Integrating Replenishment Decisions with Advance Demand Information

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There is a growing consensus that a portfolio of customers with different demand lead times can lead to higher, more regular revenues and better capacity utilization. Customers with positive demand lead times place orders in advance of their needs, resulting in *advance demand information*. This gives rise to the problem of finding effective inventory control policies under advance demand information. We show that state-dependent (s, S) and base-stock policies are optimal for stochastic inventory systems with and without fixed costs. The state of the system reflects our knowledge of advance demand information. We also determine conditions under which advance demand information has no operational value. A numerical study allows us to obtain additional insights and to evaluate strategies to induce advance demand information.
(*Stochastic Inventory Systems; Advance Demand Information; Demand Lead Times*)

1. Introduction

Different customers often have different willingness to pay for the speed with which their orders are filled. A build-to-order company may be able to improve its profits by shifting its production strategy to satisfy customers willing to pay higher prices for shorter demand lead times.[1] This requires a mixed strategy where part of the production is build to order and part is build in anticipation of orders. Similarly, a build-to-stock company may be able to increase its profits by offering price discounts to customers willing to accept longer demand lead times. This shifts the production strategy from build to stock to build to order. There is a growing consensus that manufacturers can benefit from a hybrid strategy of having a portfolio of customers with different demand lead times, see Appell et al. (2000). The hope is that such portfolios will lead to higher, more regular revenues and better capacity utilization.

Customers with positive demand lead times place orders in advance of their needs resulting in what we call *advance demand information*. In this paper we take the portfolio of customers with different demand lead times as given, and consider the resulting stochastic inventory control problems under advance demand information. We show that state-dependent policies are optimal for systems with and without fixed ordering costs. Since the expected cost of these optimal policies is essential to evaluate effectiveness of a portfolio, the results of this paper are essential to the effective design of such portfolios.

Strategies to obtain advance demand information include market segmentation where price-sensitive customers place orders in advance of their needs (see, for example, Chen 1999). Advance demand information may arise from risk-averse customers that want to minimize the risk of delivery delays; companies can induce these customers to book early by giv-

[1] A customer who places an order l units of time ahead of his needs is said to have demand lead time l. This term was coined by Hariharan and Zipkin (1995).

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ing priority to early orders. Advance demand information is often provided by supply chain partners. Ford Motor Company, for example, issues and weekly updates orders to its catalytic converter suppliers, as discussed in the "Corning Glass Works" Harvard Business School teaching case (1991). The e-commerce of customized products such as personal computers provides advance demand information for the product components. An example is Dell's cutting-edge distribution model. Under this model, consumers are allowed to customize their choice of PC online for future delivery (Hamm and Stepanek 1999). Toyota recently announced plans to make customized cars within five days, reflecting its ability to quickly respond to advance demand information (see Simison 1999). GM and Ford are scrambling to catch up on this trend. These strategies in conjunction with advances in information technology assist companies in getting a better sense of demand and its evolution over time. Advance demand information, in addition, enables companies to shift their production strategies from build to stock to build to order. In spite of the fact that many businesses operate in a dynamic environment, stochastic inventory models that incorporate advance demand information are rare.

In this paper we analyze a discrete-time, single-item, single-location, periodic-review inventory problem. At the end of period t we observe the demand vector

$$D_t = (D_{t,t}, \ldots, D_{t,t+N}),$$

where $D_{t,s}$ represents orders placed by customers during period t for future periods $s \in \{t, \ldots, t+N\}$, and N is the length of the information horizon over which we have advance demand information. In each period, a fixed cost is charged whenever an order is placed. Orders arrive after an exogenously specified lead time. Unsatisfied demands are fully backordered where backorder costs are linear.

For the positive set-up cost case, we prove that state-dependent (s, S) policies are optimal for finite-horizon problems and also for α-discounted infinite-horizon problems. The state of the system is composed of a modified inventory position that nets the known requirements and observed demands beyond the protection period.[2] Under a state-dependent (s, S) policy, an order is placed to raise the inventory position to state-dependent order-up-to level S whenever it falls to or below the state-dependent reorder point s. The proofs are based on a geometric characterization of K-convexity that simplifies the verification of the dynamic programming inductive hypothesis. The proof for the infinite horizon case is based on Iglehart (1963) and uses ideas from Veinott (1966). One important result is that if the observed demand immediately beyond the protection period is above a certain threshold, then a policy that places an order to minimize the costs of the current period is optimal. This "horizon" result allows management to act optimally without precise advance demand information and significantly reduces the computational burden of searching for state-dependent policy parameters.

For the zero set-up cost case, the optimal policy reduces to a state-dependent base-stock policy for finite-horizon and α-discounted infinite-horizon problems. We show that the base-stock level is an increasing[3] function of the observed demands beyond the protection period. For the case of stationary costs and demands, however, we show that observed demands beyond the protection period have no influence on the optimal base-stock level. This implies that management should not invest in obtaining advance demand information beyond the protection period for operational purposes.

Our computational study confirms the state-dependent nature of our results and helps quantify the value of advance demand information. Our examples indicate that systems that incorporate advance demand information have lower average inventories and lower inventory-related costs than the classical systems. The exercise of computing the benefits of advance demand information can help managers decide whether or not the benefits justify the costs of acquiring advance demand information. Our computational study also suggests certain monotonicity

[2] The protection period is defined as the lead time plus a review period.

[3] We use the terms increasing and decreasing in the weak sense, so increasing means nondecreasing.

properties of the state-dependent order-up-to level for the case of large set-up costs. This property, however, fails to hold when set-up costs are small.

1.1. Literature Review

Early references to stochastic single-item/single-location inventory problems date back to Arrow et al. (1951). We are aware of three groups of work that incorporate the dynamic nature of demand updates. The first group uses Bayesian models; to the best of our knowledge, Dvoretzky et al. (1952) are the first authors to use Bayesian models, followed by Scarf (1960), Azuory and Miller (1984), and Azuory (1985). The second group uses time-series models to subsume demand dynamics. These models arise particularly when there is a significant intertemporal correlation among the demands of consecutive periods. Examples of this work include Johnson and Thompson (1975), Miller (1989), and Lovejoy (1990). A third group of researchers have been concerned with forecast revisions. Hausman (1969) models the evolution of forecast as a quasi-Markovian or Markovian process. Heath and Jackson (1994) extend this work by modeling the evolution of forecast using martingales and coins the term Martingale Method of Forecast Evolution (MMFE). Güllü (1996) studies the optimal policy that arises under MMFE for a zero set-up cost, capacitated single-item/single-facility inventory system with zero lead times. Toktay and Wein (1999) model a production system as a single-server, discrete-time, continuous-state queue under MMFE. Gallego and Toktay (1999) characterize the form of the optimal policy in a finite production-capacity model where the fixed cost of ordering is high enough to warrant all-or-nothing ordering in each period.

Within the context of state-dependent policies the seminal paper by Song and Zipkin (1993) is relevant to our work. These authors show that optimal policies are state dependent for a continuous-time discrete-state inventory problem with Markov modulated Poisson demands. Important extensions to the work of Song and Zipkin include Sethi and Cheng (1997), Song and Zipkin (1996), and Chen and Song (1999).

There are three additional papers related to ours. Hariharan and Zipkin (1995) consider a continuous-review model where customers place orders l units of time in advance of their requirements. They show that the *demand lead time* l directly offsets the supply lead time and, as a consequence, base-stock and (s, S) policies are optimal for zero and positive ordering costs, respectively. Our model can be viewed as periodic-review generalization of theirs, and our results (appropriately interpreted) confirm the optimality of state-dependent base-stock and (s, S) policies conjectured by these authors for the case of random demand lead times. Sobel and Zhang (2001) study a finite-horizon periodic-review inventory model where, in addition to spot (stochastic) demands, there are known commitments in every period. They assume that the commitments must be honored without delay, but allow spot demands to be backordered. They show that a modified (s, S) policy is optimal. Their model differs from ours in that the commitments for the entire horizon are known at the beginning of the horizon, the lead time in their model is zero, and commitments cannot be backlogged. Finally, Brown et al. (1971) consider a model similar to ours but restrict the information horizon to be at most the length of the protection period.

The remainder of the paper is organized as follows. In §2, we introduce the necessary notation and the model of demand information. In §3, we present an alternative characterization of K-convexity and some results that simplify the proofs. In §4, we establish the optimal policies and the structural results for inventory problems with positive set-up costs both for finite-horizon problems and infinite-horizon stationary problems. In §5, we extend the results to inventory problems with zero set-up costs. In §6, we obtain additional insights to the problem through a numerical study. In §7, we conclude and suggest directions for future research. In Appendix A, we present the construction of the dynamic program. We defer all the proofs to Appendix B.

2. Model Description

This section introduces the notation and the model of advance demand information. As stated before, the vector $D_t = (D_{t,t}, \ldots, D_{t,t+N})$ represents orders placed by customers during period t for periods $s \in \{t, \ldots, t+N\}$, where N is the information horizon.

This is a random vector and its uncertainty is completely resolved at the end of period t. Notice that, at the beginning of period t, the demand to prevail in a future period $s \geq t$ can be divided into two parts: The part that is *observed* and known to us

$$O_{t,s} \equiv \sum_{r=s-N}^{t-1} D_{r,s} \tag{1}$$

and the part that is *unobserved* and not yet known to us

$$U_{t,s} \equiv \sum_{r=t}^{s} D_{r,s}. \tag{2}$$

We define $O_{t,s} \equiv 0$ for $s \geq t+N$ since we do not observe demand information beyond the information horizon.

We assume that the unobserved part $U_{t,s}$ is independent of the observed part $O_{t,s}$. While it may be desirable to model the unobserved part as dependent on the observed part, there are cases where they are naturally independent. This would be the case, for example, when customers with independent demands are segmented by their demand lead times. Making $U_{t,s}$ dependent on $O_{t,s}$ would also require a larger state space as discussed later.

Let I_t be the inventory on hand and B_t be the number of backorders at the beginning of period t. In addition to I_t and B_t, at the beginning of period t we also know the cumulative *observed* demands for periods $t, t+1, \ldots, t+N-1$ placed prior to period t. That is, we know $O_{t,s}$ for $s \in \{t, \ldots, t+N-1\}$.

We assume that an order for z_t units placed at the beginning of period t arrives at the beginning of period $t+L$ where the lead time L is assumed to be a fixed nonnegative integer. The cost of ordering $z_t \geq 0$ units in period t is given by $K_t\delta(z_t) + c_t z_t$, where $K_t \geq 0$ is the set-up cost, and $\delta(z_t) = 1$ if $z_t > 0$ and zero otherwise. This cost is realized whenever an order is placed. This assumption can be easily modified to incorporate other cases, including that in which the cost is realized at the time of delivery. We also assume that, at the end of the planning horizon, T, inventory leftovers are sold for a salvage value of c_{T+1}. In addition, backorders are satisfied by a final procurement which is equal to c_{T+1} without incurring a set-up cost. Our final assumption is $\alpha_{t+1}K_{t+1} \leq K_t$ for all t. None of these assumptions are stronger than the assumptions of classical inventory problems (see Scarf 1959, Veinott 1966, and Iglehart 1963).

To facilitate our discussion we use the term *protection-period demand* to refer to the demand over the next $L+1$ periods, e.g. $\{t, t+1, \ldots, t+L\}$. Similar to (1) and (2), at the beginning of period t the protection-period demand can also be divided into two parts: The part that has already been *observed* and known to us

$$O_t^L \equiv \sum_{s=t}^{t+L} O_{t,s}$$

and the part that is *unobserved* and not yet known to us

$$U_t^L \equiv \sum_{s=t}^{t+L} U_{t,s}.$$

The state space is given by (x_t, O_t) where

$$x_t \equiv I_t + \sum_{s=t-L}^{t-1} z_s - B_t - O_t^L \tag{3}$$

is the *modified* inventory position[4] at the beginning of period t, prior to ordering z_t and receiving z_{t-L}, net of observed protection-period demand. The second component of the state space is

$$O_t \equiv (O_{t,t+L+1}, \ldots, O_{t,t+N-1}), \tag{4}$$

which consists of cumulative observed demands for periods beyond the protection period. Notice that the state space is of dimension $1+(N-L-1)^+$. Consequently, the state space is one-dimensional whenever $N \leq L+1$.

After observing (x_t, O_t) the decision maker places an order of size $z_t \geq 0$ to be delivered at the beginning of period $t+L$. In the traditional inventory literature, where N is assumed to be zero, the current inventory position is raised to protect against the *protection-period demand*. In our case, we need to protect against the *unobserved* part of the protection-period demand.

[4] We use the term *modified* to distinguish the definition from the classical definition of inventory position, which does not include observed part of the protection-period demand (the last term in Equation (3)).

The net inventory (physical inventory minus backorders) at the *end* of period $t+L$ is equal to

$$I_t + \sum_{s=t-L}^{t} z_s - B_t - O_t^L - U_t^L = x_t + z_t - U_t^L = y_t - U_t^L,$$

where $y_t = x_t + z_t$. We assume that on-hand inventory is used to satisfy backorders, if any, and as much of the current demand as possible, and that early fulfillment of orders is not allowed. We assume that inventory holding and backorder penalty costs are charged to the inventory level at the end of the period.

The expected holding cost and penalty cost charged to period t is based on the net inventory at the end of period $t+L$.

$$\widetilde{G}_t(y_t) \equiv \frac{\alpha_{t+L}}{\alpha_t} E g_{t+L}(y_t - U_t^L)$$

where $\alpha_j = \prod_{i=t}^{j} \gamma_i$ and γ_i is the discount factor for period i, $\gamma_t = 1$ and $g_t(x)$ denotes the total holding and penalty costs based on the inventory on hand at the end of period t. We assume that g_t is convex for each t, that G_t exists, and that $\lim_{|x|\to\infty} \widetilde{G}_t(x) = \infty$. It is possible to weaken, for example, convexity to quasi convexity, but then additional assumptions are required (see Veinott 1966).

After observing $D_t = (D_{t,t}, \ldots, D_{t,t+N})$ the *modified* inventory position is updated by

$$x_{t+1} = x_t + z_t - D_{t,t} - \sum_{s=t+1}^{t+L+1} D_{t,s} - O_{t,t+L+1} \tag{5}$$

and the vector of observed demand beyond the protection period by

$$O_{t+1} = (O_{t+1,t+L+2}, \ldots, O_{t+1,t+N}) \tag{6}$$

where $O_{t+1,s} = O_{t,s} + D_{t,s}$.

A rigorous proof of the state-space reduction is given in Özer (2000). At this point we can dispose of two cases: $N = 0$ and $1 \le N \le L+1$. The rest of the paper deals with the more difficult and interesting case $N > L+1$.

For $N = 0$, notice that $O_{t,t} = 0$ for all t since nothing is observed in advance. This is the classical case studied extensively in inventory theory for which classical results have been derived (see Arrow et al. 1951, Dvoretzky et al. 1952, Scarf 1959, Veinott 1966, Porteus 1971, Iglehart 1963, and Zheng 1991).

For $1 \le N \le L+1$, although there is nontrivial information about future demands, it is subsumed in the modified inventory position. This makes the state space one dimensional, so *all* the classical results described for the case $N = 0$ apply. If ordering takes place in a period then the order quantity is increasing in the observed protection period demand. The uncapacitated version of Güllü's (1996) zero set-up costs, zero lead time MMFE model falls within this context. Extensions to positive set-up costs and positive lead times, $L \ge N-1$ also fall into this case. See also Brown et al. (1971) for an informal derivation of this result.

Our model allows what Hariharan and Zipkin (1995) refer to as demand lead times, where customers place orders l units of time in advance of their requirements, as a special case by setting $D_{t,s} = 0$ for $s \ne t+l$ and $D_{t,t+l} = X$ where X is the random number of units demanded at period t for delivery in period $t+l$. Random demand lead times can be modeled by setting $D_{t,s} = 0$ except for a randomly chosen period $r \in \{t, \ldots, t+N\}$ where $D_{t,r} = X$.

Next we provide an example to clarify the notation and to illustrate how to update demand forecasts in our model.

Example. Let $N = 2$, $L = 0$, and let t be the current period. Demand for period $t+2$ is given by $\mathcal{D}_{t+2} = D_{t,t+2} + D_{t+1,t+2} + D_{t+2,t+2}$. Assume that $E[D_{t+i,t+2}] = \lambda_i$ for $i = 0, 1, 2$. Our best estimate for $\mathcal{D}_{t+2}$ at the beginning of period t is $E[\mathcal{D}_{t+2}|\mathcal{F}_t] = \lambda_0 + \lambda_1 + \lambda_2$, where $\mathcal{F}_t$ is σ-field of events under the natural filtration of the demand process. During period t, customers place $D_{t,t+2}$ orders for period $t+2$. Thus, our forecast at the beginning of period $t+1$ is given by $E[\mathcal{D}_{t+2}|\mathcal{F}_{t+1}] = D_{t,t+2} + \lambda_1 + \lambda_2$. Similarly, at the beginning of period $t+2$ we have $E[\mathcal{D}_{t+2}|\mathcal{F}_{t+2}] = D_{t,t+2} + D_{t+1,t+2} + \lambda_2$, and finally $E[\mathcal{D}_{t+2}|\mathcal{F}_{t+3}] = \mathcal{D}_{t+2}$. Notice that $E[\mathcal{D}_t|\mathcal{F}_s]$, $s < t$ is a (Doob's) martingale, see e.g., Durrett (1996). Although the evolution of demand can be cast in a martingale framework we prefer to avoid this for the following reasons: (i) the notation and necessary background become more complicated, and (ii) martingale theorems (optional sampling, etc.) are not needed in our context.

At this point, we would like to explore the difficulties of keeping a manageable state space when the unobserved part of the demand is modeled as dependent on the observed part. A natural model would be $D_{t,s} = a_{s-t} + \rho_{s-t} D_{t-1,s} + \epsilon_{ts}$. Notice that in this case $O_{t,s}$ is not a sufficient statistics to compute the distribution of $U_{t,s}$. Indeed, $O_{t,s} = O_{t-1,s} + D_{t-1,s}$, so $O_{t,s}$ hides the value of $D_{t-1,s}$. Thus, in addition to (x_t, O_t) we would need to keep the last N components of D_{t-1} as part of the state. It is also possible to have D_t governed by a Markov chain as in Song and Zipkin (1993). This would augment the state space to include the state of the Markov chain. Finally, a parsimonious model where $U_{t,s}$ depends only on the observed part $O_{t,s} = \sum_{z=s-N}^{t-1} D_{z,s}$ imposes strong distributional assumptions on $D_{z,s}$ for $z \in \{t, \ldots, s\}$ because $U_{t,s} = \sum_{z=t}^{s} D_{z,s}$.

3. Preliminaries

In this section, we introduce a geometric characterization of K-convexity that simplify the induction arguments and a lemma that is helpful in obtaining the main results in the paper. We defer all the proofs to Appendix B.

Definition 1. Let $g: \mathcal{R} \to \mathcal{R}$ and $a \geq 0, b \geq 0$. The function g is called (a,b)-convex and denoted by $g \in C(a,b)$ if it satisfies the following inequality;

$$g(\theta x_1 + (1-\theta)x_2) \leq \theta(a + g(x_1)) + (1-\theta)(b + g(x_2)),$$
$$\text{for all} \quad x_1 \leq x_2 \quad \text{and} \quad \theta \in [0,1].$$

It is easy to show that $(0,K)$-convexity is equivalent to K-convexity as introduced by Scarf (1960; see Denardo 1982 and Porteus 1971). Paralleling the epigraph characterization of convex functions,[5] $g \in C(a,b)$ if and only if the line segment joining $(x_1, g(x_1) + a)$ and $(x_2, g(x_2) + b)$ is in the epigraph of g.

The epigraph characterization has the following geometric interpretation: A point $(x_1, g(x_1) + a)$ is said to be *visible* from $(x_2, g(x_2) + b)$ if all the intermediate points $(x, g(x)), x_1 \leq x \leq x_2$ lie below the line segment joining the two points. Visibility is a well-known concept in analysis (see, for example, Kolmogorov and Fomin 1970). The following are simple properties of (a,b)-convex functions. Part 5 of this lemma is of special interest since it helps simplify the proof of Theorem 1.

Lemma 1.
1. *$C(a,b) \subset C(a',b')$ for all $(a,b) \leq (a',b')$.*
2. *If $f \in C(a,b)$ and $g \in C(a',b')$ then for positive constants α and β, $(\alpha f + \beta g) \in C(\alpha a + \beta a', \alpha b + \beta b')$.*
3. *If $g \in C(a,b)$ and $E|g(x-D)| < \infty$, where D is a random variable, then $G(x) = Eg(x-D) \in C(a,b)$.*
4. *If $f(x,y) \in C(a,b)$ for a fixed vector y and $E_D|f(x-D,y)| < \infty$, then $F(x) = E_{D,Y} f(x-D,Y) \in C(a,b)$ where Y is a vector of random variables.*
5. *If $g \in C(a,b)$, where $a < b$, and s is such that $g(s) + a \leq g(x) + b$ for all $x \geq s$, then $f(x) = g(\max(x,s)) \in C(a,b)$.*

Definition 2. A function $f: \mathcal{R} \times \mathcal{R}^n \to \mathcal{R}$ is said to have decreasing differences in (x,θ) if it satisfies the following inequality.

$$f(x_1,\theta) - f(x_2,\theta) \leq f(x_1,\theta') - f(x_2,\theta')$$
$$\text{for all} \quad x_1 \geq x_2 \quad \text{and} \quad \theta \geq \theta'. \tag{7}$$

This concept is closely related to submodularity which is often used to show monotonicity results (see Topkis 1998 and Veinott 1980).

4. Inventory Problems with Positive Set-up Costs

The case $N \leq L+1$ reduces to a one-dimensional problem and has been dealt with. From this point on we assume that $N > L+1$. We characterize the policy that attains the minimum total cost of managing inventory over a finite-horizon problems followed by infinite-horizon problems. From now on we denote O_t by o_t when O_t is known, that is at the beginning of period t.

The functional equation for the problem is given by

$$J_t(x_t, o_t) = \min_{y_t \geq x_t} \{K_t \delta(y_t - x_t) + V_t(y_t, o_t)\}, \tag{8}$$

$$V_t(y_t, o_t) = G_t(y_t) + \alpha_{t+1} E J_{t+1}(x_{t+1}, O_{t+1}), \tag{9}$$

[5] A function g is convex if and only if the line segment joining $(x_1, g(x_1))$ and $(x_2, g(x_2))$ is in the epigraph epi(g) $\equiv \{(x,y) : y \geq g(x)\}$ of g. See Rockafellar (1997).

where $J_{T+1}(\cdot,\cdot) \equiv 0$ and $G_t(y) = (c_t - \gamma_t c_{t+1})y + \widetilde{G}_t(y)$. Notice that the expectation in (9) is with respect to the vector D_t. Appendix A gives a formal construction of this functional equation.

We rewrite the dynamic program to simplify the characterization of the optimal policy. When the optimal policy chooses not to order at the beginning of period t, the optimal value $J_t(x_t, o_t)$ is equal to $V_t(x_t, o_t)$. Therefore, Equation (8) can be expressed as

$$J_t(x_t, o_t) = V_t(x_t, o_t) + \min\{H_t(x_t, o_t), 0\},$$

where $H_t(x_t, o_t) \equiv K_t + \min_{y_t \geq x_t} V_t(y_t, o_t) - V_t(x_t, o_t)$. If $H_t(x_t, o_t) \leq 0$, then it is optimal to order. On the other hand, if $H_t(x_t, o_t) > 0$, it is not optimal to order. If $H_t(\cdot, o_t)$ has a *unique* sign change from $-$ to $+$ for every o_t then the policy has a simple form: an interval in which ordering is optimal followed by an interval in which ordering is not optimal.

LEMMA 2. *The function $H_t(\cdot, o_t)$ has a unique sign change from $-$ to $+$ if the following conditions are satisfied for any fixed vector o_t;* (i) $V_t(\cdot, o_t) \in C(0, K_t)$, (ii) *there exists a finite minimizer $S_t(o_t)$ of $V_t(\cdot, o_t)$,* (iii) *there exists $x < S_t(o_t)$ such that $V_t(x, o_t) > K_t + V_t(S_t(o_t), o_t)$.*

Define $s_t(o_t) \equiv \max\{x_t : H_t(x_t, o_t) \leq 0\}$. Clearly $s_t(o_t) \leq S_t(o_t)$ since $H_t(S_t(o_t), o_t) = K_t$.

COROLLARY 1. *If the conditions of Lemma* 2 *are satisfied then there exists a finite $s_t(o_t)$ such that it is optimal to order-up-to $S_t(o_t)$ if and only if $x \leq s_t(o_t)$. The optimal value can be written as*

$$J_t(x, o_t) = V_t(\max(s_t(o_t), x), o_t). \qquad (10)$$

The next result establishes the optimality of (s, S) policies where the policy parameters depend on advance demand information.

THEOREM 1. *The following statements are true for any fixed vector o_t:*

1. $V_t(\cdot, o_t) \in C(0, K_t)$ *and* $\lim_{|x|\to\infty} V_t(x, o_t) = \infty$.
2. *An optimal policy is defined by a state-dependent $(s_t(o_t), S_t(o_t))$-policy where*

$$S_t(o_t) = \min\{y : V_t(y, o_t) \leq V_t(x, o_t) \text{ for all } x\},$$
$$s_t(o_t) = \max\{x : H_t(x, o_t) \leq 0\}.$$

3. $J_t(\cdot, o_t) \in C(0, K_t)$ *and* $\lim_{x\to\infty} J_t(x, o_t) = \infty$, $\lim_{x\to-\infty} J_t(x, o_t) = V_t(s_t(o_t), o_t)$.

In this paper, we will refer to an inventory problem as stationary if the demand and the cost parameters are stationary, i.e., $c_j = c$, $g_j = g$, $\alpha_j = \alpha$, and $k_j = k$ and we drop the subscript from single-period cost function G. For the analysis of infinite-horizon problems we assume stationarity. Let us define

$$S^m = \min\{y : G(y) \leq G(x) \text{ for all } x\},$$
$$s^m = \max\{y \leq S^m : G(y) \geq K + G(S^m)\},$$
$$\overline{S} = \min\{y > S^m : G(y) > G(S^m) + \alpha K\}.$$

The pair (s^m, S^m) is a myopic policy for the positive set-up cost case that does not depend on advance demand information. Notice that all three points exist since G is convex with respect to y and $\lim_{|y|\to\infty} G(y) = \infty$.

THEOREM 2. *For stationary finite-horizon problems, if $o_{t,\,t+L+1} \geq (\overline{S} - s^m)$, then $S_t(o_t) = S^m$.*

The theorem proves that once the observed demand for period $t+L+1$ exceeds $\overline{S} - s^m$, the myopic order-up-to level is optimal for the stationary positive set-up cost problems. The threshold level is a function of the lower bound for the reorder point and the upper bound for the order-up-to level. Tighter bounds result in a lower threshold level. As the set-up cost increases the observed demands for the immediate period beyond the protection period need to be higher for the horizon result to hold. This result has both managerial and computational implications. Management can ignore advance demand information beyond period $t+L+1$ if the observed demand for period $t+L+1$ is sufficiently high. In particular management should concentrate in ordering, if needed, to satisfy the demand for period $t+L$, knowing that a new order will be placed in period $t+1$. The horizon result limits the need to search for state-dependent policies when the observed demand for period $t+L+1$ is sufficiently large, making it easier to compute optimal policies.

We have shown the existence and the optimality of a state-dependent $(s_t(o_t), S_t(o_t))$ policy for finite-horizon problems. Next, we establish upper and lower bounds for the optimal finite-horizon policies. We then use Heinz-Borel theorem to claim that the sequence $\{s_t(o_t), S_t(o_t)\}_{t=0}^{\infty}$ has convergent subsequences. Then we state that the limit points of this sequence satisfy the functional equation for the infinite-horizon problem and they are, therefore, optimal policies.

LEMMA 3. *For all t and any fixed vector o_t, $S^m \le S_t(o_t) \le \overline{S}$, and $s^m \le s_t(o_t)$.*

This lemma bounds the optimal policies both from below and above. The bounds presented here are similar to the ones established by Veinott (1966) and Iglehart (1963). The following lemma is necessary to prove that the limit of the optimal value for the finite-horizon has convergent subsequences.

LEMMA 4. *For all t and any fixed vector (x, o), $V_{t-1}(x, o) \ge V_t(x, o)$ and $J_{t-1}(x, o) \ge J_t(x, o)$.*

This result shows that the optimal value for a finite-horizon problem increases as the number of planning periods increases. This is quite intuitive since managing inventory for an additional period results in an additional cost. The next result establishes the optimal policies and extends the horizon result for infinite-horizon problems.

Let us now consider the finite-horizon stationary case and the limit as the horizon grows to infinity. We know that $V_t(x, o)$ depends on time to go $T-t$. We know consider the limit of the functions J_t and V_t as $T \to \infty$.

THEOREM 3. *For any fixed vector $o_t = o$,*

1. $\lim_{T\to\infty} J_t(\cdot, o)$ *exists and converges uniformly to a function $J(\cdot, o)$ and satisfies the functional Equation (8). Hence, it is the optimal value function for the infinite-horizon problem.*
2. *The sequence $\{s_t(o), S_t(o)\}$ converges to a limit point $(s(o), S(o))$. Any limit point of $S_t(o)$ is a minimizer of the function*

$$V(y, o) = G(y) + \alpha E\left\{ J\left(y - D_{t,t} - \sum_{s=t+1}^{t+L+1} D_{t,s} - o_{t,t+L+1}, O_{t+1}\right)\right\}.$$

Furthermore, function $V(\cdot, o)$ is $(0, K)$-convex and $s(o)$ is the $\max\{y : H(y, o) \le 0\}$.

3. *The policy $(s(o), S(o))$ is optimal for the infinite-horizon problem.*
4. *If $o_{t,t+L+1} \ge (\overline{S} - s^m)$ then $S(o) = S^m$, extending Theorem 2 to infinite-horizon problems.*

Lemma 3 and Theorem 3 imply Corollary 2.

COROLLARY 2. *$s^m \le s(o)$ and $S^m \le S(o) \le \overline{S}$.*

5. Inventory Problems with Zero Set-up Costs

The functional equation for the zero set-up cost is given by Equation (8) with $K_t = 0$ for all t. The following theorem summarizes our findings for the zero set-up finite-horizon problems where we do not necessarily assume stationarity.

THEOREM 4. *The following statements are true for any fixed vector o_t:*

1. *$V_t(x, o_t)$ is a convex function and* $\lim_{|x|\to\infty} \times V_t(x, o_t) = \infty$.
2. *An optimal ordering policy is a state-dependent base-stock policy where the order-up-to level is given by the smallest value of y that minimizes $V_t(y, o_t)$, i.e.*

$$y_t(o_t) = \min\{y : V_t(y, o_t) = \min_x V_t(x, o_t)\}. \tag{11}$$

3. *$J_t(x, o_t)$ is an increasing convex function.*
4. *$V_t(x, o_t)$ has decreasing differences in (x, o_t).*
5. *$y_t(o_t)$ is increasing in o_t.*
6. *$J_t(x, o_t)$ has decreasing differences in (x, o_t).*

So the optimal policy is to order whenever the modified inventory position falls below a base stock level. Part 5 of Theorem 4 shows that systems maintain higher order-up-to levels, hence higher average inventory levels, as the level of observation for future periods beyond the protection period increases. This also suggests the possibility of developing heuristics that increase the order-up-to level as o_t increases.

The following lemma for the finite-horizon stationary case shows that classical monotonicity results also hold for our model.

LEMMA 5. *$V_{t-1}(x, o) \ge V_t(x, o)$ and $J_{t-1}(x, o) \ge J_t(x, o)$. In addition $\nabla V_{t-1}(x, o) \ge \nabla V_t(x, o)$, $y_{t-1}(o) \le y_t(o)$, and $\nabla J_{t-1}(x, o) \ge \nabla J_t(x, o)$ hold for all t.*

A simple heuristic for the finite-horizon problem, which is generally not optimal, is to ignore the effect of upcoming periods and focus on minimizing the single-period cost G_t. Let

$$y^m_{\min,\,t} = \min\{y : G_t(y) = \min_x G_t(x)\},$$
$$y^m_{\min,\,t} = \max\{y : G_t(y) = \min_x G_t(x)\}.$$

Notice that the range collapses into a unique point when $G_t(\cdot)$ is strictly convex. We define $y^m_t \equiv y^m_{\min,\,t}$. For stationary problems, y^m_t is independent of t and will be denoted by y^m.

Theorem 5. *If the sequence y^m_t is nondecreasing in t then the myopic policy is optimal. In particular, for stationary problems, the base-stock level y^m is optimal for finite-horizon problems.*

This theorem parallels well-known results for the classical case without advance demand information, which is addressed by Veinott (1965). For the advance demand information case, it tells us that information *beyond* the protection period does not affect the order-up-to level when we assume stationary costs and demand distributions. Intuitively, it makes sense, in the absence of fixed costs and capacity restrictions, to order only enough to cover for the protection-period demand. This result significantly reduces the computational effort since the state space collapses to a single dimension. It also implies that management need not obtain advance demand information beyond the protection period for inventory control purposes. In addition, myopic policies are also optimal when y^m_t is increasing. This would be the case, for example, under stationary costs when demand is ramping up.

Let us now consider the finite-horizon stationary case and the limit as the horizon grows to infinity. We know that $V_t(x, o)$ depends on the time to go $T - t$. We also know that the smallest minimizer of $V_t(\cdot, o)$ is y^m which is independent of both o and t. We know consider the limit of the functions J_t and V_t as $T \to \infty$.

Theorem 6. *For any fixed vector o,*

1. $\lim_{T\to\infty} J_t(\cdot, o)$ *exists and converges uniformly to a convex function $J(\cdot, o)$. Furthermore,* $\lim_{x\to\infty} J(x, o) = \infty$.
2. $\lim_{T\to\infty} V_t(\cdot, o)$ *exists and converges uniformly to a convex function $V(\cdot, o)$. Furthermore,* $\lim_{|x|\to\infty} V(x, o) = \infty$, *so $V(\cdot, o)$ admits a finite minimizer $y(o)$.*
3. $y(o) = y^m$ *is an optimal policy for the infinite-horizon problem.*

6. Numerical Study

In this section, we provide managerial insights into our model of advance demand information. We use a backward induction algorithm to solve functional Equation (8). The basic idea of this algorithm is to solve the dynamic program starting from the very last period, which is a single-period problem, by evaluating the cost for each instance of the state space and choosing an action that minimizes the cost and repeating these steps until the first period is reached. Throughout these computations, we use the following combination of parameters.

$L = 0, N = L + 2$	$K = 0, 5, 50, 100$	$h = 1, 2, 3, 6$	$p = 1, 3, 6, 9, 19, 99$

Recall that for $N = L + 2$ the state space is two-dimensional, see the example in §2. The case $L = 0$ and $N = 2$ is the simplest case for which the problem is nontrivial, and is general enough to capture the main ideas. The demand vector for $N = 2$ is given by

$$D_t = (D_{t,\,t}, D_{t,\,t+1}, D_{t,\,t+2}).$$

In our computational study $D_{t,\,t+i}$ is modeled by Poisson distribution with mean λ_i. Due to Equations (1) and (4), vector o_t is a scalar and given by $D_{t-1,\,t+1}$. The computational effort to solve the problem optimally increases with the length of the information horizon, N. An interesting direction for future research would be the development of efficient heuristics based on the results and insights obtained in this paper.

Recall that y_t is the modified inventory position *after* ordering whereas x_t is the modified inventory position *before* ordering, i.e. $y_t = x_t + z_t$. Figures 1(A) and (B) depict the relationship between y_t and x_t with respect to observed demand information beyond the protection period, which is in this case $D_{t-1,\,t+1}$. We observe the optimality of state-dependent (s, S) policy for the positive set-up cost case, i.e., if $x_t \le s_t(D_{t-1,\,t+1})$ then order up to $S_t(D_{t-1,\,t+1})$, otherwise do nothing.

This observation is proved in Theorem 1. Notice that order-up-to level increases as the level of observed demand increases for large set-up costs.

Figure 1(A) Positive Set-up Cost

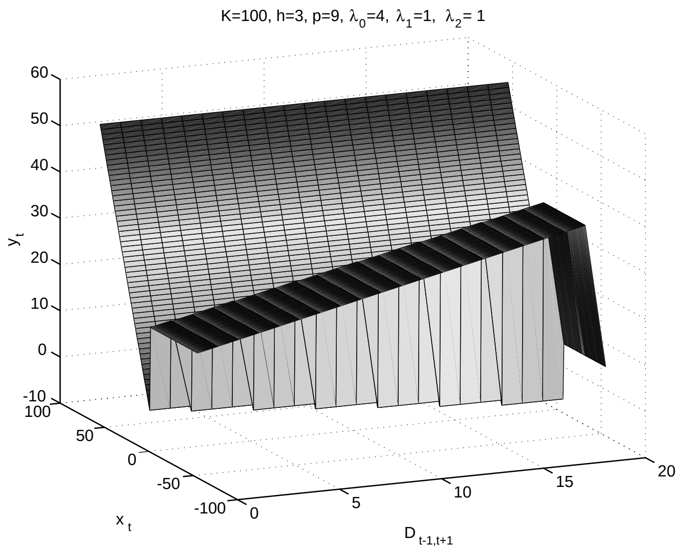
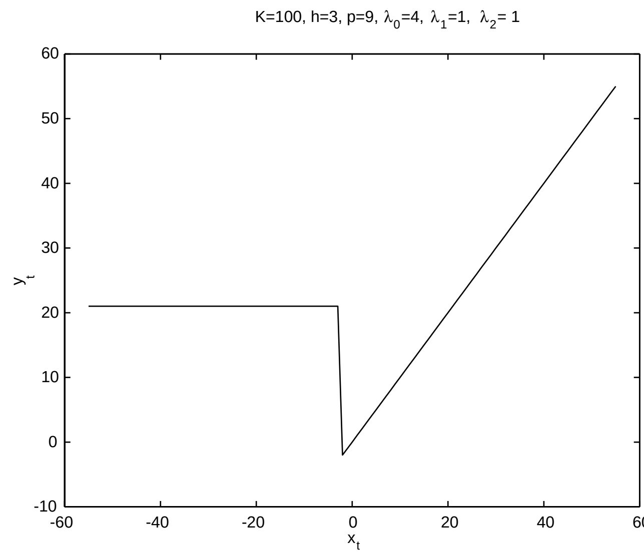


Figure 1(B) Cross-Section of (A) at $D_{t-1,t+1} = 10$



Counter examples, however, show that this monotonistic behavior is not a general property, see Table 2.

On the other hand, our extensive experiments indicate that the *reorder point* $s_t(D_{t-1,t+1})$ *decreases* as $D_{t-1,t+1}$ *increases*. We found the observation surprising, because intuition suggests that the reorder point, $s_t(D_{t-1,t+1})$, should be increasing in $D_{t,-1,t+1}$ making it more likely to place an order to cope with observed demands. Careful thought, however, reveals a more complete story. First, notice that if x_t is not too low, the holding and penalty cost of not ordering may be lower than the cost of ordering and carrying $D_{t-1,t+1}$ for one period. This suggests that at high values of $D_{t-1,t+1}$, it may be better to incur a shortage cost now rather than to place an order and carry inventory for the next period. On the other hand, for sufficiently low values of x_t and very high values of $D_{t-1,t+1}$ it is best to place two consecutive orders, which is shown in Theorem 2. In this case it is optimal, to raise the modified inventory position of the first order to minimize current costs, i.e., $S_t(D_{t-1,t+1}) = S^m$ when $D_{t-1,t+1} \geq \overline{S} - s^m$, we see a sharp drop of order-up-to-level in Figure 1 (a).

In addition to Theorem 4, Figure 2(A) and (B) clarify that order-up-to policy is optimal for the zero set-up cost case. As shown in Theorem 5, a change in $D_{t-1,t+1}$ (observed demand beyond the protection period) does not affect the order-up-to level. Information beyond the protection-period does not influence the order-up-to level, i.e., $y_t(D_{t-1,t+1}) = y^m$ is optimal. Experiments 1-6 in Table 1 confirm this point.

Our computational study enhances the sentiment that advance demand information reduces the overall system cost comprised of set-ups, holding, and shortage costs. One can also use our model to quantify the

Table 1 $K = 0, h = 1, p = 9, y_t(D_{t-1,t+1})$ **for** $D_{t-1,t+1} \in \{0, \ldots, 15\}, T = 12$

No.	λ_0	λ_1	λ_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	y^m
1	4	1	4	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
2	4	1	2	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
3	4	1	1	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
4	3	1	2	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
5	2	1	3	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
6	1	1	4	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2

Figure 2(A) **Zero Set-up Cost Cross-Section of (A) at** $D_{t-1,t+1} = 5$

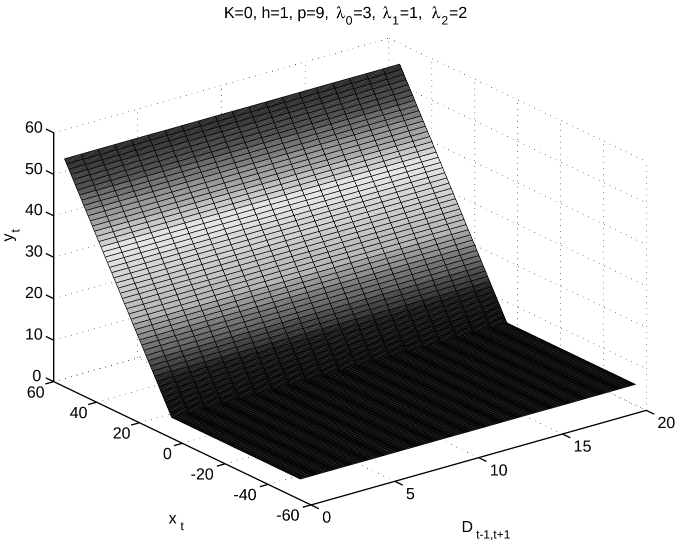
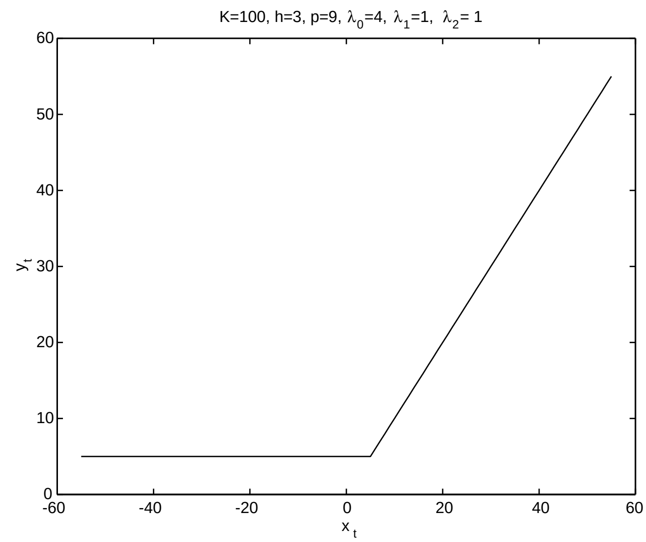


Figure 2(B) **Cross-Section of (A) at** $D_{t-1,t+1} = 5$



cost reduction due to advance demand information. Implementation of strategies to acquire this information often comes with cost. The following example illustrates how our model quantifies the trade-off between the benefits of advance demand information and cost of implementing a pricing strategy.

Recall that demand at any period for our numerical study is given by $\mathcal{D}_s = D_{s-2,s} + D_{s-1,s} + D_{s,s}$. In Table 3, we fix the mean value of the demand at $6 (= \lambda_0 + \lambda_1 + \lambda_2)$ and increase λ_2 while decreasing λ_0 (in this way, we model the case where the decision maker obtains more advance demand information). Assume that a brand manager is trying to acquire advance demand information through pricing strategies. She is willing to reduce the price of the product if customers are willing to accept future delivery. Strategies Nos. 10 through 15 in Table 3 model her aggressiveness in reducing prices. If she reduces the price dramatically, she attains the demand intensity structure of No. 15.

She can compare the benefits gained through advance demand information with the losses in profits due to aggressive price reductions and decide which strategy to implement. Figure 3(A) depicts the cross section of optimal cost functions for strategies Nos. 10-15 where the cross-section is the plane defined by $D_{t-1,t+1} = 10$. Figure 3(B) illustrates the cost reduction (of approximately 18% between experiments Nos. 10 and 15) gained through advance demand information. Notice that as she implements more aggressive strategies, the cost function decreases except when the *initial* modified inventory position is high, e.g., $x_t > S_t(D_{t-1,t+1})$, which is a transient effect for problems with several long planning horizons. It

Table 2 $K = 5$, $h = 1$, $s_t(D_{t-1,t+1})$, $S_t(D_{t-1,t+1})$ **for** $D_{t-1,t+1} \in \{0, \ldots, 15\}$, $T = 12$

No.	λ_0	λ_1	λ_2		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	s^m	S^m	$\bar{S}$
7	4	1	1	$S(\cdot)$	9	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	3	7	13
		$p = 9$		$s(\cdot)$	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4			
8	1	1	4	$S(\cdot)$	4	4	5	2	2	2	2	2	2	2	2	2	2	2	2	2	0	2	9
		$p = 9$		$s(\cdot)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1			
9	4	1	1	$S(\cdot)$	6	6	5	4	4	4	4	4	4	4	4	4	4	4	4	4	−3	4	11
		$p = 1$		$s(\cdot)$	0	1	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2			

Table 3 $K = 100$, $h = 1$, $p = 9$ $s_t(D_{t-1,t+1})$, $S_t(D_{t-1,t+1})$ **for** $D_{t-1,t+1} \in \{0, \ldots, 15\}$, $T = 12$

No.	λ_0	λ_1	λ_2		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	s^m	S^m	$\overline{S}$
10	5	1	0	$S(\cdot)$	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	−7	8	110
				$s(\cdot)$	2	2	2	2	1	1	1	1	1	1	1	1	1	0	0	0			
11	4	1	1	$S(\cdot)$	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	−8	7	108
				$s(\cdot)$	1	1	1	1	1	0	0	0	0	0	0	0	0	0	−1	−1			
12	3	1	2	$S(\cdot)$	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	−9	5	107
				$s(\cdot)$	0	0	0	0	0	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−2			
13	2	1	3	$S(\cdot)$	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	−10	4	105
				$s(\cdot)$	−1	−1	−1	−1	−1	−1	−1	−2	−2	−2	−2	−2	−2	−2	−2	−2			
14	1	1	4	$S(\cdot)$	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	−11	2	104
				$s(\cdot)$	−2	−2	−2	−2	−2	−2	−2	−2	−3	−3	−3	−3	−3	−3	−3	−3			
15	0	1	5	$S(\cdot)$	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	−12	0	100
				$s(\cdot)$	−3	−3	−3	−3	−3	−3	−3	−3	−3	−4	−4	−4	−4	−4	−4	−4			

is also evident from Table 3 that as more advance demand information is available, the order-up-to level and the reorder point decrease, suggesting a reduction in average inventory level.

7. Conclusion

In this paper we establish the form of optimal policies for a model of advance demand information. We show that the problem reduces to known results when the information horizon is shorter than the protection period. When the information horizon is longer than the protection period, a state-dependent base-stock policy is optimal in the case of zero set-up costs, and a state-dependent (s, S) policy is optimal for positive set-up costs. The policy parameters depend on the observed demands beyond the protection period.

Figure 3(A) **Optimal Cost as a Function of x_t for Experiments Nos. 10-15, $D_{t-1,t+1} = 10$**

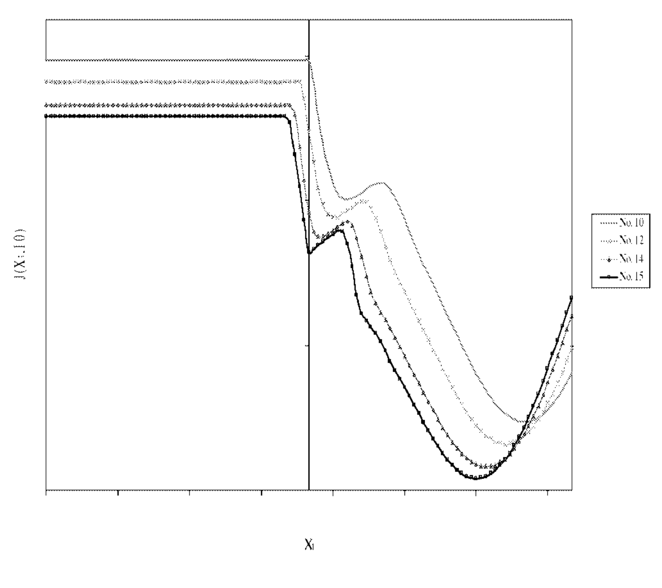
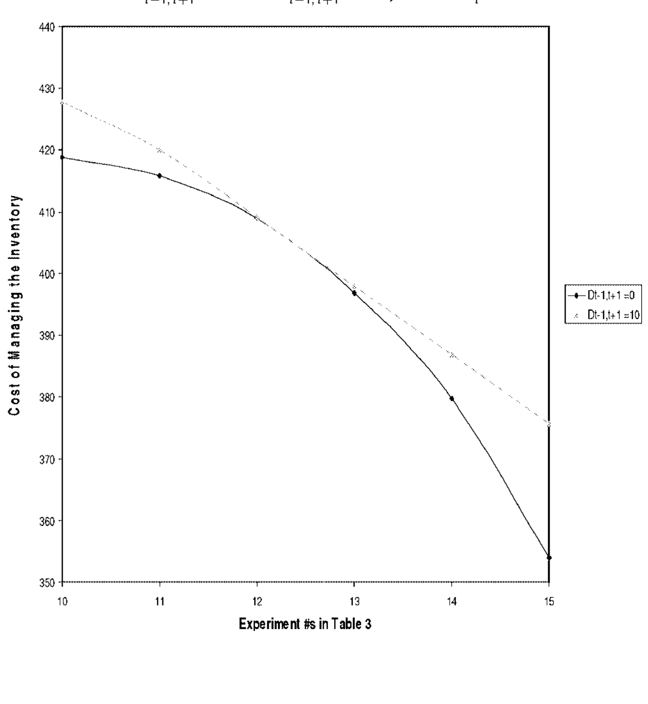


Figure 3(B) **Optimal Cost as a Function of Experiment Numbers for $D_{t-1,t+1} = 0$ and $D_{t-1,t+1} = 10$, Where $x_t = 0$**



We prove monotonicity of the base-stock levels for the zero set-up cost case, and show that a myopic policy is optimal for the infinite-horizon stationary case. For fixed set-up costs, we show that the myopic order-up-to policy is optimal when the observed demands beyond the protection period exceed a threshold level. Our numerical study indicates that system performance improves as customers place orders further into the future. It is necessary to quantify these benefits to see if they offset the cost of implementing strategies that elicit advance demand information.

There are, however, interesting avenues for further research. We wish to find conditions that guarantee monotonistic behavior of the policy parameters for the positive set-up cost case. These results, if obtained, will help in the development of efficient algorithms to solve large-scale problems to optimality or in the construction of efficient heuristics. Performance analysis of these heuristics is also an appealing research direction. We are currently investigating the optimality of state-dependent policies when the criterion is long-run average cost, and extending the analysis to the case of multiple products. Due to contractual agreements, once customers commit, they are often required to honor their obligation. Customers, however, are sometimes allowed to cancel free of charge. Exploring the issues raised here for such cases will shed more light to the use of advance demand information on inventory control problems.

Appendix A: Dynamic Programming Formulation

If the policy for the problem is specified by $Y_t = (y_t, y_{t+1}, \ldots, y_T)$, then the expected cost of inventory management due to holding, procurement, and penalty costs is

$$\tilde{J}_t(x_t, O_t | Y_t) = E\sum_{j=t}^{T} \alpha_j \{K_j \delta(y_j - x_j) + c_j(y_j - x_j) + \tilde{G}_j(y_j)\} - \alpha_{T+1} c_{T+1} E(x_{T+1} - U_{T+1}^L).$$

Notice that the expectation is taken at the beginning of period t and that the starting inventories generated by the policy Y_t are random.

By substituting the updates (5), (6) for x_{j+1}, O_{j+1} into the above equation and rearranging the terms we arrive at

$$\tilde{J}_t(x_t, O_t | Y_t) = E\sum_{j=t}^{T} \alpha_j \{K_j \delta(y_j - x_j) + G_j(y_j)\} + f(x_t, O_t),$$

where $G_j(y_j) = (c_j - \gamma_j c_{j+1}) y_j + \tilde{G}_j(y_j)$ and $f(x_t, O_t) = \sum_{j=t}^{T} \alpha_{j+1} \times c_{j+1} E(D_{j,j} + \cdots + D_{j,j+L+1} + O_{j,j+L+1}) - c_t x_t$. The last term is independent of the choice Y_t. Therefore, we define $J_t(x_t, O_t | Y_t)$ as

$$J_t(x_t, O_t | Y_t) = E\sum_{j=t}^{T} \alpha_j \{K_j \delta(y_j - x_j) + G_j(y_j)\}. \tag{12}$$

The problem is to find a policy that minimizes the cost function (12). Thus, we want

$$J_t(x_t, O_t) = \inf_{\forall Y_t \in \mathcal{A}} J_t(x_t, O_t | Y_t)$$

where $\mathcal{A}$ denote the set of all policies, including history-dependent policies. It can be shown that a Markov policy that achieves the infimum above exists. Therefore, the infimum can be replaced by the minimum. For a further discussion, refer to chapter 6 in Puterman (1994). The functional equation for the problem, thus, can be written as

$$J_t(x_t, O_t) = \min_{y_t \geq x_t} \{K_t \delta(y_t - x_t) + G_t(y_t) + \alpha_t E J_{t+1}(x_{t+1}, O_{t+1})\} \tag{13}$$

where $J_{T+1}(\cdot, \cdot) \equiv 0$ and the expectation is with respect to the vector $D_t = (D_{t,t}, D_{t,t+1}, \ldots, D_{t,t+N})$.

Appendix B: Proofs

Let $\nabla F(x, y) = F(x+1, y) - F(x, y)$.

Proof of Theorem 1. The proof is based on an induction argument built around Part 1. For $t = T$ the problem reduces to a single-period inventory problem. For this case, Part 1 is trivially satisfied since $V_T(\cdot, o_T) = G_T(\cdot) \in C(0, 0) \subset C(O, K_T)$ and $\lim_{|x|\to\infty} G_T(x) = \infty$. Assume by induction that Part 1 is true for $t = n$. Thus, $V_n(., o_n)$ has a finite minimizer, call it $S_n(o_n)$. Also there exists a $y < S_n(o_n)$ such that $V_n(y, o_n) > K_n + V_n(S_n(o_n), o_n)$. All three conditions of Lemma 2 are satisfied so H_n has a unique sign change from $-$ to $+$. Thus, the second part of the theorem is true by Corollary 1. The optimal value (10) and Lemma 1, Part 5, proves that $J_n(\cdot, o_n) \in C(0, K_n)$. The limits in the third part follow immediately from Equation (10) which proves part 3 for $t = n$. This fact implies that the limit of

$$V_{n-1}(y_{n-1}, o_{n-1}) = G_{n-1}(y_{n-1}) + \alpha_n E J_n(x_n, O_n)$$

when $|x| \to \infty$ is ∞ due to $\lim_{|x|\to\infty} G_{n-1}(x) = \infty$. It also implies that $V_{n-1}(\cdot, o_{n-1}) \in C(0, \alpha_n K_n) \subset C(0, K_{n-1})$ due to Lemma 1, Parts 2 and 4, and the convexity of G_{n-1}. Thus the first part of the theorem is also true for $t = n-1$, concluding the induction argument. □

Proof of Theorem 2. We proved the optimality of state-dependent $(s_t(o_t), S_t(o_t))$ policy in Theorem 1. If modified inventory position $x_t \leq s_t(o_t)$, then it is optimal to order-up-to $S_t(o_t)$. The corresponding optimal value is given by

$$J_t(x_t, o_t) = K_t + G(S_t(o_t)) + E J_{t+1} \times \left(S_t(o_t) - D_{t,t} - \sum_{s=t+1}^{t+L+1} D_{t,s} - o_{t,t+L+1}, O_{t+1}\right),$$

where $o_t = (o_{t,\,t+L+1}, \ldots, o_{t,\,t+N-1})$. We now demonstrate that $S_t(o_t) = S^m$ if $o_{t,\,t+L+1} \geq (\overline{S} - s^m)$. Assume for a contradiction that $S_t(o_t) > S^m$. Lemma 3 and the condition of the theorem lead to the following inequalities

$$s^m > \overline{S} - o_{t,\,t+L+1} \geq S_t(o_t) - o_{t,\,t+L+1} > S^m - o_{t,\,t+L+1}.$$

By the above inequalities we conclude that

$$EJ_{t+1}\left(S_t(o_t) - D_{t,\,t} - \sum_{s=t+1}^{t+L+1} D_{t,\,s} - o_{t,\,t+L+1}, O_{t+1}\right) = EJ_{t+1}\left(S^m - D_{t,\,t} - \sum_{s=t+1}^{t+L+1} D_{t,\,s} - o_{t,\,t+L+1}, O_{t+1}\right).$$

Since $G(S_t(o_t)) > G(S^m)$, we conclude that

$$J_t(x_t, o_t) > K_t + G(S^m) + EJ_{t+1} \times \left(S^m - D_{t,t} - \sum_{s=t+1}^{t+L+1} D_{t,\,s} - o_{t,\,t+L+1}, O_{t+1}\right),$$

contradicting the optimality of $S_t(o_t)$. □

Proof of Theorem 3. It suffices to show that the function J_t is bounded from above. Implementing the myopic order-up-to level in each period yields a trivial upper bound. We incur $K + G(S^m)$ in each period. If the initial inventory level is greater than S^m, it takes finite period of time, M, to deplete the inventory below S^m. Let us denote the cost incurred during this initial phase by C_M. Then $C_M + \alpha^M\{(K + G(S^m))/(1-\alpha)\}$ elicits a trivial upper bound. The proof of the first three parts, which implies the last part due to Theorem 2, follows the same steps as in Iglehart (1963), hence we refer the reader to his paper for the details. □

Proof of Theorem 4. We assume similar terminal conditions as in the positive set-up cost case. Hence, function $V_T(x, O_T)$ is equal to convex function $G_T(x)$ and the limit is equal to ∞ when $|x| \to \infty$. Assume by induction that the first part of the theorem is true for $t = n$. This implies the second part of the theorem; if $x \leq y_n(o_n)$ then order-up-to $y_n(o_n)$. Optimal value, thus, can be written as

$$J_n(x, o_n) = V_n(\max(y_n(o_n), x), o_n). \tag{14}$$

Equation (14) is an increasing convex function since for $x > y_n(o_n)$, $V_n(x, o_n)$ is convex and increasing due to Part 1 and 2, proving the third part of the theorem. To conclude the induction argument it suffices to show that the first part of the theorem is true for $t = n-1$. Notice that the update for the modified inventory position is linear and therefore a convex function of x. Also the composition of an increasing function and a convex function is convex (see Denardo 1982). Likewise, the convex combination of a convex function is convex. Thus $V_{n-1}(x, o_{n-1})$ is a convex function and $\lim_{|x|\to\infty} V_{n-1}(x, o_{n-1}) = \infty$ which concludes the induction argument for Parts 1, 2, and 3.

We now prove that Part 4 implies Part 5. Assume for a contradiction, there exists $o_t \geq o'_t$ such that $y_t(o_t) < y_t(o'_t)$ where $y(\cdot)$ is defined as in Equation (11). Notice that

$$0 \leq V_t(y_t(o'_t), o_t) - V_t(y_t(o_t), o_t) \leq V_t(y_t(o'_t), o'_t) - V_t(y_t(o_t), o'_t) \leq 0.$$

The first and the last inequality follow from the definition of $y_t(o'_t)$ and $y_t(o_t)$, respectively. The second inequality follows from decreasing differences. Hence, the above inequalities can only be satisfied as equalities. This implies $V_t(y_t(o'_t), o'_t) = V_t(y_t(o_t), o'_t)$. We can conclude that $y_t(o_t)$ is also a minimizer of function $V_t(\cdot, o'_t)$ and it is smaller than $y_t(o'_t)$ by assumption. This, however, contradicts the definition of $y_t(o'_t)$. Thus, Part 1 implies $y_t(o_t) \geq y_t(o'_t)$ for all $o_t \geq o'_t$. Next, we prove the theorem by induction. For $t = T$, $V_T(x, O_T) = G_T(x)$ satisfies the definition of decreasing differences, see Equation (7). Assume by induction that the first part of the theorem is true for $t = n$. This implies the second part. To prove Part 6 we investigate several cases and show that optimal value $J_n(x, o_n) = V_n(\max(y_n(o_n), x), o_n)$, has decreasing differences in (x, o_n). For ease of notation we drop the subscripts.

Case 1. If $x \geq x' > y(o) \geq y(o')$ then $J(x, o) = V(x, o)$, $J(x', o) = V(x', o)$, $J(x, o') = V(x, o')$ and $J(x', o') = V(x', o')$. The optimal value satisfies Inequality (7) due to the induction argument.

Case 2. If $y(o) \geq y(o') \geq x \geq x'$ then $J(x, o) = V(y(o), o)$, $J(x', o) = V(y(o), o)$, $J(x, o') = V(y(o'), o')$ and $J(x', o') = V(y(o'), o')$. Inequality (7) is trivially satisfied.

Case 3. If $y(o) \geq x \geq x' > y(o')$ then $J(x, o) = V(y(o), o)$, $J(x', o) = V(y(o), o)$, $J(x, o') = V(x, o')$ and $J(x', o') = V(x', o')$. Recall that $V(x, o')$ is an increasing function of x for fixed o' and $x \geq y(o')$. Thus, $V(x, o') - V(x', o') \geq 0$ for $x \geq x' > y(o')$. The inequality is again satisfied.

Case 4. If $y(o) \geq x > y(o') \geq x'$ then $J(x, o) = V(y(o), o)$, $J(x', o) = V(y(o), o)$, $J(x, o') = V(x, o')$ and $J(x', o') = V(y(o'), o')$. The inequality is satisfied since $y(o')$ is a minimizer of $V(\cdot, o')$.

Case 5. If $x > y(o) \geq x' > y(o')$ then $J(x, o) = V(x, o)$, $J(x', o) = V(y(o), o)$, $J(x, o') = V(x, o')$ and $J(x', o') = V(x', o')$. Notice that $V(x, o) - V(y(o), o) \leq V(x, o') - V(y(o), o') \leq V(x, o') - V(x', o')$ since $V(x, o)$ has decreasing differences in (x, o) and $V(x, o')$ is an increasing function for $x' > y(o')$.

All these cases show that Part 6 is true for $t = n$. Due to the definition V_{n-1}, Equation (9), and the fact that J_n has decreasing differences, we can conclude that V_{n-1} also has decreasing differences. This proves Part 4 for $t = n-1$, and concludes the induction argument. □

Proof of Theorem 5. We prove first Part 1 where $y^m_t \leq y^m_{t+1}$ for all t. For $t = T$, $V_t(y, o_t) = G_T(y)$. Therefore, any minimizer of $G_T(\cdot)$ gives an optimal order-up-to level. By the definitions of $y_T(o_T)$ and y^m_T we have $y^m_T = y_T(o_T)$. Assume by induction that $y^m_{n+1} = y_{n+1}(o_{n+1})$. Recall that $y_n(o_n) \leq y_{n+1}(o_n) = y^m_{n+1}$ holds due to Lemma 5 and the inductive hypothesis. We want to show that $y_n(o_n) = y^m_n$. Suppose first that $y_n(o_n) < y^m_n$, then $\nabla G_n(y_n(o_n)) < 0$. Also recall that $J_{n+1}(x, o_{n+1}) = J_{n+1}(y^m_{n+1}, o_{n+1})$ for all $x < y^m_{n+1}$, so $\nabla J_{n+1}(x, o_{n+1}) = 0$. Consequently, $\nabla V_n(y_n(o_n), o_n) < 0$ which contradicts the optimality of $y_n(o_n)$. Consider now the case $y^m_n < y_n(o_n)$. We have $G_n(y^m_n) \leq G_n(y_n(o_n))$ by the definition of myopic policy and $J_{n+1}(y^m_n, o) = J_{n+1}(y_n(o_n), o)$ since $y^m_n \leq y_n(o_n) \leq y^m_{n+1}$. Thus $V_n(y^m_n, o_n) \leq V_n(y_n(o_n), o_n)$, so y^m_n is a minimizer of $V_n(\cdot, o_n)$ that is strictly smaller than $y_n(o_n)$ which contradicts the definition of $y_n(o_n)$. Consequently, $y^m_n = y_n(o_n)$ as claimed. The proof for the second part of the theorem for the stationary case is entirely similar. □

Proof of Theorem 6. Similar to the fixed set-up cost, $C_M + \alpha^M\{G(y^m)/(1-\alpha)\}$ gives a trivial bound for the optimal value. The first and second parts of the theorem immediately follow from Theorem 4 and Lemma 5. The first two parts of the theorem and Theorem 5 imply the last part concluding the proof. □

Proof of Lemma 1. The first three properties can easily be shown by using the definition of (a, b)-convex functions. Hence we will only prove the last two properties.

Part 4. From the Part 3, $E[f(x-D, y)]$ is (a, b)-convex for all y. Part 2 and the fact that the expectation is a weighted sum where the weights add up to one concludes the proof for Part 4.

Part 5. Let $x_1 < x_2$. We need to show that $(x_1, f(x_1)+a)$ is visible from $(x_2, f(x_2)+b)$. This is trivial in $x_2 \le s$, so consider the case $x_1 < s < x_2$. We have to show that $(x_1, f(x_1)+a) = (x_1, g(s)+a)$ is visible from $(x_2, f(x_2)+b) = (x_2, g(x_2)+b)$. Consider the line segment joining $(s, g(s)+a)$ and $(x_2, g(x_2)+b)$. The slope of this line is greater than the slope of the line joining $(x_1, g(s)+a)$ and $(x_2, g(x_2)+b)$ since $x_1 < s$. Consequently, the line joining $(x_1, g(s)+a)$ and $(x_2, g(x_2)+b)$ lies above the line joining $(s, g(s)+a)$ and $(x_2, g(x_2)+b)$ and, hence, above the function f. Moreover, the line joining $(x_1, g(s)+a)$ and $(x_2, g(x_2)+b)$ has a nonnegative slope on account of $g(x_2)+b \ge g(s)+a$ and, as such, lies above $f(x)+a = g(s)+a$ over the interval $x_1 \le x \le s$. □

Proof of Lemma 2. We have $H_t(x, o_t) < 0$ from Part 3 and $0 < K_t = H_t(S_t(o_t), o_t)$ from Part 2. Thus, $H_t(x, o_t)$ has at least one sign change from − to +. We now argue that if $V_t(\cdot, o_t)$ is $(0, K_t)$-convex then $H_t(\dots, o_t)$ has at most one sign change from − to +. Assume for a contradiction that $H_t(x_1, o_t) > 0 > H_t(x, o_t)$ for some $x_1 < x$. This implies that there exists an $x_2 > x$ such that $V_t(x_1, o_t) < K_t + \min_{y_t \ge x_1} V_t(y_t, o_t) \le K_t + \min_{y_t \ge x} V_t(y_t, o_t) < K_t + V_t(x_2, o_t) < V_t(x, o_t)$, so this leads to $V_t(x, o_t) > \max(V_t(x_1, o_t), K_t + V_t(x_2, o_t))$, but then $V_t \notin QC(0, K_t)$, where a function f is (a, b)-quasi-convex, denoted by $f \in QC(a, b)$ if it satisfies the following inequality $f(x) \le \max(f(x_1)+a, f(x_2)+b)$ for all x such that $x_1 \le x \le x_2$. Thus, $V_t \notin C(0, K_t)$ since $C(0, K_t) \subset QC(0, K_t)$. This together with condition 1 preclude sign changes from + to −, proving that H_t has a unique sign change from − to +. □

Proof of Lemma 3. Recall from the optimal value of the finite horizon problem, equation (10), that

$$\nabla J_t(x_t, o_t) = \begin{cases} 0, & x_t < s_t(o_t), \\ \nabla V_t(x_t, o_t), & x_t \ge s_t(o_t). \end{cases} \tag{15}$$

To prove the first inequality we argue inductively that $\nabla V_t(x, o_t) < 0$ for all $x < S^m$ and any fixed vector o_t. This statement implies that the minimizer of $V_t(\cdot, o_t)$ is greater than or equal to S^m. For $t = T$ function $V_T(y, o_T) = G(y)$ is convex and reaches a minimum at S^m. Therefore, $\nabla V_T(y, o_T) < 0$ for all $y < S^m$ and for any fixed vector o_T. Assume, by induction, that $\nabla V_{t+1}(x, o_{t+1}) < 0$ for all $x < S^m$ and for all o_{t+1}. Then from equation (15), $\nabla J_{t+1}(x, o_{t+1}) \le 0$ for all $x < S^m$ and for any fixed vector o_{t+1}. Equation (9) and the previous statement imply that $\nabla V_t(x, o_t) = \{\nabla G(x) + \alpha E \nabla J_{t+1}(x - D_{t,t} - \sum_{s=t+1}^{t+L} D_{t,s} - O_{t+1,t+L+1}, O_{t+1})\} < 0$ for all o_t and $x < S^m$. This concludes the proof of the lower bound. Now, if we can show that for all $y > \overline{S} > S^m$, $V_t(y, o_t) > V_t(S^m, o_t)$ then we can claim that the global minimizer of $V_t(\cdot, o_t)$ is smaller than or equal to $\overline{S}$. Notice that,

$$\begin{aligned} V_t(y, o_t) - V_t(S^m, o_t) &\ge G(y) - G(S^m) - \alpha K \\ &\ge G(\overline{S}) - G(S^m) - \alpha K > 0. \end{aligned}$$

The first inequality can be shown along the lines of Veinott (1966, Lemma 1). The second inequality is due to the fact that function $G(\cdot)$ increases with x for all $x > S^m$. The last one is substantiated by the definition of $\overline{S}$. Thus, $S_t(o_t) \le \overline{S}$.

Recall that $s_t(o_t) = \max\{y : H_t(y, o_t) \le 0\}$. Also for all $y < s^m < S^m$,

$$\begin{aligned} V_t(y, o_t) - V_t(S_t(o_t), o_t) &\ge V_t(y, o_t) - V_t(S^m, o_t), \\ &\quad \text{since } S_t(o_t) \text{ is a minimizer,} \\ &\ge G(y) - G(S^m) > G(s^m) - G(S^m) \\ &\ge K. \end{aligned}$$

We showed above that $\nabla J_{t+1}(y, o) \le 0$ for all $y < S^m$ and any fixed vector o. Along with the definition of $V_t(\cdot, o_t)$, see Equation (9), this lemma implies the second inequality. This leads us to conclude $H_t(y, o_t) < 0$ for all $y < s^m$, which proves $s_t(o_t) \ge s^m$. □

Proof of Lemma 4. For $t = T$, the first inequality of the lemma is trivially satisfied. Assume by induction that this first inequality is true for $t = n$. Recall that $J_n(x, o) = V_n(\max(s_n(o), x), o)$. There are four cases to consider.

Case 1. If $x \ge \max\{s_n(o), s_{n-1}(o)\}$, then $J_{n-1}(x, o) - J_n(x, o) = V_{n-1}(x, o) - V_n(x, o) \ge 0$.

Case 2. If $x \le \min\{s_n(o), s_{n-1}(o)\}$, then $J_{n-1}(x, o) - J_n(x, o) = V_{n-1}(S_{n-1}(o), o) - V_n(S_n(o), o) \ge V_n(S_{n-1}(o), o) - V_n(S_n(o), o) \ge 0$. The first inequality above follows from the induction argument and the second is due to the definition of $S_n(o)$.

Case 3. If $s_{n-1}(o) \ge x \ge s_n(o)$, then $J_{n-1}(x, o) - J_n(x, o) = K + V_{n-1}(S_{n-1}(o), o) - V_n(x, o) \ge K + V_n(S_{n-1}(o), o) - V_n(x, o) \ge K + V_n(S_n(o), o) - V_n(x, o) \ge 0$ since $x \ge s_n(o)$.

Case 4. If $s_{n-1}(o) \le x \le s_n(o)$, then $J_{n-1}(x, o) - J_n(x, o) = V_{n-1}(x, o) - K - V_n(S_n(o), o) \ge V_{n-1}(x, o) - V_n(x, o) \ge 0$. The first inequality above is due to $x \le s_n(o)$.

Therefore, $J_{n-1}(x, o) \ge J_n(x, o)$. Notice that by the definition of V_{n-1} (see Equation (9)), and the previous statement, $V_{n-2}(x, o) \ge V_{n-1}(x, o)$. This concludes the induction argument. □

Proof of Lemma 5. The monotonicity of V_t and J_t follows from standard arguments of dynamic programming with nonnegative cost, namely that managing the system for one extra period cannot be done at a lower cost. It is similar to the proof of Lemma 4. We next prove the rest of the lemma using an induction argument. For $t = T$ we have $\nabla V_{T-1}(x, o) \ge \nabla V_T(x, o)$ due to the definition of V_t and the fact that J_t is an increasing function. Assume by induction that $\nabla V_{t-1}(x, o) \ge \nabla V_t(x, o)$ for some t. We will first show that this implies $y_{t-1}(o) \le y_t(o)$. Assume for a contradiction $y_{t-1}(o) > y_t(o)$.

Then we have

$$
\begin{aligned}
0 &\le \nabla V_t(y_t(o), o) \le \nabla V_t(y_{t-1}(o), o)\\
&\le \nabla V_{t-1}(y_{t-1}(o), o),\\
\nabla V_t(y_t(o)-1, o) &\le 0 \le \nabla V_t(y_{t-1}(o)-1, o)\\
&\le \nabla V_{t-1}(y_{t-1}(o)-1, o).
\end{aligned}
$$

We show above that $0 \le \nabla V_{t-1}(y_{t-1}(o), o)$ and $0 \le \nabla V_{t-1}(y_{t-1}(o) - 1, o)$. But, this contradicts the optimality of $y_{t-1}(o)$. We will now establish that $\nabla J_{t-1}(x, o) \ge \nabla J_t(x, o)$ holds for t. We consider three cases: (i) for $x \le y_{t-1}(o)$ the result holds because both sides of the inequality are zero. Similarly, for $x > y_t(o)$ the result holds because it reduces to $\nabla V_{t-1}(x, o) \ge \nabla V_t(x, o)$. Finally, for $y_{t-1}(o) < x \le y_t(o)$ we have $\nabla J_t(x, o) = 0$ and we know that $\nabla J_{t-1}(x, o) \ge 0$ since $J_t(x, o)$ is increasing in x. Finally, since $\nabla V_t(x, o) = \nabla G(x) + \alpha E \nabla J_{t+1}(x_{t+1}, O_{t+1})$ then $\nabla J_{t-1}(x, o) \ge \nabla J_t(x, o)$ implies that $\nabla V_{t-2}(x, o) \ge \nabla V_{t-1}(x, o)$ completing the induction. □

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