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INTEGRATION BY SUBSTITUTION

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ABSTRACT. The author shows how N-functions provide a natural setting in which to establish the change of variables formula for Lebesgue or Denjoy/Perron integrals. By abolishing the need to pass to the limit under the integral sign, the validity of the classical formula is significantly extended, yielding new results even in the case of Lebesgue integrals.

1. N-functions. A real-valued function on a finite interval is, by definition, an N-function if it carries null sets into null sets, the measure being Lebesgue measure (both here and in what follows).

By a theorem of Banach ([**1**, p. 169]), any continuous N-function possesses a finite derivative at each point of a set of positive measure. If this derivative, when extended by assigning it the value zero wherever it is not defined, happens to be Denjoy/Perron integrable, then its indefinite integral differs from the associated N-function at most by a constant. This remarkable result is due to S. Saks [**5**, p. 145f.], extending Bary [**2**, p. 199f.], who had proved it for the Lebesgue integral.

2. The chain rule. Serrin and Varberg [**7**] have shown that a chain rule applies to the composition of two a.e. differentiable functions when the outer function is an N-function. A slight reformulation of their result brings out a useful extension property of the chain rule derivative.

EXTENDED CHAIN RULE. *Let g be an a.e. differentiable real-valued function on the interval $[a, b]$, and let F be an a.e. differentiable N-function defined on some interval which contains the range of g. Suppose that f is any function equivalent to F'. Then the function $(f \circ g) \cdot g'$ agrees with $(F \circ g)'$ a.e. on the set where the latter exists and vanishes a.e. on the remainder of $[a, b]$.*

PROOF. Exactly the proof of Serrin and Varberg's Theorem 2, [**7**, p. 516], when their assumption of the a.e. differentiability of $F \circ g$ is dropped.

3. The change of variables formula. The foregoing considerations provide the basis for a transparent proof of a general

CHANGE OF VARIABLES FORMULA. *Let g be a continuous, a.e. differentiable N-function on $[a, b]$, and let F be a continuous, a.e differentiable N-function*

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defined on an interval which contains the range of g. Suppose that f is equivalent to F′. Then there holds

$$F \circ g(b) - F \circ g(a) = \int_a^b (f \circ g) \cdot g'$$

whenever the integrand is Lebesgue or Denjoy/Perron integrable, and the analogous formula holds on every subinterval of $[a, b]$.

PROOF. By the Extended Chain Rule, the integrand is equivalent to the function which extends $(F \circ g)'$ to all of $[a, b]$ in the trivial way. Noting that $F \circ g$ is a continuous N-function, Saks's Theorem applies: $F \circ g$ is the integral of its derivative so extended, plus a constant. This yields, by equivalence, the displayed formula. The same reasoning applies on any subinterval of $[a, b]$.

SCHOLIUM. *Take g as above and let f be Denjoy/Perron integrable, or just Lebesgue integrable, on the range of g, with indefinite integral F. Then, by* [**6**, *p.* 251], *F is a continuous, a.e. differentiable N-function, and we have the classical formula*

$$\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) \cdot g' \tag{CV}$$

whenever the integral at right exists in the Lebesgue or Denjoy/Perron sense.

REMARK. Note that the Scholium covers the case where f is Lebesgue integrable but $(f \circ g) \cdot g'$ is only Denjoy/Perron integrable, as in the example of McShane [**4**, p. 214].

NOTE ADDED IN PROOF. Priority for (CV) must be assigned to K. Krzyżewski, *On change of variable in the Denjoy-Perron integral.* I, Colloq. Math. **9** (1962), 99–104, who also established the main result of [**7**].

4. A more general formula. Continuous, a.e. differentiable N-functions belong to a more extensive class of continuous functions which have the property of transforming their set of points of nondifferentiability into a set of measure zero (which depends upon the function). This larger class has been studied by Bary and Menchoff [**2**], who showed that any such function g can be decomposed into $g = h \circ k$, where h is increasing and absolutely continuous and k is Lipschitzian. These functions g need *not* be differentiable a.e.; nevertheless, we have the

GENERALIZED CHANGE OF VARIABLES FORMULA. *Let F and f be as in the Scholium, and let g be a continuous function on* $[a, b]$ *which carries its set of points of nondifferentiabiltiy into a set of measure zero. Set* $g^* = g'$ *wherever* g' *exists, and zero elsewhere. Then*

$$\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) \cdot g^* \tag{GCV}$$

whenever the integral at right exists.

PROOF. Set $g = h \circ k$ as above, and apply (CV) first in the case of increasing absolutely continuous functions (for which it is always valid: [7, Corollary 6] or [8, p. 19]), and then apply (CV) in the form of the Scholium, or [7, Corollary 8], to get

$$\int_{g(a)}^{g(b)} f = \int_{k(a)}^{k(b)} (f \circ h) \cdot h' = \int_a^b (f \circ h \circ k)(h' \circ k) \cdot k'$$

if the last integral exists. Now the Extended Chain Rule applied to $h \circ k$ assures that $(h' \circ k) \cdot k' = g^*$ a.e., and (GCV) emerges.

REMARK. Actually, (GCV) is valid when g is *any* continuous N-function on $[a, b]$, provided both integrals exist. For a proof, see [3].

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