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INTEGRATION BY PARTS FORMULA ON SOLUTIONS TO STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMPS ON RIEMANNIAN MANIFOLDS

HIROTAKA KAI AND ATSUSHI TAKEUCHI*

Dedicated to the memory of Professor Hiroshi Kunita

Abstract. Consider solutions to Marcus-type stochastic differential equations with jumps on the bundle of orthonormal frames $O(M)$ over a Riemannian manifold M, and define the M-valued process by its canonical projection, which is parallel to the Eells-Elworthy-Malliavin construction of Brownian motions on M. In the present paper, the integration by parts formula for such jump processes is studied, and the strategy is based upon the calculus on Brownian motions via the Kolmogorov backward equations. The celebrated Bismut formula can be also obtained in our setting.

1. Introduction

Let $T > 0$ be fixed throughout the paper, and write $\mathbb{R}^m_0 := \mathbb{R}^m \backslash \{0\}$. Denote by $\nu(dz)$ a Lévy measure over $\mathbb{R}^m_0$ such that the function $|z|^2 \wedge 1$ is integrable with respect to the measure $\nu(dz)$. On a probability space $\big(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]}\big)$ with a filtration, let us introduce the following notations:

- the process $B = \big\{B_t = (B^1_t, \ldots, B^m_t)\,;\, t \in [0,T]\big\}$ is an m-dimensional Brownian motion starting from $0 \in \mathbb{R}^m$,
- $N(dt, dz)$ is a Poisson random measure over $[0,T] \times \mathbb{R}^m_0$ with the intensity measure $\widehat{N}(dt, dz) := dt\,\nu(dz)$,
- $\widetilde{N}(dt, dz) = N(dt, dz) - \widehat{N}(dt, dz)$ is the compensated Poisson random measure.

For simplicity of notations, we shall write $K_1 = \big\{z \in \mathbb{R}^m_0\,;\, |z| \le 1\big\}$ and

$$\overline{N}(dt, dz) := \widetilde{N}(dt, dz)\,\mathbb{I}_{K_1}(z) + N(dt, dz)\,\mathbb{I}_{K_1^c}(z).$$

Now, we shall introduce our framework of the differential geometry. Let (M, g) be a connected, compact and smooth Riemannian manifold of dimension m with the Levi-Civita connection $\nabla = \big\{\Gamma^i_{jk}\,;\, 1 \le i, j, k \le m\big\}$. Let $GL(M)$ be the bundle

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of linear frames on M, and $O(M)$ the submanifold of $GL(M)$ defined by

$$O(M) = \big\{ r = (x, e)\,;\, x \in M,\, e = (e_1, \ldots, e_m) : \text{ an orthonormal basis in } T_x M \big\}$$

which is called the bundle of orthonormal frames on M. The canonical projection $\pi : O(M) \to M$ is defined by $\pi(r) = x$ for $r = (x, e) \in O(M)$. Let $H_1, \ldots, H_m$ be the vector fields over $GL(M)$ such that the tangent vector $H_i(r)$ is the horizontal lift of $e_i \in T_x M$ for each $r = (x, e)$. These are called the canonical horizontal vector fields. In a local coordinate $\big(x_i, e_i^\alpha\,;\, 1 \le i,\, \alpha \le m\big)$, the vector field H_i can be expressed as

$$H_i = \sum_{\alpha=1}^{m} e_i^\alpha \frac{\partial}{\partial x_\alpha} - \sum_{\alpha,\beta,p,q=1}^{m} \Gamma^q_{\alpha\beta}(x)\, e_i^\alpha\, e_p^\beta \frac{\partial}{\partial e_p^q}. \tag{1.1}$$

Write $H = \big(H_1, H_2, \ldots, H_m\big)$. Remark that the vector fields H_i $(1 \le i \le m)$ are tangent to $O(M)$. Details on the differential geometry can be seen in Kobayashi and Nomizu [10].

Now, we shall construct stochastic processes on the manifold M. As for diffusion processes on M, especially, the Brownian motions on M, there are several approaches to do it. One of them has been known as the Eells-Elworthy-Malliavin construction. Let $r = (x, e) \in O(M)$, and $R = \big\{R_t\,;\, t \in [0, T]\big\}$ be the $O(M)$-valued process determined by the Stratonovich-type stochastic differential equation without any jumps of the form:

$$dR_t = H(R_t) \circ dB_t, \quad R_0 = r, \tag{1.2}$$

which is the diffusion process on $O(M)$ with the infinitesimal generator

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{m} H_i H_i. \tag{1.3}$$

The operator $\mathcal{L}$ is often called the horizontal Laplacian in the Bochner sense. Then, we shall define the M-valued process $X = \big\{X_t\,;\, t \in [0, T]\big\}$ by projection as

$$X_t := \pi(R_t), \tag{1.4}$$

which satisfies the equation of the form:

$$dX_t = (d\pi)_{R_t}\big(H(R_t)\big) \circ dB_t, \quad X_0 = x. \tag{1.5}$$

Here, $d\pi : T\,GL(M) \to TM$ is the tangent map of π. Denote the Laplace-Beltrami operator on M by Δ_M. Since

$$(\mathcal{A}f)(x) := \big(\mathcal{L}(f \circ \pi)\big)(r) = \frac{1}{2}\,(\Delta_M f)(x) \tag{1.6}$$

for $f \in C^\infty(M\,;\,\mathbb{R})$, and is independent of the choice of r satisfying $\pi(r) = x$, the M-valued process X is the diffusion process with the infinitesimal generator $\mathcal{A}$ by using the rotational invariance of the m-dimensional Brownian motion B. The M-valued process X is called the Brownian motion on M. Details can be seen in Hsu [6], and Ikeda and Watanabe [8].

On the other hand, it would be a natural question how we should define Lévy processes on the manifold M. There have been some works to attack such a question. Hunt [7] studied Lévy processes on Lie groups from the viewpoint of functional analysis. Moreover, Applebaum and Kunita [3] constructed the M-valued process as the solutions to jump-type stochastic differential equations driven by Lévy processes. Their equation is so-called the Marcus type one based upon the jumps by the exponential maps. See also Kunita's last book [12] in 2019. Applebaum and Estrade [2] discussed the Eells-Elworthy-Malliavin construction of an M-valued process by the canonical projection of the $O(M)$-valued process. They mentioned that the isotropic property on the Lèvy measure is one of the sufficient conditions in order to gurantee the Markovian property of the M-valued process.

In the present paper, let us consider an $O(M)$-valued process defined as the solution to Marcus-type stochastic differential equations in which jumps are determined through exponential maps on $O(M)$. Then, the M-valued process can be defined as the projection of the $O(M)$-valued process constructed above, which is almost parallel to the Eells-Elworthy-Malliavin approach on the diffusion process on M. As given a remark in Applebaum and Estrade [2], the obtained M-valued process is not always Markovian, because the process depends on the choice of the frames. The present paper shall focus on the following topics:

(i) the revisit of the work by Applebaum and Estrade [2] in order to construct the Markov process with jumps on M,
(ii) the integration by parts formula on the M-valued Lévy process.

As stated above, such kind of studies in the case without any jumps seems to be very classical, which can be seen in Hsu [6], and Ikeda and Watanabe [8]. Moreover, the detailed study with its integration by parts formula has been already done by Bismut (cf. Bismut [4] and Hsu [6]), which is called the Bismut formula after his great contribution. A similar study on the integration by parts formula for jump processes in the Euclidean space has been already obtained in Takeuchi [14]. Since the process in our situation has the Brownian part and the jump part, we have two approaches in the study of stochastic calculus : one of them is based upon the Brownian motion, and the other is to focus on the jump part. The present paper will pay attention to the effect from the diffusion terms of the equation, only. The obtained formula is quite similar to the celebrated Bismut formula for the diffusion process on M. This paper is the survey of Kai and Takeuchi [9] only on the effect from the diffusion coefficients of the equation. The formula with the effects from the jump terms, or the ones from the diffusion and the jump terms, can be seen in Kai and Takeuchi [9].

2. Jump-type Stochastic Differential Equations on Manifolds

In this section, we shall define the M-valued jump process based upon the Eells-Elworthy-Malliavin construction. For $z \in \mathbb{R}^m_0$, we shall introduce the one parameter group of diffeomorphisms $\xi^z = \big\{\xi^z_\sigma(r)\,;\,\sigma \in [0,1],\, r \in O(M)\big\}$ given by the ordinary differential equation on $O(M)$ of the form:

$$\frac{d}{d\sigma}\xi^z_\sigma(r) = H\big(\xi^z_\sigma(r)\big)\,z, \quad \xi^z_0(r) = r. \tag{2.1}$$

The solution can be also denoted by

$$\xi^z_\sigma(r) = \mathrm{Exp}(\sigma H z)(r). \tag{2.2}$$

Then, for $r = (x, e) \in O(M)$, let us consider the $O(M)$-valued process $R = \big\{R_t\,;\, t \in [0, T]\big\}$ determined by the Marcus-type stochastic differential equation on $O(M)$ formally given by

$$\begin{cases} dR_t = H(R_t) \circ dB_t + \displaystyle\int_{\mathbb{R}^m_0} \big\{\xi^z_1(R_{t-}) - R_{t-}\big\}\, \overline{N}(dt, dz), \\ R_0 = r. \end{cases} \tag{2.3}$$

Precisely, the equation (2.3) means that

$$\begin{aligned} F(R_t) = F(r) &+ \int_0^t (HF)(R_s) \circ dB_s \\ &+ \int_0^{t+}\int_{\mathbb{R}^m_0} \big\{F\big(\xi^z_1(R_{s-})\big) - F(R_{s-})\big\}\, \overline{N}(ds, dz) \\ &+ \int_0^t\int_{K_1} \Big\{F\big(\xi^z_1(R_s)\big) - F(R_s) - (HF)(R_s)\, z\Big\}\, \widehat{N}(ds, dz) \end{aligned}$$

for all $F \in C^\infty\big(O(M)\,;\,\mathbb{R}\big)$ with a compact support.

Proposition 2.1. *The process R lies in $O(M)$.*

Proof. For $1 \le i,\, j \le m$, define

$$F_{ij}(\tilde{r}) = \langle \tilde{e}_i,\, \tilde{e}_j\rangle_{T_{\tilde{x}}M}, \quad \tilde{r} = \big(\tilde{x},\, \tilde{e} = (\tilde{e}_1,\, \ldots,\, \tilde{e}_m)\big) \in GL(M),$$

where $\langle\cdot,\,\cdot\rangle_{T_{\tilde{x}}M}$ is the inner product on each tangent space $T_{\tilde{x}}M$, that is,

$$\langle \tilde{e}_i,\, \tilde{e}_j\rangle_{T_{\tilde{x}}M} = \sum_{\alpha,\,\beta=1}^m g_{\alpha\beta}(\tilde{x})\, \tilde{e}^\alpha_i\, \tilde{e}^\beta_j$$

in a local coordinate $\big(\tilde{x}_i,\, \tilde{e}^\alpha_i\,;\, 1 \le i,\, \alpha \le m\big)$. Since

$$(H_kF_{ij})(\tilde{r}) = \sum_{\alpha=1}^m \tilde{e}^\alpha_k \frac{\partial F_{ij}}{\partial \tilde{x}_\alpha}(\tilde{r}) - \sum_{\alpha,\,\beta,\,p,\,q=1}^m \Gamma^q_{\alpha\beta}(\tilde{x})\, \tilde{e}^\alpha_k\, \tilde{e}^\beta_p\, \frac{\partial F_{ij}}{\partial \tilde{e}^q_p}(\tilde{r}) = 0$$

for $1 \le i,\, j,\, k \le m$, and the initial point $r = (x,\, e)$ of the process R lies in $O(M)$, we can derive that

$$\begin{aligned} F_{ij}(R_t) = F_{ij}(r) &+ \int_0^t (HF_{ij})(R_s) \circ dB_s \\ &+ \int_0^{t+}\int_{\mathbb{R}^m_0} \big\{F_{ij}\big(\xi^z_1(R_{s-})\big) - F_{ij}(R_{s-})\big\}\, \overline{N}(ds, dz) \\ &+ \int_0^t\int_{K_1} \Big\{F_{ij}\big(\xi^z_1(R_s)\big) - F_{ij}(R_s) - (HF_{ij})(R_s)\, z\Big\}\, \widehat{N}(ds, dz) \\ = \delta_{ij} & \end{aligned}$$

for $1 \le i,\, j \le m$. The proof is completed. □

The detailed studies can be seen in Applebaum and Estrade [2], Applebaum and Kunita [3], Fujiwara [5] and Kunita [12] on the detailed studies. Moreover, the process R is Markovian, and its infinitesimal generator $\mathcal{J}$ is

$$(\mathcal{J}F)(r) = (\mathcal{L}F)(r) + \int_{\mathbb{R}^m_0} \big\{ F\big(\xi^z_1(r)\big) - F(r) - (HF)(r)\, z\, \mathbb{I}_{K_1}(z) \big\}\, \nu(dz) \quad (2.4)$$

for $F \in C^\infty\big(O(M)\,;\mathbb{R}\big)$ with a compact support.

Define the M-valued process $X = \big\{X_t\,;\, t \in [0,T]\big\}$ by $X_t = \pi(R_t)$ for $t \in [0,T]$. Then, it can be easily checked that the process X satisfies the following equation:

$$\begin{cases} dX_t = (d\pi)_{R_t}\big(H(R_t)\big) \circ dB_t + \displaystyle\int_{\mathbb{R}^m_0} \big\{\zeta^z_1(X_{t-}) - X_{t-}\big\}\, \overline{N}(dt,dz), \\ X_0 = x. \end{cases} \quad (2.5)$$

Here, $d\pi : T\,GL(M) \to TM$ is the tangent map of π, and

$$\zeta^z_\sigma(r) := \pi\big(\mathrm{Exp}(\sigma H z)(r)\big) = \mathrm{Exp}\Big(\sigma\big((d\pi)H\big)z\Big)(x)$$

for $z \in \mathbb{R}^m_0$ and $\sigma \in [0,1]$. The last equality can be justified, because

$$\big(H_i(f\circ\pi)\big)(r) = \Big(\big((d\pi)_r H_i\big) f\Big)\big(\pi(r)\big)$$

for $f \in C^\infty(M\,;\mathbb{R})$. As pointed out in Applebaum and Estrade [2], in general, the M-valued process X is not always Markovian, because it depends on the choice of the frame, which can be seen from the equation (2.5).

Theorem 2.2 (cf. Applebaum [1], Applebaum and Estrade [2], Kai and Takeuchi [9])**.** *Suppose that the Lévy measure $\nu(dz)$ is rotationally invariant. Then, the M-valued process X determined by the equation* (2.5) *is Markovian.*

Proof. We shall give the sketch of the proof only. For an orthogonal matrix $A \in \mathbb{R}^m \otimes \mathbb{R}^m$, let us write

$$eA = \big((eA)_1, \ldots, (eA)_m\big), \quad (eA)_i = \sum_{j,k=1}^m e^j_i\, A_{jk} \frac{\partial}{\partial x_k}.$$

Remark that the fibrewise action of the orthogonal group $O(m)$ to $O(M)$ is transitive. Let $Z = \big\{Z_t\,;\, t \in [0,T]\big\}$ be the m-dimensional Lévy process given by

$$Z_t = B_t + \int_0^{t+}\int_{\mathbb{R}^m_0} z\, \overline{N}(ds,dz),$$

and denote the solution to the equation (2.5) by $X_t\big((x,e),Z\big)$, in order to emphasize the dependence of the initial point (x,e) and the driving process Z. Then, since the m-dimensional Brownian motion B and the Lévy measure $\nu(dz)$ over $\mathbb{R}^m_0$ are rotationally invariant, the process $AZ = \big\{AZ_t\,;\, t \in [0,T]\big\}$ has the same law as the process Z, which implies that

$$X_t\big((x,e),AZ\big) \overset{d}{=} X_t\big((x,e),Z\big).$$

On the other hand, the uniqueness of the solutions to the equation (2.5) leads us to see that

$$X_t\big((x,e),AZ\big) \overset{d}{=} X_t\big((x,eA),Z\big).$$

So, we can get that $X_t\big((x,e),Z\big) \overset{d}{=} X_t\big((x,eA),Z\big)$, which completes the proof. $\square$

3. Main Result

Now, let us study the integration by parts formula in our setting. Before doing it, we shall prepare some studies. Suppose that the Lévy measure $\nu(dz)$ satisfies

$$\int_{K_1^c} |z|^2\,\nu(dz) < +\infty. \tag{3.1}$$

Write $H = \big(H_1,\,\ldots,\,H_m\big)$, and let $f \in C^2(M\,;\mathbb{R})$ with a compact support, and and $t \in [0,T]$. Define

$$\Phi(s,r) := \mathbb{E}\big[(f\circ\pi)(R_{t-s})\big]$$

for $s \in [0,t)$ and $R_0 = r \in O(M)$. Then, it is well known that the function Φ is in $C_b^{1,2}\big([0,t)\times O(M)\,;\mathbb{R}\big)$, and satisfies the Kolmogorov backward equation:

$$\begin{cases} \dfrac{\partial\Phi}{\partial s}(s,r) + (\mathcal{J}\Phi)(s,r) = 0, \ \ s\in[0,t) \\ \lim\limits_{s\nearrow t}\Phi(s,r) = (f\circ\pi)(r), \end{cases} \tag{3.2}$$

where $\mathcal{J}$ is defined in (2.4). Then, the Itô formula leads us to obtain that

$$\begin{aligned} \Phi(s,R_s) &= \Phi(0,r) + \int_0^s (H\Phi)(u,R_u)\,dB_u \\ &\quad + \int_0^{s+}\!\!\int_{\mathbb{R}_0^m} \Big\{\Phi\big(u,\xi_1^z(R_{u-})\big) - \Phi(u,R_{u-})\Big\}\,\widetilde{N}(du,dz) \end{aligned} \tag{3.3}$$

for $s \in [0,t)$, because the drift term is 0 by the equation (3.2). Hence, the limiting procedure as $s \nearrow t$ leads us to see that

$$\begin{aligned} f(X_t) &= \mathbb{E}\big[f(X_t)\big] + \int_0^t (H\Phi)(u,R_u)\,dB_u \\ &\quad + \int_0^{t+}\!\!\int_{\mathbb{R}_0^m} \Big\{\Phi\big(u,\xi_z^1(R_{u-})\big) - \Phi(u,R_{u-})\Big\}\,\widetilde{N}(du,dz). \end{aligned} \tag{3.4}$$

The formula (3.4) can be also regarded as the martingale representation.

Similarly to the study stated above, we can derive from the Itô formula that, for $s \in [0,t)$ and $1 \le i \le m$,

$$\begin{aligned} (H_i\Phi)(s,R_s) &= (H_i\Phi)(0,r) + \int_0^s (HH_i\Phi)(u,R_u)\,dB_u \\ &\quad + \int_0^{s+}\!\!\int_{\mathbb{R}_0^m} \big\{(H_i\Phi)\big(u,\xi_1^z(R_{u-})\big) - (H_i\Phi)(u,R_{u-})\big\}\,\widetilde{N}(du,dz) \\ &\quad + \int_0^s \left\{\frac{\partial}{\partial u}\big((H_i\Phi)(u,R_u)\big) + (\mathcal{J}H_i\Phi)(u,R_u)\right\} du. \end{aligned}$$

Here, we have to take care of the study on the last term, because $\mathcal{J}H_i\Phi$ is not always equal to $H_i\mathcal{J}\Phi$. In fact, we shall give a remark that

$$(\text{the last term}) = \int_0^s \big([\mathcal{L},H_i]\Phi\big)(u,R_u)\,du$$

$$
\begin{aligned}
&+\int_0^s\int_{\mathbb{R}_0^m}\Big\{(H_i\Phi)\big(u,\xi_1^z(R_u)\big)-\big(H_i(\Phi\circ\xi_1^z)\big)(u,R_u)\\
&\qquad\qquad -\big([Hz,H_i]\Phi\big)(u,R_u)\,\mathbb{I}_{K_1}(z)\Big\}\,\widehat{N}(du,dz)\\
&=\int_0^s\big([\mathcal{A},H_i]\Phi\big)(u,R_u)\,du,
\end{aligned}\tag{3.5}
$$

where

$$
\mathcal{A}=\mathcal{L}+\int_{\mathbb{R}_0^m}\{(\xi_1^z)^*-Hz\,\mathbb{I}_{K_1}(z)\}\,\nu(dz).
$$

Here, the symbol "$[\,\cdot\,,\,\cdot\,]$" is the Lie bracket on vector fields over $O(M)$ in the usual sense, and $(\xi_1^z)^*$ is the pullback by the function ξ_1^z. The last equality in (3.5) can be justified, because

$$
\begin{aligned}
\big(\big[(\xi_1^z)^*,H_i\big]\Phi\big)(u,R_u)&=\big((\xi_1^z)^*(H_i\Phi)\big)(u,R_u)-\big(H_i((\xi_1^z)^*\Phi)\big)(u,R_u)\\
&=(H_i\Phi)\big(u,\xi_1^z(R_u)\big)-\big(H_i(\Phi\circ\xi_1^z)\big)(u,R_u),
\end{aligned}
$$

and $\big([Hz,H_i]\Phi\big)(u,R_u)=0$ for $1\le i\le m$. Thus, we have

$$
\begin{aligned}
&(H_i\Phi)(s,R_s)\\
&=(H_i\Phi)(0,r)+\int_0^s(HH_i\Phi)(u,R_u)\,dB_u\\
&\quad+\int_0^{s+}\int_{\mathbb{R}_0^m}\Big\{(H_i\Phi)\big(u,\xi_1^z(R_{u-})\big)-(H_i\Phi)(u,R_{u-})\Big\}\,\widetilde{N}(du,dz)\\
&\quad+\int_0^s\big([\mathcal{A},H_i]\Phi\big)(u,R_u)\,du
\end{aligned}\tag{3.6}
$$

for $s\in[0,t)$. Taking the limit as $s\nearrow t$ enables us to see that the equation (3.6) can be justified for $s\in[0,t]$, because the function f is in $C^2(M\,;\mathbb{R})$ with a compact support.

Let $L=\big\{L_s\,;\,s\in[0,t]\big\}$ be the solution to the following equation:

$$
\frac{dL_s}{ds}=-L_s\,C(R_s),\quad L_0=I_m,\tag{3.7}
$$

where $I_m\in\mathbb{R}^m\otimes\mathbb{R}^m$ is the identity, $H_k^*(r)\in T_r^*O(M)$ is the dual of $H_k(r)\in T_rO(M)$, and

$$
C(r)=\begin{pmatrix}\big(H_1^*([\mathcal{J},H_1])\big)(r) & \cdots & \big(H_m^*([\mathcal{J},H_1])\big)(r)\\ \vdots & & \vdots\\ \big(H_1^*([\mathcal{J},H_m])\big)(r) & \cdots & \big(H_m^*([\mathcal{J},H_m])\big)(r)\end{pmatrix}
$$

for $r\in O(M)$. Remark that

$$
\big(V(f\circ\pi)\big)(r)=\sum_{j=1}^m\big((H_j^*(V)H_j)(f\circ\pi)\big)(r)
$$

for any smooth vector field V over $O(M)$, and that

$$\sum_{j=1}^{m} \frac{dL_s^{ij}}{ds} (H_j\Phi)(s,R_s) = -\sum_{j=1}^{m} L_s^{ij} \big([\mathcal{J}, H_j]\Phi\big)(s,R_s)$$

for $1 \leq i \leq m$. Thus, applying the Itô product formula tells us to see that

$$\begin{aligned}
&\sum_{j=1}^{m} L_s^{ij} (H_j\Phi)(s,R_s) \\
&= (H_i\Phi)(0,r) + \int_0^s \sum_{j,k=1}^{m} L_u^{ij} (H_k H_j \Phi)(u,R_u)\, dB_u^k \\
&\quad + \int_0^{s+}\int_{\mathbb{R}_0^m} \sum_{j=1}^{m} L_{u-}^{ij} \Big\{(H_j\Phi)\big(u, \xi_1^z(R_{u-})\big) - (H_j\Phi)(u,R_{u-})\Big\}\, \widetilde{N}(du,dz)
\end{aligned} \tag{3.8}$$

for $s \in [0,t]$.

Recall that $e_x = \big((e_1)_x, \ldots, (e_m)_x\big)$ is the frame in the tangent space T_xM at $x \in M$. Then, we have

Theorem 3.1 (cf. Kai and Takeuchi [9])**.** *Suppose that the Lévy measure satisfies* (3.1)*. Then, for all $f \in C^\infty(M\,;\mathbb{R})$, it holds that*

$$e_x\left(\mathbb{E}\big[f(X_t^x)\big]\right) = \mathbb{E}\left[f(X_t^x)\,\frac{1}{t}\int_0^t (L_s\, dB_s)^*\right]. \tag{3.9}$$

Proof. Our strategy to get the formula (3.9) is almost parallel to the method in Takeuchi [14] on the study of the integration by parts formula in the case of the Euclidean space. At the beginning, let us write

$$N_s := L_s \begin{pmatrix} (H_1\Phi)(s,R_s) \\ \vdots \\ (H_m\Phi)(s,R_s) \end{pmatrix}$$

for $s \in [0,t]$. Then, from the equality (3.8), we see that the process $\{N_s\,;\, s \in [0,t]\}$ is an m-dimensional $(\mathcal{F}_t)$-martingale, which implies that

$$e_x\Big(\mathbb{E}\big[f(X_t^x)\big]\Big) = H_r\Big(\mathbb{E}\big[(f\circ\pi)(R_t^r)\big]\Big) = (H\Phi)(0,r) = N_0^* = \mathbb{E}\big[N_t^*\big].$$

Here, we have used the notations X_t^x and R_t^r as seen in the above study, in order to emphasize the dependence of the initial points x and r of the processes X and R, respectively. Hence, for $v \in \mathbb{R}^m$, the Itô product rule enables us to obtain that

$$\begin{aligned}
\left\langle \Big(e_x\left(\mathbb{E}\big[f(X_t^x)\big]\right)\Big)^*, v\right\rangle_{\mathbb{R}^m} &= \big\langle \mathbb{E}\big[N_t\big], v\big\rangle_{\mathbb{R}^m} \\
&= \left\langle \mathbb{E}\left[\frac{1}{t}\int_0^t N_s\, ds\right], v\right\rangle_{\mathbb{R}^m} \\
&= \frac{1}{t}\,\mathbb{E}\left[\int_0^t \sum_{i,j=1}^{m} L_s^{ij}\,(H_j\Phi)(s,R_s)\, v_i\, ds\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{t}\,\mathbb{E}\left[\int_0^t \sum_{i=1}^m (H_i\Phi)(s,R_s)\,dB_s^i\,\left\langle \int_0^t L_s\,dB_s, v\right\rangle_{\mathbb{R}^m}\right] \\
&= \frac{1}{t}\,\mathbb{E}\left[f(X_t)\,\left\langle \int_0^t L_s\,dB_s, v\right\rangle_{\mathbb{R}^m}\right] \\
&= \left\langle \mathbb{E}\left[f(X_t)\,\frac{1}{t}\int_0^t L_s\,dB_s\right], v\right\rangle_{\mathbb{R}^m},
\end{aligned}
$$

where $\langle\,\cdot\,,\;\cdot\,\rangle_{\mathbb{R}^m}$ is the inner product in $\mathbb{R}^m$. Here, the fourth equality can be justified by the computation of the quadratic variation on stochastic integrals with respect to the Brownian motions. The proof is completed. □

Remark 3.2. The equation (3.9) in Theorem 3.1 is exactly the same representation as the celebrated Bismut formula for the Brownian motion on M (cf. Hsu [6]). The obtained result seems to be natural, because the calculus in the present paper is based upon the study focused only on the component of the Brownian motion on the Euclidean space. Moreover, the formula (3.9) can be also obtained from the viewpoint of the partial Malliavin calculus on the Wiener-Poisson space focused only on the effects from the Brownian motions on the Euclidean space (cf. Kohatsu-Higa and Takeuchi [11]).

On the other hand, under the further additional condition on the Lévy measure $\nu(dz)$, we can also obtain other types of the integration by parts formulas, which are focused on the jump term coefficients, or on the both components of the Brownian motions and the jumps, in the equation (2.5). The detailed studies can be seen in Kai and Takeuchi [9].

As one of the typical applications of Theorem 3.1, we have

Corollary 3.3. *Suppose that the Lévy measure satisfies* (3.1). *Then, for each* $0 < t \leq T$, *the probability law of the solution* X_t *is absolutely continuous with respect to the volume element on* M.

Proof. From Theorem 3.1 and the Cauchy-Schwarz inequality, we see that

$$
\begin{aligned}
\left| e_x\Big(\mathbb{E}\big[f(X_t)\big]\Big)\right| &= \left|\mathbb{E}\left[f(X_t)\,\frac{1}{t}\int_0^t (L_s\,dB_s)^*\right]\right| \\
&\leq \|f\|_\infty\,\mathbb{E}\left[\left|\frac{1}{t}\int_0^t (L_s\,dB_s)^*\right|^2\right]^{1/2},
\end{aligned}
$$

which implies that the probability law of the M-valued random variable X_t is absolutely continuous with respect to the volume element on M, via the Sobolev-type inequality. Here, $|\,\cdot\,|$ denotes the standard norm on $\mathbb{R}^m$. □

Remark 3.4. We can also obtain the higher-order integration by parts formula, similarly to the strategy stated in Theorem 3.1, or the simple computations in the framework of the Malliavin calculus on the Wiener space, which implies that the probability law of the M-valued random variable X_t admits a smooth density with respect to the volume element on M.

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