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INTEGRATING OVER HIGGS BRANCHES

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We develop some useful techinques for integrating over Higgs branches in supersymmetric theories with 4 and 8 supercharges. In particular, we define a regularized volume for hyperkahler quotients. We evaluate this volume for certain ALE and ALF spaces in terms of the hyperkahler periods. We also reduce these volumes for a large class of hyperkahler quotients to simpler integrals. These quotients include complex coadjoint orbits, instanton moduli spaces on $\mathbb{R}^4$ and ALE manifolds, Hitchin spaces, and moduli spaces of (parabolic) Higgs bundles on Riemann surfaces. In the case of Hitchin spaces the evaluation of the volume reduces to a summation over solutions of Bethe Ansatz equations for the non-linear Schrödinger system. We discuss some applications of our results.

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1. Introduction

In this note we study integrals over Higgs branches. They are kahler and hyperkahler quotients, depending on the amount of supersymmetry in the theory.

Our original motivation was inspired by the works [1] on S-duality and [2] on the action of affine algebras on the moduli spaces of instantons which have been subsequently developed in many ways [3][4][5]. In particular, the series of papers [4][6][7] arose out of attempts to give a field-theoretic interpretation of the results of [2]. Subsequently, string duality ideas turned out to be more adequate for addressing these questions [8][9] although in all cases ([4][8][9] and more recently [10]) quantum mechanics on the moduli space of instantons plays a crucial rôle.

The four-dimensional version of the WZW theory described in [4] depends on a choice of a 2-form ω. The path-integral of the gauged WZW_4 is related to the four-dimensional Verlinde number [4]. If we study the theory in the $\omega \to \infty$ limit then the Verlinde number reduces, formally, to the symplectic volume of the moduli space of instantons on a four-manifold. In some cases this moduli space is a hyperkahler quotient by the ADHM construction.

If a type IIA 5brane wraps a K3 surface then the resulting string may be identified with the heterotic string [11][12]. The natural question then arises as to whether one can wrap IIA 5branes on ALE spaces to produce other types of strings.[1] A first objection to this idea is that the infinite volume of the ALE space leads to an infinite string tension. However, it is possible that with a regularized version of the volume, e.g., that described in this paper other well-defined string theories can be obtained from wrapped IIA 5branes.

The natural hyperkahler quotients to study in the context of string duality are Hitchin spaces. They occur as spaces of collective coordinates, describing the bound states of $D2$-branes, wrapping a curve Σ in K3. It is clear that a thorough study of these spaces is important in various aspects of supersymmetric gauge theories [14][15].

In all cases we define the regularized volumes of these spaces and compute them using localization techniques. The localization with respect to different symmetries produces different formulas, thereby establishing curious identities. On this route one may get certain sum rules for solutions of Bethe Ansatz equations [16][17].

The last motivation stems from the recent studies of the Matrix theory of IIA five-branes. The Higgs branch of such theory is described by a two dimensional sigma model

[1] Such objects have recently appeared in [13].

with target space being the moduli space of instantons. The volume of target space is a characteristic of the ground state wave function. The integrals of some cohomology classes over the moduli space of instantons like those which we present below can be translated to give explicit results for correlations functions of chiral fields in $(2,0)$ six dimensional theories in the light-cone description [10].

The paper is organized as follows. In section 2 we review very briefly the localization approach and describe the quotients we are going to study. In section 3 we present our definitions of regularized integrals. Sections $4-7$ are devoted to explicit examples.

2. Quotients and Localization

2.1. Kähler and hyperkähler quotients

A kahler manifold X is by definition a complex variety with a hermitian metric g, such that the corresponding two-form $\omega(\cdot,\cdot) = g(I\cdot,\cdot)$ is closed. The complex structure I, viewed as the section of $\mathrm{End}(T^{\mathbb{C}}X)$ is covarianly closed $\nabla I = 0$, for the Levi-Civita connection ∇. A hyperkahler manifold X has three covariantly constant complex structures I, J, K which obey quaternionic algebra: $I^2 = J^2 = K^2 = IJK = -1$ which are such that in any of the complex structures the manifold X is complex. It follows that $\omega^r = g(I\cdot,\cdot)$ is the Kähler form w.r. t. complex structure I while $\omega^c = g(J\cdot,\cdot)+\sqrt{-1}g(K\cdot,\cdot)$ is a closed $(2,0)$-form i.e., a holomorphic symplectic structure. See [18] for a detailed introduction to the subject. Whenever a compact group K acts on a symplectic manifold (X,ω) preserving its symplectic structure one may attempt to define a reduced space. To this end the existence of an equivariant moment map $\mu : X \to \underline{\mathbf{k}}^*$ is helpful. The latter is defined as follows: $d\langle\mu,\xi\rangle = -\iota_{V_\xi}\omega$ where $\xi \in \underline{\mathbf{k}}$ and $V_\xi \in \mathrm{Vect}(X)$ is the corresponding vector field. In other words, $\langle\mu,\xi\rangle$ is the hamiltonian generating the action of ξ. The equivariance condition means that $\mu(g\cdot x) = Ad^*_g\mu(x)$ for any $g \in K$. The existence of μ is guaranteed in the situation where X is simply-connected while the equivariance depends on the triviality of a certain cocycle of the Lie algebra of Hamiltonian vector fields on X.

The symplectic quotient is the space $X//G = \mu^{-1}(0)/K$. In the case where K acts on kahler manifold preserving both its metric and complex structure the space $X//K$ is kahler as well. If the group K acts on a hyperkahler manifold preserving all its structures then the moment map is extended to the hyperkahler moment map: $\vec{\mu} : X \to \underline{\mathbf{k}}^* \otimes \mathbb{R}^3$, defined by

$$d\langle\vec{\mu},\xi\rangle = -\iota_{V_\xi}\vec{\omega} \tag{2.1}$$

where $\vec{\omega} \in \Omega^2(X) \otimes \mathbb{R}^3$ is the triplet of symplectic forms $g(I\cdot,\cdot), g(J\cdot,\cdot), g(K\cdot,\cdot)$. It is a theorem of [18] that the hyperkahler quotient $X////K = \vec{\mu}^{-1}(0)/K$ is itself a hyperkahler manifold.

In gauge theories with gauge group K, the quotient X/K arises as (a Higgs branch of) the moduli-space-of-vacua limit of bosonic gauge theory (whenever spontaneous symmetry breaking takes place), the kahler quotient $X//K$ occurs in the theory with 4 supercharges, the hyperkahler quotients appear in the theories with 8 supercharges.

One is tempted to define "octonionic quotients" for the purposes of the theories with sixteen supercharges, but both the notation (8 slashes) and the lack of time make us leave it for future investigations.

The original space X in gauge theories is often linear. So it is instructive to play with linear quotients first. The 0 in $\mu^{-1}(0)$ can be replaced by the central elements ζ, $\vec{\zeta}$ in $\underline{\mathbf{g}}^*, \underline{\mathbf{g}}^* \otimes \mathbb{R}^3$. In gauge theories these are called Fayet-Illiopoulos terms. The quotients $\mu^{-1}(\zeta)/K$ and $\vec{\mu}^{-1}(\vec{\zeta})/K$ will be denoted as $\mathcal{M}(\zeta)$ and $\mathcal{M}(\vec{\zeta})$ when this will not lead to confusion.

2.2. Preliminaries: Hyperkahler representations

Let K be compact group of $\dim K = k$, unitarily represented on a hermitian vector space V of complex dimension ℓ. K acts via antihermitian matrices T_A. In a natural way $V \oplus V^*$ is hyperkahler and the K-action

$$\delta_A(z^\alpha; w_\alpha) = \{(T_A)^\alpha{}_\beta z^\beta; -w_\alpha (T_A)^\alpha{}_\beta\} \tag{2.2}$$

is $\mathbb{H}$-linear. This motivates the introduction of quaternionic coordinates:

$$\begin{gathered} X^\alpha = z^\alpha + J w_\alpha = \begin{pmatrix} z^\alpha & \bar{w}^\alpha \\ -w_\alpha & \bar{z}_\alpha \end{pmatrix} \\ \delta_A X^\alpha = (\tau_A)^\alpha{}_\beta X^\beta \end{gathered} \tag{2.3}$$

where

$$(\tau_A)^\alpha{}_\beta = \begin{pmatrix} (T_A)^\alpha{}_\beta & 0 \\ 0 & \left[(T_A)^\alpha_\beta\right]^* \end{pmatrix} \tag{2.4}$$

On the linear space we have Kahler form and holomorphic symplectic form:

$$\begin{aligned} \omega^r &= \frac{i}{2} \sum [dz^\alpha d\bar{z}_\alpha + dw_\alpha d\bar{w}^\alpha] \\ \omega^c &= \sum dz^\alpha \wedge dw_\alpha \end{aligned} \tag{2.5}$$

We regard exterior derivatives as fermions:

$$dX \equiv \Psi^\alpha = \begin{pmatrix} \psi^\alpha & \bar{\chi}^\alpha \\ -\chi_\alpha & \bar{\psi}_\alpha \end{pmatrix} \tag{2.6}$$

We can also define exterior differentials transforming as a triplet under the right action of $SU(2)$. Together the four derivatives are:

$$\begin{aligned} \delta_n X &= \Psi n \\ \delta_n X^\dagger &= \bar{n}\Psi^\dagger \end{aligned} \tag{2.7}$$

here n is any quaternion. Now let ρ be antihermitian of norm 1, so $\rho = u^\dagger i\sigma_3 u$ for some unitary u. Then the holomorphic exterior differentials in the direction ρ are:

$$\begin{aligned} \partial^{(\rho)} &= \tfrac{1}{2}(d + \delta_\rho) \\ \bar{\partial}^{(\rho)} &= \tfrac{1}{2}(d - \delta_\rho) \end{aligned} \tag{2.8}$$

Finally, the (hyper)-kahler potential is

$$\kappa = \tfrac{1}{2}\mathrm{Tr} X^\dagger X \tag{2.9}$$

and satisfies:

$$\partial^{(\rho)}\bar{\partial}^{(\rho)}\kappa = \omega^{(\rho)} = -\tfrac{1}{2}\mathrm{Tr}\rho\Psi^\dagger\Psi \tag{2.10}$$

giving the Kahler form associated to the complex structure ρ.

The moment map for the K-action:

$$d\vec{\mu}_A + \iota(V_A)\vec{\omega} = 0 \tag{2.11}$$

is given explicitly by:

$$\begin{aligned} X^\dagger_\alpha (\tau_A)^\alpha{}_\beta X^\beta &= 2\begin{pmatrix} \mu^r_A & -\overline{\mu^c_A} \\ \mu^c_A & -\mu^r_A \end{pmatrix} \\ 2\mu^r_A &= \bar{z}_\alpha (T_A)^\alpha{}_\beta z^\beta - w_\alpha (T_A)^\alpha{}_\beta \bar{w}^\beta \in \sqrt{-1}\mathbb{R} \\ \mu^c_A &= w_\alpha (T_A)^\alpha{}_\beta z^\beta \end{aligned} \tag{2.12}$$

We finally construct the hyperkahler quotient. Let $\vec{\mu}^{-1}(\vec{\zeta}) = \mu_r^{-1}(\zeta^r) \cap \mu_c^{-1}(\zeta^c)$ where $\vec{\zeta}_A$ is nonzero only for central generators T_A. Then

$$\mathcal{M}(\vec{\zeta}) = \vec{\mu}^{-1}(\vec{\zeta})/K \tag{2.13}$$

is of real dimension $4\ell - 4\dim K = 4(\ell - k) = 4m$. As complex manifolds we may write [18]:

$$\mathcal{M}(\zeta^c) = \mu_c^{-1}(\zeta^c)/K_{\mathbb{C}} \tag{2.14}$$

after deleting the unstable points.

2.3. Review of CohFT approach and localization

This section is devoted to a review of mathematical and physical realizations of integration over the quotients in various (real, complex, quaternionic and octonionic) cases. Consider a manifold X with an action of a group K. Suppose that submanifold $N \subset X$ is K-invariant and is acted on by K freely. The problem we adress is how to present the integration over the quotient $\mathcal{M} = N/K$ in terms of the one over X. Let x^μ denote the coordinates on X. Their exterior derivatives are denoted as ψ^μ. Differential forms $a \in \Omega^*(X)$ on X are regarded as functions of (x, ψ). Equivariant forms are equivariant maps:

$$\alpha : \underline{\mathbf{k}} \to \Omega^*(X) \tag{2.15}$$

of the Lie algebra to the differential forms:

$$\alpha(\mathrm{Ad}_g \phi) = g^* \alpha(\phi) \tag{2.16}$$

where in the right hand side g represents the action of an element $g \in K$ on X and correspondingly on $\Omega^*(X)$. The equivariant derivative

$$D \equiv d + \iota_{V_\phi} \tag{2.17}$$

squares to zero on the space of equivariant forms, which we denote as $\Omega^*_K(X)$.

The answer to our problem is positive in the important case where the K-invariant submanifold N is realized as a set of zeroes of a K -equivariant section of a K -equivariant bundle $E \to X$: $N = s^{-1}(0)$ for $s : X \to E$. In this case it is possible to represent the integration over $\mathcal{M}$ in terms of integrals over X and some auxilliary supermanifolds. See [19][20][21][22]. Endow E with a K-invariant metric $(,)$ and let A be the connection which is compatible with it. One needs a multiplet (χ, H) of "fields" of opposite statistics taking values in the sections of E. The equivariant derivative D acts as follows:

$$\begin{aligned} D\chi =& H - \psi^\mu A_\mu \cdot \chi \\ DH =& T_\phi \cdot \chi - \psi^\mu A_\mu \cdot H - \psi^\mu \psi^\nu \mathcal{F}_{\mu\nu} \cdot \chi \end{aligned} \tag{2.18}$$

where T_ϕ represents the action of $\phi \in \underline{\mathbf{k}}$ on the fibers of E and $\mathcal{F}_{\mu\nu}$ is the curvature of A. One also needs the "projection multiplet" $(\bar{\phi}, \eta)$ with values in $\underline{\mathbf{k}}$ with the action of D as: $D\bar{\phi} = \eta, D\eta = [\phi, \bar{\phi}]$.

In the situation we are in, namely kahler and hyperkahler quotients the bundle E is the topologically trivial bundle $E = X \times \underline{\mathbf{g}}^*$ or $E = X \times \underline{\mathbf{g}}^* \otimes \mathbb{R}^3$ respectively. The section

s is simply $\mu - \zeta$ and $\vec{\mu} - \vec{\zeta}$ respectively. Let $g_{\mu\nu}$ be any K-invariant metric on X. Define a $\underline{\mathbf{k}}^*$ valued one-form β as follows: $\beta(\xi) = \frac{1}{2}g(V_\xi, \cdot)$ for any $\xi \in \underline{\mathbf{k}}$. In coordinates (x, ψ) it is simply : $\beta(\xi) = \frac{1}{2}g_{\mu\nu}\psi^\mu V_\xi^\nu$.

Equivariant cohomology of X maps to the ordinary cohomology of $\mathcal{M}$. Indeed, there is an equivariant inclusion map $i : N \to X$ hence the map $i^* : H_K^*(X) \hookrightarrow H_K^*(N)$ and by the assumption that the action of K on N is free we get an isomorphism $I : H_K^*(N) \approx H^*(\mathcal{M})$. The cohomology classes of $\mathcal{M}$ among others contain the characteristic classes of the K bundle $N \to \mathcal{M}$ which are in one-to-one correspondence with the invariant polynomials $P(\phi)$ on $\underline{\mathbf{k}}$. These correspond to the equivariant forms $P(\phi) \in \Omega^0(X)$. So let us compute the integrals over $\mathcal{M}$ of the cohomology classes which are in the image of the map $I \circ i^*$. Let $\alpha_1, \ldots, \alpha_l$ be the representatives of the classes in $H_K^*(X)$, so that $\alpha_p(\phi) \in \Omega^*(X)$. Let $\Theta_p = I \circ i^* \alpha_p$. Putting all things together we may represent the integrals of such classes over $\mathcal{M}$ as:

$$\int_{N/K} \Theta_1 \wedge \ldots \wedge \Theta_l = \int_X \frac{\mathcal{D}\chi \mathcal{D}H \mathcal{D}x \mathcal{D}\psi \mathcal{D}\phi \mathcal{D}\bar{\phi} \mathcal{D}\eta}{\mathrm{Vol}(K)} e^{iD\left(\chi \cdot s + \beta(\bar{\phi})\right)} \alpha_1(\phi) \wedge \ldots \wedge \alpha_l(\phi) \quad (2.19)$$

In case where X is a symplectic manifold then the form $\alpha = \omega + \langle \mu, \phi \rangle \in \Omega_K^*(X)$ is equivariantly closed and gives rise to a symplectic form ϖ on the reduced space $X//K$. Similarly, in the hyperkahler case the triplet of equivariantly closed forms $\vec{\alpha} = \vec{\omega} + \langle \vec{\mu}, \phi \rangle$ produce a triplet of kahler forms $\vec{\varpi}$ on the quotient $X////K$.

So the symplectic volume of the quotient is[2]

$$\int_{X//K} e^{\varpi} = \int_{\underline{\mathbf{k}}} \frac{\mathcal{D}\phi}{\mathrm{Vol}(K)} \int_{\underline{\mathbf{k}} \oplus \Pi \underline{\mathbf{k}} \oplus \underline{\mathbf{k}} \oplus \Pi \underline{\mathbf{k}}} \mathcal{D}\chi \mathcal{D}H \mathcal{D}\eta \mathcal{D}\bar{\phi} \int_X e^{\omega + i\langle \mu, \phi \rangle} e^{i\langle H, \mu \rangle + i\langle \chi, d\mu \rangle - D(\beta(\bar{\phi}))} \quad (2.20)$$

The specifics of this case is the fact that the auxilliary fields $\eta, \chi, H, \bar{\phi}$ come in a quartet and sometimes can be integrated out by introducing the D-exact term

$$e^{itD(\langle \chi, \bar{\phi} \rangle)}$$

into the exponent. In the limit $t \to \infty$ this term dominates over $D\left(\langle \chi, \mu \rangle - \beta(\bar{\phi})\right)$ and allows to get rid of the auxilliary multiplet with the result:

$$\int_{X//K} e^{\varpi} \Theta_1 \ldots \Theta_l \sim \int_{\underline{\mathbf{k}}} \frac{\mathcal{D}\phi}{\mathrm{Vol}(K)} \int_X e^{\omega + i\langle \mu, \phi \rangle} \alpha_1(\phi) \wedge \ldots \wedge \alpha_l(\phi) \quad (2.21)$$

[2] For later convenience we put an i in front of ϕ - a kind of Wick rotation

where now the integral over ϕ should be understood as the contour one and the choice of contour is the subtle memory of the eliminated quartet. See [21][23] for a thorough discussion of these matters.

Another instance where auxilliary fields come in quartets is in hyperkahler reduction. Indeed, the multiplets $(\eta, \bar{\phi}) \oplus (\chi, H)$ take values in $\underline{\mathbf{k}} \oplus \underline{\mathbf{k}} \otimes \mathbb{R}^3$ which makes possible to introduce the "mass term" of the form:

$$e^{it_1 D(\langle \chi^r, \bar{\phi} \rangle) + it_2 D(\langle \chi^c, \bar{H}^c \rangle + \langle \bar{\chi}^c, H^c \rangle)}, t_1, t_2 \to \infty$$

By taking $t_1 \to \infty$ and integrating out H^c first we arrive at the analogue of (2.21):

$$\int_{X////K} e^{\varpi^r} \Theta_1 \wedge \ldots \Theta_l =$$
$$\int_{\underline{\mathbf{k}}} \frac{\mathcal{D}\phi}{\mathrm{Vol}(K)} \int \frac{\mathcal{D}^2 \chi^c}{t_2^k} \int_X e^{\omega + \langle \mu, \phi \rangle - \frac{|\mu^c|^2}{t_2} + \langle \bar{\chi}^c, d\mu^c \rangle + \langle \chi^c, d\bar{\mu}^c \rangle + \langle \phi, [\bar{\chi}^c, \chi^c] \rangle} \alpha_1(\phi) \wedge \ldots \wedge \alpha_l(\phi) \tag{2.22}$$

Had we naively integrated out χ^c by dropping the $\langle \chi^c, d\bar{\mu}^c \rangle$ term we would get an extra determinant $\mathrm{Det}(\mathrm{ad}\phi)$ which vanishes due to the zero modes of $\mathrm{ad}(\phi)$. In fact the mass term does not allow to eliminate the whole of $\chi^c, \bar{\chi}^c$ etc. but rather only its $\underline{\mathbf{k}}/\underline{\mathbf{t}}$ part, where $\underline{\mathbf{t}}$ is the Lie algebra of the maximal torus, corresponding to ϕ[3]. So we conclude that the resulting integral without the octet of auxilliary fields may be ill-defined. In fact, it might have been ill-defined from the very begining. We haven't discussed so far the issue of compactness of the quotient space. As soon as it is compact we expect the manipulations we performed to be reasonable and leading to the correct answer. But what if it is not? We shall discuss it in the next section but here let us proceed assuming that the non-compactness can be cured in the D-invariant manner.

In this case the other side of the story is the possibility to express the integrals (2.19)(2.20) in terms of the fixed points of K action on X. This feature comes about due to the presence of the term $D(g_{\mu\nu}\psi^\mu V^\nu_{\bar{\phi}}) = g(V_\phi, V_{\bar{\phi}}) + \ldots$ in the exponential which we can scale up by any amount we wish due to its D-exactness. Therefore the integrand is peaked near the zeroes of V and can be evaluated using semi-classical approximation in the directions transverse to the fixed loci. We present the resulting formulae below.

[3] There is also a subtlety in rotating ϕ to the torus but it is irrelevant for us here.

2.4. Normal form theorems

Here we consider the hyperkahler integral (2.20) and treat it a bit differently. Namely we integrate out $\chi^r, \vec{H}, \bar{\phi}, \eta$ and get the representation for the measure using $(x, \psi, \phi, \lambda \equiv \chi^c)$ only. This exercise is a good warm-up example yet it is an illustration of the principle of "killing the quartets." Consider the setup:

$$\begin{array}{ccc} \mu_c^{-1}(\zeta^c) & \hookrightarrow & X = V \oplus V^* \\ \downarrow p & & \\ \mathcal{M}_\zeta & & \end{array} \tag{2.23}$$

A $K_{\mathbb{C}}$-equivariant tubular neighborhood of the level set will be modeled on a neighborhood of $(\mu^c)^{-1}(\zeta^c) \times \underline{\mathbf{k}}_{\mathbb{C}}$ which in turn is modelled on a neighborhood of:

$$\mathcal{M}_\zeta \times T^* K_{\mathbb{C}} \tag{2.24}$$

Let $\varpi_r, \varpi_c, \bar{\varpi}_c$ be the three kahler forms on $\mathcal{M}_\zeta$. Then in a $K_{\mathbb{C}}$- equivariant neighborhood of $\mu_c^{-1}(\zeta^c)$ we have

$$\omega_c = p^*(\varpi_c) + \sum_{A=1}^{\dim K} d\big(\langle \Theta_c^A, \mu_A^c \rangle\big) \tag{2.25}$$

where Θ_c^A is a $(1,0)$ connection form for the holomorphic principal $K_{\mathbb{C}}$ bundle $\mu_c^{-1}(\zeta^c) \to \mathcal{M}_\zeta$. We claim:

$$\begin{aligned} p^*\Big[(\varpi_c \wedge \bar{\varpi}_c)^m\Big] d\mu^{Haar}(K) = \Big[\omega_c \wedge \bar{\omega}_c\Big]^\ell \prod_{A=1}^{\dim K} \delta^{(2)}(\mu_A^c - \zeta_A^c)\delta(\mu_A^r - \zeta_A^r) \\ \int \prod_{A=1}^{k} d\lambda_A d\bar{\lambda}_A \exp\Big[\bar{\lambda}^A \lambda^B \Big(\bar{z}_\alpha \{T_B, T_A\}^\alpha{}_\beta z^\beta + w_\alpha \{T_B, T_A\}^\alpha{}_\beta \bar{w}^\beta \Big)\Big] \end{aligned} \tag{2.26}$$

Indeed, the complexified Lie algebra may be decomposed as

$$\underline{\mathbf{k}}_{\mathbb{C}} = \underline{\mathbf{k}} \otimes_{\mathbb{R}} \mathbb{C} = \underline{\mathbf{k}} \oplus i\underline{\mathbf{k}}$$

according to the decomposition of $K_{\mathbb{C}}$ into $K \cdot H$ where K is compact and H is from the coset $K \backslash K_{\mathbb{C}}$. Decompose the measure as

$$\Big[\omega_c \wedge \bar{\omega}_c\Big]^\ell = p^*\Big[(\varpi_c \wedge \bar{\varpi}_c)^m\Big]\Big[\prod_A \mid d\mu_c^A \mid^2 \prod_A \mid d\xi_c^A \mid^2\Big] \tag{2.27}$$

The action of the complex transformations on z, w is

$$\begin{aligned} \delta_{iA} z^{\alpha} &= i(T_A)^{\alpha}{}_{\beta} z^{\beta} \\ \delta_{iA} \bar{z}_{\alpha} &= i\bar{z}_{\beta}(T_A)^{\beta}_{\alpha} \\ \delta_{iA} w_{\alpha} &= -iw_{\beta}(T_A)^{\beta}_{\alpha} \\ \delta_{iA} \bar{w}^{\alpha} &= -i(T_A)^{\alpha}{}_{\beta} \bar{w}^{\beta} \end{aligned} \tag{2.28}$$

Evaluating the delta functions we get the ghost determinant:

$$\det_{AB}\left[\frac{\partial}{\partial \xi^B}\mu^R_A(\exp(i\xi^A T_A).(z,w)\right] \tag{2.29}$$

and using (2.28) to compute the variation of μ^r_A in the nilpotent directions gives the formulae. ♠

As an alternative proof we could restrict ω_r to $\mu_c = 0$ and then do ordinary Kahler reduction. The formula (2.26) is equivalent to the case $t_2 = 0$ of the formula (2.22) which is easily seen by shifting the ψ-variables[4], see the analogous manipulations in the section devoted to instanton moduli.

3. Definitions of the volume

In this section we are going to deal with the non-compactness which didn't allow us to perform a large t_2 evaluation of the formula (2.22). Indeed, $\mathcal{M}(\vec{\zeta})$ is in general noncompact, and of infinite volume in a sense of ordinary Riemannian geometry.

Suppose that the space $\mathcal{M}$ is acted on by a group H in a Hamiltonian way such that for some $\epsilon \in \underline{\mathbf{h}}$ the Hamiltonian $H_\epsilon = \langle \mu_h, \epsilon \rangle$ is sufficiently positive at infinity. Then the definition is

a.) The Hamiltonian regularized volume is:

$$\mathrm{Vol}_\epsilon(\mathcal{M}) = \int_{\mathcal{M}} e^{\varpi - H_\epsilon} \tag{3.1}$$

Consider the case where X is a linear space. This space always has a $U(1)$ symmetry which commutes with the action of the group K, namely $(z, w) \mapsto (e^{i\theta} z, e^{i\theta} w)$. This action does not preserve the hyperkahler structure but preserves the kahler form ω^r. If

[4] It has been remarked to us by A. Losev in 1995

$\zeta^c = 0$ then this action descends to the quotient space $\mathcal{M}(\zeta)$. In fact, the corresponding Hamiltonian coincides with the Kahler potential κ and it descends to the quotient even if $\zeta^c \neq 0$ since it is K -invariant. In this case the Hamiltonian regularized volume is called kahler regularized volume.

Another natural Hamiltonian is $H_\epsilon = \mathrm{Tr}\{\mu^r_A, \mu^c_B\}^c$ where $\{,\}^c$ is the Poisson bracket in the holomorphic symplectic structure. This regularization is called ghost regularization. It will in general differ from the kahler regularization if K is not simple.

We have for the hyperkahler metric: $\mathrm{vol}_g = (\varpi^r)^{2m} = (\varpi^c \wedge \bar{\varpi}^c)^m$ so the $\epsilon \to 0$ limit of (3.1) would give the Riemannian volume of $\mathcal{M}$ if only it existed. Quite analogously we can define regularized intersection numbers, taking the integrals (2.22) and inserting e^{-H_ϵ}. Notice that in order to *define* the kahler or ghost regularized volume we don't actually need ζ^c to vanish.

3.1. Doing integrals straightforwardly

Before addressing the issue of existence of Hamiltonians H_ϵ which generate some symmetries of the quotients $\mathcal{M}(\vec{\zeta})$ we can express the kahler and ghost regulated volumes as integrals over the corresponding linear space X using the normal form theorems. Applying (2.26) we have (we changed the notation $(\phi, H^c, \bar{H}^c) \to \vec{H}$):

$$\begin{gathered}\mathrm{Vol}_\epsilon(\mathcal{M}) = \int_{\mathcal{M}} (\varpi^c \wedge \bar{\varpi}^c)^m e^{-H_\epsilon} = \\ \frac{1}{\mathrm{Vol}(K)} \int_{\underline{\mathbf{k}} \otimes \mathbb{R}^3} d^3\vec{H} \int \prod_A d\chi^c_A d\bar{\chi}^c_A \int_{\hat{V} \oplus \hat{V}^*} \prod_{\alpha=1}^{\ell} d^4X^\alpha d^4\Psi^\alpha \exp\left[I_1 + I_{gh} + I_{reg}\right]\end{gathered} \tag{3.2}$$

where the action is given by

$$\begin{aligned}I_1 &= i\tfrac{1}{2}\mathrm{Tr}H^A X^\dagger \tau_A X + \tfrac{1}{2}\mathrm{Tr}\rho\Psi^\dagger\Psi + i\mathrm{Tr}H^A\zeta_A \\ I_{gh} &= \bar{\chi}^A\chi^B \left(\bar{z}_\alpha \{T_B, T_A\}^\alpha{}_\beta z^\beta + w_\alpha \{T_B, T_A\}^\alpha{}_\beta \bar{w}^\beta \right)\end{aligned} \tag{3.3}$$

and

$$I_{reg} = \begin{cases} -\epsilon \mathrm{Tr} X^\dagger X, & \text{kahler} \\ -\epsilon \mathrm{Tr} X^\dagger \tau_A \tau_A X, & \text{ghost} \end{cases}$$

and H^A, ρ, ζ_A are anti hermitian matrices

$$H^A = \begin{pmatrix} H^A_r & \overline{H^A_c} \\ -H^A_c & -H^A_r \end{pmatrix} \qquad \rho = \begin{pmatrix} \rho_r & \bar{\rho}_c \\ -\rho_c & -\rho_r \end{pmatrix} \tag{3.4}$$

(so ϕ_r, ρ_r are imaginary). Now, the above expressions are useful because they are Gaussian in X, Ψ, hence we can do the integrals. Doing the Gaussian integrals we arrive at the formula:

Theorem. The regularized symplectic volumes of Hyperkahler quotients are given by the formula

$$(2\pi)^{2\ell} \int_{\underline{\mathbf{k}}\otimes\mathbb{R}^3} \prod_{A=1}^{\dim K} d^3\vec{H}_A \int d\chi^c_A d\bar{\chi}^c_A \frac{\exp\left[i\mathrm{Tr}\vec{H}^A\vec{\zeta}_A\right]}{\det\mathcal{N}} \tag{3.5}$$

$$\mathcal{N} = \epsilon R + i\phi^A \otimes T_A + \kappa\bar{\eta}^A\eta^B 1 \otimes \{T_A, T_B\} \tag{3.6}$$

where R is the regulator matrix. It is 1 for kahler regularization and τ_A^2 for ghost regularization.

3.2. Doing integrals using localization

It turns out that one can considerably simplify the formula (3.5) in the case $\zeta^c = 0$. What is needed is the existence of a linear action of H such that μ^c is transformed non-trivially under its action. For the action of H to descend down to $\mathcal{M}$ it is sufficient for the equation $\mu^c = 0$ to be invariant under the action of H. By going to the maximal torus we may assume that H is the torus itself. In fact, we only need a one-dimensional subalgebra, generated by ϵ. With respect to this subalgebra the trivial bundle $\underline{\mathbf{k}}_{\mathbb{C}} \times X$ as well as its sub-bundle $\underline{\mathbf{t}}_{\mathbb{C}} \times X$ split as sums of line bundles. We require that the latter has no zero weight components. In this case there is a

Theorem. Suppose that $\zeta^c = 0$, and that the induced action $\Lambda_{\underline{\mathbf{k}}}(\epsilon)$ of $V_h(\epsilon)$ on $\underline{\mathbf{k}}_{\mathbb{C}}$ (which makes μ^c H-equivariant) is such that the operator $A = \Lambda_{\underline{\mathbf{k}}}(\epsilon) + \mathrm{ad}(\phi)$ is non-degenerate for generic ϕ. Then

$$\mathrm{Vol}_\epsilon(\mathcal{M}(\zeta^r) = \int_{\underline{\mathbf{k}}} \frac{\mathcal{D}\phi^r}{\mathrm{Vol}(K)} e^{i\phi^r\cdot\zeta^r} \frac{\mathrm{Det}_{\underline{\mathbf{k}}}\left(\Lambda_{\underline{\mathbf{k}}}(\epsilon) + i\mathrm{ad}\phi^r\right)}{\mathrm{Det}_{V\oplus V^*}\left(\Lambda_X(\epsilon) + i\phi^r_A T_A\right)} \tag{3.7}$$

Here we denoted by $\Lambda_X(\epsilon)$ the linear operator on X which is the derivative of $V_h(\epsilon)$ at $x = 0$. The measure in (3.7) is invariant under the adjoint action of K. It implies that the integral in (3.7) can be reduced to the integral over the maximal Cartan subalgebra $\underline{\mathbf{t}} \subset \underline{\mathbf{k}}$:

$$\mathrm{Vol}_\epsilon(\zeta^r) = \int_{\underline{\mathbf{t}}} \mathcal{D}t Z_\epsilon\left[\mathcal{M}(\zeta^r)\right](t)) \tag{3.8}$$

The measure Z_ϵ is the measure in (3.7) evaluated at the element $t \in \underline{\mathbf{k}}$ times the Vandermonde determinant and $\mathcal{D}t$ is the standard Euclidean measure. The proof of (3.7) involves some use of equivariant cohomology and all the essential ingredients are already explained in the previous section, except for the remark concerning the non-compactness of the quotient space. We said there that in order for the manipulations with "mass terms" to make sense some sort of compactness is required. Now the regularization we choose provides such a compactness in the sense that the forms we integrate exponentially fall at infinity. The formalism with equivariant cohomology must be slightly modified to take into account the action of the regulating group H. The derivative D gets promoted to $D_\epsilon = D + \iota_{V_h(\epsilon)}$. The troublesome term $D\langle \chi^c, \bar{H}^c \rangle$ becomes $D_\epsilon \langle \chi^c, \bar{H}^c \rangle$ which is non-degenerate on χ's by assumptions of the theorem.

Notice that (3.7) also suggests the interpretation of the equivariant volume of $\mathcal{M}(\vec{\zeta})$ as of the generating function of the interesection numbers of Chern classes of certain bundles over the "half" $M(\zeta)$ of $\mathcal{M}(\vec{\zeta})$ defined by the symplectic quotient of V by the action of K at the same level ζ^r.[5]

3.3. Hyperkähler regularization

The last definition is the most symmetric in a sense that it preserves the hyperkahler structure and is the direct analogue of the "equivariant volumes" of [25]. Unfortunately, this definition is rarely useful, because it requires a tri-holomorphic action on $\mathcal{M}$ of the torus of dimension $\frac{1}{4}\mathrm{dim}\mathcal{M}$ which is rather non-generic. Nevertheless, suppose that $\mathcal{M}(\vec{\zeta})$ is acted on by the torus $\mathbf{T}$ of the stated dimension and let $\vec{\epsilon} \in \underline{\mathbf{t}} \otimes \mathbb{R}^3$ (all examples studied in [26] obey this condition).

b.) Hyperkahler regularized volume:

$$\mathrm{Vol}_{\vec{\epsilon}}\left(\mathcal{M}(\vec{\zeta})\right) = \int_{\mathcal{M}(\vec{\zeta})} \mathrm{vol}_g e^{\langle \vec{\epsilon}, \vec{\mu}_t \rangle} \tag{3.9}$$

It essentially reduces to the Hamiltonian regularization in case where $\epsilon \in \underline{\mathbf{t}} \otimes (\mathbb{R} \subset \mathbb{R}^3)$.

[5] Integrals which are similar to those in this paper have been studied in [24].

3.4. Example: volume of a point

The volume of a point computed with the help of Hamiltonian regularization is instructive. We take $X = \mathbb{H} = \mathbb{C}^2 = \{(b_+, b_-)\}$ and $K = U(1)$. The standard K action is:

$$(b_+, b_-) \mapsto (e^{i\theta} b_+, e^{-i\theta} b_-)$$

and the hyperkahler moment map is given by:

$$\vec{M} = (|b_+|^2 - |b_-|^2)\vec{e}_3 + b_+ b_- \vec{e}_- + \overline{b_+ b_-}\vec{e}_+ \tag{3.10}$$

relevant integral involves first:

$$\begin{aligned} &\int d\eta d\bar{\eta} \int d^2 b_+ d^2 b_- \exp[i\vec{\phi}\cdot\vec{M} - \epsilon(\mid b_+ \mid^2 + \mid b_- \mid^2) + 2\eta\bar{\eta}(\mid b_+ \mid^2 + \mid b_- \mid^2)] \\ &= \int d\eta d\bar{\eta} \frac{1}{(\epsilon - 2\eta\bar{\eta} - i|\vec{\phi}|)(\epsilon - 2\eta\bar{\eta} + i|\vec{\phi}|)} = \frac{4\epsilon}{(\epsilon^2 + \vec{\phi}^2)^2} \end{aligned} \tag{3.11}$$

(To do the gaussian integral it is convenient to use rotational invariance to put ϕ into the 3 direction.) Finally, one must do the integral:

$$\int d^3\vec{\phi} e^{-i\vec{\phi}\cdot\zeta} \frac{4\epsilon}{(\epsilon^2 + \vec{\phi}^2)^2} = e^{-\epsilon|\vec{\zeta}|} \tag{3.12}$$

In this case all regularizations agree and produce the same result since the reduced space is point. There is not much freedom here!

4. Examples: four dimensional hyperkahler manifolds

4.1. Asymptotically locally euclidean spaces

Consider the famous ALE gravitational instantons. In the A_n case the metric on the space $X_n(\vec{\zeta})$ is given by the following explicit formula where the space is represented as an S^1 fibration over $\mathbb{R}^3$ which has singular fibers over $n+1$ point $\vec{r}_i \in \mathbb{R}^3$:

$$ds^2 = V d\vec{r}^2 + V^{-1}(d\tau + \omega)^2 \tag{4.1}$$

where $\vec{r} \in \mathbb{R}^3, V = V(\vec{r}), \omega \in \Omega^1(\mathbb{R}^3), d\omega = \star_3 dV$, and the potential V is given by:

$$V = \sum_{i=1}^{n+1} \frac{1}{|\vec{r} - \vec{r}_i|} \tag{4.2}$$

The moduli of the space are $\vec{\zeta}_i = \vec{r}_{i+1} - \vec{r}_i$, $i = 1, \ldots, n$. We now represent the A_n ALE spaces $X_n(\vec{\zeta})$ as a hyperkahler quotient, following [27]. We take

$$\begin{aligned} X &= \prod_{i=0}^{n} \mathbb{C}^2 = \{(b_{i,i+1}, b_{i+1,i}) | i = 0, \ldots n+1 \equiv 0\}, \\ K &= \prod_{i=0}^{n} U(1)_i / U(1)_d \cong \prod_{i=1}^{n} U(1)_i = \{e^{i\theta_l} | l = 0, \ldots n\} / U(1)_d, \\ H &= U(1) = \{e^{i\alpha}\} \end{aligned} \tag{4.3}$$

and the action of the groups is as follows

$$\begin{aligned} b_{i,i+1} &\to e^{i(\alpha + \theta_i - \theta_{i+1})} b_{i,i+1} \\ b_{i+1,i} &\to e^{i(\alpha + \theta_{i+1} - \theta_i)} b_{i,i+1} \end{aligned} \tag{4.4}$$

The Kähler regularized volume can be computed from our generalized integral formulae above. After some computation the integral can be reduced to the intuitively pleasing form:

$$\mathrm{Vol}_\epsilon(X_n(\zeta)) = \int_{X_n(\vec{\zeta})} e^{-\epsilon\kappa} \ \mathrm{vol}_g \tag{4.5}$$

which can be written more explicitly as

$$\mathrm{Vol}_\epsilon(X_n(\zeta)) = (2\pi) \int_{\mathbb{R}^3} d^3\vec{r} \sum_{i=1}^{n+1} \frac{1}{|\vec{r} - \vec{r}_i|} e^{-\epsilon \sum_{i=1}^{n+1} |\vec{r} - \vec{r}_i|}. \tag{4.6}$$

This expression is divergent as $\epsilon \to 0$, but it has an expansion

$$\mathrm{Vol}_\epsilon(X_n(\zeta)) = \frac{A_0}{\epsilon^2} + \frac{A_1}{\epsilon} + A_2 + \mathcal{O}(\epsilon) \tag{4.7}$$

where A_i are the analytic functions of the $\vec{r}_j$'s, invariant under the permutations, as well as the diagonal action of the group of Euclidean isometries $\mathbb{E}(3) = \mathbb{R}^3 \ltimes O(3)$. By scaling symmetry A_j scales like $A_j \to \lambda^j A_j$ under $\vec{r}_j \to \lambda \vec{r}_j$. Thus, $A_1 = 0$, $A_2 \propto \sum_j \vec{r}_j^2 - \frac{1}{n+1} \left(\sum_j \vec{r}_j \right)^2$. To fix the proportionality constant, we employ the following trick.

By our definition, the regularized "volume" is A_2. This function can be evaluated as follows. First note that the leading divergence in $\mathrm{Vol}_\epsilon(X_n(\zeta))$ for $\epsilon \to 0$ comes from the region $\vec{r} \to \infty$. The integrand simplifies in this region and we find

$$\mathrm{Vol}_\epsilon(X_n(\zeta)) = \frac{8\pi^2}{n+1} \frac{1}{\epsilon^2} + \mathcal{O}(\epsilon)$$

Next, note that for each j, $j = 1, \ldots, n+1$, the integral satisfies the differential equation:

$$
\begin{aligned}
&(\nabla_j^2 - \epsilon^2)\mathrm{Vol}_\epsilon\left(X_n\left(\zeta\right)\right) = \\
&\qquad - 8\pi^2 e^{-\epsilon\sum_i |\vec{r}_i - \vec{r}_j|} - 4\pi\epsilon \int d^3\vec{r}\, \frac{1}{|\vec{r}-\vec{r}_j|} \left(\sum_{i\neq j} \frac{1}{|\vec{r}-\vec{r}_i|}\right) \, e^{-\epsilon \sum_{i=1}^{n+1} |\vec{r}-\vec{r}_i|}
\end{aligned}
\tag{4.8}
$$

The integral on the right hand side is formally of order ϵ but due to the divergence of the integral as $\epsilon \to 0$ at large $\vec{r}$ this term is in fact

$$
-16\pi^2 \frac{n}{n+1} + \mathcal{O}(\epsilon)
$$

Using this and plugging (4.7) into (4.8) we find that $A_1(\vec{r}_j) = 0$ and that

$$
A_2 = -4\pi^2 \frac{n}{n+1}\left(\sum_j \vec{r}_j^{\,2} + \sum_{i\neq j} \kappa_{ij}\vec{r}_i \cdot \vec{r}_j\right) \tag{4.9}
$$

where κ_{ij} are some constants. Now we use the fact that the answer must be translation invariant to conclude

$$
A_2 = -4\pi^2 \left(\sum_{j=1}^{n+1} \vec{r}_j^{\,2} - \frac{1}{n+1}(\sum \vec{r}_j)^2\right) \tag{4.10}
$$

This result can be written more compactly as follows. For each component (in $\mathbb{R}^3$) of $\vec{r}_j$ form the vector in an auxiliary space $\rho = (r_1, r_2, \ldots, r_{n+1})$ (where we omit the index in $\mathbb{R}^3$ for clarity). Since (4.10) is translation invariant we can assume that $\sum_i r_i = 0$, and therefore ρ is a vector in the root space of A_n. Note that the simple roots satisfy $\alpha_i \cdot \rho = r_{i+1} - r_i = \zeta_i$, $i = 1, \ldots, n$. But that means $\rho = \sum_{i=1}^n \zeta_i \lambda_i$ where λ_i are the fundamental weights. But $\lambda_i \cdot \lambda_j = C^{ij}$, the inverse of the Cartan matrix for A_n, and hence we find

$$
\sum_{j=1}^{n+1} \vec{r}_j^{\,2} - \frac{1}{n+1}(\sum \vec{r}_j)^2 = \sum_{i,j=1}^{n} \vec{\zeta}_i C^{ij} \vec{\zeta}_j \tag{4.11}
$$

so we have

$$
\mathrm{Vol}_\epsilon(X_n) = \frac{8\pi^2}{n+1}\frac{1}{\epsilon^2} - 4\pi^2 \sum_{i,j=1}^{n} \vec{\zeta}_i C^{ij} \vec{\zeta}_j + \mathcal{O}(\epsilon^2) \tag{4.12}
$$

and hence we obtain the elegant result for the regularized volume of the A_n ALE space:

$$
-4\pi^2 \sum_{i,j=1}^{n} \vec{\zeta}_i C^{ij} \vec{\zeta}_j \tag{4.13}
$$

Remarks

1. It is worth noting that the answer depends on ζ^r. From the QFT perspective, our integral is a model for a gauge theory with complexified gauge group $K_{\mathbb{C}}$. Thus, the ζ^r dependence of the volumes is an *anomaly* in the noncompact part of the complexified gauge group $K_{\mathbb{C}}$. If we simply added ghosts we would expect slice-independence: that is the whole point of Faddeev-Popov gauge fixing. In fact, the slice parameter ζ^r for the principal $K_{\mathbb{C}}$ bundle does matter. In order to see this dependence we must regularize with ϵ. In the $\epsilon \to 0$ limit we discover, contrary to naive expectations, ζ^r dependence.
2. When $\zeta_i^c = 0$ and the points $\vec{r}_j$ lie on a single line and we can take $\vec{r}_j = (0, 0, z_j)$. In this case the integral simplifies to

$$
\begin{aligned}
&= (2\pi)^2 \int dzrdr \sum \frac{1}{\sqrt{(z-z_i)^2 + r^2}} e^{-\epsilon \sum \sqrt{(z-z_i)^2+r^2}} \\
&= \frac{(2\pi)^2}{\epsilon} \int dz e^{-\epsilon \sum |z-z_i|}
\end{aligned}
\tag{4.14}
$$

This is an elementary integral which can be done exactly. Note that although we could do the integral explicitly, in precisely this case the Kähler potential can be viewed as a Hamiltonian, generating a flow on $X_n(\zeta^r)$. So, we can apply the Duistermaat-Heckmann theorem. The fixed points are the intersection points of the spheres, where $b_{i,i+1} = b_{i+1,i} = 0$. Another way to evaluate the integral is by acting with the differential operators $\frac{d^2}{dz_j^2}$ to extract the order ϵ^0 term, which is once again quadratic in the z_i. In any case, we confirm the result (4.12). A nice interpretation of the regularized volume is that it is the area of the difference between the graphs of $H(z) = \sum_i |z - z_i|$ and $H_0(z) = (n+1)|z - z_{cm}|$ where $z_{cm} = \frac{1}{n+1}\sum_i z_i$.
3. Let us compare our Kähler regularized volume with that defined by the hyperkahler regularization of the volume. This gives

$$
\int_{X_n(\zeta)} e^{i\vec{\epsilon}\cdot(\vec{r}-\vec{r}_{cm})} \mathrm{vol}_g = \frac{8\pi^2}{\epsilon^2} \sum_i e^{i\vec{\epsilon}\cdot(\vec{r}_i - \vec{r}_{cm})} \tag{4.15}
$$

where $\epsilon = |\vec{\epsilon}|$, and we have subtracted $\vec{r}_{cm} := \frac{1}{n+1}\sum \vec{r}_j$ from the hyperkähler moment map to preserve translation invariance. If we expand in $\vec{\epsilon}$, the term of order ϵ^0 depends on the direction $\hat{\epsilon} = \frac{1}{\epsilon}\vec{\epsilon}$. Therefore, we average over directions to obtain

$$
\mathrm{Vol}_\epsilon\left(X_n\left(\zeta\right)\right) = \frac{8\pi^2(n+1)}{\epsilon^2} - \frac{4\pi^2}{3}\sum_{i,j=1}^{n} \vec{\zeta}_i C^{ij} \vec{\zeta}_j + \cdots \tag{4.16}
$$

recovering the previous answer, up to a factor of three. It seems that this factor of three has something to do with the hyperkähler moment map having three times more components then the ordinary moment map, and so this should be a universal feature of the hyperkähler versus Kähler regularisations. We do not, however, have a good understanding of this fact.

4. The hyperkähler regularized volume can also be obtained via a naive cutoff on the radial integral as follows. We begin with the naive integral for the volume:

$$\int_{X_n(\zeta)} \sqrt{g} d^4x = 2\pi \int_{\mathbb{R}^3} d^3r \sum_{j=1}^{n+1} \frac{1}{|\vec{r} - \vec{r}_j|} \tag{4.17}$$

Next, we cut off the integral at $\vec{r} \to \infty$ by putting an upper bound on the radius. More precisely, in order to preserve translation invariance we integrate over the region

$$|\vec{r} - \frac{1}{n+1} \sum \vec{r}_j| < R$$

where R is a cutoff. Next we use the elementary integral

$$\int_{|\vec{r}-\vec{b}|<R} d^3\vec{r} \frac{1}{|\vec{r} - \vec{a}|} = 2\pi R^2 - \frac{2\pi}{3} |\vec{a} - \vec{b}|^2. \tag{4.18}$$

In this way we obtain:

$$V_R = (2\pi)^2 (n+1) R^2 - \frac{(2\pi)^2}{3} \sum_{j=1}^{n+1} (\vec{r}_j - \vec{r}_{cm})^2 \tag{4.19}$$

and we recover the regularized volume:

$$-\frac{(2\pi)^2}{3} \sum_{i,j=1}^{n} C^{ij} \vec{\zeta}_i \cdot \vec{\zeta}_j \tag{4.20}$$

This is not surprising since it is related to the hyperkähler regularization by a Fourier transform.

5. A striking aspect of the regularized volumes is that they are *negative definite* functions of the moduli. Blowup up a singularity *reduces* the volume of the total space. This can be understood to be a very general feature of blowing up manifolds in toric geometry: The toric polytope of the blown up space is obtained by removing a part from the toric polytope of the original space. Indeed, this picture leads to an alternative way to compute these volumes (see the end of remark 2 above).

6. One can view ALE spaces as degenerate K3 manifolds. More precisely, near a point where K3 has a singularity of A, D, E type the relevant part of K3 looks like the corresponding ALE space. The volume of K3 is finite as it is compact manifold. Its variation near the point of degeneration may be studied using our simple integrals without knowledge of the exact metric on K3. Of course, in this case the answer essentially is given by the intersection form so there is no real simplification. But in the subsequent cases the intersection theory itself of the quotient is unknown or cumbersome.

4.2. Taub-Nut spaces

ALE spaces are special examples of *gravitational instantons* - they solve the four dimensional Einstein equations in Euclidean signature. Close relatives of ALE spaces are ALF spaces, or Taub-Nut manifolds. The metric on Taub-Nut space is of the same form (4.1) with $V = 1 + \sum_i \frac{1}{|\vec{r}-\vec{r}_i|}$. Therefore the naive regularization of the volume will lead to the same result as in the ALE case. Let us check whether it is true for the Kahler or similar regularization. Taub-Nut spaces can also be realized as hyperkahler quotients of a flat hyperkahler space [26]: one starts with $\mathbb{C}^{2n} \oplus \mathbb{R}^3 \times S^1 = \{(z^l, w_l) | l = 1, \ldots, n\} \oplus \{(t, \vec{x})\}$ which is acted on by $U(1)^n$ as follows.

$$\theta_l : (z^l, w_l) \mapsto (e^{i\theta_l} z^l, e^{-i\theta_l} w_l); \quad (t, \vec{x}) \mapsto (t + \sum_l \theta_l, \vec{x}) \tag{4.21}$$

with the moment maps:

$$\vec{\mu}_l = (|z^l|^2 - |w_l|^2 - x^r, z^l w_l - x^c, \bar{z}^l \bar{w}_l - \bar{x}^c) \tag{4.22}$$

where $\vec{x} = (x^r, x^c, \bar{x}^c)$, $x^r \in \mathbb{R}, x^c \in \mathbb{C}$. The generic Taub-Nut space $Y_n(\vec{\zeta})$ is obtained by imposing the constraints:

$$\vec{\mu}_l = \vec{\zeta}_l \tag{4.23}$$

and solving them modulo (4.21). In the case $\zeta_l^c = 0$ the regularizing $U(1)$ action can be chosen to act as follows:

$$e^{i\alpha} : (z^l, w_l) \mapsto (e^{i\alpha} z^l, e^{i\alpha} w_l), \quad l > 1; \quad (t, \vec{x}) \mapsto (t, x^r, e^{i\alpha} x^c) \tag{4.24}$$

with the Hamiltonian H:

$$H = |x^c|^2 + \sum_{l=1}^{n} |z^l|^2 + |w_l|^2 \tag{4.25}$$

We see that H does not constrain the t, x^r directions. Actually, t is gauged away by (4.21) while its x^r is constrained by (4.23). Thus we expect to get a finite answer for the regularized volume. The result is identical to (4.14).

5. Hitchin spaces and Bethe Ansatz

5.1. The setup

Consider a $G = U(N)$ gauge theory on a Riemann surface Σ of genus g. The gauge field $A = A_z dz + A_{\bar{z}} d\bar{z}$ can be thought of (locally) as a $\underline{\mathbf{g}}$-valued one-form. The topological sectors in the gauge theory (choices of a bundle E) are classified by $c_1(E) = \frac{1}{2\pi i}\int_\Sigma \mathrm{Tr} F \in \mathbb{Z}$ - a magnetic flux of the $U(1)$ part of the gauge group. If we project the abelian part out in order to study the $SU(N)/\mathbb{Z}_N$ theory, then there is $w_2(E) \in \mathbb{Z}_N$ - a discrete magnetic flux, corresponding to $\pi_1(G/U(1))$. Denote the space of all gauge fields in a given topological sector as $\mathcal{A}_p$, $p = \langle w_2(E), [\Sigma] \rangle$. The gauge group $\mathcal{G}$ acts in $\mathcal{A}_p$ in a standard fashion: $A \mapsto A^g = g^{-1} A g + g^{-1} dg$. The cotangent space $T^*\mathcal{A}_p$ is the set of pairs (A, Φ) where Φ - a Higgs field is a $\underline{\mathbf{g}}$-valued one-form, more precisely, a section of the bundle $\mathrm{ad}(E) \otimes \Omega^1(\Sigma)$, $\Phi = \phi_z dz + \phi^\dagger_{\bar{z}} d\bar{z}$. The Hitchin equations [28][29] can be thought of the hyperkahler moment map equations for the natural action $\mathcal{G}$ on $T^*\mathcal{A}_p$:

$$
\begin{aligned}
\mu^r &= F_{z\bar{z}} + [\phi_z, \phi^\dagger_{\bar{z}}] \\
\mu^c &= \bar{\partial}\phi_z + [A_{\bar{z}}, \phi_z]
\end{aligned}
\tag{5.1}
$$

The quotient $\mathcal{M} = \vec{\mu}^{-1}(0)/\mathcal{G}$ is called a Hitchin space. It is a hyperkahler manifold. By construction, it has a natural $U(1)$-action [29]:

$$
e^{i\theta} : (A, \phi_z, \phi^\dagger_{\bar{z}}) \mapsto (A, e^{i\theta}\phi_z, e^{-i\theta}\phi^\dagger_{\bar{z}}) \tag{5.2}
$$

This action preserves the form ω^r, decsending from:

$$
\omega^r = \int_\Sigma \mathrm{Tr} \delta A \wedge \delta A + \delta\Phi \wedge \delta\Phi \tag{5.3}
$$

and is generated by the hamiltonian H, descending from

$$
H = \|\Phi\|^2 = \int_\Sigma \mathrm{Tr} \phi\phi^\dagger \, d^2 z \tag{5.4}
$$

We define the regularized volume of $\mathcal{M}$ to be:

$$
V(\epsilon) = \int_{\mathcal{M}} \exp(\varpi^r - \epsilon H), \quad \epsilon > 0 \tag{5.5}
$$

Since $\mathcal{M}$ is a quotient by the gauge group one can rewrite the integral (5.5) as a partition function of a certain two dimensional gauge theory, generalizing the one studied in [21]. In the following we repeat the analysis of the sections 2, 3 in a gauge theory langauge. Recall the field content of the gauge theory, computing the volume of the moduli space of flat connections [21]:

$$A, \psi_A, \phi \tag{5.6}$$

with the nilpotent supercharge Q_0:

$$Q_0 A = \psi_A, \quad Q_0 \psi_A = d_A \phi, \quad Q_0 \phi = 0 \tag{5.7}$$

which squares to a gauge transformation $Q_0^2 = L_{V(\phi)}$ generated by ϕ. The charge Q_0 is a scalar, so ψ_A is a fermionic one-form with values in the adjoint, ϕ is a scalar with values on the adjoint (not to be confused with the Higgs one-form!). The action of the theory is

$$S_0 = \int_\Sigma \mathrm{Tr} \phi F + \frac{1}{2} \psi_A \psi_A$$

In our problem, the original set of fields is to be enlarged as we start with $T^* \mathcal{A}_p$ rather then $\mathcal{A}_p$. We add

$$\Phi, \psi_\Phi \tag{5.8}$$

to take care of that. Now we also impose three conditions rather then one $F = 0$ as it used to be in [21], so we need more Lagrange multipliers, or actually two more Q_0 - multiplets:

$$H^c, \chi^c; \quad \bar{H}^c, \bar{\chi}^c, \tag{5.9}$$

The action of Q_0 gets promoted to

$$\begin{aligned} Q_0 \Phi &= \psi_\Phi, \quad Q_0 \psi_\Phi = [\phi, \Phi], \\ Q_0 \chi^c &= H^c, \quad Q_0 H^c = [\phi, H^c] \end{aligned} \tag{5.10}$$

and S_0 generalizes to

$$\begin{aligned} \widetilde{S}_0 &= S_0 + Q(\mathcal{R}) \\ \mathcal{R} &= \int_\Sigma \mathrm{Tr}\big(\Phi \psi_\Phi + \bar{\chi}^c \mu^c + h.c.\big) \end{aligned} \tag{5.11}$$

The last modification has to do with the fact that we have an extra symmetry in the problem, namely the $U(1)$ of (5.2). Since Q_0 is the equivariant derivative it is easy to modify it to take care of (5.2):

$$Q_\epsilon = Q_0 + \epsilon \mathcal{Q},$$

where $\mathcal{Q}$ acts as follows:

$$\begin{aligned}&\mathcal{Q}A = \quad \mathcal{Q}\psi_A = \quad \mathcal{Q}\phi = 0\\&\mathcal{Q}\Phi = 0, \quad \mathcal{Q}(\psi_\Phi)_z = \phi_z, \quad \mathcal{Q}(\bar{\psi}_\Phi) = -\phi^\dagger_{\bar{z}}\\&\mathcal{Q}\chi^c = \mathcal{Q}\bar{\chi}^c = 0, \quad \mathcal{Q}H^c = \chi^c, \quad \mathcal{Q}\bar{H}^c = -\bar{\chi}^c\end{aligned} \tag{5.12}$$

Finally, the action of our gauge theory assumes the form:

$$S_\epsilon = S_0 + Q_\epsilon(\mathcal{R}) \tag{5.13}$$

with $\mathcal{R}$ still given by (5.11).

There are three ways of evaluating the partition function

$$Z(\epsilon) = \frac{1}{\mathrm{Vol}(\mathcal{G})}\int \mathcal{D}\phi\mathcal{D}A\mathcal{D}\Phi\mathcal{D}\psi_A\mathcal{D}\psi_\Phi\mathcal{D}^2\vec{H}\mathcal{D}^2\chi e^{-S_\epsilon} \tag{5.14}$$

The first and the most direct one consist of integrating out the Lagrange multiplers $\vec{H}$, thus enforcing the moment map equations and thereby reducing (5.14) to (5.5).

The second approach, similar in spirit to [21][30][31][32], uses the localization with respect to the action of $\mathcal{G}$, thereby reducing (5.14) to the sum over the fixed points of the action of $\mathcal{T}$ - the T-valued gauge transformations. This reduces the theory to the abelian one and will lead, as we will see presently, to the Bethe Ansatz equations.

The third approach uses the localization of (5.5) onto the fixed points of the (5.2), studied (for $G = SU(2)$) by Hitchin in [29]. The comparison of the second and the third approaches leads to an interesting set of identities, which we discuss only in one simple situation.

5.2. Localization by the gauge group.

Using the Q_ϵ invariance of our theory we modify the action in such a way that the integral becomes more tractable. We add our favorite mass term and throw away the terms $Q\chi^c\bar{\mu}^c$. In this way we get a new action

$$\widetilde{S}_\epsilon = S_0 + Q_\epsilon\big(\int_\Sigma \mathrm{Tr}\big(\Phi\psi_\Phi + \bar{\chi}^c\lambda^c + h.c.\big)\big)$$

which is quadratic in χ^c-λ^c, Φ-ψ_Φ and essentially quadratic in A-ψ_A. The evaluation of the partition function

$$\int \mathcal{D}(\ldots)e^{-\widetilde{S}_\epsilon}$$

proceeds as follows. Fix a gauge

$$\phi = \mathrm{diag}(l_1, \ldots, l_N), \quad \sum l_k = 0 \tag{5.15}$$

which corresponds to a decomposition of the bundle E into a sum of the line bundles

$$E = \bigoplus_k L_k \tag{5.16}$$

with the Chern numbers

$$c_1(L_k), \quad n_k = \int_\Sigma c_1(L_k) \tag{5.17}$$

which should obey the following selection rule:

$$\sum_k n_k = w_2(E) \mathrm{mod} N \tag{5.18}$$

Then the gauge field A decomposes as $A = A^{ab} + A^{\perp}$, A^{ab} being the $\underline{\mathbf{t}} = Lie(T), T = U(1)^N$ gauge field, and $A^{\perp}$ the component of A in orthogonal complement to $\underline{\mathbf{t}}$. The action is quadratic in $A^{\perp}$-$\psi_A^{\perp}$. Integrating out all these multiplets together with the Faddeev-Popov determinant for (5.15) we get the ratio of determinants:

$$\frac{\mathrm{Det}_{\Omega^0 \otimes \underline{\mathbf{g}}/\underline{\mathbf{t}}}(\mathrm{ad}\phi)}{\mathrm{Det}_{\Omega^1 \otimes \underline{\mathbf{g}}/\underline{\mathbf{t}}}(\mathrm{ad}\phi)} \frac{\mathrm{Det}_{\Omega^0 \otimes \underline{\mathbf{g}}}(\epsilon + i\mathrm{ad}\phi)}{\mathrm{Det}_{\Omega^1 \otimes \underline{\mathbf{g}}}(\epsilon + i\mathrm{ad}\phi)} \tag{5.19}$$

Here $\underline{\mathbf{g}}$ denotes the bundle $E \otimes E^* - I \equiv \oplus_{i,j} L_i \otimes L_j^{-1} - I$, $\underline{\mathbf{g}}/\underline{\mathbf{t}}$ denotes $\oplus_{i \neq j} L_i \otimes L_j^{-1}$. As usual, the determinants naively cancel, but the infinite-dimensionality of the situation makes them cancel only up to the determinants acting on the spaces of zero modes, whose dimension is given by the Riemann-Roch formula:

$$\mathrm{dim}\Omega^0 \otimes L - \mathrm{dim}\Omega^1 \otimes L = c_1(L) + 1 - g \tag{5.20}$$

Note that the similar situation was encountered in the derivation of Verlinde formula from the gauged WZW theory, using localisation techniques [32][31], so we can apply the similar technique[2006]. In our case there is an important complication, though.

[2006] We thank A. Gerasimov for the clarifying discussion on this subject, important for the remainder of this section.

The ratio of the determinants is easily calculable for the special background ϕ, namely for the covariantly constant one. In general, the regularization preserving the Q_ϵ-invariance, would produce also the A^{ab} and ψ^{ab} dependent couplings. The Q_ϵ-symmetry fixes them in terms of two functions, the induced superpotential $W(\phi)$ and the induced background charge $T(\phi)$. These two functions are calculated below. The general technique of calculating effective actions in terms of observables of topological gauge theory, using index theory, is developed in [33].

Without the detailed knowledge of W and T we can do the integral over the rest of the fields, namely, A^{ab} as well as l_k's and ψ^{ab}_A. The rest of the action together with the induced terms is given by

$$\int_\Sigma \Big[\sum_k \left(l_k + \frac{\partial W}{\partial l_k}\right) F_k + \tfrac{1}{2}\sum_{i,j}\left(\delta_{ij} + \frac{\partial^2 W}{\partial l_i \partial l_j}\right)\psi_i \wedge \psi_j\Big] + \int_\Sigma T(\vec{l})\frac{1}{8\pi}\mathcal{R}^{(2)} \qquad (5.21)$$

where $\psi^{ab}_A = \mathrm{diag}(\psi_1, \ldots, \psi_N)$, F_k is the curvature, corresponding to the k'th entry of A^{ab}, and $\mathcal{R}^{(2)}$ is the two dimensional scalar curvature. In writing (5.21) we have dropped possible Q_ϵ-exact terms, as they do not affect the result. We have (5.17):

$$\int_\Sigma F_k = 2\pi i n_k$$

Although the connection A_k in the bundle with nonvanishing n_k is not a one-form on Σ the difference of two such connections $A'_k - A_k$ is a one-form α_k. Fix a metric on Σ and let h_k be a harmonic representative of the cohomology class of F_k. Then, for any connection in this topological sector we may write

$$F_k = h_k + d\alpha_k$$

for some one-form α_k, defined up to a gauge transformation $\alpha_k \mapsto \alpha_k + d\beta_k$. Integrating α_k out together with the gauge fixing forces l_k to obey $dl_k = 0$. At the same time we can split $\psi_k = \psi^h_k + \delta\psi_k$, where ψ^h_k is a harmonic one-form, and $\delta\psi_k$ is orthogonal to the space H^1 of harmonic one-forms. The integral over $\delta\psi_k$ cancels the determinant induced by the integral over α_k. Therefore, our integral reduces to the sum over topological sectors n_k, the integral over ψ^h_k and the integral over the constant diagonal matrices ϕ with the measure

$$
\begin{aligned}
&Z_p(\epsilon) = \sum_{\vec{n}} \int_{\vec{l}} \exp\bigl(2\pi i \vec{l}\cdot\vec{n}\bigr)\nu_{\vec{l}}(\epsilon)\ \kappa^g_{\vec{l}}(\epsilon) \\
&\nu_{\vec{l}}(\epsilon,\vec{n},g) = \exp\ \Bigl(2\pi i \sum_k n_k \frac{\partial W}{\partial l_k} + (g-1)T(\vec{l})\Bigr) = \\
&\quad \prod_{i\neq j}(l_i - l_j)^{n_i - n_j + 1 - g} \prod_{i,j}\bigl(\epsilon + i(l_i - l_j)\bigr)^{n_i - n_j + 1 - g} \\
&\quad \vec{l} = (l_1,\ldots,l_N), \quad \vec{n} = (n_1,\ldots,n_N)
\end{aligned}
\tag{5.22}
$$

where

$$
\kappa_{\vec{l}}(\epsilon) = \det\left(\delta_{ij} + \frac{\partial^2 W}{\partial l_i \partial l_j}\right) \tag{5.23}
$$

is induced by the integral over the fermionic zero modes, ψ^h_k. Both the sum and integral are restricted: $\sum l = 0$, $\sum n \equiv p \bmod N$. We can relax the condition on n's by introducing a factor $e^{-2\pi i p \sum l}$ in the integral and dropping the requirement $\sum l = 0$.

Finally, we have two options. Either we first take the sum over $\vec{n}$ or we first take the integral over $\vec{l}$. The summation over $\vec{n}$ is easily performed if we notice that (5.22) can be rewritten as

$$
\begin{aligned}
&Z_p(\epsilon) = \int_{\vec{l}} \mathcal{D}\vec{l}\quad e^{-2\pi i p \sum_k l_k}\ \nu^{1-g}_{\vec{l}}\ \kappa_{\vec{l}}(\epsilon)^g\ \sum_{\vec{n}} \prod_{k=1}^{N} e^{2\pi i n_k \chi_k} \\
&\chi_k = l_k + \frac{N+1}{2} + \frac{\partial W}{\partial l_k} = l_k + \frac{N+1}{2} + \frac{1}{2\pi i}\sum_{j\neq k} \log \frac{\epsilon + i(l_k - l_j)}{\epsilon - i(l_k - l_j)} \\
&\nu_{\vec{l}} = \epsilon^N \prod_{i<j} l^2_{ij}(\epsilon^2 + l^2_{ij}), \quad l_{ij} = l_i - l_j
\end{aligned}
\tag{5.24}
$$

The sum over n_k leads to the delta function on χ_k supported at integers, or, in other words, the integral in (5.24) localizes onto the solutions of the equations[2006]:

$$
-e^{2\pi i l_k} = \prod_{j\neq k} \frac{l_k - l_j - i\epsilon}{l_k - l_j + i\epsilon} \tag{5.25}
$$

A look at [16], Eqs. (228) and a formula below Eq. (291) there suggests that this equation (5.25) is nothing but the Bethe Ansatz Equation for the Non-linear Schrödinger model (NLS) ! Under this identification ϵ maps to the coupling $\mathbf{g}$ of the NLS Hamiltonian:

$$
H^{NLS} = \int dx \Bigl(|\partial_x \psi|^2 + \mathbf{g}(\psi^*)^2(\psi)^2\Bigr) \tag{5.26}
$$

[2006] It was noticed in [34] that the original version of this paper missed the $\kappa_{\vec{l}}(\epsilon)$ factor in (5.22).

In the strong NLS coupling limit $\mathbf{g} \to \infty$, $\epsilon \to \infty$ and (5.25) turns into $l_k \in \mathbb{Z} + \frac{N}{2}$, with the usual fermionic exclusion principle.

The second option is to integrate $\vec{l}$ out keeping $\vec{n}$ fixed and then to sum over $\vec{n}$. This approach seems to be equivalent to

5.3. Localization via $U(1)$ action.

The fixed points of the $U(1)$ action (5.2) on $\mathcal{M}$ necessarliy have $\mathrm{Tr}\Phi^r = 0$, i.e. they form a submanifold of the *nilpotent cone* $\mathcal{N} \subset \mathcal{M}$. Let $\mathcal{S}_\alpha$ be a connected component of the set of fixed points. As explained in [29] the bundle E splits into a sum of line bundles:

$$E = \oplus \mathcal{L}_k$$

The contribution of $\mathcal{S}_\alpha$ is the standard integral

$$Z_\alpha = e^{-\epsilon d_\alpha} \int_{\mathcal{S}_\alpha} \frac{e^{\varpi^r}}{\mathrm{Eu}_\epsilon(\mathcal{N}_\alpha)}$$

where d_α is the value of $||\Phi||^2$ on $\mathcal{S}_\alpha$ (it is easy to express it in terms of $c_1(\mathcal{L}_k)$), $\mathcal{N}_\alpha$ is the normal bundle to $\mathcal{S}_\alpha$ and Eu_ϵ is its equivariant Euler class. By summing up Z_α we get another expression for (5.24) thus establishing a curious identity obeyed by solutions of Bethe Ansatz equations (5.25). We hope to elaborate on this point in future publications.

6. Volumes of the moduli spaces of instantons

6.1. Gauge theory approach

Let X be a hyperkahler manifold of dimension $4l$ and let $\mathcal{A}$ denotes the space of gauge fields in some principal G-bundle E over X. Let $\vec{\omega}$ be a triple of Kähler forms on X and $\vec{\pi}$ be the corresponding triple of Poisson structures. Then $\mathcal{A}$ carries the following (formal) Kähler forms:

$$\vec{\Omega} = \int_X \langle \vec{\pi}, \mathrm{Tr}\delta A \wedge \delta A \rangle \mathrm{dvol}_g \tag{6.1}$$

where dvol_g is the metric volume form and $\langle , \rangle$ is the usual contraction. They are invariant under the gauge group action:

$$A \to A^g = g^{-1}dg + g^{-1}Ag, \tag{6.2}$$

and the hyperkahler moment map is

$$\vec{\mu} = \langle \vec{\pi}, F \rangle \tag{6.3}$$

The hyperkähler reduction of the space $\mathcal{A}$ by the action of the gauge group K produces a manifold $\mathcal{M}$ which is infinite-dimensional unless $l = 1$.

Suppose X is a compact four-dimensional hyperkahler manifold, i.e. $K3$ or T^4. Then the result of the hyperkahler reduction of the space of the gauge fields produces a finite-dimensional hyperkahler space, which is nothing but the moduli space $\mathcal{M}_k$ of instantons on X of a given instanton charge k. This space posesses a natural compactification and has finite volume. It depends on the volume of X. In fact, one can deduce from [35] that

$$\mathrm{Vol}(\mathcal{M}_k) = \frac{1}{N!}\mathrm{Vol}(X)^N \tag{6.4}$$

where $\mathrm{Vol}(X)$ is computed with the help of the Kahler metric and N is the quarter of the real dimension of $\mathcal{M}_k$.

As mentioned above, one of the motivations for this work is the desire to understand the QFT formula for the number of "four-dimensional conformal blocks" [4]:

$$\sum_i (-1)^i \dim H^i(\mathcal{M}^+(c_1, c_2); \mathcal{L}_\omega) = \int_{\mathcal{M}^+(c_1,c_2)} \mathrm{ch}(\mathcal{L}_\omega) Td(T^{1,0}\mathcal{M}^+) \tag{6.5}$$

This is related to the gauged WZW_4 theory. Let us rescale the form $\omega = k\omega_0$ and take the limit $k \to \infty$. In that case (6.5) becomes: $k^{\dim \mathcal{M}^+/2}\mathrm{Vol}_{\omega_0}(\mathcal{M}^+)$. In the WZW_2 theory this leads to expressions for the symplectic volume of the moduli space of flat connections in terms of ζ-functions [21]. In the WZW_4 theory from the path integral expression we obtain:

$$\begin{gathered} Z_\omega^{GWZW_4} \to k^{\dim \mathcal{M}^+/2} \int \frac{[dAd\psi d\chi dH d\varphi]}{\mathrm{vol}\mathcal{G}} \\ \exp\Biggl\{ -\frac{i}{4\pi} \int_{X_4} \mathrm{Tr}\Bigl[\bar{H}^{0,2}F^{2,0} + \varphi\omega_0 \wedge F^{1,1} + H^{2,0}F^{0,2} \\ + \omega_0 \psi \wedge \bar{\psi} + \bar{\chi}^{0,2} D_A \psi + \chi^{2,0} \bar{D}_A \bar{\psi} \Bigr] \Biggr\} \end{gathered} \tag{6.6}$$

This theory has been considered in [36][37] (where it was called holomorphic Yang-Mills theory). On a hyperkahler manifold we can identify the $(2,0), (0,2)$ Donaldson fermions with η:

$$\begin{aligned} \chi^{2,0} &= \omega^c \eta \\ \chi^{0,2} &= \bar{\omega}^c \bar{\eta} \end{aligned} \tag{6.7}$$

Integrating out $\psi, \bar{\psi}$ from the HYM lagrangian we produce all parts of the above action for computing volumes described above.

If the manifold X is non-compact it is necessary to regularize the volume to get a well-defined result. One natural regularization uses the Kähler potential [38]:

$$\kappa \to \int_{X_4} \kappa(x^\mu) \mathrm{Tr} F^2 \tag{6.8}$$

where $\kappa(x^\mu)$ is the Kahler potential of X, equal to $\sum \mid \vec{r} - \vec{r}_i \mid$ for ALE space.

Now suppose that X is $\mathbb{R}^4$ or an ALE manifold. The theory with (6.8) added to the action reduces, upon localization to the moduli space of instantons, to the "Kähler regularization" used in this paper.

In case where X is ALE manifold there is another natural term to add to the Lagrangian:

$$\int \kappa_0(x) \mathrm{Tr} F \wedge F + \lim_{r \to \infty} \int_{S^3_r} \iota(\frac{\partial}{\partial r})[\omega \mathrm{Tr}(\epsilon F)]$$

where $\kappa_0(x)$ is the Kähler potential of X and ϵ is a generic diagonal matrix. As shown by Nakajima, this generates a torus action on the moduli space of instantons on $X_n(\vec{\zeta})$. The fixed points are the fully reducible connections, i.e., the theory can be abelianized. On the other hand, the moduli spaces of instantons on $\mathbb{R}^4$ and ALE manifolds can be described using ADHM construction. The latter realizes the moduli space as a finite-dimensional hyperkähler quotient. We may therefore apply our techniques directly in finite-dimensional case. We shall do it in the next section.

6.2. Volumes of ADHM moduli spaces

In the case where the space X is non-compact the moduli space of instantons on X might have a nice finite-dimensional description. For example, the instantons on $\mathbb{R}^4$ with the gauge group $U(k)$ are described by the hyperkahler quotient with respect to the group $K = U(N)$ of the space $V \oplus V^*$ where $V = \underline{\mathbf{k}} \oplus \mathbb{C}^N \otimes \mathbb{C}^k$, with $\mathbb{C}^N$ being the fundamental representation of K. The FI term ζ vanishes in this case. We can consider a slightly deformed space with $\zeta^r \neq 0$. Applying (3.7) we arrive at the formula for the regularized volume:

$$\mathrm{Vol}_{N,k}(\zeta^r, \epsilon) = 2^N \int \prod_{l=1}^{N} \frac{e^{i\zeta^r \phi_l} d\phi_l}{\epsilon \left(\phi_l(\phi_l + 2i\epsilon)\right)^k} \prod_{i<j} \frac{\phi_{ij}^2(\phi_{ij}^2 + 4\epsilon^2)}{(\epsilon^2 + \phi_{ij}^2)^2} \tag{6.9}$$

It is interesting to study this integral in large N limit. We can rewrite it (after a shift: $\phi_l \to \phi_l + i\epsilon$) as a partition function of a gas of N particles on a line with the pair-wise repulsive potential

$$V^{rep}(x) = \log\left(1 + \frac{\epsilon^2}{x^2}\right)\left(1 - \frac{3\epsilon^2}{x^2 + 4\epsilon^2}\right) \tag{6.10}$$

in the external potential, attracting to the origin in the limit $\zeta^r = 0$:

$$V^{ext}(x) = k\log(\epsilon^2 + x^2) \tag{6.11}$$

Assuming the existence of the density of the eigenvalues in the large N limit we get the following equation for the critical point

$$\sum_{i<j} V^{rep}(\phi_{ij})' = -V^{ext}(\phi_i)' \Rightarrow$$

$$v.p. \int \rho(x)dxV^{rep}(x-y)' = \frac{2k}{N\epsilon^2}\frac{y}{y^2+\epsilon^2} \tag{6.12}$$

This equation can be solved in a standard way using the Fourier transform. It seems that the non-trivial limit exists only if N and k are taken to infinity with k/N fixed.

The moduli space of instantons on $\mathbb{R}^4$ is acted on by the rotation group $Spin(4) = SU(2)_L \times SU(2)_R$. The Cartan subalgebra of its Lie algebra is two dimensional and is spanned by two elements (ϵ_1, ϵ_2) which generate rotations in two orthogonal planes in $\mathbb{R}^4$. The corresponding equivariant volume is a slight modification of (6.9):

$$\mathrm{Vol}_{\epsilon_1,\epsilon_2}(\zeta^r) = \frac{(\epsilon_1+\epsilon_2)^N}{\epsilon_1^N\epsilon_2^N}\int\prod_{l=1}^{N}\frac{e^{i\zeta^r\phi_l}d\phi_l}{(\phi_l(\phi_l+\epsilon_1+\epsilon_2))^k}\prod_{i\neq j}\frac{\phi_{ij}(\phi_{ij}+\epsilon_1+\epsilon_2)}{(\phi_{ij}+\epsilon_1)(\phi_{ij}+\epsilon_2)} \tag{6.13}$$

Remark. In the case $N = 1$ $\zeta \neq 0$ the hyperKähler quotient coincides with Hilbert scheme $(\mathbb{C}^2)^{[N]}$ of points on $\mathbb{C}^2$. The integral (6.13) can be further evaluated by residues, which turn out to be enumerated by all Young tableaux Y of length N, and the contribution of a given Young tableau is computed as follows. The tableau encodes the partition of $N = \nu_1 + \ldots + \nu_l$ with the condition that $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_l$. Let us enumerate the boxes in the tableau by the pairs of integers (m,n), where $1 \leq m \leq l$, $1 \leq n \leq \nu_m$. Then the integrand in (6.13) has a pole at

$$\phi_{m,n} = -\epsilon_1(m-1) - \epsilon_2(n-1)$$

Moreover the residue can be rather easily evaluated using the results of [5].

Another example where the moduli space of instantons is given by the hyperkahler quotient is the instantons on ALE spaces. For concreteness we consider the $U(k)$ instantons on A_{n-1} ALE space. The topology of the gauge bundle is specified by the vectors $\vec{w}$ and $\vec{v}$ which enter Kronheimer-Nakajima construction [39]. We present the result.

$$\mathrm{Vol}_{n;k,\vec{w},\vec{v}}(\vec{\zeta},\epsilon) =$$
$$e^{-\sum_i \zeta_i v_i} \int \prod_{i=1}^{n} \left[\prod_{\alpha=1}^{v_i} \left(\frac{e^{i\zeta_i \phi_i^\alpha} d\phi_i^\alpha}{(\epsilon^2 + (\phi_i^\alpha)^2)^{w_i}} \frac{\prod_{\alpha<\beta} (\phi_i^{\alpha\beta})^2 \left((\phi_i^{\alpha\beta})^2 + 4\epsilon^2 \right)}{\prod_{\beta=1}^{v_{i+1}} \left((\phi_i^\alpha - \phi_{i+1}^\beta)^2 + \epsilon^2 \right)} \right) \right] \tag{6.14}$$

where $\phi_i^{\alpha\beta} = \phi_i^\alpha - \phi_i^\beta$.

7. Regularized volumes of quiver varieties

The formulae (6.14)(6.9) can be readily generalized to cover the case of general quiver variety (again with the restriction $\zeta^c = 0$) [39][2]. So assume that a quiver Γ is given. Let Ω denote the set of its oriented edges. There is an involution $\iota : \Omega \to \Omega$ which sends an oriented edge to the same edge of opposite orientation. Let $Vert$ be the set of its vertices v. There are two maps $s, t : \Omega \to Vert$, assigning to an edge its begining (source) and the end (target). To each vertex a hermitian vector space L_v and to every element ω of Ω a vector space $H_\omega = \mathrm{Hom}(L_{s(\omega)}, L_{t(\omega)})$ are assigned. Let $l_v = \mathrm{dim} L_v$. The space

$$\mathcal{V}_\Gamma = \bigoplus_\omega H_\omega \tag{7.1}$$

is naturally acted on by the group

$$\mathcal{G}_\Gamma = \times_v U(L_v) \tag{7.2}$$

and this action preserves the hyperkahler structure on $\mathcal{V}_\Gamma$. The quiver variety $\mathcal{X}_\Gamma(V, \vec{\zeta})$ is defined for $V \subset Vert$, $\vec{\zeta} : V \to \mathbb{R}^3$. It is the hyperkahler quotient of $\mathcal{V}_\Gamma$ with respect to the subgroup $\mathcal{G}_{V,\Gamma}$ of (7.2):

$$\mathcal{G}_{V,\Gamma} = \times_{v \in V} U(L_v) \tag{7.3}$$

Let

$$l_v^\vee = \sum_{\omega : s(\omega) = v; t(\omega) \in Vert \backslash V} l_{t(\omega)} \tag{7.4}$$

Applying (3.7) we arrive immediately at:

$$\mathrm{Vol}_\epsilon(\mathcal{X}_\Gamma(V,\vec{\zeta})) =$$

$$e^{-\sum_v \zeta_v^r l_v} \int \prod_{v\in V}\prod_{\alpha=1}^{l_v} \left(\frac{e^{i\zeta_v^r \phi_v^\alpha} d\phi_v^\alpha}{(\epsilon^2+(\phi_v^\alpha)^2)^{l_v^\vee}} \frac{\prod_{l_v\geq\beta>\alpha}(\phi_v^{\alpha\beta})^2\left(\epsilon^2+(\phi_v^{\alpha\beta})^2\right)}{\prod_{\omega\in\Omega:s(\omega)=v,t(\omega)\in V}\prod_{\beta=1}^{l_{t(\omega)}}\left(\epsilon^2+(\phi_v^\alpha-\phi_{t(\omega)}^\beta)^2\right)^{\frac{1}{2}}} \right) \tag{7.5}$$

This example covers a lot of interesting hyperkahler manifolds, complex coadjoint orbits of $GL_N(\mathbb{C})$ among them, for example $T^*\mathbb{CP}^N$ with Calabi metric and deformed cotangent bundles to generalized flag varieties (see for example [2]).

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