INTEGRATION OF THE COMPOSITE MESH TECHNIQUE WITH THE MULTIGRID METHOD *

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Abstract. The Composite Finite Element Mesh method is useful for discretization error estimation and, in addition, for solution improvement with no appreciable increment in the computational cost. The technique consists in redefine over a given mesh the linear operator that arises from the discretization of a partial differential equation. This operator is modified according to an appropriate linear combination between the operators of the given mesh and of a coarse mesh, which must be a coarsening of the first one. On the other hand, Multigrid methods solve a linear system using systems of several sizes resulting from different discretization levels. This feature motivates the study of the application of the Multigrid strategy in conjunction with the Composite Mesh technique. In this work, we propose a scheme for solving the linear system arising from the Composite Mesh strategy using a Multigrid method. Particularly, we use a geometric version of the Multigrid technique. The proposal is based on a modification of the operators in some levels of the Multigrid algorithm in order to achieve both, the advantages of Multigrid for linear systems resolution and the solution improvement that could be achieved with the Composite Mesh strategy. Several elliptic test problems with analytical solution are presented, where the convergence rates are analyzed and the decrease in discretization errors is quantified.

Key words.
Composite Mesh, Multigrid, Discretization errors.

1. Introduction

In this work, an integration of the Multigrid (MG) technique with the Composite Mesh (CM) strategy is presented. In the CM method each component of the 'mixture' represents a mesh with different approximation error. Given two finite element meshes with nodes in common, an appropriate linear combination between the discrete operators that arise on each grid could give a better solution than the solutions obtained from each mesh individually without incrementing the computational cost [2]. The resultant linear system obtained with the CM technique can be solved by means of a direct method or an iterative smoothing method which, besides of being easy to implement, have the smoothing property, which allows the removal of high-frequency components of the error in the first iterations leaving practically unchanged the less oscillating ones. In order to modify the smoothing methods to reduce simultaneously all the components of the error, a suitable initial approximation to the solution for the relaxation scheme is needed. This approximation is obtained by performing some iterations on a coarse mesh because the process have fewer variables to be updated and, furthermore, it will have a convergence rate marginally improved. Assuming that a relaxation scheme has been applied until only the low-frequency components of the error are left, it is observed how these components look like on a coarser mesh.

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Since these components are less smooth in the coarse grid, then it will be necessary to perform a new relaxation in order to reduce them. Faced with the question of how to move from one mesh to the other one and relax, the developing of the MG technique was started [1, 4, 10, 15, 20, 21].

In this paper, an integration of the Geometric Multigrid and Composite Mesh methods is presented. The technique is applied to some elliptic test problems with exact solution on unstructured meshes, where discretization errors are analyzed.

2. Multigrid method

Let Ω a bounded domain in $\mathbb{R}^d$ with boundary $\partial\Omega$, $d=1,2,3$ being the spatial dimension. Consider the following differential equation with homogeneous boundary conditions

$$\begin{aligned} Lu &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

Assume that the operator L is self-adjoint, *i.e.* $(Lu,v)=(u,Lv)$ for any $u,v\in H\subset L_2(\Omega)$ and that is positive in the sense that $(Lu,u)>0$ for all $u\in H$, $u\neq 0$, where the subspace H contains smooth functions which vanish on $\partial\Omega$. With these properties, to solve the differential equation (2.1) is formally equivalent to minimize the quadratic functional

$$F(u)\equiv\frac{1}{2}(Lu,u)-(f,u),\quad u\in H \tag{2.2}$$

The problem can be rewritten in compact form as follows

$$u=\underset{v\in H}{\operatorname{argmin}}F(v) \tag{2.3}$$

which denotes finding the argument that minimizes F over all the functions in H.

Given a triangulation of Ω with size h, denoted by Ω_h, let H^h the finite subspace of H consisting on the functions v_h which are continuous in Ω, polynomial in each element and vanish on the boundary domain. The discrete problem is written as follows

$$u_h=\underset{v_h\in H^h}{\operatorname{argmin}}F(v_h) \tag{2.4}$$

Since the operator L is positive and self-adjoint, the problems of determining the function $u\in H$ which satisfies (2.3) and $(Lu,v)=(f,v)$ for all $v\in H$ are equivalent [4]. Thus, to solve (2.4) is equivalent to find $u_h\in H^h$ so that

$$(Lu_h,v_h)=(f,v_h)\quad \text{for all } v_h\in H^h \tag{2.5}$$

In order to solve (2.5) in its weak form, let ε_i^h the function of a base of H^h, with ε_i^h such that $\varepsilon_i^h(N_j)=\delta_{ij}$, N_j being the nodes of Ω_h. These basis functions are chosen as the test functions v_h. Then, after assembling all the rows of the matrix and their corresponding right sides, the resulting equation system $\mathbf{A}_h\mathbf{u}_h=\mathbf{f}_h$ is obtained.

Given two meshes, where the first is a homogeneous refinement of the second one, the two-grid iteration is the basis for building the MG method. In summary, this iteration is composed by the following steps:

- **Pre-smoothing step:** A few iterations of some iterative method such as Gauss-Seidel or damped Jacobi are performed in order to smooth the residue $\mathbf{r}_h = \mathbf{f}_h - \mathbf{A}_h \bar{\mathbf{u}}_h$, where $\bar{\mathbf{u}}_h$ is an approximation to the solution of the linear system. This step will reduce the high-frequency components of the error but no completely the low-frequency ones. The reduction of the oscillatory errors can be achieved carrying out local changes in the following way: $\bar{u}_h \leftarrow \bar{u}_h - s\varepsilon_i^h$, where $s \in \mathbb{R}$ is a suitable size step. The choice of s is carried out in the sense of minimizing the functional over all the possible choices, *i.e.*

$$s = \underset{t\in\mathbb{R}}{\operatorname{argmin}} F(\bar{u}_h - t\varepsilon_i^h), \qquad i = 1, 2, \ldots \tag{2.6}$$

- **Coarse grid correction:** The residue is projected on the coarse mesh, where the error equation minimizing its low-frequency components is solved. The abstract formulation for this process is the following: Let $H^H \subset H^h$ the space of the coarse grid, *i.e.* the set of piecewise polynomial functions associated with a standard coarse grid Ω_H. The goal is to correct the approximation $\bar{u}_h$ with a function $\bar{u}_H \in H^H$ that will approximate the new smooth error. The correction is $\bar{u}_h \leftarrow \bar{u}_h + \bar{u}_H$. The choice of $\bar{u}_H$ is made in order to obtain the best correction on the coarse mesh in a sense of functional minimizing, *i.e.*

$$\bar{u}_H = \underset{w_H \in H^H}{\operatorname{argmin}} F(\bar{u}_h + w_H) \tag{2.7}$$

- **Prolongation step:** The solution on the coarse grid is transferred to the fine mesh, where it is used to improve the smoother approximation of the first step.
- **Post-smoothing step:** Finally, the approximation computed in the last step is smoothed in order to remove any remaining high-frequency error component.

In the process of adding a function $\bar{u}_H \in H^H$ to a function $\bar{u}_h \in H^h$, appropriate coefficients must be found in order to write $\bar{u}_H$ as a function of H^h. Then, the prolongation operator I_H^h is sought to carry over $\bar{u}_h = I_H^h \bar{u}_H$ [4].

To determine the coarse grid operator $\mathbf{A}_H$ corresponding to the operator $\mathbf{A}_h$, it is necessary to work with nodal vectors and to transfer the minimization principle to matrix terms. Let N_i, $i = 1, \ldots, M$ the nodes of the mesh Ω_h, thus the system $\mathbf{A}_h \mathbf{u}_h = \mathbf{f}_h$ is equivalent to the minimization problem

$$\mathbf{u}_h = \underset{\mathbf{v}_h \in \mathbb{R}^M}{\operatorname{argmin}} F^h(\mathbf{v}_h) \tag{2.8}$$

where $F^h(\mathbf{v}_h) \equiv (\mathbf{A}_h \mathbf{v}_h, \mathbf{v}_h)/2 - (\mathbf{f}_h, \mathbf{v}_h)$. Similarly, the problem of the coarse mesh correction is equivalent to the matrix minimization principle[1]

$$\bar{\mathbf{u}}_H = \underset{\mathbf{w}_H \in \mathbb{R}^{\bar{M}}}{\operatorname{argmin}} F^h(\bar{\mathbf{u}}_h + \mathbf{I}_H^h \mathbf{w}_H) \tag{2.9}$$

But $F^h(\bar{\mathbf{u}}_h)$ is independent of $\mathbf{w}_H$ since $F^h(\bar{\mathbf{u}}_h + \mathbf{I}_H^h \mathbf{w}_H) = F^h(\bar{\mathbf{u}}_h) + F^H(\mathbf{w}_H)$. Therefore, to minimize $F^h(\bar{\mathbf{u}}_h + \mathbf{I}_H^h \mathbf{w}_H)$ is equivalent to minimize $F^H(\mathbf{w}_H)$ on the vectors $\mathbf{w}_H \in \mathbb{R}^{\bar{M}}$.

[1] Here we assume that the coarse mesh Ω_H have $\bar{M}$ nodes.

Then, carrying out the calculations $\mathbf{A}_H = (\mathbf{I}_H^h)^T \mathbf{A}_h \mathbf{I}_H^h$, is obtained the version of $\mathbf{A}_h$ in the coarse mesh. The matrix $(\mathbf{I}_H^h)^T$ takes a vector from the fine grid and leads to a vector in the coarse one, *i.e.* is a restriction operator. Hence, it makes sense having $\mathbf{I}_h^H = (\mathbf{I}_H^h)^T$. Finally, let $\mathbf{r}_h = \mathbf{f}_h - \mathbf{A}_h \bar{\mathbf{u}}_h$ and $\mathbf{f}_H = \mathbf{I}_h^H \mathbf{r}_h$.

The MG algorithm is obtained by applying the two grid iteration recursively. Specifically, the problem in the coarse mesh is solved by a two-grid iteration involving a coarser mesh. This procedure is applied at each mesh level until a coarse enough grid is obtained, in which the system could be solved by a direct method. A MG iteration from the finest grid to the coarser one and returning to the fine one, is called *cycle*. The exact structure of a cycle depends on the value of γ, the number of iterations of two grids in each intermediate step. The case $\gamma = 1$ is called *V-cycle*, while $\gamma = 2$ is called *W-cycle*.

3. Composite Mesh technique

The goal of this work is to solve linear problems arising from the application of the CM technique using the MG method. The composite finite element mesh applied to elliptic problems can be used to improve the numerical solution without an appreciable increment in the computational cost and also to estimate the discretization error [2, 16, 17]. In the h version, the method consists in replacing the discrete operator for a given mesh (fine mesh), by a linear combination of the operators computed by using that mesh and a coarser mesh with nodes in common with the first one. In this case, the interpolation polynomials retain the same degree in both meshes. Then, assuming that the fine mesh is obtained from the homogeneous refinement of a given grid, the connection between both meshes is forced in the shared nodes. The participation factor of each mesh in the compound model, *i.e.* the coefficient in the linear combination between the meshes, is introduced in such a way to minimize the discretization error.

Let Ω_H a discretization of the problem domain Ω and Ω_h the mesh obtained by a homogeneous refinement of Ω_H. Grids Ω_H and Ω_h have discretization sizes H and h, respectively. Applying the Finite Element Method (FEM) for the discretization of problem (2.1), the systems of equations $\mathbf{A}_h \mathbf{u}_h = \mathbf{f}_h$ and $\mathbf{A}_H \mathbf{u}_H = \mathbf{f}_H$ are obtained with the meshes Ω_h and Ω_H, respectively. Now, we define the discrete operator $\mathbf{A}_{Hh}$ as the matrix with the coefficients of $\mathbf{A}_H$ and the same size as $\mathbf{A}_h$ where, in order to obtain the required size, null coefficients are imposed for the nodes belonging to the fine mesh but not to the coarse mesh. In an analogous way, the vector $\mathbf{f}_{Hh}$ is defined. The approximate solution by the CM method $\mathbf{u}_{Hh}$ is obtained from the following system [17]

$$[\alpha \mathbf{A}_h + (1-\alpha)\mathbf{A}_{Hh}]\mathbf{u}_{Hh} = \alpha \mathbf{f}_h + (1-\alpha)\mathbf{f}_{Hh} \tag{3.1}$$

where the coefficient α depends on the regularity of the exact solution of the problem [2].

The asymptotic error of the FEM numerical approximation has the form [17]

$$\|u - u_h\| = Ch^p + \mathcal{O}(h^q) \tag{3.2}$$

where C is a constant and $q > p$. Then, an extrapolation analysis of the error leads to the following estimation [2, 17, 18]

$$\alpha = \frac{(H/h)^p}{(H/h)^p - 1} \tag{3.3}$$

The improvement introduced by the CM method with respect to the FEM solution is verified in the nodal values of the solution and, thus, it must be evaluated using a discrete norm of the error. Let N_i, $i=1,\ldots,M$ the nodes of the fine mesh Ω_h and H^h the discrete space associated with the mesh Ω_h. The interpolant $\pi_h u$ of u in the space H^h is defined as

$$\pi_h u(N_i) = u(N_i), \quad i=1,\ldots,M \tag{3.4}$$

Then, the solution of the system (3.1) is a better approximation to $\pi_h u$ in H^h than the u_h solution given by FEM. This fact is verified in the tests presented in section 5. For the tests, we use in the analysis the standard L_∞ norm ($\|\mathbf{a}\|_\infty = \max_i |a_i|$) and the euclidean norm ($\|\mathbf{a}\|_2 = \left(\sum_i a_i^2\right)^{1/2}$) for vectors. The nodal vector error $\mathbf{e}$ is computed as the difference $\mathbf{u}-\mathbf{u}^h$, where $\mathbf{u}\in\mathbb{R}^M$, $u_i = u(N_i)$ and $\mathbf{u}^h$ are the nodal values of some approximation to u in the H^h space.

4. Scheme of Multigrid with mesh composition

As pointed out above, the main objective of this paper is the integration of the MG and CM techniques for solving elliptic problems. Suppose that there are n levels involved at each MG iteration and that the corresponding mesh to the j-th level is obtained from the $(j-1)$-th homogeneous refinement of a given initial mesh (corresponding to level 1). Now, we introduce m mix levels, $1\le m<n$, in which the standard linear operators of the MG method are replaced by the linear combination proposed in the CM strategy. In the k-th mix level, $1\le k\le m$, the grids of levels $n-k+1$ and $n-k$ take part in the 'mixture'. In other words, for the level k, $\mathbf{A}_h = \mathbf{A}_{n-k+1}$, where $\mathbf{A}_{n-k+1}$ is the corresponding matrix of the level $n-k+1$, and $\mathbf{A}_{Hh}$ is obtained completing appropriately with zeros the matrix $\mathbf{A}_{n-k}$ in order to reach the dimension of $\mathbf{A}_{n-k+1}$ (see equation (3.1)). Figure 4.1 outlines the strategy.

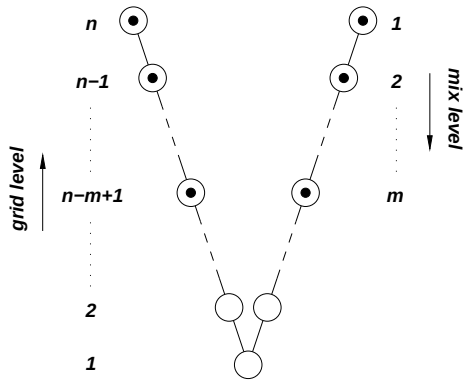

FIG. 4.1. *Multigrid for Composite Mesh (V-cycle).*

With the same set of parameters (smoother, number of pre- and post-smoothing steps, etc.) the resolution of each level increases the computational cost if in such a level the grid 'mixture' is introduced. This occur since the mixture makes to grow the matrix bandwidth of a given level. Therefore, the number of mix levels m should be kept at the minimum possible while keeping the full error reduction attainable by the CM technique. As will be shown in the numerical examples, the finer the (finest) mesh, the higher the decreasing in the nodal error produced by the CM strategy

with respect to FEM. Hence when the grid levels are increased, it is expected that the levels of mixing need to be increased because the FEM and CM nodal solutions departs away as the discretization size is decreased. We named the strategy presented MGCM, for Multigrid with Composite Mesh.

5. Test Problems

In order to show the effectiveness of the proposed MGCM strategy, we will consider three different 2D cases with analytical solution.

5.1. Poisson problem with constant coefficients

Consider the following problem of Poisson in a quadrangular domain

$$\begin{aligned} -\Delta u(x,y) &= f(x,y), \quad \text{in } \Omega = (0,1)\times(0,1) \\ u(x,y) &= 0, \quad \text{on } \partial\Omega \end{aligned} \tag{5.1}$$

The source term is such that the exact solution is given by the following expression

$$u(x,y) = 31250x(x-1)(x-\frac{3}{5})(x-\frac{1}{5})(x-\frac{9}{10})y(y-1)(y-\frac{4}{5})(y-\frac{1}{5})(y-\frac{1}{10})$$

In this case, the domain is discretized by means of an unstructured grid with 82 triangular elements and 52 nodes. This grid is the coarsest mesh of the problem and the grid sequence is obtained by homogeneous refinement. The parameters selected for this problem consist in three steps of pre and post smoothing of damped Jacobi with relaxation parameter 0.7. The participation factor for the mesh composition is obtained taking into account the regularity of the exact solution, giving $\alpha = 4/3$. The tolerance applied to the residue for the convergence of the linear system is 1×10^{-6}.

Error results in L_∞ and euclidean norms are presented in table 5.1 for V-cycle and in table 5.2 for W-cycle. The number of iterations to achieve the convergence, the number of grid levels used and a measure of the elapsed computational time are shown in each table. In the MGCM case, the number of mix levels utilized is also included in the tables. These tables show how the error decreases with the grid composition in both, infinity and euclidean norms. The increasing computational time of the MGCM method is due to the fact that the matrix of the system is less sparse than in the 'standard' MG case. Regarding the behavior of the proposed strategy in section 4 for solving the CM problem with MG, we conclude that the MGCM method preserves the features of the original Multigrid method. This assertion is based on that the number of iterations needed to reach the convergence in the cases with and without 'mixture' seems to remain constant. Figure 5.1 presents the norm of the residue as a function of the iteration number in the case of five grid levels for the V and W-cycles. As could be observed the curves for the MG and MGCM methods seem to be superimposed, with a slightly difference for the W-cycle.

For the V-cycle with six grid levels, two levels of mesh composition are used in order to achieve the full error reduction that the CM technique should give. Nevertheless, for the W-cycle only one mix level was sufficient in all the cases solved. In figures 5.2 and 5.3 the nodal error as a function of the discretization size (h) for both norms is presented. In this figures, we refer with FEM and CM as the MG and MGCM solutions, respectively. A noticeable feature is the error reduction rate of the CM technique.

REMARK 5.1. *Suppose we want to compute a solution by MG (FEM) with a nodal error such that its euclidean norm is 2.827×10^{-6}. Extrapolating the data from figure 5.2 it is concluded that it would be necessary to use a mesh with a discretization size $h \approx$*

MG					
grid levels	iterations	$\|\mathbf{e}\|_\infty$	$\|\mathbf{e}\|_2$	time [s]	mix levels
3	7	3.3960×10^{-3}	5.1424×10^{-2}	1.42×10^{-1}	-
4	7	9.5195×10^{-4}	2.5609×10^{-2}	3.42×10^{-1}	-
5	7	2.8171×10^{-4}	1.2796×10^{-2}	1.38	-
6	7	8.1310×10^{-5}	6.4094×10^{-3}	5.13	-
MGCM					
grid levels	iterations	$\|\mathbf{e}\|_\infty$	$\|\mathbf{e}\|_2$	time [s]	mix levels
3	7	1.1824×10^{-3}	4.9490×10^{-3}	9.26×10^{-2}	1
4	7	1.6160×10^{-4}	8.7621×10^{-4}	2.88×10^{-1}	1
5	7	2.1730×10^{-5}	1.5758×10^{-4}	1.83	1
6	7	2.7157×10^{-6}	5.1402×10^{-5}	5.09	1
6	8	2.8270×10^{-6}	2.8249×10^{-5}	5.73	2

TABLE 5.1. *Results of the V-cycle for the Poisson problem with constant coefficients.*

MG					
grid levels	iterations	$\|\mathbf{e}\|_\infty$	$\|\mathbf{e}\|_2$	time [s]	mix levels
3	6	3.3959×10^{-3}	5.1423×10^{-2}	1.26×10^{-1}	-
4	5	9.5185×10^{-4}	2.5605×10^{-2}	3.06×10^{-1}	-
5	5	2.8159×10^{-4}	1.2787×10^{-2}	1.29	-
6	5	8.1190×10^{-5}	6.3915×10^{-3}	6.08	-
MGCM					
grid levels	iterations	$\|\mathbf{e}\|_\infty$	$\|\mathbf{e}\|_2$	time [s]	mix levels
3	6	1.1824×10^{-3}	4.9485×10^{-3}	1.26×10^{-1}	1
4	6	1.6181×10^{-4}	8.7539×10^{-4}	3.47×10^{-1}	1
5	5	2.1912×10^{-5}	1.5506×10^{-4}	1.52	1
6	5	2.8553×10^{-6}	2.7476×10^{-5}	5.23	1

TABLE 5.2. *Results of the W-cycle for the Poisson problem with constant coefficients.*

2.17×10^{-4}. Assuming that the computational time t_C of the MG method is $\mathcal{O}(h^2)$, which could be obtained from data in tables 5.1 or 5.2, we obtain $t_C \approx 4.26\times10^{3}$ sec. for $h \approx 2.17\times10^{-4}$ in the V-cycle case. The same error level is reached with the MGCM strategy in 5.73 sec (742 times faster). Of course, if another norm is used the FEM solution could present an error less than that of CM solution. For example, in $L_2(\Omega)$ the error for both, FEM and CM solutions, is $\mathcal{O}(h^2)$.

5.2. Poisson problem with variable coefficients

In the second test, we consider the following elliptic problem with variable coef-

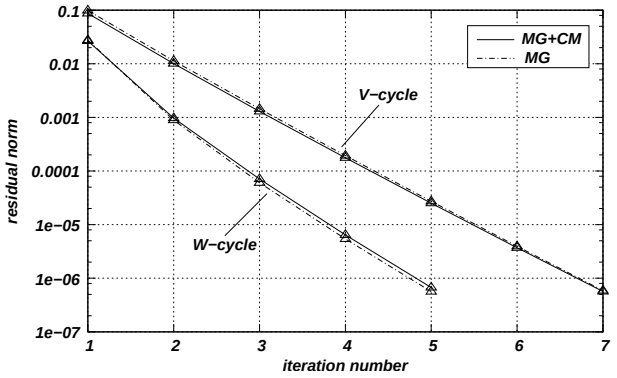

FIG. 5.1. *Residual norm as a function of the iteration number for the Poisson problem with constant coefficients with five grid levels.*

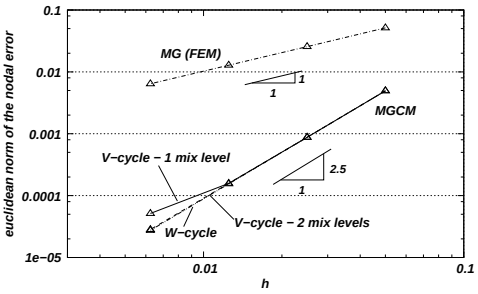

FIG. 5.2. *Euclidean norm of the nodal error as a function of the discretization size h for the Poisson problem with constant coefficients.*

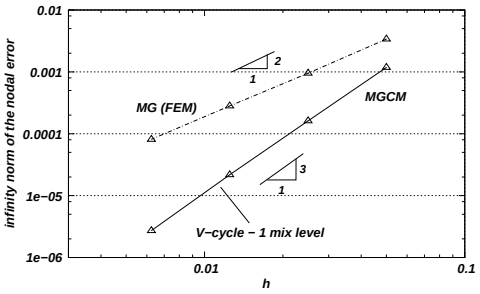

FIG. 5.3. *Infinity norm of the nodal error as a function of the discretization size h for the Poisson problem with constant coefficients.*

ficients in the unit square:

$$
\begin{aligned}
-\boldsymbol{\nabla}\cdot(\mu(x,y)\boldsymbol{\nabla}u) &= f(x,y), \quad \text{in } (0,1)\times(0,1)\\
u(0,y) &= 0, \quad 0\le y\le 1\\
u(x,0) &= \sin(5\pi x), \quad 0\le x\le 1\\
u(x,1) &= -\sin(5\pi x), \quad 0\le x\le 1\\
u(1,y) &= 0, \quad 0\le y\le 1
\end{aligned}
\tag{5.2}
$$

where $\mu(x,y)=1+xy^2$ and $u(x,y)=\sin(5\pi x)\cos(3\pi y)$. The mesh and the whole set of parameters are the same as the previous test.

In tables 5.3 and 5.4 we present the results obtained for the V-cycle and W-cycle, respectively. These tables show how the nodal error decreases with the 'mixture' in the infinity and euclidean norms. In this case the convergence rates of the proposed MGCM strategy and the standard MG method are similar, in particular when only one mix level is applied (see also figure 5.4).

Figures 5.5 and 5.6 show the euclidean and infinity norms of the nodal error as a function of the discretization size. In this case, is mandatory the use of two mix levels for the V-cycle with six grid levels in order to achieve the error reduction rate.

MG					
grid levels	iterations	$\|\mathbf{e}\|_\infty$	$\|\mathbf{e}\|_2$	time [s]	mix levels
3	7	9.5902×10^{-3}	1.3441×10^{-1}	7.52×10^{-2}	-
4	6	2.4065×10^{-3}	6.6750×10^{-2}	2.03×10^{-1}	-
5	6	6.0359×10^{-4}	3.3383×10^{-2}	9.09×10^{-1}	-
6	6	1.5232×10^{-4}	1.6833×10^{-2}	3.93	-
MGCM					
grid levels	iterations	$\|\mathbf{e}\|_\infty$	$\|\mathbf{e}\|_2$	time [s]	mix levels
3	7	1.6852×10^{-3}	1.1015×10^{-2}	7.66×10^{-2}	1
4	6	1.9785×10^{-4}	1.7776×10^{-3}	2.09×10^{-1}	1
5	6	2.4413×10^{-5}	3.3890×10^{-4}	9.66×10^{-1}	1
6	6	4.9948×10^{-6}	3.5801×10^{-4}	4.06	1
6	8	3.0296×10^{-6}	5.2584×10^{-5}	5.96	2

TABLE 5.3. *Results of the V-cycle for the Poisson problem with variable coefficients.*

MG					
grid levels	iterations	$\|\mathbf{e}\|_\infty$	$\|\mathbf{e}\|_2$	time [s]	mix levels
3	6	9.5901×10^{-3}	1.3441×10^{-3}	9.47×10^{-2}	-
4	5	2.4048×10^{-3}	6.6705×10^{-2}	2.68×10^{-1}	-
5	5	6.0180×10^{-4}	3.3285×10^{-2}	1.13	-
6	5	1.5048×10^{-4}	1.6634×10^{-2}	4.67	-
MGCM					
grid levels	iterations	$\|\mathbf{e}\|_\infty$	$\|\mathbf{e}\|_2$	time [s]	mix levels
3	6	1.6852×10^{-3}	1.1015×10^{-2}	9.54×10^{-2}	1
4	6	1.9806×10^{-4}	1.7835×10^{-3}	3.23×10^{-1}	1
5	5	2.4445×10^{-5}	3.0136×10^{-4}	1.14	1
6	5	3.0380×10^{-6}	5.4106×10^{-5}	4.28	1

TABLE 5.4. *Results of the W-cycle for the Poisson problem with variable coefficients.*

5.3. Laplace problem in a L-shaped domain

Let the Laplace problem in the L-shaped domain $\Omega=(-1,0)\times(-1,1)\cup(0,1)\times(0,1)$ where the exact solution is given in polar coordinates by $u(r,\phi)=r^{2/3}\sin\left(\frac{2}{3}\phi\right)$.

The mesh of the first level is an unstructured grid with 82 triangular elements and 52 nodes. Again, we use three steps of pre and post smoothing of damped Jacobi with factor 0.7, and a tolerance of 1×10^{-6} for the residual convergence. The participation factor for the mesh composition is obtained taking into account the regularity of the exact solution[2] as explained in [16], resulting $\alpha=2^{5/3}/(2^{5/3}-1)$.

The standard L_∞ and euclidean norms of the error are presented in tables 5.5 and 5.6 for the V-cycle and W-cycle, respectively. Again, a reduction of the error norm can be reached when the MGCM technique is applied. However, there is a decrease in the convergence rate of the MGCM method compared to MG since, except in the V-cycle case with six grid levels, the former one needs one additional iteration

[2] $u(r,\phi)\in H^{\frac{5}{3}-\epsilon}$ [6, 8].

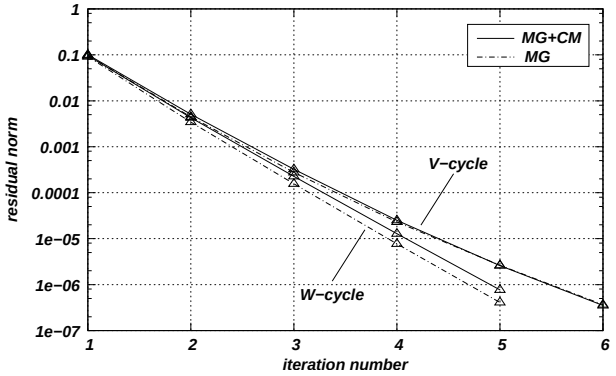

FIG. 5.4. *Residual norm as a function of the iteration number for the Poisson problem with variable coefficients using five grid levels.*

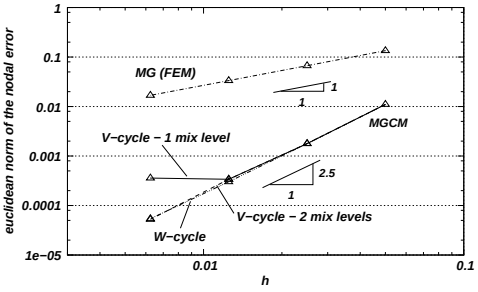

FIG. 5.5. *Euclidean norm of the nodal error as a function of the discretization size h for the Poisson problem with variable coefficients.*

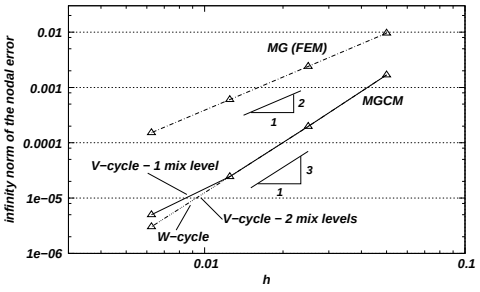

FIG. 5.6. *Infinity norm of the nodal error as a function of the discretization size h for the Poisson problem with variable coefficients.*

than the last one to reach the convergence. The relative reduction between MG and MGCM errors is smaller than in the previous examples due to the lower regularity in the exact solution. Because that the singularity of the analytical solution governs the error, the solutions of the MG and MGCM methods share the same error reduction rate in both, euclidean and infinity norms as could be noted from figures 5.7 and 5.8. Although the reduction in the error between the FEM and CM solutions seems to be small, consider for instance the value of h required to achieve a nodal error with a euclidean norm of 8.476×10^{-4} using MG (FEM). According to figure 5.7, we would need a mesh with $h \approx 2.0688 \times 10^{-3}$. But the computational time would amount to 109 sec.[3], *i.e.* 8.5 times more than the time required by the MGCM method.

6. Conclusions

In this work we present a scheme for solving a Composite Mesh problem using the Geometric Multigrid method. The matrix system and the right hand side vector of some levels in the MG algorithm are redefined introducing the proper definitions given by the CM technique. The proposed strategy was tested in two-dimensional elliptic problems with analytical solution. These test problems shown that the new strategy preserves the main features of both methods. First, using the correct number

[3]Here we assume that computational time behaves as $t_C \approx Ch^2$, where C is a constant.

MG					
grid levels	iterations	$\|\mathbf{e}\|_\infty$	$\|\mathbf{e}\|_2$	time [s]	mix levels
3	7	4.8705×10^{-3}	2.8198×10^{-2}	1.79×10^{-1}	-
4	7	3.0792×10^{-3}	2.2843×10^{-2}	7.93×10^{-1}	-
5	7	1.9430×10^{-3}	1.8408×10^{-2}	2.87	-
6	7	1.2253×10^{-3}	1.4858×10^{-2}	1.20×10^{1}	-
MGCM					
grid levels	iterations	$\|\mathbf{e}\|_\infty$	$\|\mathbf{e}\|_2$	time [s]	mix levels
3	8	3.3756×10^{-3}	8.9248×10^{-3}	2.16×10^{-1}	1
4	8	2.1293×10^{-3}	6.6774×10^{-3}	9.39×10^{-1}	1
5	8	1.3423×10^{-3}	5.0972×10^{-3}	3.43	1
6	7	8.4760×10^{-4}	4.1603×10^{-3}	1.28×10^{1}	1

TABLE 5.5. *Results of the V-cycle for the Laplace problem in a L-shaped domain.*

MG					
grid levels	iterations	$\|\mathbf{e}\|_\infty$	$\|\mathbf{e}\|_2$	time [s]	mix levels
3	7	4.8699×10^{-3}	2.8189×10^{-2}	2.58×10^{-1}	-
4	6	3.0783×10^{-3}	2.2813×10^{-2}	1.00	-
5	6	1.9418×10^{-3}	1.8324×10^{-2}	3.53	-
6	6	1.2239×10^{-3}	1.4652×10^{-2}	1.46×10^{1}	-
MGCM					
grid levels	iterations	$\|\mathbf{e}\|_\infty$	$\|\mathbf{e}\|_2$	time [s]	mix levels
3	8	3.3754×10^{-3}	8.9220×10^{-3}	2.99×10^{-1}	1
4	7	2.1290×10^{-3}	6.6688×10^{-3}	1.17	1
5	7	1.3418×10^{-3}	5.0748×10^{-3}	4.29	1
6	7	8.4546×10^{-4}	3.9145×10^{-3}	1.78×10^{1}	1

TABLE 5.6. *Results of the V-cycle for the Laplace problem in a L-shaped domain.*

of mix levels, it is possible to achieve error reductions similar to those obtained when the linear system arising from the application of the CM strategy is solved by a direct method. On the other hand, the convergence rates reached by the MGCM technique were close to those presented by the MG method, particularly when exact solution of the problem is smooth enough.

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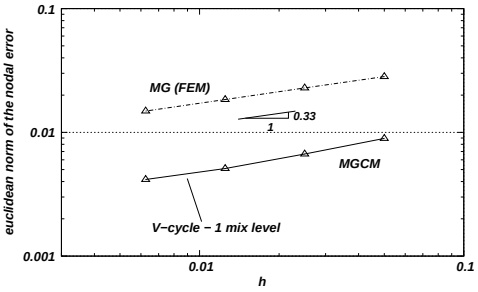

FIG. 5.7. *Euclidean norm of the nodal error as a function of the discretization size h for the Laplace problem in a L-shaped domain.*

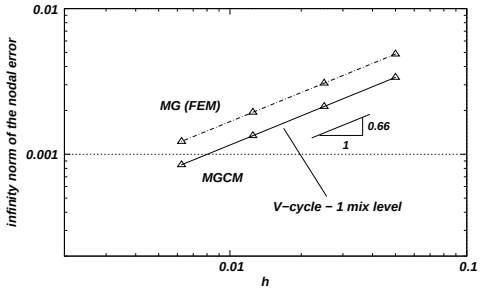

FIG. 5.8. *Infinity norm of the nodal error as a function of the discretization size h for the Laplace problem in a L-shaped domain.*

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