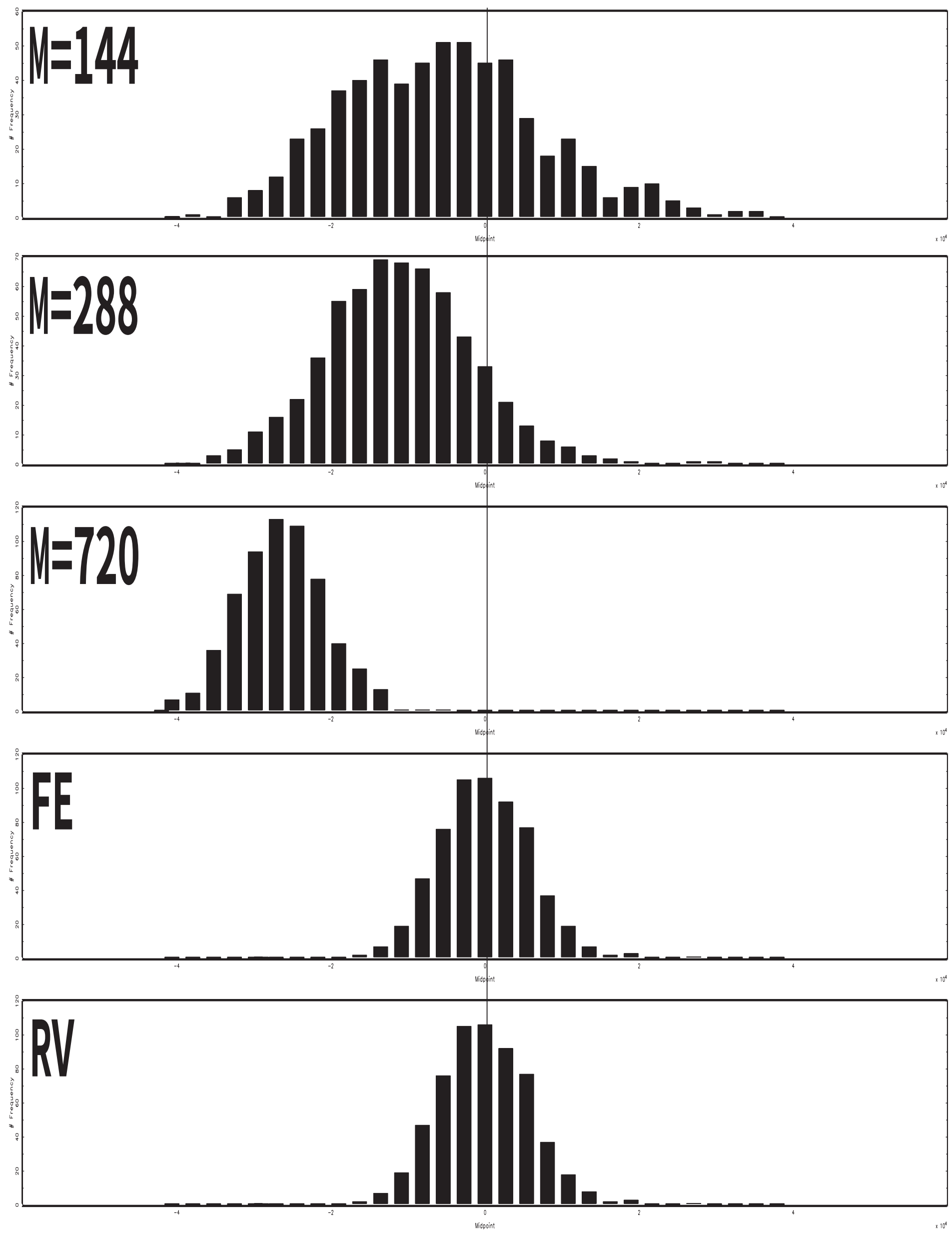
■ Time difference $t_i - t_{i-1}$ are drawn from an exponential distribution with mean qual to 45 seconds:[*1]

(2) $$F(t_i - t_{i-1}) = 1 - e^{-\lambda(t_i - t_{i-1})}$$

where $\lambda = 1/45$.

■ Distributions of $\hat{\sigma}^2(M) - \int \sigma^2(t)dt (M = 144, 288, 720)$, $\hat{\sigma}_F^2 - \int \sigma^2(t)dt$, and $\hat{\sigma}^2 - \int \sigma^2(t)dt$

[*1] Barucci and Renò (2002); Aït-Sahalia and Mykland (2003); Engle and Russell (1998)



	-error	-bias (1)
$\hat{\sigma}^2(144)$	5684	5420
	(13000)	(502.0)
$\hat{\sigma}^2(288)$	10960	10900
	(9628)	(766.2)
$\hat{\sigma}^2(720)$	26480	27320
	(5762)	(1529)
$\hat{\sigma}^2_F$	2.830	
	(5954)	
$\hat{\sigma}^2$	2.864	
	(5954)	

	computational time
FE	1116.42 seconds
RV	0.25 seconds

Conclusion

- Linear interpolation bias of LRV
- Use RV rather than FE in the scaler case

Remaining Works

- Conjectures
- multivariate case

Acknowledgements

- Thank you very much for your attention.

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